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Noncommutative Quantumelectrodynamics from Seiberg-Witten Maps to All Orders in $\theta^{\mu\nu}$

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Zusammenfassung

Die wichtigste Motivation dieser Arbeit war es eine nichtkommutative Erweiterung der Quantenelektrodynamik (QED) zu entwickeln, die auch für eine zeitartige Nichtkommutativität, das heißt Nichtvertauschbarkeit von Orts und Zeit Koordinaten, physikalisch interpretierbar bleibt.

Unser Modell basiert im Wesentlichen auf zwei Annahmen. Die erste Annahme hat mit der Raumzeit selbst zu tun und ist der Grund warum man von „nichtkommutativen“ Theorien spricht. Wir fordern, dass zwei Raumzeitkoordinaten nicht mehr miteinander kommutieren sollen. Diese nichtkommutative Raumzeit kann man nun dadurch realisieren, dass man in einer gegebenen Wirkung alle Punktprodukte durch Moyal-Weyl Sternprodukte ersetzt. Die nach dieser Ersetzung erhaltene Wirkung ist dann nicht mehr invariant unter der ursprünglichen, sondern unter der nichtkommutativen Eichtransformation.
In der zweite Annahme fordern wir, dass eben diese nichtkommutative Wirkung, die wir erhalten haben, nicht nur invariant unter nichtkommutativen sondern auch unter den gewöhnlichen Eichtransformationen sein soll. Dass die letzte Forderung tatsächlich Sinn macht und eine Wirkung existiert, die invariant unter beiden Eichtransformationen ist, zeigten Seiberg und Witten [1]. Der Grund warum man die zweite Annahme fordert, liegt auf der Hand. Man erhält eine nichtkommutative Eichtheorie, die aber die kommutativen Eichstrukturen aufweist. Um der zweiten Annahme zu genügen, muss man die Felder in der nichtkommutativen Wirkung durch die sogenannten Seiberg-Witten Abbildungen ersetzen. Nachdem man diese Wirkung eichfixiert hat, erhält man die Wirkung (2.31), auf der unser Modell basiert.

Wir wollen in dieser Arbeit das Hochenergieverhalten dieses Modells untersuchen. Deswegen ist es für unsere Zwecke nicht ausreichend, wenn die Wirkung nur bis zu einer endlichen Ordnung im nichtkommutativen Parameter $\theta^{\mu\nu}$ entwickelt ist. Wir benötigen eine Wirkung, in der alle Ordnungen von $\theta^{\mu\nu}$ resummiert sind. Das Moyal-Weyl Sternprodukt ist in allen Ordnungen in $\theta^{\mu\nu}$ bekannt. Das Problem vor dem wir standen war es, die benötigten Seiberg-Witten Abbildungen in allen Ordungen im nichtkommutativen Parameter zu finden. Diesem Problem widmeten wir uns in Kapitel 3. Basierend auf der Arbeit von Barnich, Brandt and Grigoriev [2] konnten wir diejenigen Seiberg-Witten Abbildungen in allen Ordnungen in $\theta^{\mu\nu}$ be-

stimmen, die nötig waren um den Streuprozess der Elektronen-Positronen Paar Vernichtung auf Born Niveau zu berechnen.

Aber bevor wir diese Berechnung in Angriff nahmen, untersuchten wir in Kapitel 4 die Seiberg-Witten Abbildung für das Eichfeld. Es stellte sich nämlich heraus, dass die Seiberg-Witten Abbildungen im Allgemeinen nicht eindeutig sind. Wie wir feststellten, führen diese Mehrdeutigkeiten tatsächlich zu unterschiedlichen Streuquerschnitten und somit zu unterschiedlichen Observablen. Was auf den ersten Blick als Nachteil erscheinen mag, beinhaltet aber auch eine Chance. Man kann diese Mehrdeutigkeiten dazu benutzen, um ein physikalisch sinnvolles Modell zu erstellen.

Basierend auf den Berechnungen aus Kapitel 3 und den Erkenntnissen aus Kapitel 4 bestimmten wir die Feynman Regeln, die zu diesem Modell gehören. Mit den Feynman Regeln berechneten wir dann in Kapitel 6 die Elektronen-Positronen Paar Vernichtung $e^-e^+ \to \gamma\gamma$ auf Born Niveau. Anhand dieses Streuprozesses untersuchten wir dann das Hochenergieverhalten (tree level unitarity) dieses Modells. Das Ergebnis war, dass das Modell, zumindest für diesen konkreten Prozess, „tree level“ unitär ist, bzw. gemacht werden kann. Die Vorderung nach Unitarität schränkte die Mehrdeutigkeit der Seiberg-Witten Abbildung des Eichfeldes teilweise ein. Trotz dieser Einschränkung der Mehrdeutigkeit blieb der differentielle Wirkungsquerschnitt divergent für hohe Schwerpunktsenergien. Aber die eigentliche physikalische Observable, nämlich der integrierte Wirkungsquerschnitt, wird konstant. Das heißt, dass man die Unschärfe in der Schwerpunktsenergie als auch die Unschärfe in den Impulsen berücksichtigen muss, um einen Wirkungsquerschnitt zu erhalten, der „tree level“ unitär ist.

Wir haben somit in dieser Arbeit eine nichtkommutative abelsche Eichtheorie mit Seiberg-Witten Abbildungen entwickelt, die in allen Ordnungen im nichtkommutativen Parameter resummiert ist. Anhand des Prozesses der Elektronen-Positronen Paar Vernichtung konnten wir zeigen, dass dieses Modell „tree level“ unitär ist.

Contents

Chapter 1

Introduction

Up to now there are four fundamental forces known, namely gravitation, electromagnetism, the weak and the strong force. The latter three are described by local quantum field theories which together form the so-called standard model of elementary particle physics. These quantum field theories are formulated on a static four dimensional Minkowski spacetime, where the dynamical variables are the quantized fields which are functions of the spacetime coordinates. In gravitation, which is described by the theory of general relativity, the spacetime coordinates themselves are the dynamical variables. Therefore the spacetime has a totally different meaning as in the standard model of elementary particle physics. In other words, in the standard model spacetime is the stage where the quantum fields interact, whereas in general relativity the spacetime is an actor itself.

Naturally the question arises how a theory may look like which combines general relativity and the principles of quantum theory. If we think of classical mechanics one quantizes this theory by introducing a non-vanishing commutation relation among the dynamical variables, which are the position and the momentum. This of course implies that the variables become suitable operators. Adapting this procedure to the theory of general relativity would lead to operator valued coordinates. The set of all those coordinates is called quantum or noncommutative spacetime. At this admittedly naive level one already sees a sign for a connection between noncommutative spacetime and quantum gravity.
In fact, it was shown that noncommutative spacetime is necessary in order to prevent a gravitational collapse from vacuum fluctuations near the Planck length [3] . So it seems that the spacetime has to become noncommutative near the Planck scale.

The string theory, which is a candidate for a theory of quantum gravity, also suggest that the spacetime cannot be smooth for arbitrary short distances. This leads to an uncertainty of the observable spacetime which can again be represented by operator-like spacetime coordinates. Furthermore Seiberg and Witten found that the dynamics of open strings ending on D-branes and which are embedded in a magnetic background field can be described by a quantum field theory on a noncommutative spacetime, too [1]. This discovery was the reason for the renewed attention to noncommutative field theories in 2000.

Taking all the above hints into account we could conclude that a noncommutative spacetime appears to be a general feature of any quantum theory of gravity. So it is reasonable to investigate quantum field theories on a noncommutative spacetime. But independently of what we have stated so far, studying noncommutative quantum field theories leads to a deeper understanding of the concepts of quantum field theories. This fact alone justifies the study of this kind of extension of quantum field theory.

Historically, the idea that the spacetime may be noncommutative goes back to Heisenberg. In a correspondence with Peierls 1930 [4] he proposed that it could be possible to circumvent the infinities present in quantum field theories by introducing a non-vanishing commutator between two spacetime points. In 1947, Snyder wrote a paper [5] where he presented an ansatz to introduce a minimal length by introducing noncommuting coordinates. Again, the motivation was to get rid of the infinities present in quantum electrodynamics. This ansatz was extended to curved spacetimes in the same year by Yang [6]. The problem of the ansatz was that it explicitly breaks the translational invariance of the theory. In addition at the same time renormalization theory succeeded by the experimental confirmation of Schwinger's calculation of the anomalous magnetic moment of the electron. Therefore this idea was not longer pursued.

On the mathematical side von Neumann was one of the first who studied the quantum "space" which is the phase space of quantum mechanics. Due to Heisenberg's uncertainty relation, points become meaningless so that one has to replace points by so called "Planck cells". Von Neumann called the geometry of such a space "pointless geometry" which was later on labeled as "noncommutative geometry". After some time this field was revived in the 1980's by Connes, Woronowicz and Drinfeld. Especially the work of Connes provides the mathematical background for the works on noncommutative quantum field theories.

In the 1990s Fredenhagen et.al. [7] and Filk [8] began to investigate field theories on quantum spaces. Wess et.al. also studied quantum spaces and deformed algebras [9]. The fact that noncommutative field theory got the attention it has nowadays is mainly due to the already mentioned paper of Seiberg and Witten where they showed that a kind of noncommutative gauge field theory appears as a low energy limit of certain string theories.

Now let us come to the often used words "noncommutative spacetime" and the question what this notion implies. In general "noncommutative spacetime" only means that two spacetime coordinates do not commute, i.e. their commutator is non-zero

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathbb{F}_{\mu\nu}(\hat{x})\,, \tag{1.1}$$

where the function $\mathbb{F}_{\mu\nu}$ can be an arbitrary function of the coordinate $\hat{x}_\mu$. The coordinates are self-adjoint operators so that the spacetime manifold becomes a Hilbert space of states where the union of the spectra of all the spacetime operators creates the observable spacetime. Obviously, if the spectrum of the operators is discrete or continuous one also gets a discrete or continuous space.

Usually three classes of deformed spacetimes can be distinguished, namely the canonical or θ-deformed spacetime with a constant r.h.s. of the commutation relation, a Lie algebra deformed spacetime where the r.h.s. depends linearly on the coordinate and a q-deformed spacetime with a quadratic dependency on the coordinate. The most simple one is a constant r.h.s. of the commutation relation

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}\,\hat{\Theta}_{\mu\nu}\,, \tag{1.2a}$$

where we extract the imaginary unit for convenience. This kind of noncommutative spacetime forms the basis of our work. In addition we assume that the nested commutator of the $\hat{x}$ vanishes

$$\big[[\hat{x}_\mu, \hat{x}_\nu], \hat{x}_\rho\big] = \mathrm{i}\big[\hat{\Theta}_{\mu\nu}, \hat{x}_\rho\big] = 0\,. \tag{1.2b}$$

This additional assumption has an important consequence as we will explain in the next chapter. The introduction of a non-vanishing commutation relation (1.2a) poses the question wether theories based on this relation break Lorentz invariance. The answer is somewhat subtle. If we Lorentz transform the whole noncommutative spacetime, the matter content as well as the observer himself, the physics remains unchanged. Simply because the matter content remains constant relative to the spacetime and the observer. In terms of this so-called "observer" Lorentz transformation the Lorentz invariance is satisfied. Instead if one only transforms the matter content the physics changes in respect of the observer. In terms of this so-called "particle" Lorentz transformation the Lorentz invariance is broken.

Nevertheless by assuming the relations (1.2) it is clear that the ordinary quantum field theory has to be changed. So the question arises how a quantum field theory may looks like on a noncommutative spacetime. The decisive answer is given by the Gel'fand-Naimark theorem [10]. In substance the theorem states that the algebra of functions on a manifold includes all information about the manifold itself. Therefore it is completely sufficient to consider only the algebra of functions on a manifold. This means that in the case of a noncommutative spacetime we have a noncommuting algebra of functions rather than a commuting algebra.

So how can a noncommuting algebra of functions be realised? The first way, which was mainly developed by Fredenhagen et.al. [3] was to deal with fields as functions of the noncommutative spacetime coordinates. They worked out a mathematical framework where they give answer to questions concerning for example the noncommutative analogue of the four dimensional spacetime integration.

The other way, done by Filk [8], is based on the fact that the product of functions of noncommuting variables can be realised by a noncommutative product of functions of commuting variables. This leads to fields as a function of the ordinary commuting coordinates but with a deformed product. For the case of a constant commutator between spacetime coordinates, this deformed product becomes the well known Moyal-Weyl star-product

$$(\phi_1 * \phi_2)(x) = \phi_1(x)\,\mathrm{e}^{\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu}\,\phi_2(x)\,. \tag{1.3}$$

A different ansatz arose in the 2000s, where Chaichian et.al. assumed a twisted Poincaré symmetry as the fundamental symmetry of the spacetime [11]. A twisted Poincaré symmetry is the symmetry of the twisted product of two Poincaré groups, whereas a twisted product, is essentially a direct product of groups. One ends up with a twist product which for the simplest case is again the Moyal-Weyl star-product. Therefore the non-vanishing commutator of two spacetime coordinates is obtained as the consequence of the assumed quantum spacetime symmetry.

In this work we will follow the ansatz of Filk meaning that in the first step we create our quantum field theory on a noncommutative spacetime by simply replacing the point-like products between the fields by an appropriate deformed product. Because in this work we consider only the θ-deformed spacetime the deformed product is just the already mentioned Moyal-Weyl star-product (1.3). Therefore the noncommutative action is the ordinary one but with the star-product instead of the ordinary product.

In the case of gauge theories this leads to deformed gauge transformations because one also has the star-product between the gauge parameter and the gauge field. One problem with such theories is to construct an appropriate noncommutative extension of the standard model which is not in contradiction to current observations. For example in such a noncommutative quantum electrodynamics (NCQED) only the charges ± 1 and 0 are allowed which of course would forbid quarks [12, 13]. A more grave problem of all noncommutative theories which are in all orders in the noncommutative parameter is the existence of UV/IR-mixing [14]. UV/IR-mixing is a phenomenon which appears in loop diagrams of noncommutative theories. Namely one gets as a result in one-loop calculations besides the ordinary UV divergent parts also UV finite parts which are instead divergent in the limit of zero momenta. This UV/IR-mixing is the reason why those theories are not renormalizable at least in the usual sense [15].

Studying a low energy limit of some string theories, Seiberg and Witten discovered that one gets ordinary and noncommutative fields by regularizing a two-dimensional field theory in different ways. Thus, they conclude that there has to exist a transformation between these two theories in such a way that the standard Yang-Mills gauge invariance maps to the gauge invariance of noncommutative Yang-Mills theory. In a first step one may try to redefine the gauge field and the gauge parameter separately from each other. But if this would be possible, the noncommutative gauge group has to be isomorphic to the commutative one. But the gauge group for the commutative abelian gauge theory is abelian while the gauge group for the noncommutative abelian gauge theory is nonabelian. An abelian group can never be isomorphic to an nonabelian group. Therefore one must simultaneously redefine the gauge field and the gauge parameter in a consistent manner.

There are two ways known up to now how to get the redefined gauge field and the gauge parameter. First, as has been done by Seiberg and Witten [1], one can extract the redefined gauge field and gauge parameter which depend on the noncommutative parameter order by order of the noncommutativity simply by comparing the noncommutative with the ordinary gauge transformation. If one has the first order in the redefined field and parameter one can reinterpret these maps as the generating functional (with respect to a change of the noncommutativity) of the exact, i.e. the all order, maps. This results in differential equations which describe how the redefined gauge field and gauge parameter should change when the noncommutativity is varied. These so called Seiberg-Witten maps were used by Wess et.al. [16] to build a noncommutative standard model (NCSM) in first order in the noncommutative parameter. They circumvent the problem of integer charges by using the freedom introduced by the maps. Because up to now the Seiberg-Witten maps were known only up to the first order in the noncommutative parameter, the NCSM was only an effective theory so that one couldn't tackle questions such as high energy behavior, unitarity, renormalizability and so on.

The second way to get this differential equations was taken by Barnich, Brandt and Grigoriev [2]. They construct noncommutative Yang-Mills theory as a consistent deformation of standard Yang-Mills theory. Consistent deformation means, that the action and the gauge transformation are not deformed independently of each other. They are deformed in such a way that the deformed action is invariant under the deformed gauge transformation. They used the antifield-antibracket formalism to get the differential equations Seiberg and Witten found. The advantage of their work was the construction of recursive solutions for the maps not only with respect to the noncommutative parameter but also with respect to the gauge field. The last one is so important because this recursion supplies a map in all orders in the noncommutative parameter.

Having a noncommutative theory with Seiberg-Witten maps in all orders in the noncommutative parameter one can study questions which could not be answered in an effective theory. And exactly those questions underlie this thesis: What is the high energy behavior of the theory? Is this theory unitary?

At the end of this introduction we want ask if it is possible to measure effects predicted from noncommutative theories in the near future. Up to now no measurements exist which verify any prediction coming from noncommutative theories. From Large Electron Positron (LEP) collider experiments the noncommutative scale has to be at least $\gtrsim 140\,\mathrm{GeV}$ [17]. Phenomenological studies for the Large Hardron Collider (LHC) can push the exclusion limit above the TeV scale [18]. But generically one has to expect the noncommutative scale near the Planck scale which is in four-dimensional spacetime of the order of $10^{19}\,\mathrm{GeV}$. The only hope one can have to see noncommutative effect in the TeV region is the existence of extra dimensions. Such extra dimensional models can push the Planck scale down to the TeV region which is reachable by the LHC.

Anyway, the purpose of our investigation is to clarify basic properties of the noncommutative theory with Seiberg-Witten maps, necessary for a consistent quantum mechanical interpretation of these models.

Chapter 2

Technical Basics

In this chapter we want to prepare the technical basics needed in this thesis. Namely we will first look at how an explicit expression for the Moyal-Weyl star-product can be obtained. Secondly, we will ask whether the different, but in the commutative case equivalent, perturbation series expansions of the coupling constant are equivalent in the noncommutative case. Then we will construct a noncommutative action which is invariant under the noncommutative as well as under the commutative gauge transformation. Afterwards we will discuss how different representations of the gauge generators affect the model. Before we will present our final model we will elaborate on the question how the nonlocal property of the noncommutative model affects the time-ordering operator.

2.1 The Moyal-Weyl Star-Product

As we already mentioned in the introduction, we take a canonical noncommutativity as the basis of our model , i.e. a constant commutator. The two commutation relations (1.2) define an algebra $\mathcal{A}$ of all selfadjoint operators $\hat{x}_\mu$ of a Hilbert space $\mathcal{H}$. Because of the second equation (1.2b), $\hat{\Theta}_{\mu\nu} \in \mathcal{A}$ is in the center of this algebra. Hence, for a given irreducible representation of $\hat{\Theta}_{\mu\nu}$ one can replace that operator valued matrix by a pure c-number valued matrix

$$\hat{\Theta}_{\mu\nu} = \theta_{\mu\nu} \cdot \mathbb{1} \,, \quad \text{with } \mathbb{1} \in \mathcal{A} \,. \tag{2.1}$$

This matrix $\theta_{\mu\nu}$ has the dimension of an inverse squared length which sets the length scale where noncommutative effects become important.

After having fixed the commutation relation of the spacetime operators we have to ask how we can incorporate the noncommutative spacetime in a quantum field theory. As mentioned, we will follow Filk [8] who stated that instead of considering products of fields which are functions of the noncommutative spacetime operators one can study deformed noncommutative products of ordinary fields, equivalently. In our case of a canonical noncommutativity the deformed product is given by the Moyal-Weyl star-product.

Let us derive an explicit representation of this product following the discussion in [19]. In order to be able to tackle the question how the deformed noncommutative product looks like, we have to construct a field which is a function of the noncommutative hermitian spacetime operators. We do this by means of Weyl quantization of the ordinary field. This looks like a twofold Fourier transformation whereas one transformation includes the noncommutative hermitian spacetime operators

$$\begin{aligned}\phi(\hat{x}) := \frac{1}{(2\pi)^4}\int \mathrm{d}^4p \int \mathrm{d}^4x \; \mathrm{e}^{\mathrm{i}\,p\hat{x}}\,\mathrm{e}^{-\,\mathrm{i}\,px}\,\phi(x) \\ = \frac{1}{(2\pi)^4}\int \mathrm{d}^4x \int \mathrm{d}^4p \; \mathrm{e}^{-\,\mathrm{i}\,px}\,\hat{T}(p)\phi(x)\,. \quad (2.2)\end{aligned}$$

The operator $\hat{T}(p) := \exp(\mathrm{i}\,p\hat{x})$ we introduced above has some useful properties, namely

$$\hat{T}^\dagger(p) = \hat{T}(-p)\,, \quad (2.3a)$$

$$\hat{T}(p)\hat{T}(q) = \mathrm{e}^{-\,\mathrm{i}\,p\wedge q}\,\hat{T}(p+q)\,, \quad (2.3b)$$

$$\mathrm{tr}\,\hat{T}(p) = (2\pi)^4\delta^{(4)}(p)\,, \quad (2.3c)$$

with the antisymmetric "wedge"-product

$$p\wedge q := \frac{1}{2}p_\mu\theta^{\mu\nu}q_\nu = -q\wedge p\,. \quad (2.4)$$

The $\wedge$-product is of course antisymmetric because of the antisymmetry of $\theta^{\mu\nu}$. The first property is obvious. The second one follows from the Baker-Campbell-Hausdorff formula and the property (1.2b) of the algebra we consider, i.e. the nested commutator of the hermitian operators vanishes. The third property (2.3c) is not as obvious as the other two but the expression is not really surprising. Namely, the trace over elements $\hat{x}_\mu$ of the Hilbert space $\mathcal{H}$ corresponds to the integral over the spacetime in Minkowski space

$$\mathrm{tr}\,\hat{T}(p) \,\widehat{=} \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}\,px} = (2\pi)^4\delta^{(4)}(p)\,. \quad (2.5)$$

Nevertheless we will derive this expression in appendix B.1.

Now we are able to calculate the trace of a noncommutative field

$$\mathrm{tr}\,\phi(\hat{x}) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4x \int \mathrm{d}^4p \; \mathrm{e}^{-\,\mathrm{i}\,px}\,\phi(x)\,\mathrm{tr}\,\hat{T}(p) = \int \mathrm{d}^4x\,\phi(x)\,, \quad (2.6)$$

where we used the above trace property (2.3c). In principle, one has to be careful in the first step, where one swaps the trace and the integral. But these two commute as stated in [19].

After all the work we have done so far we only need one last step to be able to determine the deformed product, namely the inverse map of (2.2)

$$\phi(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \; \mathrm{e}^{\mathrm{i}\, px} \,\mathrm{tr}\left[\phi(\hat{x})\hat{T}^\dagger(p)\right] . \tag{2.7}$$

With this map we define the Moyal-Weyl product as the product built up from the point-wise product of two functions of the hermitian variables

$$\begin{aligned}
\left(\phi_1 * \phi_2\right)(x) :=& \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \; \mathrm{e}^{\mathrm{i}\, px} \,\mathrm{tr}\left[\phi_1(\hat{x})\phi_2(\hat{x})\hat{T}^\dagger(p)\right] \\
=& \frac{1}{(2\pi)^8} \int \mathrm{d}^4 p \int \mathrm{d}^4 p_1 \int \mathrm{d}^4 p_2 \; \mathrm{e}^{\mathrm{i}\, px}\, \tilde{\phi}(p_1)\tilde{\phi}(p_2) \,\mathrm{tr}\left[\hat{T}(p_1)\hat{T}(p_2)\hat{T}^\dagger(p)\right] \\
=& \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p_1 \int \mathrm{d}^4 p_2 \; \mathrm{e}^{\mathrm{i}(p_1+p_2)x}\, \tilde{\phi}(p_1)\tilde{\phi}(p_2)\, \mathrm{e}^{-\,\mathrm{i}\, p_1 \wedge p_2} .
\end{aligned} \tag{2.8}$$

As usual, the tilded functions are the Fourier transforms of the untilded function. Actually, the position space representation of the $*$-product is the most commonly used one. It is just the Fourier transform of the above result

$$\left(\phi_1 * \phi_2\right)(x) = \phi_1(x)\, \mathrm{e}^{\frac{\mathrm{i}}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu}\, \phi_2(x) . \tag{2.9}$$

Let us discuss our result. First of all, one can directly derive from (2.8) that

$$\int \mathrm{d}^4 x\, (\phi_1 * \phi_2)(x) = \int \mathrm{d}^4 x\, \phi_1(x)\phi_2(x) . \tag{2.10}$$

This leads to one important feature of noncommutative field theories, namely that the free part of such a theory is equal to the commutative one.
Second, the generalisation of the Moyal-Weyl product is straight-forward for the case of n fields, which is

$$\begin{aligned}
\left(\phi_1 * \phi_2 * \ldots * \phi_n\right)(x) :=& \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \; \mathrm{e}^{\mathrm{i}\, px} \,\mathrm{tr}\left[\phi_1(\hat{x})\phi_2(\hat{x})\ldots\phi_n(\hat{x})\hat{T}^\dagger(p)\right] \\
=& \prod_{i=1}^{n}\left[\frac{1}{(2\pi)^2}\int \mathrm{d}^4 p_i \; \mathrm{e}^{\mathrm{i}\, p_i x}\, \tilde{\phi}_i(p_i)\right] \mathrm{e}^{-\,\mathrm{i}\,\varphi(p_1,p_2,\ldots,p_n)} ,
\end{aligned} \tag{2.11}$$

with the noncommutative phase

$$\varphi(p_1, p_2, \ldots, p_n) := \sum_{i<j} p_i \wedge p_j . \tag{2.12}$$

In the general case, the property (2.10) leads to a cyclical symmetry of the $*$-product

$$\int \mathrm{d}^4 x\, \left(\phi_1 * \phi_2 * \ldots * \phi_n\right)(x) = \int \mathrm{d}^4 x\, \left(\phi_2 * \ldots * \phi_n * \phi_1\right)(x) , \tag{2.13a}$$

such that one can replace one of the $*$-products by the ordinary one

$$\int \mathrm{d}^4x \; (\phi_1 * \ldots * \phi_n)(x) \\ = \int \mathrm{d}^4x \; (\phi_1 * \ldots * \phi_i)(x)\,(\phi_{i+1} * \ldots * \phi_n)(x) \,. \quad (2.13b)$$

In the case of field theories one can, roughly speaking, neglect *one* $*$-product for each summand in the action.

2.2 Perturbation Theory

In ordinary local quantum field theory there exists more than one way to realise a perturbation solution. One way is given by the Hamiltonian approach which starts with the interaction Hamiltonian and leads to the Dyson series. This approach is manifestly unitary. By Legendre transformation the Dyson series merges into the Gell-Mann-Low formula

$$\langle 0|\,\mathrm{T}\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}t\, H_\mathrm{I}}\,|0\rangle = \langle 0|\,\mathrm{T}\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, \mathcal{H}_\mathrm{I}}\,|0\rangle = \langle 0|\,\mathrm{T}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_\mathrm{I}}\,|0\rangle \;, \quad (2.14)$$

which finally depends on the interaction Lagrangian. The crucial point is that the Legendre transformation is only defined for finite many time derivatives[1]. The reason is that one needs the conjugate momenta of the field. However, it is completely unclear how to define the conjugate momenta of a field with infinitely many derivatives. Another way to realise a perturbation solution is the Yang-Feldman ansatz which basically solves the interacting field equations perturbatively.

The question is if these different ways remain equivalent in nonlocal theories. So far a final answer cannot be given. Hence, one has to make a decision what ansatz one wants to choose. We will base our perturbation series on the Gell-Mann-Low formula

$$S(i \to f) = \langle f|\,\mathrm{T}\left[e^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_I(x)}\right]|i\rangle \;, \quad (2.15)$$

which contains the interaction part $\mathcal{L}_I$ of the Lagrange density $\mathcal{L}$.

We will consider a version of the noncommutative quantum electrodynamics (QED), because it is the simplest gauge theory. And this is essential for our considerations.

[1] In the case of higher order time derivatives one can eliminate all higher order time derivatives by recursively applying the equation of motion [20].

2.3 Lorentz Invariance

If one introduces a non-vanishing commutation relation such as (1.2), the question arises how the Lorentz invariance of a theory is affected by this assumption. In principle there are three possibilities how $\hat{\Theta}^{\mu\nu}$ can behave under Lorentz transformations.

1. The noncommutative parameter is an invariant tensor under Lorentz transformation
$$\Lambda^{\mu}_{\mu'}\Lambda^{\nu}_{\nu'}U\hat{\Theta}^{\mu'\nu'}U^{-1} = \hat{\Theta}^{\mu\nu} .$$
This implies that the elements of the tensor have to be operator valued objects which is not compatible with (2.1).

2. $\hat{\Theta}^{\mu\nu}$ is a constant matrix. This would mean that the noncommutative parameter would be constant in every coordinate system. Thus the observables would depend on the Lorentz frame in which one does the calculations. Obviously, this makes no sense for a physically meaningful theory.

3. The last possibility assumes that $\hat{\Theta}^{\mu\nu}$ is a Lorentz tensor, so that the noncommutativity transforms like a tensor. Relative to the whole spacetime $\hat{\Theta}^{\mu\nu}$ is constant and as a result it implicitly breaks the ("particle") Lorentz invariance of the action.

Under the assumption (2.1) the only reasonable choise is to assume that $\hat{\Theta}^{\mu\nu}$ and thus $\theta^{\mu\nu}$ is a Lorentz tensor. Thus one can contract $\theta^{\mu\nu}$ with two 4-vectors to obtain a scalar. But for phenomenological studies one has to assign fixed values to $\theta^{\mu\nu}$, which then breaks Lorentz invariance.

2.4 The Noncommutative Action

After we have derived a representation of the Moyal-Weyl star-product and clarified which ansatz of perturbation expansion we will use, we now want to construct the fundamental action of our theory.

2.4.1 Ordinary Quantum Electrodynamics

Before we come to the action of the noncommutative quantum electrodynamics we first summarize some properties of the ordinary quantum electrodynamics because the noncommutative action is based on the commutative one.

The action of quantum electrodynamics (QED) is

$$S = \int \mathrm{d}^4x \left[\bar{\psi}(x)\left(\mathrm{i}\,\not{D}(x) - m\right)\psi(x) - \frac{1}{4g^2}F^{\mu\nu}(x)F_{\mu\nu}(x)\right] , \tag{2.16}$$

which is invariant under the local abelian gauge transformation

$$\begin{aligned} \delta_a\psi(x) &= \mathrm{i}\, a(x)\psi(x)\,, \\ \delta_a\bar{\psi}(x) &= -\,\mathrm{i}\,\bar{\psi}(x)a(x)\,, \\ \delta_a A_\mu(x) &= \partial_\mu a(x) - \mathrm{i}[A_\mu(x), a(x)] = \partial_\mu a(x)\,, \\ \delta_a F_{\mu\nu}(x) &= \mathrm{i}[a(x), F_{\mu\nu}(x)] = 0\,, \end{aligned} \tag{2.17}$$

where $a(x)$ is the spacetime dependent gauge parameter. The covariant derivative and the field-strength tensor are defined as usual

$$D_\mu = \partial_\mu - \mathrm{i}\, A_\mu\,, \qquad F_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{2.18}$$

Note that the covariant derivative satisfies the Jacobi identity

$$\big[D_\mu, [D_\nu, D_\rho]\big] + \big[D_\nu, [D_\rho, D_\mu]\big] + \big[D_\rho, [D_\mu, D_\nu]\big] = 0 \tag{2.19}$$

and as a consequence of that identity the field-strength tensor satisfies the Bianchi identity

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0\,. \tag{2.20}$$

Based on the above action (2.16) we will now construct the noncommutative action.

2.4.2 Noncommutative Quantum Electrodynamics

After introducing the noncommutative algebra (1.2) for the coordinates, the action becomes a product of fields of these hermitian spacetime operators. However, as we mentioned in section 2.1 one gets an equivalent noncommutative action by replacing every ordinary product by the Moyal-Weyl star-product which leads to the action

$$S^* = \int \mathrm{d}^4x \left[\left(\bar{\hat{\psi}} * (\mathrm{i}\,\hat{\not{D}} - m) * \hat{\psi}\right)(x) - \frac{1}{4g^2}\,\mathrm{tr}\left[\left(\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu}\right)(x)\right]\right]\,. \tag{2.21}$$

Consequently, this action is now invariant under the noncommutative gauge transformation

$$\begin{aligned} \hat{\delta}_{\hat{a}}\hat{\psi}(x) &= \mathrm{i}\left(\hat{a} * \hat{\psi}\right)(x)\,, & \hat{\delta}_{\hat{a}}\bar{\hat{\psi}}(x) &= -\,\mathrm{i}\left(\bar{\hat{\psi}} * \hat{a}\right)(x)\,, \\ \hat{\delta}_{\hat{a}}\hat{A}_\mu(x) &= \partial_\mu\hat{a}(x) - \mathrm{i}[\hat{A}_\mu \overset{*}{,} \hat{a}](x)\,, & \hat{\delta}_{\hat{a}}\hat{F}_{\mu\nu}(x) &= \mathrm{i}[\hat{a} \overset{*}{,} \hat{F}_{\mu\nu}](x)\,, \end{aligned} \tag{2.22}$$

where the graded star commutator $[\cdot \overset{*}{,} \cdot]$ is defined as

$$[A \overset{*}{,} B] = A * B - (-1)^{|A||B|} B * A\,. \tag{2.23}$$

The ghost field C has an odd Grassmann parity $|C| = 1$. All other fields we deal with are Grassmann even, i.e. the parity of them is zero.

Due to the star-product the defining commutator of the field-strength tensor is now also a star commutator

$$\hat{F}_{\mu\nu} = \mathrm{i}[\hat{D}_\mu \stackrel{*}{,} \hat{D}_\nu] = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}[\hat{A}_\mu \stackrel{*}{,} \hat{A}_\nu]\,, \tag{2.24}$$

so that this commutator of two gauge fields doesn't vanish although we have abelian gauge fields. As an important consequence of this non vanishing commutator such theories generally possess couplings among photons, i.e. there is a 3- and 4-photon vertex. As in the commutative case, the Jacobi and the Bianchi identities are satisfied in the noncommutative case.

2.4.3 Seiberg-Witten maps

Based on the noncommutative action (2.21) one could proceed by gauge fixing this action ending up with a noncommutative QED which is invariant under noncommutative BRST transformations [21, 22]. The same holds also for arbitrary $U(N)$ gauge theories all of which have the drawbacks mentioned in the introduction. For a purely space-like noncommutativity ($\theta_{0i} = 0$) this class of theories doesn't possess any problems with unitarity [23]. However, in the time-like case ($\theta_{0i} \neq 0$) which we are interested in, it turns out that those theories cannot be simultaneously gauge invariant and unitary at least with a perturbation series based on the Gell-Mann-Low formula[2].

We will postulate an additional assumption which our noncommutative theory has to possess and which was first considered by Seiberg and Witten [1]. Namely, they discovered that one gets ordinary and noncommutative fields[3] by regularizing a two-dimensional field theory in different ways. Thus, they conclude that there has to exist a transformation between these two fields in such a way that the standard Yang-Mills gauge invariance maps to the gauge invariance of noncommutative Yang-Mills theory. This transformation is not just a redefinition of the fields. The reason is simple, namely in the case of QED the commutative gauge group is abelian (2.17) whereas the noncommutative one is nonabelian (2.22). An abelian group can never be isomorphic to a noncommutative one. Instead one must simultaneously redefine the gauge field and the gauge parameter in a consistent manner.

With the above said, we assume in addition to the fundamental assumption (1.2) which leads to the action (2.21) the additional postulation that the above action should not only be invariant under the noncommutative gauge transformation (2.22) but also under the commutative one (2.17). So there must exist maps, known as Seiberg-Witten maps,

$$\hat{\psi}(\psi, A)\,, \qquad \hat{\bar{\psi}}(\bar{\psi}, A)\,, \qquad \hat{A}_\mu(A)\,, \qquad \hat{F}_{\mu\nu}(F, A)\,, \qquad \hat{a}(a, A)\,, \tag{2.25}$$

[2] There is also a proposal [24] to get an unitary and gauge invariant result by solving the equations of motion perturbatively (Yang-Feldman ansatz).

[3] By a "noncommutative field" we mean a field which transforms under the noncommutative gauge transformation and by an "ordinary field" we mean a field which transforms under the commutative gauge transformation, respectively.

in such a manner that the gauge equivalence equations for each of the fields

$$\begin{aligned}
\hat{\delta}_{\hat{a}(a,A)}\hat{\psi}(\psi, A) &\stackrel{!}{=} \delta_a\hat{\psi}(\psi, A)\,, \qquad \hat{\delta}_{\hat{a}(a,A)}\bar{\hat{\psi}}(\bar{\psi}, A) \stackrel{!}{=} \delta_a\bar{\hat{\psi}}(\bar{\psi}, A)\,, \\
\hat{\delta}_{\hat{a}(a,A)}\hat{A}_\mu(A) &\stackrel{!}{=} \delta_a\hat{A}_\mu(A)\,, \qquad \hat{\delta}_{\hat{a}(a,A)}\hat{F}_{\mu\nu}(F, A) \stackrel{!}{=} \delta_a\hat{F}_{\mu\nu}(F, A)\,, \\
\hat{\delta}_{\hat{a}(a,A)}\hat{a}(a, A) &\stackrel{!}{=} \delta_a\hat{a}(a, A)\,,
\end{aligned} \tag{2.26}$$

are satisfied. The commutative and the noncommutative gauge transformation of the gauge parameter are

$$\delta_a a(x) = \frac{\mathrm{i}}{2}[a(x), a(x)] = 0\,, \qquad \hat{\delta}_{\hat{a}}\hat{a}(x) = \frac{\mathrm{i}}{2}[a \stackrel{*}{,} a](x)\,. \tag{2.27}$$

From the above discussion it should be clear that the Seiberg-Witten maps depend not only on the appropriate commutative field but also on the commutative gauge field A_μ. Thus when δ acts on a noncommutative field, this means that it basically acts on the commutative fields from which the noncommutative one is built up, by using the chain rule.

The result of the Seiberg-Witten maps to first order in $\theta^{\mu\nu}$ are

$$\hat{\psi}(\psi, A) = \psi - \frac{1}{2}\theta^{\rho\sigma}A_\rho(\partial_\sigma\psi) + \mathcal{O}(\theta^2)\,, \tag{2.28a}$$

$$\bar{\hat{\psi}}(\bar{\psi}, A) = \bar{\psi} - \frac{1}{2}\theta^{\rho\sigma}A_\rho(\partial_\sigma\bar{\psi}) + \mathcal{O}(\theta^2)\,, \tag{2.28b}$$

$$\hat{A}_\mu(A) = A_\mu + \frac{1}{2}\theta^{\rho\sigma}(\partial_\rho A_\mu + F_{\rho\lambda})A_\sigma + \mathcal{O}(\theta^2)\,, \tag{2.28c}$$

$$\hat{F}_{\mu\nu}(F, A) = F_{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma}\big[2F_{\mu\rho}F_{\nu\sigma} - A_\rho(D_\nu + \partial_\nu)F_{\mu\nu}\big] + \mathcal{O}(\theta^2)\,, \tag{2.28d}$$

$$\hat{a}(a, A) = a - \frac{1}{2}\theta^{\rho\sigma}A_\rho(\partial_\sigma a) + \mathcal{O}(\theta^2)\,. \tag{2.28e}$$

One can derive these maps order by order in $\theta^{\mu\nu}$ by using a polynomial ansatz and comparing both sides of the appropriate gauge equivalence equations.

How to calculate the maps in all orders in the noncommutative parameter, was one of the main challenges of this thesis and will be the topic of the next chapter.

2.4.4 Noncommutative Action with Seiberg-Witten maps

At this stage there are two different ways to proceed. One way is to first gauge fix the action, which is then invariant under noncommutative BRST transformation

$$\hat{\gamma}\hat{A}_\mu = \partial_\mu\hat{C} - \mathrm{i}[\hat{A}_\mu \stackrel{*}{,} \hat{C}]\,, \qquad \hat{\gamma}\hat{C} = \frac{\mathrm{i}}{2}[\hat{C} \stackrel{*}{,} \hat{C}]\,, \qquad \hat{\gamma}\hat{\psi} = \mathrm{i}\,\hat{C} * \hat{\psi}\,, \tag{2.29}$$

and then to replace the fields by their Seiberg-Witten maps. This is the composition of the upper and the right arrow in the commutative diagram 2.1 which leads to the gauge fixed action

$$
\begin{aligned}
S^*_{\text{g.f., SW}} = \int \mathrm{d}^4x \, \Big[& \hat{\bar{\psi}} * (\mathrm{i}\,\hat{D\!\!\!/} - m) * \hat{\psi} - \frac{1}{4g^2}\,\mathrm{tr}\left[\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu}\right] \\
& - \frac{1}{2\xi g^2}(\partial\hat{A}) * (\partial\hat{A}) + \hat{\bar{C}} * \partial^\mu\partial_\mu\hat{C} - \mathrm{i}\;\mathrm{tr}\left[\hat{\bar{C}} * \partial_\mu[\hat{A}^\mu \stackrel{*}{,} \hat{C}]\right]\Big] \,, \qquad (2.30)
\end{aligned}
$$

with the freely choosable parameter ξ. All symbols with a hat in the above action as well as in the following equations denote the appropriate Seiberg-Witten map. The other way would be to first replace the fields by their

$$
\begin{tikzcd}
S^*(\hat{A},\hat{\psi}) \arrow[r, "\text{g.f.}"] \arrow[d, "\text{SW}"] & S^*_{\text{g.f.}}(\hat{A},\hat{\psi}) \arrow[d, "\text{SW}"] \\
S^*_{\text{SW}}(A,\psi) \arrow[r, "\text{g.f.}"] & S^*_{(\text{g.f., SW})}(A,\psi)
\end{tikzcd}
$$

Figure 2.1: Equivalence of the commutative and noncommutative gauge-fixed action

Seiberg-Witten maps and then gauge fix the resulting action, which gives

$$
\begin{aligned}
S^*_{\text{SW, g.f.}} = \int \mathrm{d}^4x \, \Big[& \hat{\bar{\psi}} * (\mathrm{i}\,\hat{D\!\!\!/} - m) * \hat{\psi} - \frac{1}{4g^2}\,\mathrm{tr}\left[\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu}\right] \\
& - \frac{1}{2\xi g^2}(\partial A)(\partial A)\Big] \,. \qquad (2.31)
\end{aligned}
$$

The action obtained this way is invariant under the commutative BRST transformation

$$
\gamma C = 0\,, \qquad \gamma A_\mu = \partial_\mu C\,, \qquad \gamma\psi = \mathrm{i}\,C\psi\,. \qquad (2.32)
$$

The gauge equivalence equations ensure that $S^*_{\text{g.f., SW}}$ and $S^*_{\text{SW, g.f.}}$ describe equivalent physics, i.e.

$$
\langle 0|S^*_{\text{g.f., SW}}|0\rangle = \langle 0|S^*_{\text{SW, g.f.}}|0\rangle \,, \qquad (2.33)
$$

which basically should be picture by the commutative diagram 2.1.

So we are free to choose the action which implicates the fewest work for calculating the cross sections. Because in the action (2.31) the ghost fields decouple, as in the ordinary QED, one doesn't need to calculate the ghost-photon vertex which also means that one has to calculate fewer Feynman diagrams. Thus we will choose this action to calculate our cross section in chapter 6.

2.5 Choice of the Representation

In ordinary gauge theories the trace of different representations of the generators are equal up to a reparametrisation of the fields so that the explicit choice of a representation leaves the physical observables invariant.

In noncommutative theories this is no longer true. Consider the case where the $U(1)$ generator T^a is the Pauli matrix $T^a = \sigma_3$. This leads to a vanishing 3-gauge boson coupling, because

$$\mathrm{tr}[\sigma_3\sigma_3\sigma_3] = 0\,. \tag{2.34}$$

Of course such a model differs from models with, for example $T^a = 1$, where the 3-photon vertex exists. Furthermore the commutator of two gauge fields does not close in the Lie algebra. Namely one gets for $T^a = \sigma_3$

$$\begin{aligned}[][A^a_\mu T^a \stackrel{*}{,} A^b_\nu T^b] = \frac{1}{2}\{A_\mu \stackrel{*}{,} A_\nu\}[T^a, T^b] \\ + \frac{1}{2}[A_\mu \stackrel{*}{,} A_\nu]\{T^a, T^b\} = [A_\mu \stackrel{*}{,} A_\nu]\delta^{ab}\,,\end{aligned} \tag{2.35}$$

where δ^{ab} is an element of the enveloping algebra.

So in general, different representations lead to different physical models so that one has to specify the trace in the Maxwell part of the action. The representations are constrained by the commutative limit. That implies that the trace of the product of two generators has to be one: $\mathrm{tr}\, T^a T^a = 1$. The freedom we have in the choice of the representation leads to different 3-photon couplings. This freedom we parametrise by the factor $k_{\gamma\gamma\gamma} := \mathrm{tr}[T^a T^a T^a]$ to which the 3-photon vertex is proportional.

2.6 Time-Ordering

Another question will arise in time-like noncommutative theories ($\theta^{0i} \neq 0$) in all orders in the noncommutative parameter. Namely how one has to deal with the time-ordering operator present in the Gell-Mann-Low formula (2.15) if one has infinitely many time derivatives.

At first sight one would make an error if one interchanges the time derivatives with the time-ordering operator because they do not commute. But in ordinary field theory in the case of finitely many derivatives one has to interchange them in order to get a covariant propagator [25]. The difference between interchanging and non interchanging is a term which is proportional to the Dirac delta distribution

$$\begin{aligned}\langle 0|\,\mathrm{T}[\partial_\mu\phi(x)\partial_\nu\phi(y)]\,|0\rangle - \partial^x_\mu\partial^y_\nu\,\langle 0|\,\mathrm{T}[\phi(x)\phi(y)]\,|0\rangle \\ = -\,\mathrm{i}\,g_{\mu 0}g_{\nu 0}\delta^{(4)}(x-y)\,.\end{aligned} \tag{2.36}$$

Hence, to get a covariant result one defines a modified time-ordering operator $\tilde{\mathrm{T}}$ (eq. (6-60) in [25]) so that

$$\langle 0| \tilde{\mathrm{T}}\{\partial_\mu\phi(x)\partial_\nu\phi(y)\} |0\rangle := \partial^x_\mu\partial^y_\nu \langle 0| \mathrm{T}[\phi(x)\phi(y)] |0\rangle \; . \tag{2.37}$$

The question is of course what has to be done in the case of noncommutative theories or in theories with infinitely many time derivatives.

In many publications the authors usually used the modified time-ordering operator, i.e. they interchanged the operator with the derivatives. But this leads in the case of time-like noncommutative theories to problems with unitarity [26]. In order to fix this problem there was a proposal [27] not to interchange the time-ordering operator with the time derivatives. This leads to the so called time-ordered perturbation theory (TOPT). However, in the case of noncommutative abelian gauge theories without Seiberg-Witten maps we have shown earlier [28] that with TOPT the Ward identities couldn't be satisfied. Hence, those gauge theories have no consistent interpretation. There are other proposals [29, 30] which suggest a different time-ordering operator to get rid of the unitary problem of noncommutative time-like theories. Yet, up to now it is not known if these proposals are successful.

To illustrate what one typically gets in noncommutative theories if one doesn't interchange the time derivatives with the time ordering operator we have calculated a simple example, which can be found in appendix B.2.

As we mentioned, the above problem only occurs in the case of time-like noncommutativity. The purely space-like case ($\theta^{0i} = 0$) has not infinitely many time derivative so the point is moot. But because we explicitly want to investigate time-like noncommutative gauge theories we have to make a decision. Before a statement can be made whether the noncommutative time-like QED with Seiberg-Witten maps satisfies the unitarity or not, it is reasonable to interchange the time derivatives with the time-ordering operator, thus using the modified operator. Although the modified time-ordering operator is used, we will nevertheless use the symbol T for it.

2.7 The Model

Now it is time to summarise the decisions we made so far and which fix our model:

1. We consider an abelian gauge theory on a flat spacetime but where the spacetime coordinates are hermitian operators which satisfy (1.2). This leads to a noncommutative QED where the products in the action are replaced by the Moyal-Weyl star product (2.21).

2. Afterwards we postulate that the noncommutative action (2.21) should not only be invariant under the noncommutative but also under the commutative gauge transformation which leads to the Seiberg-Witten

maps. These maps are replacing the appropriate fields present in the action which gives, after gauge fixing, the basic action (2.31) of our theory.

3. The starting point of our perturbation theory will be the Gell-Mann-Low formula (2.15).

4. In the calculation of n-point functions we use the modified time-ordering operator, i.e. we interchange the time derivatives with the time-ordering operator.

Chapter 3

Seiberg-Witten Maps

In order to construct the Lagrange density for the noncommutative QED which we want to consider, we need the Seiberg-Witten maps of the gauge and matter fields in all orders in the noncommutative parameter $\theta^{\mu\nu}$. For the calculation of these maps one also needs the Seiberg-Witten map of the ghost field. The problem is that up to now the exact Seiberg-Witten maps are not known and it is totally unclear how to get a closed solution. So what can we do? All phenomenological studies of this class of theories use an effective theory, that means an expansion in orders in $\theta^{\mu\nu}$. But this is not sufficient for the question we want to tackle. As already mentioned, we want to analyse the unitarity of this theory. Thus we need the Seiberg-Witten maps to all orders in $\theta^{\mu\nu}$ and not up to some finite order. To illustrate this, consider

$$\sum_{i=0}^{n} \frac{(-1)^i}{(2i+1)!}(p \wedge q)^{2i+1} \quad \text{vs.} \quad \sin(p \wedge q)\,. \tag{3.1}$$

A cross section which is proportional to the series would be divergent for $p, q \to \infty$ for any finite n, whereas the sine is bounded.

The solution to this problem arises if one realises that the maps don't only build a power series in $\theta^{\mu\nu}$ but also in the gauge field A_μ. With this knowledge we can formally expand the unknown exact Seiberg-Witten maps of the ghost, matter and gauge field in powers of the gauge field

$$\hat{C}(C, A, \theta) = \sum_{k=1}^{\infty} C^{[k]}(C, A, \theta)\,, \qquad C^{[k]} \propto A^{k-1}\,, \qquad C^{[1]} = C\,, \tag{3.2a}$$

$$\hat{\psi}(\psi, A, \theta) = \sum_{k=1}^{\infty} \psi^{[k]}(\psi, A, \theta)\,, \qquad \psi^{[k]} \propto A^{k-1}\,, \qquad \psi^{[1]} = \psi\,, \tag{3.2b}$$

$$\hat{A}_\mu(A, \theta) = \sum_{k=1}^{\infty} A_\mu^{[k]}(A, \theta)\,, \qquad A_\mu^{[k]} \propto A^{k}\,, \qquad A_\mu^{[1]} = A_\mu\,. \tag{3.2c}$$

To understand how this can be a way out of the dilemma, let us look at the process $e^+e^- \to \gamma\gamma$ at Born level. This process has two external and one internal photon (in noncommutative theories one has in general a 3-photon vertex and therefore a s-channel diagram is allowed). As a consequence, a vertex with two matter fields and more than two gauge fields does not contribute to the process. The same holds for a 4-photon vertex. If we translate this to the level of the Seiberg-Witten maps we only need the matter map up to $\psi^{[3]}$ and the gauge field map up to $A_\mu^{[2]}$. We actually need only the second order in the gauge map because each photon of the gauge map couples either to another photon or to two fermions.

The conclusion is that for a given process in a fixed order of the loop expansion one only needs a finite power of the gauge field. This means that it is sufficient to tackle the above mentioned questions if we calculate the maps recursively order by order in the gauge field whereas each order has to be in all orders in the noncommutative parameter.

Now we have circumvented the problem of having to calculate the exact maps but we still have to derive the monomials in the above series up to the desired power, i.e. the $C^{[k]}$s, $\psi^{[k]}$s and $A_\mu^{[k]}$s. Fortunately the answer for $A_\mu^{[k]}$ and $C^{[k]}$ was provided by Barnich, Brandt and Grigoriev in [2]. They showed that

$$A_\mu^{[k]} = -\rho^{[0]}\left(\gamma^{[1]}A_\mu^{[k-1]} - \partial_\mu C^{[k]} + \mathrm{i}\sum_{l=1}^{k-1}\left[A_\mu^{[l]} \overset{*}{,} C^{[k-l]}\right]\right)\,, \qquad k \geq 2\,,\; A_\mu^{[1]} = A_\mu \tag{3.3a}$$

and

$$C^{[k]} = -\rho^{[0]}\left(\gamma^{[1]}C^{[k-1]} - \frac{\mathrm{i}}{2}\sum_{l=1}^{k-1}\left[C^{[l]} \overset{*}{,} C^{[k-l]}\right]\right)\,,\; k \geq 2\,,\; C^{[1]} = C \tag{3.3b}$$

are the desired recursive relations needed in order to calculate $A_\mu^{[k]}$ and $C^{[k]}$. With this knowledge it was an easy task to find the appropriate relation for the matter field

$$\psi^{[k]} = -\rho^{[0]}\left(\gamma^{[1]}\psi^{[k-1]} - \mathrm{i}\sum_{l=1}^{k-1} C^{[l]} * \psi^{[k-l]}\right)\,,\; k \geq 2\,,\; \psi^{[1]} = \psi\,. \tag{3.3c}$$

That these recursive relations indeed lead to the maps satisfying the gauge equivalence equations is proofed in appendix C.5. Of course what remains is to define the two operators $\gamma^{[1]}$ and $\rho^{[0]}$, where the latter is the crucial one. But first let us introduce $\gamma^{[1]}$ which is nothing else but the quadratic part of the commutative BRST differential $\gamma = \gamma^{[0]} + \gamma^{[1]}$. If one applies this operator to the three fields A_μ, C and ψ one gets

$$\begin{aligned} \gamma A_\mu &= D_\mu C = \partial_\mu C - \mathrm{i}\,[A_\mu, C] = \partial_\mu C\,, \\ \gamma C &= \frac{\mathrm{i}}{2}[C, C] = 0\,, \\ \gamma\psi &= \mathrm{i}\, C\psi\,, \end{aligned} \tag{3.4}$$

where in the noncommutative case one only has to replace the ordinary commutator by the star commutator (2.29). From the above equations we can read off, how the linear and quadratic term act on the fields, namely

$$\gamma^{[0]}A_\mu = \partial_\mu C\,, \qquad \gamma^{[0]}C = 0\,, \qquad \gamma^{[0]}\psi = 0\,, \tag{3.5}$$

and

$$\gamma^{[1]}A_\mu = 0\,, \qquad \gamma^{[1]}C = 0\,, \qquad \gamma^{[1]}\psi = \mathrm{i}\,C\psi\,, \tag{3.6}$$

respectively. The above relations are only valid for the abelian case. For nonabelian gauge theories one has to change these equations appropriately.

If one looks at the recursive equations (3.3) one can check that the r.h.s. of the equations have the right field dependencies after $\gamma^{[1]}$ has been applied to the fields. By "right field dependencies" we mean that all terms in the outer parenthesis have to have exactly the field content needed in order to get only terms for the order in the gauge field under consideration.

Now we come to the contracting homotopy operator $\rho^{[0]}$. In order to understand the origin of this operator we have to introduce the basic mathematics of cohomology.

3.1 Basics on Cohomology[1]

Let $\mathcal{F}$ be a vector space over the field $\mathbb{K}$, such that $f^i \in \mathcal{F} \Rightarrow \lambda_i f^i \in \mathcal{F},\ \lambda_i \in \mathbb{K}$. Then D is called a differential in $\mathcal{F}$ when it is a linear operation which squares to zero

$$D\,:\,\mathcal{F}\to\mathcal{F}\,, \quad D(\lambda_i f^i) = \lambda_i(Df^i)\,, \quad D^2 = 0\,. \tag{3.7}$$

$f\in\mathcal{F}$ is called "cocycle of D" if $Df=0$ and $f\in\mathcal{F}$ is called "coboundary of D" if $f = Dg$ for some $g\in\mathcal{F}$. Due to $D^2=0$ one sees directly that every coboundary is a cocycle. The interesting cocycles are those which are not coboundaries. This set of cocycles is called the cohomology of D. Or more precisely the cohomology $H(D,\mathcal{F})$ is the quotient space of cocycles and coboundaries of D

$$H(D,\mathcal{F}) = \frac{\text{cocycle of } D}{\text{coboundaries of } D}\,. \tag{3.8}$$

Now let us suppose that the vector space $\mathcal{F}$ decomposes into eigenspaces $\mathcal{F}_\lambda$ of an operator N

$$\mathcal{F} = \bigoplus_\lambda \mathcal{F}_\lambda, \quad \mathcal{F}_\lambda = \{f\in\mathcal{F}\,:\,Nf=\lambda f\}\,. \tag{3.9}$$

[1]This section based on lectures from Brandt [31]

Then the elements of f decompose into eigenfunctions of N

$$\forall f \in \mathcal{F} : \quad f = \sum_{\lambda} f_{\lambda}, \quad N f_{\lambda} = \lambda f_{\lambda} . \tag{3.10}$$

An operation $B : \mathcal{F} \to \mathcal{F}$ whose anticommutator with D yields N

$$N = BD + DB \, , \quad \lambda f_{\lambda} = N f_{\lambda} = (BD + DB) f_{\lambda} \, , \tag{3.11}$$

is called a contracting homotopy for N. Roughly speaking, the operator B is a kind of an "inverse" operator to the differential D.

Now, let us relate the above abstract notation with the notation we used so far. We can identify the differential D by the linear part of the BRST differential $\gamma^{[0]}$ and the contracting homotopy B/λ by $\rho^{[0]}$

$$D \leftrightarrow \gamma^{[0]} \, , \qquad\qquad \frac{B}{\lambda} \leftrightarrow \rho^{[0]} \, . \tag{3.12}$$

We will identify the eigenspaces $\mathcal{F}_{\lambda}$ in the next section because we first need new coordinates in the space of fields and their derivatives.

3.2 The Contracting Homotopy

Before we come to the explicite formula for the contracting homotopy, we introduce for convenience new coordinates in the space of fields and their derivatives. These new coordinates y^i, z^i and w^i form a basis in such a way, that the basis vectors are totally symmetric with respect to the indices of the fields and their derivatives

$$\begin{aligned} \left\{y^i\right\} &= \left\{\partial_{(\nu_1} \dots \partial_{\nu_n} A_{\mu)}\right\} \, , \\ \left\{z^i\right\} &= \left\{\partial_{(\nu_1} \dots \partial_{\nu_n} \partial_{\mu)} C\right\} = \left\{\partial_{\nu_1} \dots \partial_{\nu_n} \partial_{\mu} C\right\} \, , \\ \left\{w^i\right\} &= \left\{C, \partial_{(\nu_1} \dots \partial_{\nu_n} F_{\mu)\rho}, \psi, \partial_{\nu_1} \dots \partial_{\nu_n} \psi\right\} \, , \end{aligned} \tag{3.13}$$

where the field-strength tensor is the ordinary abelian one $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The parenthesis around the indices denote the total symmetrisation with respect to the indices (for a precise definition see appendix A.3). The above variables are independent and complete in the sense that every local function of fields and their derivatives can be uniquely expressed in terms of them. For example one can express a gauge field with n derivatives by

$$\partial_{\mu_1} \dots \partial_{\mu_n} A_{\rho} = \partial_{(\mu_1} \dots \partial_{\mu_n} A_{\rho)} + \frac{n}{n+1} \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} F_{\mu_n)\rho} \, , \tag{3.14}$$

where the first term on the r.h.s. is element of $\{y^i\}$ and the second term is element of $\{w^i\}$, respectively (cf. appendix C.1).

Now we can express the eigenspaces $\mathcal{F}_\lambda$ explicitely using these new variables, namely

$$\mathcal{F}_\lambda = \text{spam}\left(\left\{\left(y^i\right)^\alpha \left(z^i\right)^{\lambda-\alpha} \left(w^i\right)^\beta\right\}\right), \tag{3.15}$$

where $\alpha = 1, \ldots, \lambda$ and $\beta \in \mathbb{N}$. This means that the function $f_\lambda(y, z, w) \in \mathcal{F}_\lambda$ is composed of λ y^is and z^is and arbitrarily many w^is. The eigenvalue λ just counts the number of the gauge and ghost fields. For example a term contributes to $C^{[3]}$ is $A_{\mu_1}(\partial_{(\nu_1} A_{\mu_2)})(\partial_{\nu_2} C) \in \mathcal{F}_3$.

Using the new coordinates Barnich, Brandt and Grigoriev [2] found an explicite expression of the contracting homotopy

$$\rho^{[0]} f(y, z, w) := \int_0^1 \frac{\mathrm{d}t}{t} \left[y^i \frac{\partial}{\partial z^i} f\right](ty, tz, w)\,. \tag{3.16}$$

The integration over t extracts the inverse of the eigenvalue λ which is defined within the operator $\rho^{[0]}$ (3.12). So essentially the contracting homotopy replaces a z^i by a y^i whereas $\gamma^{[0]}$ does the inverse. Thus, $\rho^{[0]}$ is indeed a kind of an inverse operator to the BRST differential.

In order to check that the above definition is indeed the contracting homotopy of the operator $\rho^{[0]}\gamma^{[0]} + \gamma^{[0]}\rho^{[0]}$ the equation

$$\left(\rho^{[0]}\gamma^{[0]} + \gamma^{[0]}\rho^{[0]}\right) f(y, z, w) = f(y, z, w) \tag{3.17}$$

has to be verified as it is shown in appendix C.4. In [2] the r.h.s. of the above equation possesses an additional term $-f(0, 0, w)$ which is zero in all the cases we consider. The term becomes relevant when f depends only on the coordinate w which actually is not the case in the calculations we consider. Note that the ghost number of the homotopy operator is one, hence $\rho^{[0]}$ anticommutes with the BRST differential $\gamma^{[0]}$ and with the ghost fields. Actually, the reason why the above operators anticommute is not that they have ghost number one but that they have an odd Grassmann parity. But because we never deal with objects which have an odd Grassmann parity and a ghost number of zero or vice versa, we can use "ghost number" as a synonym for "Grassmann parity".

Now we have all ingredients in order to be able to calculate the first recursive step, i.e. to calculate the maps $C^{[2]}$, $\psi^{[2]}$ and $A_\mu^{[2]}$, which is the subject of the next sections.

3.3 The Leading Order Ghost Field Map

Using the calculation of $C^{[2]}$ as an example we demonstrate how a calculation can be done. We start with $C^{[2]}$ not only because it is the simplest example but also because we need this map in order to be able to calculate the maps for the matter and the gauge field. For $k = 2$ equation (3.3b) becomes

$$C^{[2]} = -\rho^{[0]} \left[\gamma^{[1]} C - \frac{\mathrm{i}}{2} [C \overset{*}{;} C] \right] = -\rho^{[0]} (C \sin \wedge_{12} C) \,, \tag{3.18}$$

where the first term is zero due to $\gamma^{[1]} C = 0$. The bidifferential $\wedge_{12}$ which is proportional to $\theta^{\mu\nu}$ is defined by

$$A \wedge_{12} B := \frac{1}{2} (\partial_\mu A) \theta^{\mu\nu} (\partial_\nu B) \,. \tag{3.19}$$

The first step is to change the basis of the fields and its derivatives to the symmetrised variables y^i, z^i and w^i, defined above. To be able to perform the basis transformation we first have to rewrite the sine in its series representation

$$C^{[2]} = -\rho^{[0]} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (C \wedge_{12}^{2n+1} C) \,. \tag{3.20}$$

With the definition of $\wedge_{12}$ we can write each of the above summand in the new coordinate basis

$$C^{[2]} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\theta^{\mu_1 \nu_1}}{2} \cdots \frac{\theta^{\mu_{2n+1} \nu_{2n+1}}}{2} \rho^{[0]} \left[(\partial_{(\mu_1} \ldots \partial_{\mu_{2n+1})} C)(\partial_{(\nu_1} \ldots \partial_{\nu_{2n+1})} C) \right] \,. \tag{3.21}$$

Obviously, both ghost fields and their derivatives are elements of $\{z^{2n+1}\}$. Although we don't have to symmetrise the indices of the partial derivatives for this special case we nevertheless write them down in order to illustrate where they would be.

Now, we let $\rho^{[0]}$ operate on the ghost field or more precisely on the z^{2n+1}. Therefore we need (3.16) which gives

$$\begin{aligned} C^{[2]} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\theta^{\mu_1 \nu_1}}{2} \cdots \frac{\theta^{\mu_{2n+1} \nu_{2n+1}}}{2} \\ \int_0^1 \frac{\mathrm{d}t}{t} [(t\partial_{(\mu_1} \ldots \partial_{\mu_{2n}} A_{\mu_{2n+1})})(t\partial_{(\nu_1} \ldots \partial_{\nu_{2n+1})} C) \\ - (t\partial_{(\mu_1} \ldots \partial_{\mu_{2n+1})} C)(t\partial_{(\nu_1} \ldots \partial_{\nu_{2n}} A_{\nu_{2n+1})})] \,. \end{aligned} \tag{3.22}$$

Because the ghost number of $\rho^{[0]}$ is one it anticommutes with C such that the second term gets a minus sign. Let us look more precisely at the result we have so far. Firstly, the integral over t is trivial and is just one-half.

Secondly, the first and second terms of our result are equal if we interchange the indices $\mu_i \leftrightarrow \nu_i$, which leads to one minus sign for each exchange due to the antisymmetry of $\theta^{\mu\nu}$. And because we have $2n+1$ swaps we obtain an overall minus sign which cancels the minus sign of the second term. Thirdly, if we look at the sum of the symmetrised gauge field we notice that all terms are equal. This is due to the fact that the corresponding indices on the ghost field side are totally symmetric. If we take all three observations into account we obtain finally

$$C^{[2]} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\theta^{\mu\nu}}{2} A_\mu \wedge_{12}^{2n} (\partial_\nu C) = -\frac{1}{2}\theta^{\mu\nu} A_\mu \frac{\sin\wedge_{12}}{\wedge_{12}} (\partial_\nu C)\,, \quad (3.23)$$

which is a quite simple result. Of course this map depends on a ghost and a gauge field as it should be because of the recursive equation (3.3b). The series expansion of the map begins with the first order in $\theta^{\mu\nu}$ and has only odd powers of the noncommutative parameter. Obviously, the map $C^{[2]}$ vanishes in the commutative limit $|\theta^{\mu\nu}| \to 0$, as we expected.

The operator-like functions are defined by their series representation as we implicitly already assumed. In order to be able to keep formulas compact and easy to read we introduce an abbreviation for this function

$$*_{\rm s}\,(\wedge_{12}) := \frac{\sin\wedge_{12}}{\wedge_{12}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \wedge_{12}^{2n} \quad (3.24)$$

in analogy to the Moyal-Weyl star-product. In the following we will neglect the argument of the $*_{\rm s}$-function because it is, at least in this chapter, always $\wedge_{12}$.

In order to convince us that the map given above really satisfies the gauge equivalence equation we will check this. What we have to show is that

$$\hat{\gamma}\hat{C}(A,C) = \gamma\hat{C}(A,C)\,, \quad (3.25)$$

namely that the noncommutative gauge transformation on the l.h.s. is equivalent to the commutative one on the r.h.s. With (3.4) we get for the noncommutative BRST transformation of the ghost field

$$\hat{\gamma}\hat{C}(A,C) = \frac{\mathrm{i}}{2}[\hat{C} \stackrel{*}{,} \hat{C}]\,. \quad (3.26)$$

Now let us replace the exact map $\hat{C}$ by the first and second order map. It should be clear that on both sides of the equation one has to have the same field content in order to satisfy this equation. Therefore in the star commutator on the l.h.s. one only has to consider the first order ghost field C because the commutator is then already of second order in the ghost field. Hence, the above equation reads up to the second order

$$\frac{\mathrm{i}}{2}[C \stackrel{*}{,} C] = -C \sin\wedge_{12} C = -\frac{1}{2}\theta^{\mu\nu}(\partial_\mu C) *_{\rm s} (\partial_\nu C) = \gamma C^{[2]}\,, \quad (3.27)$$

which is correct by the definition of (3.24).

3.4 The Leading Order Matter Field Map

After we have seen how the leading order ghost field map can be derived we now want to perform a similar calculation for the matter field $\psi^{[2]}$. The calculation is as simple as that of the previous section, therefore we present it briefly. We have to determine

$$\psi^{[2]} = -\rho^{[0]}\left(\gamma^{[1]}\psi - \mathrm{i}\, C * \psi\right) , \tag{3.28}$$

where we know from (3.4) that $\gamma^{[1]}\psi = \mathrm{i}\, C\psi$. As in the calculation of the $C^{[2]}$ we have no gauge field which could mark one of the indices. This means that the above term in the parenthesis is already symmetric in its indices of the derivatives. If we express the star-product by its series representation (2.9), we get

$$\begin{aligned}\psi^{[2]} &= -\rho^{[0]}\left(\mathrm{i}\, C\psi - \mathrm{i}\sum_{n=0}^{\infty}\frac{1}{n!}C(\mathrm{i}\wedge_{12})^n\psi\right)\\ &= \mathrm{i}\,\rho^{[0]}\left(\sum_{n=1}^{\infty}\frac{\mathrm{i}^n}{n!}(\partial_{\mu_1}\ldots\partial_{\mu_n}C)\frac{\theta^{\mu_1\nu_1}}{2}\cdots\frac{\theta^{\mu_n\nu_n}}{2}(\partial_{\nu_1}\ldots\partial_{\nu_n}\psi)\right) .\end{aligned} \tag{3.29}$$

It is important that we have only ghosts with at least one derivative in the above calculation. Namely this fact is required by the proof that the recursive equation satisfies the gauge equivalence equation (for details see appendix C.5.3). Now, let us apply $\rho^{[0]}$. The integral over t is again trivial, namely it is just unity. Because we already have a symmetric term we directly replace the ghost field and its derivatives by the symmetrised gauge field

$$\rho^{[0]}(\partial_{\mu_1}\ldots\partial_{\mu_n}C) = \partial_{(\mu_1}\ldots\partial_{\mu_{n-1}}A_{\mu_n)} . \tag{3.30}$$

Summing over n leads to the desired matter field map

$$\psi^{[2]} = -\frac{1}{2}\theta^{\mu\nu}A_\mu\sum_{n=1}^{\infty}\frac{(\mathrm{i}\wedge_{12})^{n-1}}{n!}(\partial_\nu\psi) = -\frac{1}{2}\theta^{\mu\nu}A_\mu *_{\mathrm{e}} (\partial_\nu\psi) . \tag{3.31}$$

With $*_{\mathrm{e}}$ we denote the above sum, namely

$$*_{\mathrm{e}}(\wedge_{12}) := \frac{\mathrm{e}^{\mathrm{i}\wedge_{12}}-1}{\mathrm{i}\wedge_{12}} = \sum_{n=1}^{\infty}\frac{\mathrm{i}^{n-1}}{n!}\wedge_{12}^{n-1} . \tag{3.32}$$

In contrast to the map $C^{[2]}$ the series expansion of the matter field map contains even as well as odd powers of $\theta^{\mu\nu}$. In the commutative limit the matter map is the identity, i.e. $\psi^{[2]} = 0$.

Let us check the gauge equivalence equation which in the case of the matter map is

$$\hat{\gamma}\hat{\psi}(A,\psi) = \gamma\hat{\psi}(A,\psi) . \tag{3.33}$$

If one wants to check that the l.h.s. and the r.h.s. are equal, one has to be careful to consider all terms. Because terms with different gauge fields mix after the application of γ. This is due to the fact that the nonlinear part of the BRST differential is non-zero $\gamma^{[1]}\psi = \mathrm{i}\, C\psi$. In our case we get for the above equation

$$\mathrm{i}\, C * \psi = \mathrm{i}\, C\psi - \frac{1}{2}\theta^{\mu\nu}(\partial_\mu C) *_{\mathrm{e}} (\partial_\nu \psi)\,, \tag{3.34}$$

which is indeed equal. One can easily apply the whole procedure to the case of $\hat{\bar{\psi}}$. We will not separately discuss this here.

3.5 The Leading Order Gauge Field Map

After we have calculated the two simplest examples we can now tackle the next more complicated calculation, namely the map $A_\lambda^{[2]}$.

Why is the calculation of the gauge map more complicated than the one for the ghost and the matter map? The answer is quite simple, namely because the gauge field has an Lorentz index. This index breaks the symmetry of the indices under permutation. So in this calculation we have a qualitative difference to the previous two.

Let us begin with the calculation. First of all remember that $\gamma^{[1]} A_\mu = 0$. In section 3.3 we have calculated the second term $C^{[2]}$ so that we obtain

$$\begin{aligned} A_\lambda^{[2]} &= -\rho^{[0]} \left[\gamma^{[1]} A_\lambda - \partial_\lambda C^{[2]} + \mathrm{i}\, [A_\lambda \stackrel{*}{,} C]\right] \\ &= -\rho^{[0]} \left[-\partial_\lambda \left(-\frac{1}{2}\theta^{\mu\nu} A_\mu *_{\mathrm{s}} (\partial_\nu C)\right) - 2A_\lambda \sin \wedge_{12} C\right] \\ &= -\frac{1}{2}\theta^{\mu\nu}\rho^{[0]} \left[(-2\partial_\mu A_\lambda + \partial_\lambda A_\mu) *_{\mathrm{s}} (\partial_\nu C) + A_\mu *_{\mathrm{s}} (\partial_\lambda \partial_\nu C)\right]\,. \end{aligned} \tag{3.35}$$

We have written the above equation in such a manner that we have only one operator like product, namely the $*_{\mathrm{s}}$ which we already had in the map for the ghost field $C^{[2]}$. Generically we can split up our work into three pieces

$$\theta^{\mu\nu}\rho^{[0]} \left[(\partial_\mu A_\lambda) \wedge_{12}^{2n} (\partial_\nu C)\right]\,, \tag{3.36a}$$

$$-\frac{1}{2}\theta^{\mu\nu}\rho^{[0]} \left[(\partial_\lambda A_\mu) \wedge_{12}^{2n} (\partial_\nu C)\right]\,, \tag{3.36b}$$

$$-\frac{1}{2}\theta^{\mu\nu}\rho^{[0]} \left[A_\mu \wedge_{12}^{2n} (\partial_\lambda \partial_\nu C)\right]\,, \tag{3.36c}$$

where we already have written the n-th summand of the series (3.24) of the $*_{\mathrm{s}}$ operator. Note that the detailed appearance of the three parts is not fixed because one can express the gauge fields by the field-strength tensor and vice versa. But if one switches over to the independent variables y^i, z^i and w^i one gets an unique expression.

As an example we now want to calculate the first part. The calculation of the other two expressions contains no conceptual differences so that we will present just the result after this calculation. Since we have to write the expression in the brackets in terms of the new coordinates y^i, z^i and w^i we have to split up the gauge field with its derivatives into a symmetric part and a part which contains the field-strength tensor

$$\partial_{\mu_1}\ldots\partial_{\mu_{2n+1}}A_\lambda = \partial_{(\mu_1}\ldots\partial_{\mu_{2n+1}}A_{\lambda)} + \frac{2n+1}{2n+2}\partial_{(\mu_1}\ldots\partial_{\mu_{2n}}F_{\mu_{2n+1})\lambda}\,. \tag{3.37}$$

We can always perform such a decomposition because the new variables form a complete basis, as already mentioned above. It is obvious that the first term on the r.h.s. of the above equation is element of $\{y^i\}$ whereas the second term is an element of $\{w^i\}$. The factor

$$\frac{2n+1}{2n+2} \tag{3.38}$$

in front of the field-strength tensor needs a more detailed discussion. To see that this factor is indeed the correct one let us first determine the number of different terms of $\partial_{(\mu_1}\ldots\partial_{\mu_{2n+1}}A_{\lambda)}$. We have $(2n+2)!$ terms where $(2n+1)!$ of them are equal. So each different term comes with a factor $(2n+2)^{-1}$. But in order to get the l.h.s. of (3.37) we need additional terms from the second expression, namely

$$\frac{2n+1}{2n+2}\partial_{\mu_1}\ldots\partial_{\mu_{2n+1}}A_\lambda\,. \tag{3.39}$$

On the other hand the unwanted terms which we obtain from the symmetrisation have to vanish. This cancellation is only possible if the appropriate term in the expression containing the field-strength tensor comes with a factor $-(2n+2)^{-1}$. This leads to the factor (3.38). A detailed calculation can be found in appendix C.1.

Thus, with the knowledge of (3.37) we have the first part of our calculation. Namely we can let $\rho^{[0]}$ operate, which basically means, that

$$\partial_{(\nu_1}\ldots\partial_{\nu_{2n}}\partial_{\nu)}C \to \partial_{(\nu_1}\ldots\partial_{\nu_{2n}}A_{\nu)}\,. \tag{3.40}$$

The integration over t is still trivial. One only has to be careful with the w^is because within the definition of the operator $\rho^{[0]}$ (3.16) they don't have a t in front of them. Therefore one obtains a factor one-half for $\rho^{[0]}[yz]$ and unity for $\rho^{[0]}[wz]$, respectively. If we combine what we found so far we get

$$\theta^{\mu\nu}\rho^{[0]}\left[(\partial_\mu A_\lambda)\wedge_{12}^{2n}(\partial_\nu C)\right] = \frac{\theta^{\mu_1\nu_1}}{2}\cdots\frac{\theta^{\mu_{2n}\nu_{2n}}}{2}\left[\frac{1}{2}\partial_{(\mu_1}\ldots\partial_{\mu_{2n}}\partial_\mu A_{\lambda)}\right.$$

$$+\frac{2n+1}{2n+2}\partial_{(\mu_1}\dots\partial_{\mu_{2n}}F_{\mu)\lambda}\Big]\,\partial_{(\nu_1}\dots\partial_{\nu_{2n}}A_{\nu)}\,. \quad (3.41)$$

The next step is to get rid of the symmetrisation parenthesis. But before we do this we rewrite the two terms in the bracket

$$\begin{aligned}\frac{1}{2}\partial_{(\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_{\lambda)} &+ \frac{2n+1}{2n+2}\partial_{(\mu_1}\dots\partial_{\mu_{2n}}F_{\mu)\lambda}\\ &= \frac{1}{2}\frac{4n+3}{2n+2}\partial_{\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_\lambda - \frac{1}{2}\frac{2n+1}{2n+2}\partial_\lambda\partial_{(\mu_1}\dots\partial_{\mu_{2n}}A_{\mu)}\,, \quad (3.42)\end{aligned}$$

so that we only have to consider one term where the gauge field carries the "external" index λ and another term where the gauge field has only one of the "internal" indices $\mu_1\dots\mu_{2n}\mu$. In the above equation we used the identity (3.37) and

$$\begin{aligned}\partial_{(\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_{\lambda)} = \frac{1}{2n+2}&\partial_{\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_\lambda\\ &+\frac{2n+1}{2n+2}\partial_\lambda\partial_{(\mu_1}\dots\partial_{\mu_{2n}}A_{\mu)} \quad (3.43)\end{aligned}$$

(cf. appendix C.1). Note that we would get just $\partial_{\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_\lambda$ if we didn't have the factor one-half coming from the integration over the parameter t.

Now we consider the rewritten terms times the rightmost one (3.41), which comes from the ghost field. The first one is easy, namely

$$\begin{aligned}&[\partial_{\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_\lambda][\partial_{(\nu_1}\dots\partial_{\nu_{2n}}A_{\nu)}]\\ &\qquad= [\partial_{\mu_1}\dots\partial_{\mu_{2n}}\partial_\mu A_\lambda][\partial_{\nu_1}\dots\partial_{\nu_{2n}}A_\nu]\,. \quad (3.44)\end{aligned}$$

We can neglect here the symmetrisation because all summands are equal. For the second product we get two different terms

$$\begin{aligned}&[\partial_\lambda\partial_{(\mu_1}\dots\partial_{\mu_{2n}}A_{\mu)}][\partial_{(\nu_1}\dots\partial_{\nu_{2n}}A_{\nu)}]\\ &\qquad= \frac{1}{2n+1}[\partial_\lambda\partial_{\mu_1}\dots\partial_{\mu_{2n}}A_\mu][\partial_{\nu_1}\dots\partial_{\nu_{2n}}A_\nu]\\ &\qquad\quad+\frac{2n}{2n+1}[\partial_\lambda\partial_{\mu_1}\dots\partial_{\mu_{2n}}A_\mu][\partial_{\nu_1}\dots\partial_{\nu_{2n-1}}\partial_\nu A_{\nu_{2n}}]\,, \quad (3.45)\end{aligned}$$

because there is obviously only one term proportional to $(\partial_\lambda A_\mu)\wedge_{12}^{2n}A_\nu$ and $2n$ terms of the type $(\partial_\lambda\partial_{\mu_1}A_{\mu_2})\wedge_{12}^{2n-1}(\partial_{\nu_2}A_{\nu_1})$. In the above two expressions we have implicitly assumed that the indices μ_i and ν_i are contracted with $\theta^{\mu_i\nu_i}$.

If we summarize the calculation for the first part (3.36a) we obtain the following result

$$\rho^{[0]}\left[(\partial_\mu A_\lambda)\wedge_{12}^{2n}(\partial_\nu C)\right] = \frac{1}{2}\frac{1}{2n+2}\Big[(4n+3)(\partial_\mu A_\lambda)\wedge_{12}^{2n}A_\nu$$

$$- (\partial_\lambda A_\mu) \wedge_{12}^{2n} A_\nu - \frac{\theta^{\rho\sigma}}{2} 2n(\partial_\lambda \partial_\mu A_\rho) \wedge_{12}^{2n-1} (\partial_\sigma A_\nu) \Bigg] . \quad (3.46)$$

Can we understand why three different terms appear in our result? Let us look at the number of fields with an index and at the number of "external" indices. In our case we have two gauge fields where one has an "internal" index and the other one has an "external" index λ. So how many different terms can be obtained with these given number of gauge fields and external indices? The answer is of course three, namely the three given in our result. One can ask why our result has the maximum number of different terms. We know that the operator $\rho^{[0]}$ implies the change to the symmetric variables y^i, z^i and w^i. So the result is symmetric in all indices and thus contains all possible index combinations. This means that one gets actually all index structures i.e. tensor structures which are non zero.

We will now discuss the sums. In order to get more manageable functions of the bidifferential $\wedge_{12}$ we will perform the sums. Therefore we have to consider the series representation of three functions involved. The series representation of the $*_\mathrm{s}$ we already know. In addition we also get a function which contains the cosine and which we will call $*_\mathrm{c}$ and a function which contains an integral sine which we will abbreviate with $*_\mathrm{si}$

$$*_\mathrm{s}(\wedge_{12}) := \frac{\sin \wedge_{12}}{\wedge_{12}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \wedge_{12}^{2n} , \quad (3.47a)$$

$$*_\mathrm{c}(\wedge_{12}) := 2\frac{1 - \cos \wedge_{12}}{\wedge_{12}^2} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{2n+2} \wedge_{12}^{2n} , \quad (3.47b)$$

$$*_\mathrm{si}(\wedge_{12}) := \frac{\mathrm{si} \wedge_{12}}{\wedge_{12}} = \frac{1}{\wedge_{12}} \int_0^{\wedge_{12}} \mathrm{d}t \, \frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{2n+1} \wedge_{12}^{2n} . \quad (3.47c)$$

Note that the $*$-functions are defined in such a manner that they become unity for the commutative limit $|\theta^{\mu\nu}| \to 0$, or in other words the power series of the functions starts with unity. With these three functions we can perform all sums, so that we get the following final result for the first part

$$\theta^{\mu\nu} \rho^{[0]} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[(\partial_\mu A_\lambda) \wedge_{12}^{2n} (\partial_\nu C) \right] = \theta^{\mu\nu} \Bigg[(\partial_\mu A_\lambda) \left(*_\mathrm{s} - \frac{*_\mathrm{c}}{4} \right) A_\nu$$
$$-(\partial_\lambda A_\mu) \frac{*_\mathrm{c}}{4} A_\nu - \frac{1}{4} \theta^{\rho\sigma} (\partial_\lambda \partial_\mu A_\rho) \frac{*_\mathrm{s} - *_\mathrm{c}}{\wedge_{12}} (\partial_\sigma A_\nu) \Bigg] , \quad (3.48)$$

where we again neglect the argument of the $*$-functions.

As mentioned we will only give the result for the second and the third part of the expression for the gauge field map $A_\lambda^{[2]}$ without a detailed calculation. For the second part (3.36b) one obtains

$$
\begin{aligned}
-\frac{1}{2}\theta^{\mu\nu}\rho^{[0]}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[(\partial_\lambda A_\mu)\wedge_{12}^{2n}(\partial_\nu C)\right] \\
= -\frac{1}{2}\theta^{\mu\nu}\Bigg[-\frac{1}{4}(\partial_\mu A_\lambda+\partial_\lambda A_\mu)*_{\mathrm{c}} A_\nu+(\partial_\lambda A_\mu)*_{\mathrm{si}} A_\nu \\
-\frac{1}{4}\theta^{\rho\sigma}(\partial_\lambda\partial_\mu A_\rho)\frac{2*_{\mathrm{si}}-*_{\mathrm{c}}-*_{\mathrm{s}}}{\wedge_{12}}(\partial_\sigma A_\nu)\Bigg]
\end{aligned}
\tag{3.49}
$$

and the third part (3.36c) is just

$$
\begin{aligned}
-\frac{1}{2}\theta^{\mu\nu}\rho^{[0]}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[A_\mu\wedge_{12}^{2n}(\partial_\lambda\partial_\nu C)\right] \\
= -\frac{1}{2}\theta^{\mu\nu}\Bigg[-\frac{1}{4}(\partial_\mu A_\lambda+\partial_\lambda A_\mu)*_{\mathrm{c}} A_\nu \\
-\frac{1}{4}\theta^{\rho\sigma}(\partial_\lambda\partial_\mu A_\rho)\frac{*_{\mathrm{s}}-*_{\mathrm{c}}}{\wedge_{12}}(\partial_\sigma A_\nu)\Bigg]\,.
\end{aligned}
\tag{3.50}
$$

As one can see we need the above introduced $*_{\mathrm{si}}$-function only for the second part. And in this part the $*_{\mathrm{si}}$-function appears only at two positions. We stress this fact, because this function will play a crucial role in the next chapter.

We now have all three parts of the gauge field map. So the last step is to combine these three parts. What one gets is a quite short and compact expression but, as we will see, not the simplest one possible. We will come to the question of ambiguities in the Seiberg-Witten maps in the next chapter. But up to now we obtained the solution of the recursion relation (3.3a) in the leading order for the gauge field map, namely

$$
\begin{aligned}
A_\lambda^{[2]} = \frac{1}{2}\theta^{\mu\nu}\left[2(\partial_\mu A_\lambda)*_{\mathrm{s}} A_\nu-(\partial_\lambda A_\mu)*_{\mathrm{si}} A_\nu\right] \\
-\frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\lambda\partial_\mu A_\rho)\frac{*_{\mathrm{s}}-*_{\mathrm{si}}}{\wedge_{12}}(\partial_\sigma A_\nu)\,.
\end{aligned}
\tag{3.51}
$$

If we look at the result we see that the complete map for the gauge field is independent of the $*_{\mathrm{c}}$-function. This is no longer true if one consider nonabelian gauge theories. In this case one gets not only the $*_{\mathrm{c}}$-function but also a function which contains the integral cosine analogue to $*_{\mathrm{si}}$ (for more details, see [32]). We also see that we have all non-zero indices combinations i.e. tensor structures which are possible as we already motivated earlier. Note that the first term in the result begins with order θ where the second term begins with order θ^3. As a trivial consequence the map $A_\mu^{[2]}$ becomes zero if we take the commutative limit $|\theta^{\mu\nu}|\to 0$, as it should be. Otherwise we would have made an error. Further more the result has the property that it includes only odd powers of $\theta^{\mu\nu}$ so that the map is an odd function of the noncommutative parameter. This is of course due to the presence of only the $*_{\mathrm{s}}$ and the $*_{\mathrm{si}}$-function which are both even functions.

As we did in the previous section we again want to check our result by plugging the gauge field map $A_\lambda^{[2]}$ into the gauge equivalence equation for the gauge field

$$\hat{\gamma}\hat{A}_\lambda(A) = \gamma\hat{A}_\lambda(A) \tag{3.52}$$

and verify that this equation is satisfied. Again let us rewrite this equation with the knowledge of (3.4), which leads to

$$\partial_\lambda C^{[2]}(A,C) - \mathrm{i}[A_\lambda \stackrel{*}{,} C] = \gamma A_\lambda^{[2]}(A)\,, \tag{3.53}$$

where we have only the ordinary field in the star commutator because all others would generate terms with more than two fields. Now let us first look at the r.h.s. part which contains the sine integral, i.e. the $*_{\mathrm{si}}$-function. After a short and simple calculation one sees that these two terms with the $*_{\mathrm{si}}$-function vanish exactly. This is an important observation regarding the question of ambiguities of the Seiberg-Witten maps. For the remaining terms with the $*_{\mathrm{s}}$-function and therefore for the whole r.h.s. of the above equation one gets

$$\begin{aligned}
&\frac{1}{2}\theta^{\mu\nu}\left[(\partial_\mu\partial_\lambda C) *_{\mathrm{s}} A_\nu + (2\partial_\mu A_\lambda - \partial_\lambda A_\mu) *_{\mathrm{s}} (\partial_\nu C)\right] \\
&\qquad = -\frac{1}{2}\theta^{\mu\nu}\left[A_\mu *_{\mathrm{s}} (\partial_\nu\partial_\lambda C) + \partial_\lambda A_\mu *_{\mathrm{s}} (\partial_\nu C)\right] + 2A_\lambda \sin\wedge_{12} C\,,
\end{aligned} \tag{3.54}$$

which is indeed equal to the l.h.s. of (3.53).

3.6 Higher Order Maps

Until now we have seen the calculations of the leading order Seiberg-Witten maps. The calculation of the maps in the next higher order in the gauge field becomes much more complicated. One reason is that one has more different terms. The other complicating factor is that there are more fields with an index, i.e. gauge fields. In particular the gauge field complicates the calculation because the vector property of this field destroys the symmetry with respect to the indices. So the symmetrisation and in the end the summation is much more complicated. But what we can relatively easily consider are all the different tensor or index structures one can get. Thus, the main work will be to get the functions in front of these structures explicitly, which we will not calculate at this place but which can be found in appendix C.3.

What is the tensor structure of the ghost map $C^{[3]}$? We know that it depends on one ghost field and two gauge fields. This means that we have two distinguished indices. It is not hard to find that the next to leading order ghost map has five different terms, namely

$$
\begin{aligned}
C^{[3]}(A,C) = {} & \frac{1}{2}\theta^{\mu_1\nu_1}F^C_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}A_{\nu_1}C \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^C_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}A_{\mu_2})(\partial_{\nu_2}A_{\nu_1})C \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^C_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}A_{\mu_2})A_{\nu_1}(\partial_{\nu_2}C) \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^C_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}(\partial_{\nu_1}A_{\mu_2})(\partial_{\nu_2}C) \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^C_{\mathrm{V}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}A_{\mu_2}(\partial_{\nu_1}\partial_{\nu_2}C)\,,
\end{aligned} \tag{3.55}
$$

where the functions F^C are given in appendix C.3. The arguments of these functions are the bidifferentials which extended to three versions because we have three fields now. The definition should be clear, namely

$$\wedge_{12}ABC := \frac{\theta^{\mu\nu}}{2}(\partial_\mu A)(\partial_\nu B)C\,, \tag{3.56a}$$

$$\wedge_{13}ABC := \frac{\theta^{\mu\nu}}{2}(\partial_\mu A)B(\partial_\nu C)\,, \tag{3.56b}$$

$$\wedge_{23}ABC := \frac{\theta^{\mu\nu}}{2}A(\partial_\mu B)(\partial_\nu C)\,. \tag{3.56c}$$

Thus, the index of the $\wedge$ operators indicates on which field the partial derivatives act.

In the case of the matter field map we get the same tensor structure but of course different functions labeled by F^ψ

$$
\begin{aligned}
\psi^{[3]}(A,\psi) = {} & \frac{1}{2}\theta^{\mu_1\nu_1}F^\psi_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}A_{\nu_1}\psi \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^\psi_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}A_{\mu_2})(\partial_{\nu_2}A_{\nu_1})\psi \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^\psi_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}A_{\mu_2})A_{\nu_1}(\partial_{\nu_2}\psi) \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^\psi_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}(\partial_{\nu_1}A_{\mu_2})(\partial_{\nu_2}\psi) \\
& + \frac{1}{4}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}F^\psi_{\mathrm{V}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}A_{\mu_2}(\partial_{\nu_1}\partial_{\nu_2}\psi)\,.
\end{aligned} \tag{3.57}
$$

3.7 The Calculation in Three Steps

If we go back and remember the example calculations we can extract three steps which one has to go through in every calculation:

1. One has to replace the gauge fields and their derivatives by the symmetrised one, i.e. one has to introduce the coordinates y^i, z^i and w^i.

In order to do this order by order one has to rewrite the functions of the bidifferential in their series representation.

2. Afterwards one can perform the contracting homotopy $\rho^{[0]}$ which basically replaces one ghost field by a gauge field.

3. After this the sums should be perform and expressed, if possible, in terms of known functions.

This sounds not very complicated and actually it isn't for the first recursive step. But already the maps of the next recursive step become very lengthy. We will see that the level of complexity can be reduced if one takes ambiguities into account. Yet, even then, the result is still not simple.

Chapter 4

Ambiguities

In the previous chapter we learned how to calculate the Seiberg-Witten maps in all orders in the noncommutative parameter but order by order in the gauge field. In this chapter we want to discuss an alternative way to construct the Seiberg-Witten maps presented by Zumino et.al. [33]. The big problem of this ansatz is that it only works for the first recursive step, i.e. for the $C^{[2]}$, $\psi^{[2]}$ and $A_\mu^{[2]}$ maps. To illustrate the break down of this ansatz we calculated the next-to-leading order ghost field map in appendix D. The result of this calculation is that the map doesn't satisfy the gauge equivalence equation (2.26). But as we will see by using the ansatz from Zumino et.al. we will obtain a different result for the leading order gauge field map. And this will give us a hint how the ambiguity for $A_\mu^{[2]}$ looks like.

4.1 Alternative Ansatz

The authors of [33] replaced the noncommutative parameter $\theta^{\mu\nu} \to t\theta^{\mu\nu}$ in order to be able to differentiate with respect to the order in the noncommutativity. Therefore the Moyal-Weyl star-product now reads

$$*(t) = \mathrm{e}^{\mathrm{i}\, t \wedge_{12}}\,, \qquad \frac{\partial * (t)}{\partial t} = \mathrm{i} \wedge_{12} * (t)\,. \tag{4.1}$$

The starting point of their recursive equations are the "differential evolution equations"

$$\dot{C}(t) = \frac{1}{4}\theta^{\mu\nu}\left[\partial_\mu C \overset{*(t)}{,} A_\nu\right]_+\,, \tag{4.2a}$$

$$\dot{A}_\lambda(t) = -\frac{1}{4}\theta^{\mu\nu}\left[A_\mu \overset{*(t)}{,} \partial_\nu A_\lambda + F_{\nu\lambda}\right]_+\,, \tag{4.2b}$$

where the dot on the l.h.s. denotes the differentiation with respect to the parameter t. These "differential evolution equations" arise out of the cohomological approach the authors developed in [34] and continued to work out in [33]. Note that for $\dot{C}(t=0)$ and $\dot{A}_\lambda(t=0)$ one gets the Seiberg-Witten maps in first order in $\theta^{\mu\nu}$ (cf. (2.28)). With (4.2) Zumino et.al. propose to get the n-th order result in the noncommutative parameter by calculating the n-th derivative with respect to the parameter t at the point $t=0$

$$C^{(n)} := \frac{1}{n!} \left.\frac{\partial^n C(t)}{\partial t^n}\right|_{t=0}, \qquad A_\lambda^{(n)} := \frac{1}{n!} \left.\frac{\partial^n A_\lambda(t)}{\partial t^n}\right|_{t=0}, \tag{4.3}$$

where $C^{(0)} = C$ and $A_\lambda^{(0)} = A_\lambda$. The terms for $n=1$ are obtained from (4.2) for $t \to 0$. The terms with higher derivatives of t can be derived by recursively inserting the terms (4.2). The exact Seiberg-Witten map is then given by the sum over all n. Nevertheless we showed by direct calculation that the next-to-leading order ghost field map is not a solution to the gauge equivalence equation (cf. appendix D). Even if we are not able to calculate the Seiberg-Witten maps to all orders in the gauge field and in the noncommutative parameter we can nevertheless calculate the maps in all orders in $\theta^{\mu\nu}$ and the gauge field up to a fixed order.

Let us look at the recursive algorithm in more detail. If we first consider the derivative of the star-product we see that for each derivative we get one order in $\theta^{\mu\nu}$ in the limit $t \to 0$

$$\left.\frac{\partial^n * (t)}{\partial t^n}\right|_{t=0} = (\mathrm{i} \wedge_{12})^n \,. \tag{4.4}$$

Thus in order to obtain all orders in $\theta^{\mu\nu}$ we have to consider all derivatives with regard to the star-product. If we now look at the derivatives with regard to the fields than we see that for each derivative we get an additional gauge field or in other words, the order in the gauge field increases by every derivative with regard to a field. This is due to the fact that the derivative of the fields depends nonlinear on themselves as one can see in (4.2).

4.2 Leading Order Gauge Field Map

Now let us calculate the first order in the gauge field map which we will denote as $\tilde{A}_\lambda^{[2]}$ in order to distinguish it from the map we calculated in the chapter before

$$\hat{A}_\lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{\partial^n A_\lambda(t)}{\partial t^n}\right|_{t=0} = A_\lambda + \tilde{A}_\lambda^{[2]} + \mathcal{O}(A^3) \,. \tag{4.5}$$

This map contains two gauge fields so that we don't have to consider derivatives with respect to the fields because the first derivative depends already on two fields. Therefore we only have to consider the derivatives regarding the star-product. In addition we can also neglect the star commutator in the field-strength tensor because otherwise we would again get more than two fields. With this knowledge we obtain for the desired map

$$\tilde{A}_\lambda^{[2]} = -\frac{1}{4}\theta^{\mu\nu}\sum_{n=1}^{\infty}\frac{1}{n!}\left(A_\mu\frac{\partial^n *(t)}{\partial t^n}(2\partial_\nu A_\lambda - \partial_\lambda A_\nu) + (2\partial_\nu A_\lambda - \partial_\lambda A_\nu)\frac{\partial^n *(t)}{\partial t^n}A_\mu\right)_{t=0}, \quad (4.6)$$

where we already expanded the anticommutator. Computing first the derivatives and afterwards performing the sum over all the terms we find with (4.4)

$$\sum_{n=1}^{\infty}\frac{1}{n!}(\mathrm{i}\wedge_{12})^n = \frac{\mathrm{e}^{\mathrm{i}\wedge_{12}}-1}{\mathrm{i}\wedge_{12}} = *_{\mathrm{e}}(\wedge_{12})\,. \quad (4.7)$$

If we plug this into our calculation, we get the very compact result

$$\begin{aligned}\tilde{A}_\lambda^{[2]} &= -\frac{1}{4}\theta^{\mu\nu}\left(A_\mu *_{\mathrm{e}} (2\partial_\nu A_\lambda - \partial_\lambda A_\nu) + (2\partial_\nu A_\lambda - \partial_\lambda A_\nu) *_{\mathrm{e}} A_\mu\right)\\ &= -\frac{1}{2}\theta^{\mu\nu}A_\mu *_{\mathrm{s}} (2\partial_\nu A_\lambda - \partial_\lambda A_\nu)\,, \end{aligned} \quad (4.8)$$

where we have used that $*_{\mathrm{e}}(\wedge_{12}) + *_{\mathrm{e}}(-\wedge_{12}) = 2 *_{\mathrm{s}} (\wedge_{12})$.

Like in the last chapter we want to check the gauge equivalence equation for $\tilde{A}_\lambda^{[2]}$. The equation which has to be satisfied is the same as for the gauge field map we calculated in the previous chapter (3.53), up to the r.h.s.

$$\partial_\lambda C^{[2]}(A, C) + 2A_\lambda \sin\wedge_{12} C = \gamma\tilde{A}_\lambda^{[2]}(A)\,. \quad (4.9)$$

Let us take a closer look at the r.h.s. of the above equation. Due to the absence of the sine integral function $*_{\mathrm{si}}$ we have only one term on which the operator γ acts

$$\begin{aligned}&-\frac{1}{2}\theta^{\mu\nu}\gamma\left[A_\mu *_{\mathrm{s}} (2\partial_\nu A_\lambda - \partial_\lambda A_\nu)\right]\\ &\quad = -2C\sin\wedge_{12} A_\lambda - \frac{1}{2}\theta^{\mu\nu}\left[-(\partial_\mu C) *_{\mathrm{s}} (\partial_\lambda A_\nu) + A_\mu *_{\mathrm{s}} (\partial_\nu\partial_\lambda C)\right]\,. \end{aligned} \quad (4.10)$$

As one can see this is again exactly the l.h.s. of the gauge equivalence equation. Thus two maps for the same field are known both satisfying the gauge equivalence equation. Remember that this is the only relevant constraint which has to be satisfied by a Seiberg-Witten map. This leads us directly to the next section where we will take a closer look at this subject.

4.3 Comparison

Let us take a closer look at the Seiberg-Witten map of the gauge field in leading order. First we recall our results of the two maps (3.51) and (4.8)

$$A_\lambda^{[2]} = \frac{1}{2}\theta^{\mu\nu}\left[2(\partial_\mu A_\lambda) *_{\mathrm{s}} A_\nu - (\partial_\lambda A_\mu) *_{\mathrm{si}} A_\nu\right] - \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\lambda\partial_\mu A_\rho)\frac{*_{\mathrm{s}} - *_{\mathrm{si}}}{\wedge_{12}}(\partial_\sigma A_\nu)\,, \tag{4.11a}$$

$$\tilde{A}_\lambda^{[2]} = -\frac{1}{2}\theta^{\mu\nu}A_\mu *_{\mathrm{s}} (2\partial_\nu A_\lambda - \partial_\lambda A_\nu)\,, \tag{4.11b}$$

where the first one was obtained from the recursive algorithm developed by Brandt et.al. [2] and the second one from Zumino et.al. [33]. As we mentioned at the end of section 3.5 the terms which contain the sine integral function $*_{\mathrm{si}}$ cancel each other if the BRST operator γ acts on them. This is due to the tensor structure of the fields and their derivatives and has nothing to do with the operator function between the two fields. So nothing prevents us from writing an "almost" arbitrary function instead of the sine integral. Hence this map will still be a Seiberg-Witten map. To be concrete the map

$$A_\lambda^{[2]} = \frac{1}{2}\theta^{\mu\nu}\left[2(\partial_\mu A_\lambda) *_{\mathrm{s}} A_\nu - (\partial_\lambda A_\mu) *_{\mathrm{f}} A_\nu\right] - \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}(\partial_\lambda\partial_\mu A_\rho)\frac{*_{\mathrm{s}} - *_{\mathrm{f}}}{\wedge_{12}}(\partial_\sigma A_\nu) \tag{4.12}$$

with the "almost" arbitrary function $*_{\mathrm{f}}$ is also a Seiberg-Witten map. By "almost arbitrary" we mean those functions $*_{\mathrm{f}}$ which multiplied by $\theta^{\mu\nu}$ vanish for $|\theta^{\mu\nu}| \to 0$, since our theory should become the ordinary QED for the commutative limit i.e. a vanishing noncommutativity. Nevertheless, we have infinitely many different functions namely all functions whose coefficient of the Laurent series with negative powers are identically zero.

This means, that the two maps (4.11) are only special cases of the general map (4.12). Namely if we choose $*_{\mathrm{f}} = *_{\mathrm{s}}$ we get the map (4.11a) and for $*_{\mathrm{f}} = *_{\mathrm{si}}$ we get (4.11b).

Now we come to the question wether the above general map is really the most general map. The answer is yes, at least if one thinks about possible tensor structures which cancel after applying the BRST operator γ. The reason is that the formalism from Brandt et.al. provides us with all possible tensor structures since we have to symmetrise our fields and their derivatives with regard to the indices. Therefore we can be sure that we really get all possible terms. Of course, all terms which depend only on z^i and w^i give also zero if one applies $\gamma^{[0]}$, but those terms don't exist. The reason is, that we explicitly consider summands of a series expansion (3.2) of the gauge field. Therefore each summand, beside the trivial zeroth order one, contains at least one gauge field, i.e. one y^i.

What happens in the case of the ghost and matter field maps? In the leading order case there is only one map for each, because they have only one gauge field. This is not sufficient to get terms which are non-zero and which become zero after γ is applied. Consequently, the alternative ansatz leads to the same maps as the one of Brandt et.al.. Of course, if one considers the next higher maps $C^{[3]}$ and $\psi^{[3]}$ then each of them has two gauge fields and accordingly one gets ambiguities as in the case of the leading order gauge field map.

As we will see each choice of a function $*_{\mathrm{f}}$ will lead to a different physical theory. This means that the observables depend on the concrete choice of this function. At first sight this sounds not very good but tinking a little more deeper we have a freedom which we will possibly can use in order to build a theory which satisfies physically important properties such as unitarity.

Chapter 5

Feynman Rules

After calculating the Seiberg-Witten maps in chapter 3 we have discussed the ambiguities of these maps in the previous chapter. There we found that the leading order maps for the ghost field and for the matter field are unique. For the gauge field map which is not unique we found an explicit expression for the most general map. In the following we will consider this general map in order to be able to see how the Feynman rules depend on the ambiguities of the gauge map. The dependence of the Feynman rules on the ambiguities will then result in a dependency of the cross section on the ambiguities as well. We will get a whole class of theories with different physical observables.

The next step which we will discuss in this chapter is to calculate the Feynman rules which follow from the NCQED Lagrangian (2.31). What we have to do is to take the maps we obtained in the last two chapters and insert them into the Lagrangian. Then we have to go through all the terms and collect those terms which contribute to the appropriate Feynman vertex. After obtaining all relevant terms we can change to the momentum space where the Feynman vertices are formulated. Note that in this chapter we choose all momenta as incoming momenta.

Let us now replace the exact Seiberg-Witten maps by the expansion with respect to the gauge field in order to be able to combine all terms which contribute to a given vertex. If we do this for the Maxwell part of the NCQED action we get

$$\mathrm{i}\int \mathrm{d}^4x\,\hat{\bar{\psi}} * (\mathrm{i}\,\hat{\not{D}} - m) * \hat{\psi} = \mathrm{i}\int \mathrm{d}^4x\,\left[\mathcal{L}_{\mathrm{f\bar{f}}} + \mathcal{L}_{\mathrm{f\bar{f}g}} + \mathcal{L}_{\mathrm{f\bar{f}gg}}\right] + \mathcal{O}(A^3)\,, \qquad (5.1)$$

where the different pieces of the Lagrangian combine the terms with the same number of gauge fields

$$\mathcal{L}_{\mathrm{f\bar{f}}} = \bar{\psi}(\mathrm{i}\,\not{\partial} - m) * \psi\,, \qquad (5.2\mathrm{a})$$

$$\mathcal{L}_{\mathrm{f\bar{f}g}} = \bar{\psi} * \not{A} * \psi + \bar{\psi}(\mathrm{i}\,\not{\partial} - m) * \psi^{[2]} + \bar{\psi}^{[2]}(\mathrm{i}\,\not{\partial} - m) * \psi\,, \qquad (5.2\mathrm{b})$$

$$\begin{aligned}\mathcal{L}_{\mathrm{f\bar{f}gg}} &= \bar{\psi} * \not{A}^{[2]} * \psi + \bar{\psi} * \not{A} * \psi^{[2]} + \bar{\psi}^{[2]} * \not{A} * \psi \\ &\quad + \bar{\psi}(\mathrm{i}\not{\partial} - m) * \psi^{[3]} + \bar{\psi}^{[3]}(\mathrm{i}\not{\partial} - m) * \psi + \bar{\psi}^{[2]}(\mathrm{i}\not{\partial} - m) * \psi^{[2]} . \qquad (5.2c)\end{aligned}$$

Of course, due to (2.13b) one can neglect one $*$-product in each term. One important consequence is that the two-point functions, i.e. the propagators, are the same as in the commutative theories. This ensures that the theory for large distances is independent of the noncommutative spacetime. This property of the theory is demonstratively understandable, namely the fields far before and far after the interaction are considered to be free fields, i.e. (almost) plane waves. Hence they should not depend on the structure of the spacetime at very small distances.

An important feature of noncommutative theories with Seiberg-Witten maps is the existence of the terms $\mathcal{L}_{\mathrm{f\bar{f}gg}}$ where two matter and two gauge fields couple. These so called contact terms are unique to the kind of noncommutative theories we consider. Actually they have to be there. Otherwise the action wouldn't be BRST invariant.

We stopped at order $\mathcal{A}^2$ because higher order terms in the gauge field would result in vertices which we don't need for the tree level calculation for the process $e^+e^- \to \gamma\gamma$ which will be considered in the next chapter.

We can of course also expand the gauge boson part of the action in powers of the gauge field

$$\begin{aligned}\mathrm{i}\int \mathrm{d}^4x \left[-\frac{1}{4g^2}\,\mathrm{tr}\left[\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu}\right] - \frac{1}{2\xi g^2}(\partial A)(\partial A)\right] \\ = \mathrm{i}\int \mathrm{d}^4x\, \left[\mathcal{L}_{\mathrm{gg}} + \mathcal{L}_{\mathrm{ggg}}\right] + \mathcal{O}(A^4) . \qquad (5.3)\end{aligned}$$

As mentioned above, the term which is bilinear in the gauge field is the same as in ordinary QED, namely

$$\begin{aligned}\mathcal{L}_{\mathrm{gg}} &= -\frac{1}{2g^2}\left[(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\mu A^\nu)(\partial_\nu A_\mu) + \frac{1}{\xi}(\partial A)^2\right] \\ &= \frac{1}{2g^2} A_\mu \left[\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\nu , \qquad (5.4a)\end{aligned}$$

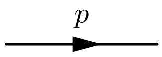
with the gauge parameter ξ. Note that the last equal sign holds only under the space time integral, because we performed an integration by parts and neglected one $*$-product. Now we come to the 3-photon vertex. Here we have two different terms. First we get a term which is proportional to the star commutator which we also have in noncommutative theories without Seiberg-Witten maps. And we have a term which comes from the bilinear term by replacing one gauge field by its leading order Seiberg-Witten map $A_\mu^{[2]}$. Since we have two fields which we can replace we get the corresponding term twice. Thus, if we combine what we said, the Lagrangian becomes

$$\mathcal{L}_{\mathrm{ggg}} = -\frac{k_{\gamma\gamma\gamma}}{g^2}\Bigg\{ -\mathrm{i}(\partial^\mu A^\nu) * [A_\mu \overset{*}{,} A_\nu] \\ - A_\mu \left[\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right] * A_\nu^{[2]} \Bigg\}\,. \quad (5.4b)$$

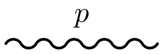
Note that the gauge parameter appears in the 3-photon vertex but nevertheless any observable still has to be independent of the gauge parameter. In the explicit calculation within the next chapter we will see this cancellation for the process we consider. We have absorbed the freedom in the choice of the trace in the parameter $k_{\gamma\gamma\gamma}$, introduced in section 2.5.

5.1 Propagators

First let us look at the two propagators. As already stated the Feynman propagators are the same as in the ordinary QED. Nevertheless for completeness we will also present them. The propagators of the fermion and the gauge boson are



$$= \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\,\epsilon} =: D_{\mathrm{f}}(p)\,,$$

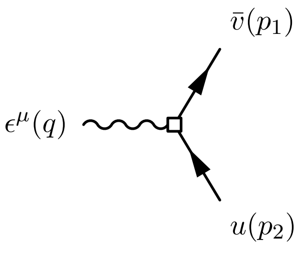


$$= \frac{-\,\mathrm{i}}{p^2 + \mathrm{i}\,\epsilon}\left[g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\frac{p^\mu p^\nu}{p^2}\right] =: D_{\mathrm{g}}^{\mu\nu}(p)\,.$$

In the next sections we will take a closer look at the calculation of the Feynman vertices of our noncommutative theory.

5.2 The $\mathrm{f\bar{f}g}$ Vertex

The first and also simplest vertex which already exists in the ordinary QED is the $\mathrm{f\bar{f}g}$ vertex. As already mentioned in the introduction of this chapter we choose all momenta to be incoming so that a derivative with respect to a field gives $-\,\mathrm{i}$ times the momenta.



$$= V^{\mu}_{\mathrm{f\bar{f}g}}(p_1, p_2, q)$$

What we have to do now is to calculate the vertex function $V^{\mu}_{\mathrm{f\bar{f}g}}$.

The relevant terms for this vertex are given in (5.2b). The first term is the vertex of the noncommutative QED without Seiberg-Witten maps. It is well known and also easy to calculate, namely one obtains the ordinary vertex of the commutative QED times a phase factor

$$\mathrm{i}\,\bar{\psi} * \not{A} * \psi \to \mathrm{i}\, g\gamma^{\mu}\, \mathrm{e}^{\mathrm{i}\, q\wedge p_1}\,, \tag{5.5}$$

where q is the momenta of the gauge boson and p_1 is the momenta of the anti-fermion as pictured in the above graph. The $\wedge$-product is defined in (2.4). If we look at the above equation (5.5) one may ask why we only have the momenta of the photon and the anti-fermion in the phase and not the momenta of the fermion. The answer is quite simple, namely we could neglect either the first or the second $*$-product in the vertex. The underlying reason is the energy-momentum conservation in the vertex. Therefore one can arbitrarly choose one of the two momenta, but one has to pay attention that one has the correct sign.

The second and third term are new contributions to the $\mathrm{f\bar{f}g}$ vertex coming from the Seiberg-Witten maps. If we explicitely write down the second term we get

$$\begin{aligned} -\frac{\mathrm{i}}{2}\theta^{\mu\nu}\bar{\psi}(\mathrm{i}\,\not{\partial} - m)\,(A_{\mu} *_{\mathrm{e}} (\partial_{\nu}\psi)) \\ \to -\frac{\mathrm{i}}{2} g *_{\mathrm{e}} ((-\mathrm{i}\, q) \wedge (-\mathrm{i}\, p_2))\,(\theta(-\mathrm{i}\, p_2))^{\mu}(\not{q} + \not{p}_2 - m) \\ = \frac{1}{2} g *_{\mathrm{e}} (p_2 \wedge q)\,(p_2\theta)^{\mu}(-\not{p}_1 - m)\,, \end{aligned} \tag{5.6}$$

where we have again used the antisymmetry of the $\wedge$-product and the energy-momentum conservation $q + p_2 = -p_1$.

For the third term the procedure is the same as for the second so that we can combine our result

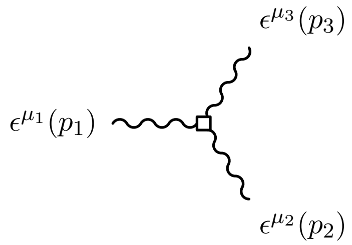
$$\begin{aligned} V^{\mu}_{\mathrm{f\bar{f}g}}(p_1, p_2, q) = \mathrm{i}\, g \Big[\gamma^{\mu}\, \mathrm{e}^{\mathrm{i}\, q\wedge p_1} - \frac{\mathrm{i}}{2} *_{\mathrm{e}} (q \wedge p_1)\,(p_1\theta)^{\mu}(\not{p}_2 - m) \\ - \frac{\mathrm{i}}{2} *_{\mathrm{e}} (p_2 \wedge q)\,(p_2\theta)^{\mu}(-\not{p}_1 - m)\Big]\,. \end{aligned} \tag{5.7}$$

As one can already see from the Lagrangian $\mathcal{L}_{\mathrm{f\bar{f}g}}$ the terms coming from the maps are proportional to the equation of motion of the fermions. In momentum space the equations of motion lead to the factor $(\not{p}_2 - m)$ or $(-\not{p}_1 - m)$. If these momenta are on-shell the corresponding part of the vertex becomes zero.

Let us look what happens if we take the commutative limit $|\theta^{\mu\nu}| \to 0$. The phase in the first term then becomes just unity, because the exponent becomes zero. In the second and third term the $*_{\mathrm{e}}$-function becomes unity so that this term vanishes in this limit. Therefore just $\mathrm{i}\, g\gamma^{\mu}$ remains which is the ordinary QED vertex, as it should be.

5.3 The ggg Vertex

Now we come to the 3-gauge boson vertex which is totally new for non-commutative abelian gauge theories, but is well known from ordinary non-abelian theories like QCD.



$$= V_{\text{ggg}}^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)$$

There are two sources of the gauge boson self-coupling in noncommutative QED. First due to the star commutator in the definition of the field-strength tensor (2.18), where one has a non-vanishing commutator of the gauge boson. The second reason is the presence of the Seiberg-Witten map $A_\mu^{[2]}$ which is also responsible for the photon self coupling. As we already mentioned in section 2.5, we can choose the representation of the $U(1)$ generators such that the trace over the product of three generators becomes zero. The consequence would be that with this special choice the 3-photon vertex is absent. But we want to consider the whole class of such theories so that we absorb this freedom in the parameter $k_{\gamma\gamma\gamma}$.

If we now want to calculate the Feynman vertex $V_{\text{ggg}}^{\mu_1\mu_2\mu_3}$ of the 3-photon self coupling we first have to find all terms which are relevant for this vertex, i.e. all terms with three gauge fields. This are exactly the terms of the Lagrangian $\mathcal{L}_{\text{ggg}}$ (5.4b). The first part is the part of the vertex which is also present in the noncommutative theories without Seiberg-Witten maps. The result is known, namely

$$\begin{aligned} -\,\mathrm{i}(\partial^\mu A^\nu)[A_\mu \stackrel{*}{,} A_\nu] \rightarrow -2\,\mathrm{i}\,g^3 \sin(-p_2 \wedge p_3) p_1^{\mu_2} g^{\mu_1\mu_3} =: V_{\text{ggg},1}^{\mu_1\mu_2\mu_3} \\ +\ \text{all permutations of } \{(\mu_1, p_1), (\mu_2, p_2), (\mu_3, p_3)\}\ . \quad (5.8) \end{aligned}$$

We have this sum of all permutations due to the indistinguishability of the three photons. The sine has its origin in the star commutator where the tensor structure is known from ordinary QCD, for example.

The second term is new in theories with Seiberg-Witten maps. It is the Lagrangian of the free theory where one gauge field has been replaced by the Seiberg-Witten map $A_\mu^{[2]}$. Thus, this part is proportional to the equation of motion of a photon. Due to the structure of the Lagrangian we can split up our calculation into two parts. The first part is the calculation of the bracket which is in essence the propagator

$$
\begin{aligned}
-\left[\partial^2 g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] \to \left[p_1^2 g^{\mu_1\nu} - \left(1-\frac{1}{\xi}\right)p_1^{\mu_1}p_1^{\nu}\right]& \\
= \mathrm{i}\, p_1^4 D_{\mathrm{g}}^{\mu_1\nu}(p_1) =: \mathrm{i}\, V_{\mathrm{g}}^{\mu_1\nu}(p_1)\,.& \quad (5.9)
\end{aligned}
$$

The second part is also straightforward to calculate. It is just the next to leading order map of the gauge field

$$
\begin{aligned}
A_\nu^{[2]} \to -\mathrm{i}\, g^2 \Bigg[&*_{\mathrm{s}}(-p_2\wedge p_3)(p_2\theta)^{\mu_3} g_\nu^{\mu_2} - \frac{1}{2} *_{\mathrm{f}}(-p_2\wedge p_3)\theta^{\mu_2\mu_3} p_{2,\nu} \\
&- \frac{1}{4}(p_2\theta)^{\mu_3}(\theta p_3)^{\mu_2}\frac{*_{\mathrm{s}}(-p_2\wedge p_3) - *_{\mathrm{f}}(-p_2\wedge p_3)}{p_2\wedge p_3} p_{2,\nu}\Bigg] =: V^{\mu_2\mu_3}_{\nu,\mathrm{ggg},2} \\
&+ \text{ all permutations of } \{(\mu_1,p_1),(\mu_2,p_2),(\mu_3,p_3)\}\,. \quad (5.10)
\end{aligned}
$$

We also used in the above calculation the overall energy momentum conservation at this vertex, namely $p_1+p_2+p_3=0$.

Now let us combine our results. The whole 3-photon vertex is then

$$
\begin{aligned}
V_{\mathrm{ggg}}^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = &-\frac{\mathrm{i}}{g^2}k_{\gamma\gamma\gamma} \\
&\left[V_{\mathrm{ggg},1}^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) + \mathrm{i}\, g\, \{V_{\mathrm{g}}(p_1)V_{\mathrm{ggg},2}(p_1,p_2,p_3)\}^{\mu_1\mu_2\mu_3}\right] \\
&+ \text{ all permutation of } \{(\mu_1,p_1),(\mu_2,p_2),(\mu_3,p_3)\}\,. \quad (5.11)
\end{aligned}
$$

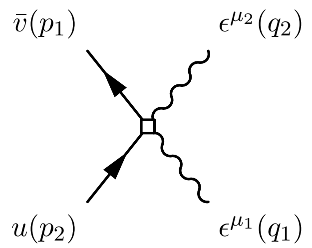
Note that the coupling constant g in front of the p_1^4 comes from the leftmost gauge field in the second term of (5.4b).

Depending on how one chooses $*_{\mathrm{f}}$ it is possible to cancel either the second or third term in (5.10). But one of these two terms is always present, at least if the 3-photon vertex exists, i.e. $k_{\gamma\gamma\gamma}\neq 0$. Note that we have a part, namely the terms with the $*_{\mathrm{s}}$-function whose series expansion starts with order $\theta^{\mu\nu}$ just as $V_{\mathrm{ggg},1}^{\mu_1\mu_2\mu_3}$ does. We also have the terms with the freely choosable function $*_{\mathrm{f}}$ whose series expansion can start with an arbitrary non-negative power in $\theta^{\mu\nu}$. So the order at which the second term of (5.10) begins is until now freely choosable. But as we will see in chapter 7 we obtain a restriction to the minimum power at which the series expansion of the $*_{\mathrm{f}}$-function has to start. The above mentioned restriction to non-negative powers in $\theta^{\mu\nu}$ has its origin in the fact that we want to get a vanishing 3-photon vertex in the commutative limit (cf. chapter 4).

If we look at the terms in the 3-photon vertex for the limit $|\theta^{\mu\nu}|\to 0$ we see that the term $V_{\mathrm{ggg},1}$ becomes zero because of the sine. The other term also vanishes because the series expansion of the function $*_{\mathrm{s}}$ and $*_{\mathrm{f}}$ starts, at least, with a constant. Therefore in the commutative limit these functions times $\theta^{\mu\nu}$ give always zero, so that the whole vertex vanishes.

5.4 The $\mathrm{f\bar{f}gg}$ Vertex

In this section we want to discuss the $\mathrm{f\bar{f}gg}$-vertex which has the particularity that it is only present in noncommutative theories with Seiberg-Witten maps. This is not really amazing because the only new structure in abelian noncommutative theories without the maps is the commutator of two gauge fields which appears in the field-strength tensor. Of course this cannot produce a vertex with two fermions and two photons.



$$= V^{\mu_1\mu_2}_{\mathrm{f\bar{f}gg}}(p_1, p_2, q_1, q_2)$$

In (5.2c) we have combined the relevant terms. So we have six parts which we have to calculate. The first three parts come from the interaction vertex of the ordinary QED in which one has replaced one field by the appropriate leading order Seiberg-Witten map. The last three parts come from the free Maxwell part. One important consequence is that these three parts are proportional to the equation of motion of the matter field. So if the fermion and the anti-fermion are on the mass shell, most of these terms vanish. Only one term coming from the sixth part is non-zero in this case.

Now let us look somewhat closer at the first part $\bar{\psi}\not{A}^{[2]} * \psi$. This part of the vertex contains basically a phase which comes from the $*$-product and the map $A^{[2]}_\lambda$. Thus we get for the first part the following expression

$$\begin{aligned}\bar{\psi}\not{A}^{[2]} * \psi \to \mathrm{i}\, g^2 \Big[&-(q_1\theta)^{\mu_2}\gamma^{\mu_1} *_{\mathrm{s}} (q_1 \wedge q_2) + \frac{1}{2}\theta^{\mu_1\mu_2} *_{\mathrm{f}} (q_1 \wedge q_2)\not{q}_1 \\ &+ \frac{1}{4}(q_1\theta)^{\mu_2}(\theta q_2)^{\mu_1}\frac{(*_{\mathrm{s}} - *_{\mathrm{f}})(q_1 \wedge q_2)}{q_1 \wedge q_2}\not{q}_1 \Big]\, \mathrm{e}^{\mathrm{i}\, p_1 \wedge p_2} =: g^2 V^{\mu_1\mu_2}_{\mathrm{f\bar{f}gg},1}\,, \end{aligned} \tag{5.12}$$

where we neglect the terms which come from the exchange $(q_1, \mu_1) \leftrightarrow (q_2, \mu_2)$. Note that the derivative in the $*$-product acts on the left only on the gauge field and not on the anti-fermion.

The next two parts are calculated quickly. One obtains for the second and the third part

$$\bar{\psi}\not{A} * \psi^{[2]} \to \frac{\mathrm{i}}{2} g^2 \gamma^{\mu_1}(\theta p_2)^{\mu_2} *_{\mathrm{e}} (p_2 \wedge q_2)\, \mathrm{e}^{-\mathrm{i}\, p_1 \wedge q_1} =: g^2 V^{\mu_1\mu_2}_{\mathrm{f\bar{f}gg},2}\,, \tag{5.13a}$$

$$\bar{\psi}^{[2]}\not{A} * \psi \to \frac{\mathrm{i}}{2} g^2 \gamma^{\mu_2}(\theta p_1)^{\mu_1} *_{\mathrm{e}} (q_1 \wedge p_1)\, \mathrm{e}^{-\mathrm{i}\, q_2 \wedge p_2} =: g^2 V^{\mu_1\mu_2}_{\mathrm{f\bar{f}gg},3}\,. \tag{5.13b}$$

Let us look at the sixth part $\bar{\psi}^{[2]}(\mathrm{i}\,\not{\partial} - m)\psi^{[2]}$. Here we have two leading order matter maps. Because the partial derivative $\not{\partial}$ acts not only on the matter field within the $\psi^{[2]}$ but also on the gauge field one gets a term which is not proportional to the equation of motion. Therefore the sixth part is given by

$$\begin{aligned}\bar{\psi}^{[2]}(\mathrm{i}\,\not{\partial} - m)\psi^{[2]} \to -\frac{1}{4}g^2(\theta p_1)^{\mu_1}(\theta p_2)^{\mu_2}\\ *_{\mathrm{e}}(q_1 \wedge p_1) *_{\mathrm{e}} (p_2 \wedge q_2)\,(\not{q}_2 + \not{p}_2 - m) =: g^2 V^{\mu_1\mu_2}_{\mathrm{f\bar{f}gg},6}\,, \quad (5.14)\end{aligned}$$

where the term mentioned above is proportional to $\not{q}_2$.

Now we come to the remaining two parts which depend on the next to leading order matter map $\psi^{[3]}$ (3.57). Let us first look at the derivative $\not{\partial}$ within the two parts. If the matter map $\psi^{[3]}$ is on the r.h.s. then the partial derivative becomes the sum of the two photon momenta and the fermion momentum. With the use of the four momentum conservation $p_1+p_2+q_1+q_2 = 0$ we get in this case $(-\not{p}_1 - m)$. If the map is on the l.h.s. we get for the partial derivative just $(\not{p}_2 - m)$. So all these terms are proportional to the equation of motion of either the matter or anti-matter field

$$\bar{v}(p_1)(-\not{p}_1 - m) = 0\,, \qquad \text{for } p_1^2 = m^2\,, \quad (5.15a)$$

$$(\not{p}_2 - m)u(p_2) = 0\,, \qquad \text{for } p_2^2 = m^2\,. \quad (5.15b)$$

For the case that one fermion or anti-fermion is on-shell the corresponding equation becomes zero (5.15).

The functions F^{ψ} of the bidifferentials, present in the map $\psi^{[3]}$, become now a function of the appropriate momenta. Note that these functions are purely real so if we complex conjugate these functions we only have to complex conjugate the arguments. Therefore, for the fourth part we get

$$\begin{aligned}&\bar{\psi}(\mathrm{i}\,\not{\partial} - m)\psi^{[3]}\\ &\to \frac{g^2}{2}(-\not{p}_1 - m)\Big[F^{\psi}_{\mathrm{I}}(-\mathrm{i}\,q_1\wedge q_2, -\mathrm{i}\,q_1\wedge p_2, -\mathrm{i}\,q_2\wedge p_2)\frac{\theta^{\mu_1\mu_2}}{2}\\ &\quad - F^{\psi}_{\mathrm{II}}(-\mathrm{i}\,q_1\wedge q_2, -\mathrm{i}\,q_1\wedge p_2, -\mathrm{i}\,q_2\wedge p_2)\frac{(q_1\theta)^{\mu_2}}{2}\frac{(\theta q_2)^{\mu_1}}{2}\\ &\quad - F^{\psi}_{\mathrm{III}}(-\mathrm{i}\,q_1\wedge q_2, -\mathrm{i}\,q_1\wedge p_2, -\mathrm{i}\,q_2\wedge p_2)\frac{(q_1\theta)^{\mu_2}}{2}\frac{(\theta p_2)^{\mu_1}}{2}\\ &\quad - F^{\psi}_{\mathrm{IV}}(-\mathrm{i}\,q_1\wedge q_2, -\mathrm{i}\,q_1\wedge p_2, -\mathrm{i}\,q_2\wedge p_2)\frac{(\theta q_2)^{\mu_1}}{2}\frac{(\theta p_2)^{\mu_2}}{2}\\ &- F^{\psi}_{\mathrm{V}}(-\mathrm{i}\,q_1\wedge q_2, -\mathrm{i}\,q_1\wedge p_2, -\mathrm{i}\,q_2\wedge p_2)\frac{(\theta p_2)^{\mu_1}}{2}\frac{(\theta p_2)^{\mu_2}}{2}\Big] =: g^2 V^{\mu_1\mu_2}_{\mathrm{f\bar{f}gg},4}\,.\end{aligned} \quad (5.16)$$

The only difference between the function F^{ψ} of the forth and the fifth part is that the arguments of the functions F have to be complex conjugates and that one has to replace the momenta p_2 of the fermion by the momenta of the anti-fermion p_1 and vice versa. By doing that we get for the fifth part

$$
\begin{aligned}
&\bar{\psi}^{[3]}(\mathrm{i}\,\not{\partial} - m)\psi \\
&\qquad \to \frac{g^2}{2}(\not{p}_2 - m)\Bigg[F^{\psi}_{\mathrm{I}}(\mathrm{i}\,q_1 \wedge q_2, \mathrm{i}\,q_1 \wedge p_1, \mathrm{i}\,q_2 \wedge p_1)\frac{\theta^{\mu_1\mu_2}}{2} \\
&\qquad\quad - F^{\psi}_{\mathrm{II}}(\mathrm{i}\,q_1 \wedge q_2, \mathrm{i}\,q_1 \wedge p_1, \mathrm{i}\,q_2 \wedge p_1)\frac{(q_1\theta)^{\mu_2}}{2}\frac{(\theta q_2)^{\mu_1}}{2} \\
&\qquad\quad - F^{\psi}_{\mathrm{III}}(\mathrm{i}\,q_1 \wedge q_2, \mathrm{i}\,q_1 \wedge p_1, \mathrm{i}\,q_2 \wedge p_1)\frac{(q_1\theta)^{\mu_2}}{2}\frac{(\theta p_1)^{\mu_1}}{2} \\
&\qquad\quad - F^{\psi}_{\mathrm{IV}}(\mathrm{i}\,q_1 \wedge q_2, \mathrm{i}\,q_1 \wedge p_1, \mathrm{i}\,q_2 \wedge p_1)\frac{(\theta q_2)^{\mu_1}}{2}\frac{(\theta p_1)^{\mu_2}}{2} \\
&\quad - F^{\psi}_{\mathrm{V}}(\mathrm{i}\,q_1 \wedge q_2, \mathrm{i}\,q_1 \wedge p_1, \mathrm{i}\,q_2 \wedge p_1)\frac{(\theta p_1)^{\mu_1}}{2}\frac{(\theta p_1)^{\mu_2}}{2}\Bigg] =: g^2 V^{\mu_1\mu_2}_{\bar{\mathrm{f}}\mathrm{fgg},5}\,. \qquad (5.17)
\end{aligned}
$$

If we combine our results we obtain the final expression for the $\bar{\mathrm{f}}$fgg vertex, namely

$$
\begin{aligned}
V^{\mu_1\mu_2}_{\bar{\mathrm{f}}\mathrm{fgg}}(p_1, p_2, q_1, q_2) = \mathrm{i}\,g^2 \sum_{n=1}^{6} V^{\mu_1\mu_2}_{\bar{\mathrm{f}}\mathrm{fgg},n}(p_1, p_2, q_1, q_2) \\
+ (q_1, \mu_1) \leftrightarrow (q_2, \mu_2)\,, \qquad (5.18)
\end{aligned}
$$

where we now have considered also the terms which one gets by exchanging the two photons.

Again if we look at the commutative limit we know from the discussions in the last two sections that the first, second, third and sixth term vanish. Now what is with the functions F^{ϕ} in the fourth and fifth term? As one can see the series expansions of these functions have only non-negative powers of $\theta^{\mu\nu}$ so that these function will vanish, too. This means that the whole contact vertex vanishes for $|\theta^{\mu\nu}| \to 0$, as expected.

5.5 Remarks

In this chapter we have calculated three Feynman vertices which allow us to calculate almost every two to two cross sections at Born level. The vertex which was not calculated is the 4-photon vertex. The reason is that the needed next to leading order gauge field map $A^{[3]}_{\mu}$ becomes very complex and lengthy at least in the general form. The problem is that we do not know how we can get a more compact and thus more manageable map.

Nevertheless with the Feynman rules derived in this chapter we can calculate the pair annihilation process $e^+e^- \to \gamma\gamma$ which we will tackle in the next chapter.

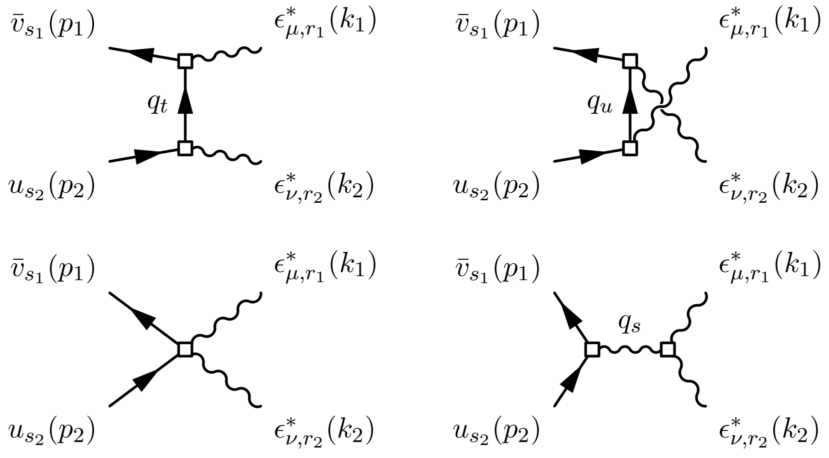
Chapter 6

Scattering Process

After we had calculated the Seiberg-Witten maps of the various fields in chapter 3 we were able to derive the Feynman rules in the preceding chapter. The next step is to examine a concrete scattering process. We choose the electron-position annihilation process where a fermion and an anti-fermion pair annihilate into two photons. We choose this process because already at Born level two new vertices appear, namely the 3-photon and the contact vertex.

6.1 The Amplitude of $f\bar{f} \to \gamma\gamma$

As we just said we want to calculate the amplitude for the process $f\bar{f} \to \gamma\gamma$. In addition to the t- and u-channel which already exist in the ordinary QED one also gets a s-channel and a contact vertex.



The contact vertex, also denoted as c-channel, is totally new and is a particularity of noncommutative theories with Seiberg-Witten maps whereas the s-channel is already present in noncommutative theories without the maps.

We consider an incoming fermion with momentum p_2 and spin s_2 and an incoming anti-fermion with momentum p_1 and spin s_1. These fermions annihilate into two outgoing photons with momentum k_1 and polarisation r_1 and with momentum k_2 and polarisation r_2, respectively. The momentum flow inside the t-, u- and s-channel is denoted by q with an appropriate lower index. These internal momenta depend on the external one and are defined as usual, namely

$$q_s = p_1 + p_2\,, \quad p_t = p_2 - k_2\,, \quad q_u = p_2 - k_1\,, \quad p_1 + p_2 = k_1 + k_2\,. \tag{6.1}$$

With the Feynman rules calculated in the previous chapter we can write down the corresponding amplitudes to the above Feynman graphs

$$\begin{aligned}\mathcal{A}_t^{\mu\nu}(s_1,s_2)\epsilon^*_{\mu,r_1}(k_1)\epsilon^*_{\nu,r_2}(k_2) &= \bar{v}_{s_1}(p_1)V^{\mu}_{\bar{\mathrm{f}}\mathrm{fg}}(p_1,q_t,-k_1)\epsilon^*_{\mu,r_1}(k_1)\\ &\qquad D_{\mathrm{f}}(q_t)\epsilon^*_{\nu,k_2}(k_2)V^{\nu}_{\bar{\mathrm{f}}\mathrm{fg}}(-q_t,p_2,-k_2)u_{s_2}(p_2)\,,\end{aligned} \tag{6.2a}$$

$$\begin{aligned}\mathcal{A}_u^{\mu\nu}(s_1,s_2)\epsilon^*_{\mu,r_1}(k_1)\epsilon^*_{\nu,r_2}(k_2) &= \bar{v}_{s_1}(p_1)V^{\mu}_{\bar{\mathrm{f}}\mathrm{fg}}(p_1,q_u,-k_2)\epsilon^*_{\mu,k_2}(k_2)\\ &\qquad D_{\mathrm{f}}(q_u)\epsilon^*_{\nu,k_1}(k_1)V^{\nu}_{\bar{\mathrm{f}}\mathrm{fg}}(-q_u,p_2,-k_1)u_{s_2}(p_2)\,,\end{aligned} \tag{6.2b}$$

$$\begin{aligned}\mathcal{A}_c^{\mu\nu}(s_1,s_2)\epsilon^*_{\mu,r_1}(k_1)\epsilon^*_{\nu,r_2}(k_2) &= \bar{v}_{s_1}(p_1)\epsilon^*_{\mu,k_1}(k_1)\\ &\qquad V^{\mu\nu}_{\bar{\mathrm{f}}\mathrm{fgg}}(p_1,p_2,-k_1,-k_2)\epsilon^*_{\nu,k_2}(k_2)u_{s_2}(p_2)\,,\end{aligned} \tag{6.2c}$$

$$\begin{aligned}\mathcal{A}_s^{\mu\nu}(s_1,s_2)\epsilon^*_{\mu,r_1}(k_1)\epsilon^*_{\nu,r_2}(k_2) &= \bar{v}_{s_1}(p_1)V^{\rho}_{\bar{\mathrm{f}}\mathrm{fg}}(p_1,p_2,-q_s)u_{s_2}(p_2)\\ &\qquad D_{\mathrm{g},\rho\sigma}(q_s)V^{\sigma\mu\nu}_{\mathrm{ggg}}(q_s,-k_1,-k_2)\epsilon^*_{\mu,k_1}(k_1)\epsilon^*_{\nu,k_2}(k_2)\,,\end{aligned} \tag{6.2d}$$

which we now want to consider in detail. In the following we won't write out the spin index of the amplitudes, which is always s_1 and s_2.

6.2 The t- and u-Channel

First of all let us look somewhat closer at the t-channel amplitude. It contains two $\bar{\mathrm{f}}\mathrm{fg}$ vertices where the lower one includes the incoming fermion with momentum p_2 the outgoing photon with $-k_2$ and an outgoing anti-fermion with momentum $-q_t = k_2 - p_2$. The intermediate fermion is off the mass shell where the incoming and outgoing particles are on-shell. With these momenta the vertex becomes

$$\begin{aligned}V^{\nu}_{\bar{\mathrm{f}}\mathrm{fg}}(q_t,p_2,-k_2) &= \mathrm{i}\,g\gamma^{\nu}\,\mathrm{e}^{-\mathrm{i}\,k_2\wedge p_2}\\ &\quad + \frac{g}{2} *_{\mathrm{e}} (k_2\wedge p_2)\,(k_2\theta)^{\nu}(\not{p}_2 - m) + \frac{g}{2} *_{\mathrm{e}} (k_2\wedge p_2)\,(p_2\theta)^{\nu}(-\not{k}_2)\,,\end{aligned} \tag{6.3}$$

where the second term is proportional to $\not{p}_2 - m$ which is zero due to the equation of motion if one multiplies this expression with the spinor $u(p_2)$. In an analogue way we get for the upper vertex with the momentum q_t flowing now into the vertex

$$\begin{aligned}V^{\mu}_{\bar{\mathrm{f}}\mathrm{fg}}(p_1, q_t, -k_1) &= \mathrm{i}\, g\gamma^{\mu}\, \mathrm{e}^{\mathrm{i}\, p_1\wedge k_1} \\ &+ \frac{g}{2} *_{\mathrm{e}} (p_1\wedge k_1)\,(p_1\theta)^{\mu}\not{q}_1 + \frac{g}{2} *_{\mathrm{e}} (p_1\wedge k_1)\,(k_1\theta)^{\nu}(-\not{p}_1 - m)\,, \quad (6.4)\end{aligned}$$

where the last term becomes again zero because the external anti-fermion is on-shell. After knowing the two vertices of the t-channel, we can multiply them with the propagator. After some simplifications we obtain four compact terms for the t-channel, namely

$$\begin{aligned}\mathcal{A}^{\mu\nu}_t = -g^2 \Bigg[&\frac{\mathrm{i}}{4} *_{\mathrm{e}} (p_1\wedge k_1) *_{\mathrm{e}} (k_2\wedge p_2)\,(p_1\theta)^{\mu}(p_2\theta)^{\nu}\bar{v}_{s_1}(p_1)\not{q}_2 u_{s_2}(p_2) \\ &+ \frac{1}{2} *_{\mathrm{e}} (p_1\wedge k_1)\,\mathrm{e}^{\mathrm{i}\, k_2\wedge p_2}(p_1\theta)^{\mu}\bar{v}_{s_1}(p_1)\gamma^{\nu}u_{s_2}(p_2) \\ &+ \frac{1}{2} *_{\mathrm{e}} (k_2\wedge p_2)\,\mathrm{e}^{\mathrm{i}\, p_1\wedge k_1}(p_2\theta)^{\nu}\bar{v}_{s_1}(p_1)\gamma^{\mu}u_{s_2}(p_2) \\ &+ \frac{\mathrm{i}}{t}\,\mathrm{e}^{\mathrm{i}(p_1\wedge k_1 + k_2\wedge p_2)}\,\bar{v}_{s_1}(p_1)\gamma^{\mu}(\not{p}_t + m)\gamma^{\nu}u_{s_2}(p_2)\Bigg]\,. \quad (6.5)\end{aligned}$$

Note that the last term is exactly the same as in noncommutative theories without Seiberg-Witten maps. If we look at the limit $|\theta^{\mu\nu}| \to 0$ we see that the first three terms vanish while the phase in the last term becomes just unity so that we get the ordinary QED as it should be.

The u-channel is identical to the t-channel if one exchanges (k_1, μ) with (k_2, ν). So we give only the result, which is

$$\begin{aligned}\mathcal{A}^{\mu\nu}_u = -g^2 \Bigg[&\frac{\mathrm{i}}{4} *_{\mathrm{e}} (p_1\wedge k_2) *_{\mathrm{e}} (k_1\wedge p_2)\,(p_1\theta)^{\nu}(p_2\theta)^{\mu}\bar{v}_{s_1}(p_1)\not{q}_1 u_{s_2}(p_2) \\ &+ \frac{1}{2} *_{\mathrm{e}} (p_1\wedge k_2)\,\mathrm{e}^{\mathrm{i}\, k_1\wedge p_2}(p_1\theta)^{\nu}\bar{v}_{s_1}(p_1)\gamma^{\mu}u_{s_2}(p_2) \\ &+ \frac{1}{2} *_{\mathrm{e}} (k_1\wedge p_2)\,\mathrm{e}^{\mathrm{i}\, p_1\wedge k_2}(p_2\theta)^{\mu}\bar{v}_{s_1}(p_1)\gamma^{\nu}u_{s_2}(p_2) \\ &+ \frac{\mathrm{i}}{u}\,\mathrm{e}^{\mathrm{i}(p_1\wedge k_2 + k_1\wedge p_2)}\,\bar{v}_{s_1}(p_1)\gamma^{\nu}(\not{p}_u + m)\gamma^{\mu}u_{s_2}(p_2)\Bigg]\,. \quad (6.6)\end{aligned}$$

6.3 The s-Channel

Now we come to the s-channel. Here, we have one $\bar{\mathrm{f}}\mathrm{fg}$-vertex and a 3-photon vertex which is complex at first sight due to the indistinguishability of the three photons, i.e. of all the permutations of the momenta and the indices. But if one performs the calculation of $V^{\sigma\mu\nu}_{\mathrm{ggg}}(q_s, -k_1, -k_2)$ one sees that many terms vanish because the two external photons are on-shell, i.e. $k_{1,2}^2 = 0$ and physically polarized, i.e. $k_{1,2}^\mu \epsilon_\mu^*(k_{1,2}) = 0$. But before we look at the 3-photon vertex let us write down the f$\bar{\text{f}}$g-vertex which in the case of the s-channel is only the term with the phase factor, since now both fermions are on-shell and thus the second and third term are zero

$$V^\rho_{\mathrm{f\bar{f}g}}(p_1, p_2, -q_s) = \mathrm{i}\, g\gamma^\rho\, \mathrm{e}^{\mathrm{i}\, p_1\wedge p_2} \ . \tag{6.7}$$

Coming to the 3-photon vertex we first want to count the number of terms we have overall. The vertex (5.11) is composed out of two parts $V_{\mathrm{ggg},1}$ and $V_{\mathrm{g}} \cdot V_{\mathrm{ggg},2}$. Due to the permutations of the photons each part will give 3! terms. Thus altogether we have twelve terms. But as we already said above, a lot of terms will vanish, because the two external photons are on the mass shell and physically polarized. In the first part of the 3-photon vertex one only use that the external photons are physically polarized. With this property we get for the part which is independent of the Seiberg-Witten maps

$$\begin{aligned} &V^{\sigma\mu\nu}_{\mathrm{ggg},\,1}(q_s, -k_1, -k_2) + \text{all permutations} \\ &\qquad = 2\,\mathrm{i}\, g^3 \sin(k_1\wedge k_2)\left[2k_1^\nu g^{\mu\sigma} - 2k_2^\mu g^{\nu\sigma} + (k_2 - k_1)^\sigma g^{\mu\nu}\right] \ . \end{aligned} \tag{6.8a}$$

This is basically the s-channel amplitude of the NCQED without the maps. The second part of the 3-photon vertex becomes quite a bit larger, so that we will spit this part into three terms. The first term contains only the $*_{\mathrm{s}}$-function

$$\begin{aligned} &-\mathrm{i}\, g V_{\mathrm{g}}^{\sigma\alpha}(q_s)\left(-\mathrm{i}\, g^2 *_{\mathrm{s}} (-k_1\wedge k_2)(-k_1\theta)^\nu g^\mu_\alpha\right) + \text{all permutations} \\ &\qquad = -\mathrm{i}\, g^3 *_{\mathrm{s}} (k_1\wedge k_2)\left[\left\{2(k_1k_2)g^{\mu\sigma} - \left(1-\frac{1}{\xi}\right) q_s^\mu q_s^\sigma\right\}(k_1\theta)^\nu \right. \\ &\qquad\qquad \left. + \left\{2(k_1k_2)g^{\nu\sigma} - \left(1-\frac{1}{\xi}\right) q_s^\nu q_s^\sigma\right\}(k_2\theta)^\mu\right] \ . \end{aligned} \tag{6.8b}$$

Because $V_{\mathrm{g}}^{\mu\nu}(p)$ is proportional to the fourth power of its argument, it will vanish if p becomes an on-shell photon momentum. So only the terms which are proportional to $V_{\mathrm{g}}^{\sigma\alpha}(q_s)$ survive. In the term, which is proportional to $1-\xi^{-1}$, the external momenta are multiplied with each of the two polarisation vectors. Thus, one product becomes always zero.

The second term contains only the $*_{\mathrm{f}}$-function

$$\begin{aligned} &-\mathrm{i}\, g V_{\mathrm{g}}^{\sigma\alpha}(q_s)\left(\frac{\mathrm{i}}{2} g^2 *_{\mathrm{f}} (-k_1\wedge k_2)\theta^{\mu\nu}(-k_1)_\alpha\right) + \text{all permutations} \\ &\qquad = -\frac{\mathrm{i}}{2} g^3 (k_1k_2)\theta^{\mu\nu}\left[*_{\mathrm{f}}(k_1\wedge k_2)\left\{2k_2^\sigma - \left(1-\frac{1}{\xi}\right) q_s^\sigma\right\}\right. \end{aligned}$$

$$- *_{\mathrm{f}}(-k_1 \wedge k_2)\left\{2k_1^\sigma - \left(1-\frac{1}{\xi}\right) q_s^\sigma\right\}\Bigg] \,. \tag{6.8c}$$

Again only the terms with the internal momentum in V_{g} survive. What one can also see is that the function $*_{\mathrm{f}}$ is present once with the positive and once with the negative argument. This is due to the fact that the function $*_{\mathrm{f}}$ is arbitrary therefore it is not known if it is an even or odd function or a mixture of both.

Now we come to the last term, namely the term which contains the difference of the $*_{\mathrm{s}}$- and $*_{\mathrm{f}}$-function. The function V_{g} is contracted with the same momenta as in the previous cases so that we can take over this part

$$\begin{aligned} &- \mathrm{i}\, g V_{\mathrm{g}}^{\sigma\alpha}(q_s)\left(\frac{\mathrm{i}}{4} g^2 \frac{(*_{\mathrm{s}} - *_{\mathrm{f}})(-k_1 \wedge k_2)}{k_1 \wedge k_2}(k_1\theta)^\nu(\theta k_2)^\mu(-k_1)_\alpha\right) \\ &\qquad + \text{all permutations} = -\frac{\mathrm{i}}{4} g^3 (k_1 k_2)(k_1\theta)^\nu(\theta k_2)^\mu \\ &\qquad \Bigg[\frac{(*_{\mathrm{s}} - *_{\mathrm{f}})(k_1 \wedge k_2)}{k_1 \wedge k_2}\left\{2k_2^\sigma - \left(1-\frac{1}{\xi}\right) q_s^\sigma\right\} \\ &\qquad\quad - \frac{(*_{\mathrm{s}} - *_{\mathrm{f}})(-k_1 \wedge k_2)}{k_1 \wedge k_2}\left\{2k_1^\sigma - \left(1-\frac{1}{\xi}\right) q_s^\sigma\right\}\Bigg] \,. \end{aligned} \tag{6.8d}$$

Of course we know that the $*_{\mathrm{s}}$-function is even with respect to its argument so that we could combine this function but to keep the existing structure we leave them as they are.

Note that all terms which come from the Seiberg-Witten maps and which are nonzero at the end include the Mandelstam variable s on the numerator which cancels the s^{-1} of the photon propagator. So these terms do no longer depend on the pole of the s-channel. As we shall see, we will get the same expressions form the contact term up to the factor $k_{\gamma\gamma\gamma}$. For a special choice of this parameter this will lead to a cancellation of these terms. This means, that only the first term, namely the one already present in NCQED without Seiberg-Witten maps, would survive in this special case.

After we have calculated the two vertices of the s-channel we will combine them. But before we do this, let us look at those terms which contain a $\not{q}_s$. After a short calculation one sees that these terms vanish due to the equation of motion of the fermions

$$\bar{v}_{s_1}(p_1)\not{q}_s u_{s_2}(p_2) = \bar{v}_{s_1}(p_1)(\not{p}_1 + m + \not{p}_2 - m)u_{s_2}(p_2) = 0 \,. \tag{6.9}$$

Of course this also allows us to exchange $\not{k}_1$ by $-\not{k}_2$ due to momentum conservation. If we use the above equation we could again simplify our result, namely all terms in (6.8d) which are proportional to q_s^ρ vanish.

After all the calculations done in this section we finally combine all terms with the same spinor structure. The first part is

$$\mathcal{A}^{\mu\nu}_{s,\mathrm{I}} = g^2 k_{\gamma\gamma\gamma} \left[-\frac{4}{s} \sin(k_1 \wedge k_2) g^{\mu\nu} + \frac{1}{2} \left\{ *_{\mathrm{f}}(k_1 \wedge k_2) + *_{\mathrm{f}}(-k_1 \wedge k_2) \right\} \theta^{\mu\nu} \right.$$
$$\left. + \frac{1}{4} \left\{ \frac{(*_{\mathrm{s}} - *_{\mathrm{f}})(k_1 \wedge k_2)}{k_1 \wedge k_2} + \frac{(*_{\mathrm{s}} - *_{\mathrm{f}})(-k_1 \wedge k_2)}{k_1 \wedge k_2} \right\} (\theta k_2)^\mu (k_1 \theta)^\nu \right]$$
$$\mathrm{e}^{\mathrm{i}\, p_1 \wedge p_2}\, \bar{v}_{s_1}(p_1) \not{k}_2 u_{s_2}(p_2)\,, \quad (6.10\mathrm{a})$$

which is the only part where the arbitrary function $*_{\mathrm{f}}$ appears. The second and third part contain only the sine and the $*_{\mathrm{s}}$-function

$$\mathcal{A}^{\mu\nu}_{s,\mathrm{II}} = g^2 k_{\gamma\gamma\gamma} \left[-\frac{4}{s} \sin(k_1 \wedge k_2) k_1^\nu + *_{\mathrm{s}}(k_1 \wedge k_2)(k_1 \theta)^\nu \right]$$
$$\mathrm{e}^{\mathrm{i}\, p_1 \wedge p_2}\, \bar{v}_{s_1}(p_1) \gamma^\mu u_{s_2}(p_2)\,, \quad (6.10\mathrm{b})$$

$$\mathcal{A}^{\mu\nu}_{s,\mathrm{III}} = g^2 k_{\gamma\gamma\gamma} \left[\frac{4}{s} \sin(k_1 \wedge k_2) k_2^\mu - *_{\mathrm{s}}(k_1 \wedge k_2)(\theta k_2)^\mu \right]$$
$$\mathrm{e}^{\mathrm{i}\, p_1 \wedge p_2}\, \bar{v}_{s_1}(p_1) \gamma^\nu u_{s_2}(p_2)\,, \quad (6.10\mathrm{c})$$

where the complete s-channel amplitude is then of course $\mathcal{A}^{\mu\nu}_s = \mathcal{A}^{\mu\nu}_{s,\mathrm{I}} + \mathcal{A}^{\mu\nu}_{s,\mathrm{II}} + \mathcal{A}^{\mu\nu}_{s,\mathrm{III}}$. Now let us discuss the above result. If we look at the s-channel pole, we observe that it appears only in terms which are proportional to the sine. All terms coming from the Seiberg-Witten maps contain no s-channel pole. Hence, they are similar to a Feynman graph without a propagator i.e. to a contact graph. And as we will see in the next section the contact amplitude contains many similar terms. What we also observe is that $\mathcal{A}_{s,\mathrm{III}}$ becomes $\mathcal{A}_{s,\mathrm{II}}$ if one exchanges (μ, k_1) with (ν, k_2) and vice versa. The first part remains the same under this exchange. This is not really amazing and actually it has to be because the two outgoing photons are indistinguishable. Therefore the whole s-channel is symmetric under the exchange of $(\mu, k_1) \leftrightarrow (\nu, k_2)$, as expected.

6.4 The Contact Vertex

In this section we will calculate the last contribution to the amplitude which we have to consider, namely the contact vertex $V^{\mu\nu}_{\bar{\mathrm{f}}\mathrm{fgg}}(p_1, p_2, -k_1, -k_2)$. As mentioned during the calculation of the Feynman rules in the previous chapter the fifth, sixth and one half of the fourth term of the vertex becomes zero. The reason is that these terms are proportional to the equations of motion of the fermions. This holds of course only if the fermions are on-shell as it is the case in our calculation. Before we will discuss the first part of the vertex let us write down the second, third and the remaining half of the fourth part. The calculation is straightforward so that we just present the result

$$\mathcal{A}^{\mu\nu}_{c,\mathrm{II}} = -\frac{g^2}{2}\Big[\mathrm{e}^{\mathrm{i}\,p_1\wedge k_1} *_{\mathrm{e}} (k_2\wedge p_2)\,(\theta p_2)^\nu \bar{v}_{s_1}(p_1)\gamma^\mu u_{s_2}(p_2) + \mathrm{e}^{\mathrm{i}\,p_1\wedge k_2} *_{\mathrm{e}} (k_1\wedge p_2)\,(\theta p_2)^\mu \bar{v}_{s_1}(p_1)\gamma^\nu u_{s_2}(p_2)\Big]\,, \quad (6.11\mathrm{a})$$

$$\mathcal{A}^{\mu\nu}_{c,\mathrm{III}} = -\frac{g^2}{2}\Big[\mathrm{e}^{\mathrm{i}\,k_2\wedge p_2} *_{\mathrm{e}} (p_1\wedge k_1)\,(\theta p_1)^\mu \bar{v}_{s_1}(p_1)\gamma^\nu u_{s_2}(p_2) + \mathrm{e}^{\mathrm{i}\,k_1\wedge p_2} *_{\mathrm{e}} (p_1\wedge k_2)\,(\theta p_1)^\nu \bar{v}_{s_1}(p_1)\gamma^\mu u_{s_2}(p_2)\Big]\,, \quad (6.11\mathrm{b})$$

$$\mathcal{A}^{\mu\nu}_{c,\mathrm{IV}} = \mathrm{i}\,\frac{g^2}{4}\Big[*_{\mathrm{e}} (p_1\wedge k_1) *_{\mathrm{e}} (k_2\wedge p_2)\,(\theta p_1)^\mu(\theta p_2)^\nu \bar{v}_{s_1}(p_1)\not{k}_2 u_{s_2}(p_2) + *_{\mathrm{e}} (p_1\wedge k_2) *_{\mathrm{e}} (k_1\wedge p_2)\,(\theta p_1)^\nu(\theta p_2)^\mu \bar{v}_{s_1}(p_1)\not{k}_1 u_{s_2}(p_2)\Big]\,. \quad (6.11\mathrm{c})$$

Each term above contains two summands which differ by the interchange $(\mu, k_1) \leftrightarrow (\nu, k_2)$. Of course the reason is, that the photons are bosons.

Now let us look at the first term of our t- and u-channel (6.5) and (6.6). These terms are exactly the terms we obtained in $\mathcal{A}_{c,\mathrm{IV}}$ but with the opposite sign. Not only this but also the second and the third term in the t- and u-channel are exactly the same terms we have in the part of the c-channel amplitude $\mathcal{A}_{c,\mathrm{II}}$ and $\mathcal{A}_{c,\mathrm{III}}$, again with the opposite sign. So if we merge $\mathcal{A}_t$ and $\mathcal{A}_u$ with the appropriate terms of $\mathcal{A}_c$ then only two terms survive, namely

$$\mathcal{A}^{\mu\nu}_{t+c} = \frac{\mathrm{i}}{t} g^2\, \mathrm{e}^{\mathrm{i}(p_1\wedge k_1 + k_2\wedge p_2)}\, \bar{v}_{s_1}(p_1)\gamma^\mu(\not{p}_t - m)\gamma^\nu u_{s_2}(p_2)\,, \quad (6.12)$$

$$\mathcal{A}^{\mu\nu}_{u+c} = \frac{\mathrm{i}}{u} g^2\, \mathrm{e}^{\mathrm{i}(p_1\wedge k_2 + k_1\wedge p_2)}\, \bar{v}_{s_1}(p_1)\gamma^\nu(\not{p}_u - m)\gamma^\mu u_{s_2}(p_2)\,. \quad (6.13)$$

This means that the only terms which are non-zero are precisely those which don't have contributions from the Seiberg-Witten maps, i.e. the amplitudes $\mathcal{A}_{t+c}$ and $\mathcal{A}_{u+c}$ are exactly the corresponding t- and u-channel amplitudes of the noncommutative QED without Seiberg-Witten maps.

The question which now arises concerns the remaining part of the c-channel, namely $\mathcal{A}_{c,\mathrm{I}}$, and the s-channel. To answer this question we first have to calculate the remaining $\mathcal{A}_{c,\mathrm{I}}$ which we split up into three pieces

$$\mathcal{A}^{\mu\nu}_{c,\mathrm{I}a} = -g^2\, \mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}\, [*_{\mathrm{s}}(k_1\wedge k_2)(k_1\theta)^\nu \bar{v}_{s_1}(p_1)\gamma^\mu u_{s_2}(p_2) + *_{\mathrm{s}}\,(k_1\wedge k_2)(k_2\theta)^\mu \bar{v}_{s_1}(p_1)\gamma^\nu u_{s_2}(p_2)]\,, \quad (6.14\mathrm{a})$$

$$\mathcal{A}^{\mu\nu}_{c,\mathrm{I}b} = \frac{g^2}{2}\, \mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}\, \{*_{\mathrm{f}}(k_1\wedge k_2) + *_{\mathrm{f}}(-k_1\wedge k_2)\}\, \theta^{\mu\nu} \bar{v}_{s_1}(p_1)\not{k}_1 u_{s_2}(p_2)\,, \quad (6.14\mathrm{b})$$

$$\mathcal{A}^{\mu\nu}_{c,\mathrm{I}c} = \frac{g^2}{4}\,\mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}\left\{\frac{(*_\mathrm{s}-*_\mathrm{f})(k_1\wedge k_2)}{k_1\wedge k_2}+\frac{(*_\mathrm{s}-*_\mathrm{f})(-k_1\wedge k_2)}{k_1\wedge k_2}\right\}$$
$$(k_1\theta)^\nu(\theta k_2)^\mu\bar{v}_{s_1}(p_1)\not{k}_1 u_{s_2}(p_2)\,. \quad (6.14c)$$

Now let us combine the above three pieces with the corresponding parts of the s-channel. Since $\mathcal{A}_s$ is proportional to the parameter $k_{\gamma\gamma\gamma}$ it is in general not possible to have a cancellation as we had in the case of the t- and u-channel. But if we combine $\mathcal{A}_{c,\mathrm{I}b}$, $\mathcal{A}_{c,\mathrm{I}c}$ and $\mathcal{A}_{s,\mathrm{I}}$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{s+c,\mathrm{I}} = -g^2\,\mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}\Bigg[&-k_{\gamma\gamma\gamma}\frac{4}{s}\sin(k_1\wedge k_2)g^{\mu\nu}\\ &+\frac{1}{4}(k_{\gamma\gamma\gamma}-1)\left\{\frac{(*_\mathrm{s}-*_\mathrm{f})(k_1\wedge k_2)}{k_1\wedge k_2}+\frac{(*_\mathrm{s}-*_\mathrm{f})(-k_1\wedge k_2)}{k_1\wedge k_2}\right\}(\theta k_2)^\mu(k_1\theta)^\nu\\ &+\frac{1}{2}(k_{\gamma\gamma\gamma}-1)\left\{*_\mathrm{f}(k_1\wedge k_2)+*_\mathrm{f}(-k_1\wedge k_2)\right\}\theta^{\mu\nu}\Bigg]\bar{v}_{s_1}(p_1)\not{k}_1 u_{s_2}(p_2)\,, \quad (6.15a)\end{aligned}$$

we see that the second and third term in the above expression vanish for the special choice $k_{\gamma\gamma\gamma}=1$. The only term we would then have is the term which has no contribution from the Seiberg-Witten map. Thus we obtain again only the contribution which we already had in the noncommutative QED without Seiberg-Witten maps. The same is true for the other two parts. So if we combine $\mathcal{A}_{c,\mathrm{I}a}$ with $\mathcal{A}_{s,\mathrm{II}}$ and $\mathcal{A}_{s,\mathrm{III}}$ we obtain

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{s+c,\mathrm{II}} = -g^2\,\mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}\Bigg[&k_{\gamma\gamma\gamma}\frac{4}{s}\sin(k_1\wedge k_2)k_1^\nu\\ &-(k_{\gamma\gamma\gamma}-1)*_\mathrm{s}(k_1\wedge k_2)(k_1\theta)^\nu\Bigg]\bar{v}_{s_1}(p_1)\gamma^\mu u_{s_2}(p_2)\,, \quad (6.15b)\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{\mu\nu}_{s+c,\mathrm{III}} = -g^2\,\mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}\Bigg[&-k_{\gamma\gamma\gamma}\frac{4}{s}\sin(k_1\wedge k_2)k_2^\mu\\ &+(k_{\gamma\gamma\gamma}-1)*_\mathrm{s}(k_1\wedge k_2)(\theta k_2)^\mu\Bigg]\bar{v}_{s_1}(p_1)\gamma^\nu u_{s_2}(p_2)\,. \quad (6.15c)\end{aligned}$$

Thus we have for the s-channel plus the appropriate c-channel amplitude three parts with different spinor structures where in each part one term is proportional to $k_{\gamma\gamma\gamma}$ and one is proportional to $(k_{\gamma\gamma\gamma}-1)$. As mentioned the terms which are proportional to $k_{\gamma\gamma\gamma}$ are exactly those which have no map dependencies.

6.5 Differential Cross Section

Before we come to the differential cross section we will summarise the whole amplitude of the process $e^+e^-\to\gamma\gamma$ at Born level, which is

$$\mathcal{A}^{\mu\nu} = \mathcal{A}^{\mu\nu}_{t+c}+\mathcal{A}^{\mu\nu}_{u+c}+\mathcal{A}^{\mu\nu}_{s+c,\mathrm{I}}+\mathcal{A}^{\mu\nu}_{s+c,\mathrm{II}}+\mathcal{A}^{\mu\nu}_{s+c,\mathrm{III}}\,, \quad (6.16)$$

where the above amplitudes are listed in (6.12), (6.13) and (6.15). This model is gauge invariant by construction and thus the amplitude satisfies the Ward identity. The unphysical photon polarisation modes coming from the s-channel are absorbed by parts of the contact vertex.

With this amplitude the unpolarized differential cross section is in the massless limit

$$\frac{\mathrm{d}\sigma(s_1, s_2, r_1, r_2)}{\mathrm{d}\Omega} = \frac{\left|\mathcal{A}^{\mu\nu}(s_1, s_2)\epsilon^*_{\mu,r_1}(k_1)\epsilon^*_{\nu,r_2}(k_2)\right|^2}{64\pi^2 s}\,. \tag{6.17}$$

To clarify that the amplitude depends on the spins of the fermions we explicitly write down this dependency. The massless limit is sufficient because the noncommutative scale is at least around the TeV scale.

6.6 Discussion

For the special choice of our free parameter $k_{\gamma\gamma\gamma}$ we get an surprising result: independently of the function $*_{\mathrm{f}}$ we get exactly the amplitude of the NCQED without Seiberg-Witten maps. This means that the NCQED without Seiberg-Witten maps is a special case of the NCQED with Seiberg-Witten maps, at least for the considered process. This is a remarkable result which gives us the opportunity to compare our results to that of the NCQED without the maps by only setting $k_{\gamma\gamma\gamma}$ to unity.

In general one can say that the trigonometric functions present in $\mathcal{A}^{\mu\nu}_{s+c}$ will lead to an oscillating cross section. The frequency of these oscillations depends on the square of the center of mass energy, since the argument contains two photon momenta. In addition to the trigonometric functions there are only phase factors present in the amplitude. So one would naively assume that the cross section is well-behaved and converges like s^{-1} for $s \to \infty$. But there exist cases where the cross section diverges, at least at first sight. The investigation of theses cases is the topic of the next chapter.

Chapter 7

Tree Level Unitarity

In this chapter we want to answer the question whether the cross section calculated in the previous chapter is unitary in the sense of tree-level-unitarity and wether the function $*_{\mathrm{f}}$ can be constrained if we postulate the tree level unitarity for the cross section. If so, how does these constraints look like?

First of all let us clarify what we mean by tree level unitarity. By tree level unitarity we mean that the Froissart-Martin bound [35, 36] is satisfied, which basically states that a total cross section must not increase faster than

$$\sigma_{\mathrm{tot}}(s) \leq \mathrm{const} \cdot \log^2 \frac{s}{s_0}, \quad \text{for } s \to \infty\,, \tag{7.1}$$

where $\sqrt{s}$ is the center-of-mass energy.

Before we will think about the question wether the cross section satisfies tree level unitarity and how the $*_{\mathrm{f}}$-function can be constrained from the above bound let us first introduce some notation. In analogy to the usual decomposition of the field-strength tensor of ordinary electrodynamics into an electric and magnetic field we want introduce the vectors $\vec{E}$ and $\vec{B}$ which are build up out of $\theta^{\mu\nu}$ in the following manner

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}\,. \tag{7.2}$$

In order to obtain concrete expressions for the four momenta we introduce the center-of-mass system as our coordinate system. The momenta of the fermions are assumed to be along the z-axis. In this frame they are

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)\,, \qquad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)\,. \tag{7.3}$$

Consequently, the momenta of the outgoing photons are

$$k_1 = \frac{\sqrt{s}}{2}(1, \sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,, \tag{7.4a}$$

$$k_2 = \frac{\sqrt{s}}{2}(1, -\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, -\cos\vartheta)\,, \tag{7.4b}$$

where ϑ is the polar and φ is the azimuthal angle. Hence the argument of the trigonometric functions and the $*_\mathrm{f}$-function can be written as

$$k_1 \wedge k_2 = \omega(\vartheta, \varphi, \vec{E})s = \big(E_3\cos\vartheta + \big[E_1\cos\varphi + E_2\sin\varphi\big]\sin\vartheta\big)\ s\,, \tag{7.5}$$

where we have separated the angle function ω from s.

The argument $k_1\theta k_2$ is the only one which appears in the trigonometric functions and the $*_\mathrm{f}$-function (6.12), (6.13) and (6.15). The other momentum combinations, such as $p_1\wedge p_2$ or $k_1\wedge p_2$, appear only in various phase factors. It is obvious that phase factors can't influence the high energy behavior of an amplitude. Hence they aren't relevant for the following considerations.

7.1 Constraints

Let us remember what we know so far about the function $*_\mathrm{f}$. We have already forbidden that this function contains negative powers of $\theta^{\mu\nu}$. Otherwise the Seiberg-Witten maps wouldn't have a commutative limit. We also observe that the amplitude of the scattering process discussed in the previous chapter includes only the even part of the $*_\mathrm{f}$ function, i.e. only the sum $*_\mathrm{f}(k_1\wedge k_2) + *_\mathrm{f}(-k_1\wedge k_2)$ appears. This means that we can only constrain the even part of the function, whereas the odd part can't be constrained from this process at Born level.

How can the function $*_\mathrm{f}$ be additionally constrained? A simple constraint comes from the term

$$\frac{(*_\mathrm{s} - *_\mathrm{f})(k_1\wedge k_2)}{k_1\wedge k_2} + \frac{(*_\mathrm{s} - *_\mathrm{f})(-k_1\wedge k_2)}{k_1\wedge k_2}\,, \tag{7.6}$$

which is part of $\mathcal{A}^{\mu\nu}_{s+c,\mathrm{I}}$ (6.15). This term has to remain finite in the limit $k_1\wedge k_2 \to 0$. Thus in order to keep the above term finite, the series expansion of $*_\mathrm{f}$ has to start with unity, simply because the series expansion of the $*_\mathrm{s}$-function starts with unity. In particular, this means that the function $*_\mathrm{f}$ can't be chosen to be identically zero!

Now we come to the asymptotic behavior of the $*_\mathrm{f}$-function. Let us look at the last term of $\mathcal{A}^{\mu\nu}_{s+c,\mathrm{I}}$, which only depends on $*_\mathrm{f}$, namely

$$\mathcal{A}^{\mu\nu}_{*_\mathrm{f}} = -\frac{g^2}{2}\,\mathrm{e}^{\mathrm{i}\,p_1\wedge p_2}(k_{\gamma\gamma\gamma} - 1) \\ \big[*_\mathrm{f}(k_1\wedge k_2) + *_\mathrm{f}(-k_1\wedge k_2)\big]\theta^{\mu\nu}\bar{v}(p_1)\not{k}_1 u(p_2)\,. \tag{7.7}$$

The part of the cross section depending on the square of the absolute value of this part of the amplitude is

$$\frac{\mathrm{d}\sigma_{*_\mathrm{f}}(s)}{\mathrm{d}\Omega} = N\,s\,\left|*_\mathrm{f}(\omega s) + *_\mathrm{f}(-\omega s)\right|^2\,, \tag{7.8}$$

where we implicitly have summed over all spin and polarisation states. The factor N is independent of s, namely

$$N = \frac{g^4(k_{\gamma\gamma\gamma} - 1)^2 \sin^2(\vartheta)(\vec{E}^2 + \vec{B}^2)}{64\pi^2}\,.$$

Let us first consider the case $\omega \neq 0$. In order that the part (7.8) of the cross section satisfies the Froissart-Martin bound the even part of $*_{\mathrm{f}}$ has to satisfy

$$|\,*_{\mathrm{f}}\,(s) + *_{\mathrm{f}}(-s)| \leq \mathrm{const} \cdot \frac{\log(s)}{\sqrt{s}}, \quad \text{for } s \to \infty\,. \tag{7.9}$$

One may ask if it is sufficient to look only at this part of the cross section. The answer is yes, because if this part satisfies the bound (7.1) than the remaining parts will do that, too. This happens simply because if the square of the absolute value of a function satisfies a given bound then the function times a bounded function satisfies that bound, too. The same holds also for the difference of two functions. If we go through the terms of the amplitude (6.12), (6.13) and (6.15) we see that there are, besides various phase factors, only sine and $*_{\mathrm{s}}$-functions which are both bounded. Hence all other functions which have their origin in the Seiberg-Witten maps are bounded for $s \to \infty$, at least for $\omega \neq 0$.

One could object that the above cross section is the differential cross section and not the total one as in the definition of the bound (7.1). An integration can't lead to a worse asymptotic behavior of the expression (7.8), because the mean value theorem for integration states that the integral over a non-divergent function integrated over a finite interval is finite. And if the bound (7.9) is satisfied then the cross section is finite, at least if one excludes the t- and u-channel poles, i.e. for $\cos(\vartheta) \in \,]-1,1[\,$. Indeed the integration will enhance the convergence of the integrand, as we will see below.

These two constraints are all constraints one can obtain from the scattering process discussed in the previous chapter. As expected the functions $*_{\mathrm{s}}$ as well as $*_{\mathrm{si}}$ satisfy the above constraints which is consistent with (4.11).

7.2 Irregularities

In the above considerations we assumed that $\omega \neq 0$. Now let us consider the case where $\omega = 0$. Naively this leads to a total cross section which diverges like s. In order to see this behavior, let us split up the complete amplitude (6.12), (6.13) and (6.15) into a part which is regular and one which is divergent for $s \to \infty$

$$\mathcal{A}^{\mu\nu} = \mathcal{A}^{\mu\nu}_{\mathrm{reg}} + \mathcal{A}^{\mu\nu}_{\mathrm{div}}\,, \tag{7.10}$$

where the divergent part is

$$
\begin{aligned}
\mathcal{A}^{\mu\nu}_{\text{div}} = -g^2\, \mathrm{e}^{\mathrm{i}\, p_1\wedge p_2}(k_{\gamma\gamma\gamma}-1) \\
\Bigg[\frac{1}{2}\left\{*_{\mathrm{f}}(k_1\wedge k_2) + *_{\mathrm{f}}(-k_1\wedge k_2)\right\}\theta^{\mu\nu}\bar{v}(p_1)\not{k}_1 u(p_2) \\
- *_{\mathrm{s}}(k_1\wedge k_2)(k_1\theta)^{\nu}\bar{v}(p_1)\gamma^{\mu}u(p_2) + *_{\mathrm{s}}(k_1\wedge k_2)(\theta k_2)^{\mu}\bar{v}(p_1)\gamma^{\nu}u(p_2)\Bigg]\,. \qquad (7.11)
\end{aligned}
$$

As an example let us assume $\vec{E}=0$. The differential cross section of $\mathcal{A}_{\text{div}}$ then becomes

$$
\frac{\mathrm{d}\sigma_{\text{div}}}{\mathrm{d}\Omega} = \frac{g^4}{128\pi^2 s}(k_{\gamma\gamma\gamma}-1)^2(B_1^2+B_2^2)s^2\,, \qquad (7.12)
$$

which means that the differential cross section diverges like s. And thus the cross section would violate tree level unitarity. So we have to investigate the cases where ω becomes close to zero. Appart from the case $\vec{E}=0$ there exist values for the angles ϑ and φ where ω can vanish, too.

7.3 The Case $\vec{E}=0$

Let us first consider the case $\vec{E}=0$. From (7.5) we see that for a vanishing $\vec{E}$ the angle function ω becomes zero and as a result the cross section naively diverges. The way out of the problem is to consider small discrepancies of the center-of-mass frame and the laboratory frame. In any experiment, the energy is in principle not a delta-like distribution. So one has to integrate over small variations. To realize such variations one can integrate over a small boost, what we indeed want to do. The connection between the center-of-mass frame and the laboratory frame in the case of $e^+e^-\to\gamma\gamma$ is the usual one, namely

$$
s^\star = x_1x_2 s\,, \qquad \cos\vartheta^\star = \frac{\cos\vartheta-\beta}{1-\beta\cos\vartheta}\,, \qquad \varphi^\star=\varphi\,.
$$

Beside the energy and the scattering angles we also have to boost $\theta^{\mu\nu}$

$$
\begin{aligned}
E_1^\star &= \gamma(E_1-\beta B_2)\,, & B_1^\star &= \gamma(B_1+\beta E_2)\,,\\
E_2^\star &= \gamma(E_2+\beta B_1)\,, & B_2^\star &= \gamma(B_2-\beta E_1)\,,\\
E_3^\star &= E_3\,, & B_3^\star &= B_3\,,
\end{aligned}
$$

where the Lorentz factor and the velocity are defined as usual

$$
\gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \qquad \beta = \frac{x_2-x_1}{x_2+x_1} = \frac{y_1}{y_2}\,.
$$

The variables with the star $\star$ belong to the center-of-mass frame whereas the variables on the r.h.s. of the above equations belong to the laboratory frame.

Since we consider only small boosts we can expand the argument of the trigonometric functions and the $*_{\mathrm{f}}$-function in powers of $y_1 = x_2 - x_1$. Thus in the laboratory frame, we have for the argument

$$\begin{aligned} k_1 \wedge k_2 &= \frac{1}{2} s y_1 (y_2^2 - y_1^2)(B_2 \cos\varphi - B_1 \sin\varphi) \sqrt{\frac{1-\cos^2\vartheta}{(y_2 - y_1\cos\vartheta)^2}} \\ &= \frac{1}{8} s |\sin\vartheta| (B_2\cos\varphi - B_1\sin\varphi) y_1 y_2 + \mathcal{O}\left(y_1^2\right) + \mathcal{O}(\vec{E}) \,. \end{aligned} \tag{7.13}$$

If we choose for the arbitrary function $*_{\mathrm{f}} = *_{\mathrm{s}}$, the squared of the divergent part of the amplitude is

$$\frac{\mathrm{d}\sigma^\star_{\mathrm{div}}}{\mathrm{d}\Omega} = \frac{c_2}{s} \left(\frac{\sin\bigl(c_1 s y_1 y_2 + \mathcal{O}(y_1^2)\bigr)}{c_1 s y_1 y_2 + \mathcal{O}(y_1^2)} \right)^2 s^2 y_2^2 (2y_1^2 + y_2^2) \,, \tag{7.14}$$

where the two factors

$$\begin{aligned} c_1 &:= \frac{1}{4} |\sin\vartheta| (B_2\cos\varphi - B_1\sin\varphi) \,, \\ c_2 &:= \frac{1}{64\pi^2} \frac{g^4}{32} (k_{\gamma\gamma\gamma} - 1)^2 (B_1^2 + B_2^2) \,, \end{aligned}$$

are independent of s. We can choose $*_{\mathrm{s}}$ for the arbitrary function $*_{\mathrm{f}}$ without loss of generality, because we consider this function when its argument is around zero, i.e. $\omega \approx 0$ (cf. section 7.1). And because $*_{\mathrm{f}}$ as well as $*_{\mathrm{s}}$ start their series expansion with unity we can set $*_{\mathrm{f}} = *_{\mathrm{s}}$. With the integration over y_1 from $-y_2$ to y_2 and over y_2 from 0 to 2 one covers the whole critical region, which is $y_1 = 0$. In the case of $c_1 \neq 0$ this leads to

$$\begin{aligned} \frac{\mathrm{d}\sigma_{\mathrm{div}}}{\mathrm{d}\Omega} \approx \int_0^2 \mathrm{d}y_2 \int_{-y_2}^{y_2} \mathrm{d}y_1 \, \frac{c_2}{s} \left(\frac{\sin c_1 s y_1 y_2}{c_1 y_1 y_2} \right)^2 y_2^2 (2y_1^2 + y_2^2) = \frac{c_2}{c_1} \Biggl[*_{\mathrm{si}}(8c_1 s) \\ + \frac{1}{c_1} \frac{2 + \cos(8c_1 s)}{s} + \frac{1}{8c_1^2} \frac{\sin(8c_1 s)}{s^2} - \frac{1}{2c_1^2} \frac{*_{\mathrm{si}}(8c_1 s)}{s^2} \Biggr] \,, \end{aligned} \tag{7.15}$$

where the approximation means that we neglect the $\mathcal{O}(y_1^2)$ terms present in (7.14). This approximation is sufficient for our considerations even if we integrate y_1 over the interval $[-y_2, y_2]$. The reason is that except for the critical point $y_1 = 0$ the integrand is convergent for $s \to \infty$ and thus not relevant for our discussion. The important case occurs when y_1 is around zero. And for this case the above approximation is sufficient.

If we finally consider the limit $s \to \infty$ one sees that only the first term survives. So the high energy limit of the cross section is constant

$$\lim_{s\to\infty} \frac{\mathrm{d}\sigma_{\mathrm{div}}}{\mathrm{d}\Omega} = 2\pi \frac{c_2}{|c_1|} \,. \tag{7.16}$$

Now let us consider the case where $c_1^\star = 0$. This case occurs if the azimuthal angle becomes

$$\varphi_{\mathrm{div}} = \arctan\left(\frac{B_2}{B_1}\right) . \tag{7.17}$$

In addition, $c_1^\star$ becomes zero for $\vartheta = 0, \pi$. But then the photons would be scattered exactly along the beam axis. And this case is in principle undetectable.

In the case where the azimuthal angle becomes φ_{div} one has to integrate over a small interval over this angle. The reason for doing this is that one has no sharp momenta of the outgoing photons, so that the angles are blurred. After this integration one still has a well behaved, i.e. in a Taylor series expandable, function. Thus the integration over the boost parameters finally gives a constant cross section in the high energy limit.

7.4 Irregular Angles

The case $\vec{E} = 0$ is not the only one for which the argument can become zero. Namely if the polar angle is given by

$$\vartheta_{\mathrm{div}}(\varphi) = \arccos\left(-\frac{E_1\cos(\varphi) + E_2\sin(\varphi)}{\sqrt{E_3^2 + \big(E_1\cos(\varphi) + E_2\sin(\varphi)\big)^2}}\right) , \tag{7.18}$$

then ω becomes zero, too. The set of all points $(\varphi, \vartheta_{\mathrm{div}}(\varphi))$ we call irregular points.

Let us look somewhat closer at the above relation between ϑ_{div} and the azimuthal angle φ. First, let us determine the codomain of ϑ_{div}. Or in other words, are there polar angles where ω is nonzero for all azimuthal angles? To answer this question let us look at figure 7.1 where we plotted ϑ_{div} over φ for $\vec{E} = (0.3, 0.7, 1.0)$. One sees that if the polar angle lies between the dashed lines one has two azimuthal angles for which $\omega = 0$. If $\cos(\vartheta_{\mathrm{div}}) \approx \pm 0.61$ then there is only one φ. But if the polar angle is above or below the dashed lines one finds no azimuthal angles for which ω becomes zero.

Of course, the position of the dashed lines depends on $\vec{E}$, because the two maxima are at the azimuthal angles

$$\varphi_{\mathrm{min}} = \arccos\left(\frac{E_1}{\sqrt{E_1^2 + E_2^2}}\right) , \qquad \varphi_{\mathrm{max}} = -\arccos\left(-\frac{E_1}{\sqrt{E_1^2 + E_2^2}}\right) .$$

If we insert these two angles in (7.18) we obtain

$$\cos\big(\vartheta_{\mathrm{div}}(\varphi_{\mathrm{max,\,min}})\big) = \pm\frac{\sqrt{E_1^2 + E_2^2}}{\sqrt{E_1^2 + E_2^2 + E_3^2}} . \tag{7.19}$$

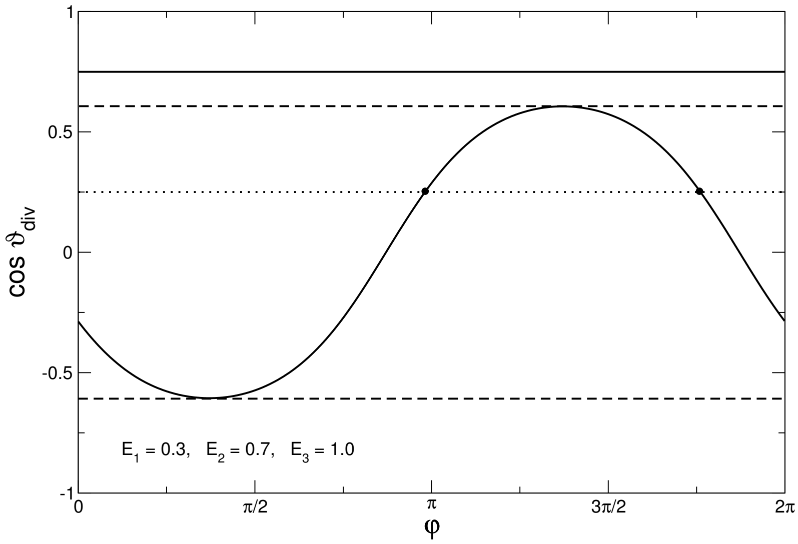


Figure 7.1: The function $\cos(\vartheta_{\text{div}}(\varphi))$ for $\vec{E} = (0.3, 0.7, 1.0)$. For a given polar angle ϑ one obtains either one (dashed line), two (dotted line) or no (drawn through line) azimuthal angle φ for which the differential cross section diverges

As one can see, the argument of the arccos is always in the interval $[-1, 1]$. For the case where $E_3 \to 0$ the dashed lines tend to ± 1. And if $E_1 = E_2 = 0$ then there exist only one ϑ_{div} independent of φ. But in general there exists either one, two or no azimuthal angles where, for a given ϑ, the differential cross section diverges.

In the simplest case, namely when ϑ can't become zero for any φ, then ω is always non-zero, i.e. the cross section at this polar angle is always convergent.

The first problematic case is when the polar angle is between the two dashed lines. The differential cross section for this case diverges at two azimuthal angles. But if one integrates over the azimuthal angle the resulting cross section becomes constant for $s \to \infty$. And as we already said, in general one has to integrate over a small angle, because the momenta of the photons can only be measured with finite precision. To clarify how an integration can cancel a divergent part of the integrand, let us consider the function

$$f(\omega, s) = \left(\frac{\sin(\omega s)}{\omega s} \right)^2 s \,,$$

which diverges like s for $\omega = 0$ and $s \to \infty$. If we integrate this function over ω around zero, the limit $s \to \infty$ becomes finite

$$\lim_{s\to\infty} \int_{-\epsilon}^{\epsilon} \mathrm{d}\omega\, f(\omega, s) = \lim_{s\to\infty} \int_{-s\epsilon}^{s\epsilon} \mathrm{d}x \left(\frac{\sin(x)}{x}\right)^2 = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\sin(x)}{x}\right)^2 = \pi\,,$$

where we substitute $x = \omega s$. Thus the smearing enhances the convergent behavior of the integrand.

Now we come to the case where the dotted line tends to one of the dashed lines. Then the two azimuthal angles tends to one point, namely to φ_{min} or φ_{max}. At such a point the degree of the divergence is doubled with respect to the case where the divergent points were separated. In order to illustrate this behavior let us look at the following integral

$$F(y) = \int_{-1}^{1} \mathrm{d}x \frac{1}{(x-y)(x+y)}\,, \quad y \in (-1, 1)\,. \tag{7.20}$$

For $y \neq 0$ the Cauchy principal value of this integral is finite, namely

$$F(y) = 2y^{-1} \operatorname{arctanh}\big(y^{-1}\big)\,, \quad y \neq 0\,. \tag{7.21}$$

But if y becomes zero the integrand becomes $-x^{-2}$. Thus the integral diverges.

The way out of this dilemma is straightforward. Namely one has to integrate not only over φ but also over ϑ. After these two integrations the total cross section becomes constant for $s \to \infty$.

7.5 Illustration

After this somewhat technical discussion we now want to illustrate what we have said in the previous section.

But before we are able to plot the differential cross section we first have to fix some parameters. We choose

$$\vec{E} = \frac{1}{\mathrm{TeV}^2}(0.3, 0.7, 1.0)\,, \quad \vec{B} = (0,0,0)\,, \quad k_{\gamma\gamma\gamma} = 0.5\,, \quad *_{\mathrm{f}} = *_{\mathrm{s}}\,. \tag{7.22}$$

Hence, we assume a purely time-like noncommutativity with a noncommutative scale of 1 TeV. For our purpose we can set $\vec{B} = 0$ without loss of generality because ω does not depend on the $\vec{B}$ components of $\theta^{\mu\nu}$. In order that the terms proportional to $k_{\gamma\gamma\gamma}$ as well as those proportional to $(k_{\gamma\gamma\gamma} - 1)$ have the same weight we choose $k_{\gamma\gamma\gamma}$ to be one-half. For the function $*_{\mathrm{f}}$ we again choose $*_{\mathrm{s}}$. And as already stated we can do this without loss of generality. Finally we set the electric charge to $g = \sqrt{4\pi\alpha} = 0.308$.

With the parameters fixed, we now can generate plots. First let us look at the φ dependency of the differential cross section for $\sqrt{s} = 10\,\mathrm{TeV}$. We

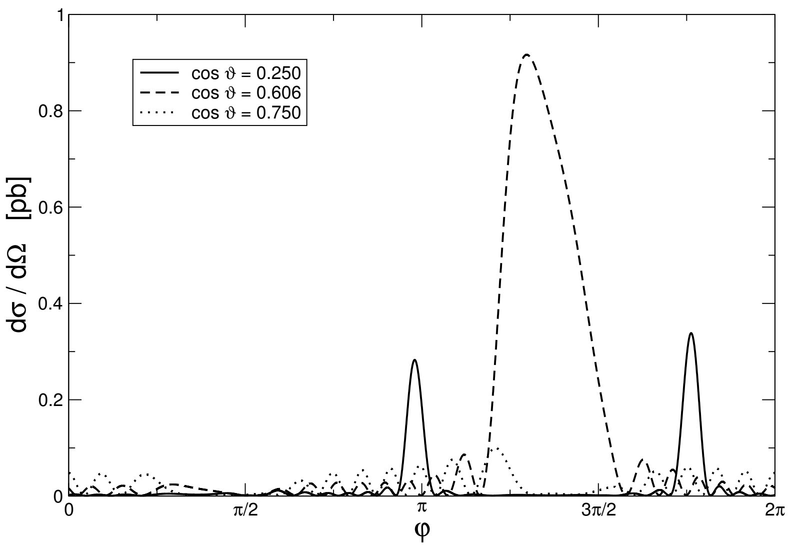


Figure 7.2: The differential cross section plotted over the azimuthal angle φ for $\sqrt{s} = 10\,\text{TeV}$. For different polar angles ϑ the differential cross section has either one, two or no divergent peaks.

choose $\cos\vartheta$ in such a way, that we have either one, two or no divergences. One can impressively see that one has two divergent peaks for the case $\cos\vartheta = 0.25$ and one big peak for the case $\cos\vartheta = 0.606$ where the two peaks coincide. For $\cos\vartheta = 0.75$ no divergent angle exists thus for this case no peak is present.

In the next plot we choose a center-of-mass energy of $50\,\text{TeV}$. One sees that the peaks become more narrow and higher. The maximum of the peaks depend on s, as expected. The small plot in figure 7.3 zooms in around $\varphi = 3\pi/8$. By magnifying the y-axis by a factor of 10^4 the different curves can be resolved. One observes the fast oscillations of the cross section relative to the oscillations in figure 7.2. The reason is that the argument of the trigonometric functions and the $*_{\mathrm{f}}$-function is proportional to s, i.e. the frequency depends quadratically on the center-of-mass energy.

Figure 7.4 shows the differential cross section for three different energies plotted over $\cos\vartheta$. In this graph we integrated over the whole range of φ. From the previous section we would assume that the cross section is convergent for $\cos\vartheta$ within the interval $]\,0.606, 1\,[$. For $|\cos\vartheta| \in\,]\,0, 0.606\,[$ we assume a constant cross section with respect to s because in this interval we integrated over the two divergent peaks (cf. figure 7.2 or 7.3). In the case where one has only the single peak, the integration over φ is not sufficient to obtain a constant cross section. Actually the generated plot in figure 7.3 shows all the expected behavior. Looking at the amplitude of the peaks

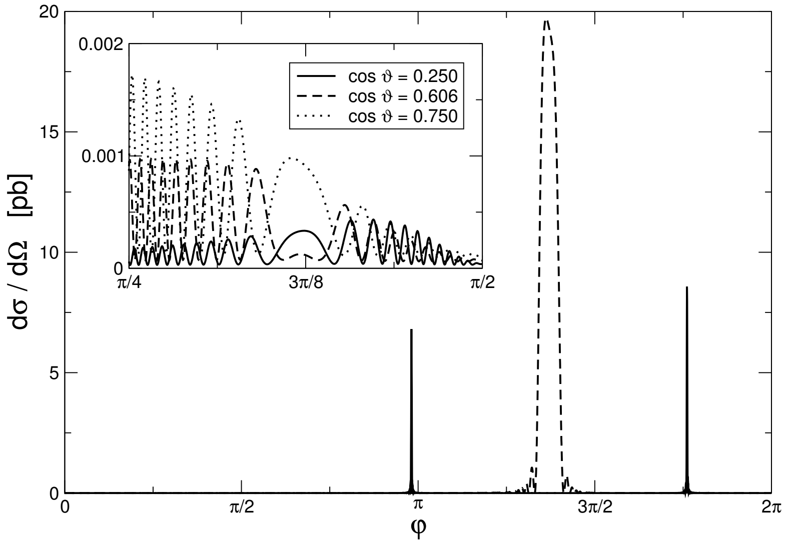


Figure 7.3: The differential cross section plotted over the azimuthal angle φ for $\sqrt{s} = 50\,\text{TeV}$. For different polar angles ϑ the differential cross section has either one, two or no divergent peaks. In the zoomed plot the oscillations become visible.

for $\sqrt{s} = 10\,\text{TeV}$ and for $50\,\text{TeV}$ one observes that for the amplitude in the latter case is approximately 25 times greater than the one for $10\,\text{TeV}$. Exactly this behavior we do expect because the peaks diverge like s.

The last plot presented here shows the total cross section plotted over the center-of-mass energy $\sqrt{s}$. As we can impressively see the cross section converges to a constant value, for high energies. The frequency of the oscillations also increases as mentioned earlier.

As we have seen, the total cross section is non-divergent for $s \to \infty$. We have explained and illustrated that every integration leads to a less divergent behavior of the differential cross section. We showed that if one takes two integrations into account, namely the integration over the azimuthal and polar angle, one ends up with a total cross section which converges to a constant. Actually, in order to take the intrinsic uncertainty of the outgoing momenta completely into account one has to integrate also over the polar angle. So the Froissart-Martin bound (7.1) and thus the tree level unitarity can always be satisfied if the function $*_{\rm f}$ itself satisfies the constraints we found and which are summarized below.

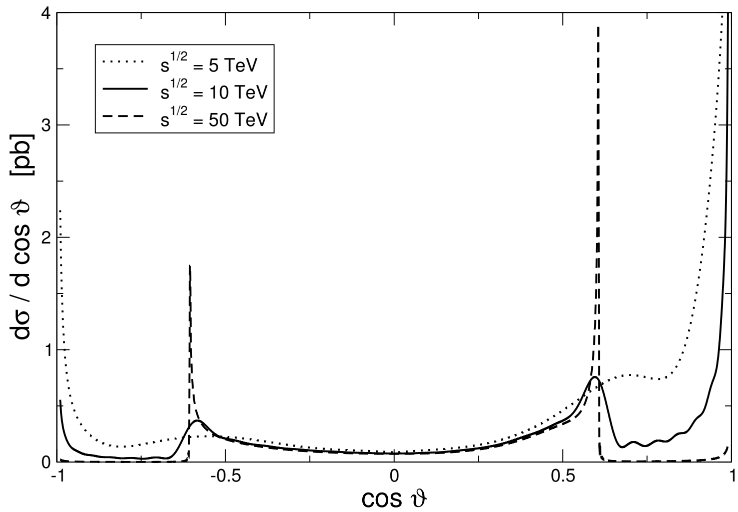


Figure 7.4: The differential cross section plotted over the polar angle ϑ and integrated over the azimuthal angle for various center-of-mass energies $\sqrt{s}$. To the left and to the right of the peaks the differential cross section remains bounded for high energies. Between the two peaks the differential cross section is almost constant. The peaks diverge with s.

7.6 Discussion

First let us combine our knowledge about the function $*_\mathrm{f}$. From the commutative limit and from the request of tree level unitarity we can restrict the function $*_\mathrm{f}$ to functions of the following form

1. The series expansion of $*_\mathrm{f}$ has to start with unity.

2. The asymptotic behavior of the even part of $*_\mathrm{f}$ is given by

$$| *_\mathrm{f} (x) + *_\mathrm{f}(-x)| \leq \frac{\log(x)}{\sqrt{x}} \quad \text{for } x \to \infty\,. \tag{7.23}$$

3. We can't constrain the odd part of $*_\mathrm{f}$.

In section 7.3 we discussed the cases where the angle function ω becomes zero. Naively the cross section diverges like s for $s \to \infty$. This would lead to a theory which violates tree level unitarity and thus this theory can't be interpreted in a consistent manner. But if one uses the intrinsic uncertainty in the energy and the uncertainty in the momenta of the photons, one ends up with a cross section which is constant for s goes to infinity. Thus tree level unitarity is satisfied.

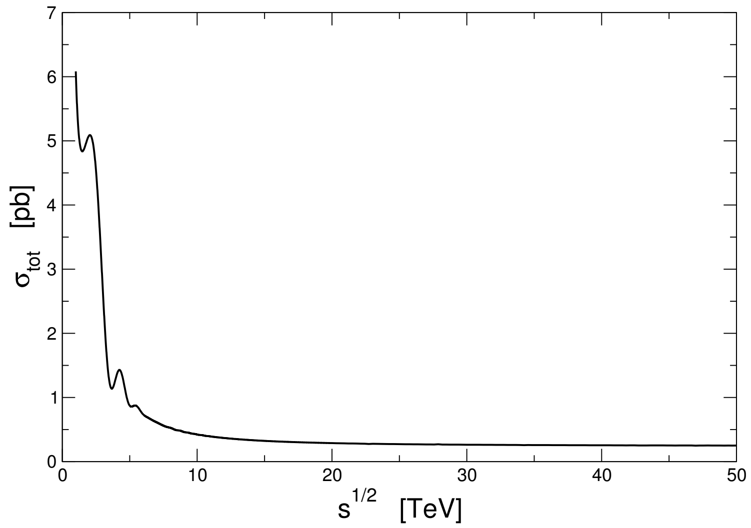


Figure 7.5: The total cross section plotted over the center-of-mass energy $\sqrt{s}$. The frequency of the oscillations increase with the energy.

Indeed, one has to take all uncertainties into account in order to get rid of all the divergences. It seems that this is the price one has to pay if one allows a noncommutative spacetime. But perhaps these problems appear only because we used plane waves for the description of our particles. This is a simplification which provides the right observables, at least in the commutative case. But actually one has wave packages and not plane waves. On the other hand the differential cross section is not an observable itself. What one basically measures is the differential cross section integrated over some angle $d\Omega$. Or analogous, one measure the differential cross section convoluted with some function. The smearing coming from the integration over the scattering angles is similar to the smearing coming from the consideration of wave packages. So it is permitted to assume that the calculations done with wave packages will lead to the same results we got by considering integrations over the angles.

One can ask if the optical theorem could be in general satisfied if one has such trouble satisfying tree level unitarity. The question is justified. Namely the NCQED without the Seiberg-Witten maps is tree level unitary but violates the optical theorem at one loop. Thus the optical theorem is a more stringent condition as the tree level unitarity. Another factor which supports the argument is that the considered cross section becomes the one of the NCQED without Seiberg-Witten maps for $k_{\gamma\gamma\gamma} = 1$. Thus one can ask whether it is possible that for $k_{\gamma\gamma\gamma} \neq 1$ the NCQED with Seiberg-Witten maps can satisfy the optical theorem. By calculating the photon self-energy one can check the optical theorem. With the vertices derived in chapter 5 we calculated some of the necessary loops in appendix F. The first results, although incomplete, don't raise the hope that this model satisfies the optical theorem.

On the other hand one still has the freedom in the choice of the function $*_\mathrm{f}$ which may be used to obtain a cross section which satisfies the optical theorem. But in the end, one has to complete the necessary calculations to finally answer this question.

Chapter 8

Summary and Outlook

The basic question which drove our whole work was to find a meaningful noncommutative gauge theory even for the time-like case ($\theta^{0i} \neq 0$). In order to be able to tackle questions regarding unitarity, it is not sufficient to consider theories which include the noncommutative parameter only up to a finite order. The reason is that in order to investigate tree-level unitarity or the optical theorem in loops one has to know the behavior of the noncommutative theory for center-of-mass energies much greater than the noncommutative scale. Therefore an effective theory, that is by construction only valid up to the noncommutative scale, isn't sufficient for our purpose.

Our model is based on two fundamental assumptions. The first assumption is given by the commutation relations (1.2). This led to the Moyal-Weyl star-product (2.9) which replaces all point-like products between two fields. The second assumption is to assume that the model built this way is not only invariant under the noncommutative gauge transformation but also under the commutative one. In order to obtain an action of such a model one has to replace the fields by their appropriate Seiberg-Witten maps. We chose the gauge fixed action (2.31) as the fundamental action of our model.

After having constructed the action of the NCQED including the Seiberg-Witten maps we were confronted with the problem of calculating the Seiberg-Witten maps to all orders in $\theta^{\mu\nu}$. By means of [2] we could calculate the Seiberg-Witten maps order by order in the gauge field, where each order in the gauge field contains all orders in the noncommutative parameter (cf. chapter 3). By comparing the maps with the result we obtained from an alternative ansatz [34], we realized that already the simplest Seiberg-Witten map for the gauge field is not unique. In chapter 4 we examined this ambiguity, which we could parametrised by an arbitrary function $*_{\mathrm{f}}$.

The next step was to derive the Feynman rules for our NCQED. One finds that the propagators remain unchanged so that the free theory is equal to the commutative QED. The fermion-fermion-photon vertex contains not only a phase factor coming from the Moyal-Weyl star-product but also two additional terms which have their origin in the Seiberg-Witten maps. Beside the 3-photon vertex which is already present in NCQED without Seiberg-Witten maps and which has also additional terms coming from the Seiberg-Witten maps, too, one has a contact vertex which couples two fermions with two photons.

After having derived all the vertices we calculated the pair annihilation scattering process $e^+e^- \to \gamma\gamma$ at Born level. By choosing the parameter $k_{\gamma\gamma\gamma} = 1$ (cf. section 2.5), we found that the amplitude of the pair annihilation process becomes equal to the amplitude of the NCQED without Seiberg-Witten maps. This means that, at least for this process, the NCQED excluding Seiberg-Witten maps is only a special case of NCQED including Seiberg-Witten maps.

On the basis of the pair annihilation process, we afterwards investigated tree-level unitarity. In order to satisfy the tree-level unitarity we had to constrain the arbitrary function $*_{\mathrm{f}}$. We found that the series expansion of $*_{\mathrm{f}}$ has to start with unity. In addition, the even part of the function must not increase faster than $s^{-1/2}\log(s)$ for $s \to \infty$, whereas the odd part of the $*_{\mathrm{f}}$-function can't be constrained, at least by the process we considered. By assuming these constrains for the $*_{\mathrm{f}}$-function, we could show that tree-level unitarity is satisfied if one incorporates the uncertainties present in the energy and the momenta of the scattered particles, i.e. the uncertainties of the center-of-mass energy and the scattering angles. This uncertainties are not exclusively present due to the finite experimental resolution. A delta-like center-of-mass energy as well as delta-like momenta are in general not possible because the scattered particles are never exact plane waves.

Some open questions still remain which have to be worked out in the future.

One question arose within the calculation of the amplitude of the pair annihilation process. Namely, is it a general feature that one can always find the theories without Seiberg-Witten maps as a special case of the theories with Seiberg-Witten maps? There are some hints which suggest that this is not a general feature. Namely, if one has fermions off the mass shell, then the contributions from the maps $\psi^{[3]}$ in the contact vertex wouldn't vanish. And probably these contributions can't be canceled by other terms. So there would be differences between the NCQED with and without Seiberg-Witten maps, even for $k_{\gamma\gamma\gamma} = 1$.

Another question arose from the discussion at the end of chapter 7. Namely, are the uncertainties of the center-of-mass energy and the scattering angles always sufficient to obtain a cross section which satisfies tree-level unitarity? Or are there processes or loop contributions which remain divergent after the various integrations? We expect that tree-level unitarity is always satisfied, because where should additional divergences come from?

We used tree-level unitarity to constrain the arbitrary function $*_{\mathrm{f}}$. Naturally, the question arises if this function can be further constrained by considering other scattering processes or higher order contributions in perturbation theory? We also know that the higher order Seiberg-Witten maps $\psi^{[3]}$ and especially $A_\mu^{[3]}$ contains new arbitrary functions. So can these functions also be constrained by tree-level unitarity or other conditions? It is very likely that one can constrain new arbitrary functions analogously to the $*_{\mathrm{f}}$-function. But it is doubtful if one can find additionally constraints for $*_{\mathrm{f}}$ in other processes.

Another whole class of questions concerns the optical theorem. As already stated, we weren't able to do the full calculation in order to check the optical theorem by direct calculation of the photon self-energy. The complexity of the map $A_\mu^{[3]}$ was the main obstacle to carry out the calculations. Unfortunately we can't give a complete and final answer to the question whether the NCQED including Seiberg-Witten maps is unitary in the sense of the optical theorem.

It would be very interesting to study all the open questions in the future in order to obtain a consistent time-like noncommutative gauge theory.

Appendix A

Notations and Definitions

In this appendix we will summerize our notations and definitions.

A.1 Units, Special Relativity and Noncommutativity

We will work in "natural" units, where

$$\hbar = c = 1 \,. \tag{A.1}$$

As a result of this system of units

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} \,. \tag{A.2}$$

The metric tensor is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{A.3}$$

where Greek indices run over $0, 1, 2, 3$. Four-vectors are defined by

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x}) \,, \qquad x_\mu = g_{\mu\nu} x^\mu = (x^0, -\vec{x}) \,. \tag{A.4}$$

The noncommutative parameter $\theta_{\mu\nu}$ is assumed to be constant and real. By definition

$$[\hat{x}_\mu, \hat{x}_\nu] =: \mathrm{i}\, \theta_{\mu\nu} \,, \tag{A.5}$$

it is antisymmetric. In analogy to the field-strength tensor one can decompose the noncommutative parameter into electric and magnetic components

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} . \tag{A.6}$$

The elements of $\vec{E}$ are also called time-like components of $\theta_{\mu\nu}$ and the elements of $\vec{B}$ are called space-like components of $\theta_{\mu\nu}$, respectively.
In position space the wedge-product is a bidifferential and is defined by

$$A \wedge_{12} B := \frac{\theta^{\mu\nu}}{2}(\partial_\mu A)(\partial_\nu B)\,. \tag{A.7a}$$

We find it convenient to introduce

$$\wedge_{12} ABC := \frac{\theta^{\mu\nu}}{2}(\partial_\mu A)(\partial_\nu B)C\,, \tag{A.7b}$$

$$\wedge_{13} ABC := \frac{\theta^{\mu\nu}}{2}(\partial_\mu A)B(\partial_\nu C)\,, \tag{A.7c}$$

$$\wedge_{23} ABC := \frac{\theta^{\mu\nu}}{2}A(\partial_\mu B)(\partial_\nu C)\,. \tag{A.7d}$$

It carries always two indices which label the position of the quantities on which the derivatives act. In momentum space the wedge-product appears only between two four-momenta. It carries no index and is defined as

$$p \wedge q := p_\mu \theta^{\mu\nu} q_\nu = -q \wedge p\,. \tag{A.8}$$

Due to the antisymmetry of $\theta^{\mu\nu}$, the $\wedge$-product is antisymmetric, too.
By a tilded momenta we denote the contraction of the momenta with the noncommutative parameter from the right $\tilde{p}^\mu := (p\theta)^\mu$.
With the above definition of the wedge-product one can also define functions of this object by their series expansion. The most important one ist the Moyal-Weyl star-product

$$(\phi_1 * \phi_2)(x) = \phi_1(x)\, \mathrm{e}^{\mathrm{i}\wedge_{12}}\, \phi_2(x)\,. \tag{A.9}$$

In analogy to the Moyal-Weyl star-product we define the following functions

$$*_{\mathrm{e}}(\wedge_{12}) := \frac{\mathrm{e}^{\mathrm{i}\wedge_{12}} - 1}{\mathrm{i}\wedge_{12}}\,, \tag{A.10a}$$

$$*_{\mathrm{s}}(\wedge_{12}) := \frac{\sin(\wedge_{12})}{\wedge_{12}}\,, \tag{A.10b}$$

$$*_{\mathrm{si}}(\wedge_{12}) := \frac{\mathrm{si}(\wedge_{12})}{\wedge_{12}}\,, \quad \mathrm{si}(x) = \int_0^x \mathrm{d}t\, \frac{\sin t}{t}\,. \tag{A.10c}$$

By "leading order Seiberg-Witten map" we mean always the leading order map in the gauge field, which is either $C^{[2]}$, $A_\mu^{[2]}$ or $\psi^{[2]}$.

A.2 Graded Star Commutator

With the definition of the Moyal-Weyl star-product we define the graded star commutator $[\cdot \overset{*}{,} \cdot]$

$$[A \overset{*}{,} B] := A * B - (-1)^{|A||B|} B * A\,, \tag{A.11}$$

where $|\cdot|$ is the Grassmann parity

$$|A| := \begin{cases} 0 & \text{for } A \text{ Grassmann even} \\ 1 & \text{for } A \text{ Grassmann odd} \end{cases}. \tag{A.12}$$

The ghost field C, the contracting homotopy operator $\rho^{[0]}$ and the BRST differential γ are the only Grassmann odd quantities that appear in this work. All other fields and quantities are Grassmann even.

A.3 Symmetrisation Parenthesis

The "symmetrisation parenthesis" are defined by

$$i_{(1} \ldots i_{n)} := \frac{1}{n!} \sum_{\sigma \in S^n} i_{\sigma(1)} \ldots i_{\sigma(n)}\,, \tag{A.13}$$

where S^n is the set of all symmetric n tuple. For the example $n = 3$ we have

$$i_{(1} i_2 i_{3)} = \frac{1}{3!} (i_1 i_2 i_3 + i_1 i_3 i_2 + i_2 i_1 i_3 + i_2 i_3 i_1 + i_3 i_1 i_2 + i_3 i_2 i_1)\,. \tag{A.14}$$

A.4 Multi-Index Notation

With an upper bold index we label a multi-index

$$x^{\boldsymbol{\alpha}[i,j]} := x^{\alpha_i} \ldots x^{\alpha_j} \text{ for } i \le j\,, \quad x^{\boldsymbol{\alpha}[i,j]} := 1 \text{ for } i > j\,. \tag{A.15}$$

For a tensor we define the multi-index by

$$\theta^{\boldsymbol{\alpha}[i,j]\boldsymbol{\beta}[i,j]} := \theta^{\alpha_i \beta_i} \ldots \theta^{\alpha_j \beta_j} \text{ for } i \le j\,. \tag{A.16}$$

If the multi-index has only one argument then the lower index starts with zero

$$x^{\boldsymbol{\alpha}[n]} := x^{\alpha_0} \ldots x^{\alpha_n}\,. \tag{A.17}$$

A.5 Miscellaneous Notation

By $[a, b]$ with $a, b \in \mathbb{R}$ we denote the closed interval including a, b.
By $]\,a, b\,[$ with $a, b \in \mathbb{R}$ we denote the open interval excluding a, b.
The Heaviside or step function is defined by

$$H(x) := \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}. \tag{A.18}$$

Appendix B

Traces and Time Ordering

B.1 Traces

The following considerations are based on [19]. For simplicity we make the following consideration for the two dimensional case. In this case the noncommutative matrix is

$$\theta_{\mu\nu} = \frac{1}{\Lambda_{\rm NC}^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \tag{B.1}$$

which leads to the commutation relation

$$[\hat{x}_1, \hat{x}_2] = \mathrm{i}\,\frac{1}{\Lambda_{\rm NC}^2} \,. \tag{B.2}$$

The above relation is similar to that of ordinary quantum mechanics if we identify $\hat{x}_1 = \hat{q}$, $\hat{x}_2 = \hat{p}$ and $1/\Lambda_{\rm NC}^2 = \hbar$. Thus we can define the eigenstates of $\hat{x}_1$ and $\hat{x}_2$ by

$$\hat{x}_1 \left|x\right\rangle = x \left|x\right\rangle \,, \qquad \hat{x}_2 \left|p\right\rangle = \frac{p}{\Lambda_{\rm NC}^2} \left|p\right\rangle \,. \tag{B.3}$$

The complete states are normalized to the Dirac delta distribution as usual

$$\begin{aligned} \langle x|x'\rangle &= \delta(x - x')\,, \quad \int \mathrm{d}x \; \left|x\right\rangle \left\langle x\right| = 1\,, \\ \langle p|p'\rangle &= \delta(p - p')\,, \quad \int \mathrm{d}p \; \left|p\right\rangle \left\langle p\right| = 1\,, \end{aligned} \tag{B.4}$$

and

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}\,px} \,. \tag{B.5}$$

With the above properties it is straightforward to calculate the matrix elements of $\hat{T}(p) = \exp(\mathrm{i}\,p\hat{x})$, namely

$$\left\langle x'\right| \hat{T}(p) \left|x''\right\rangle = \delta\left(\frac{p_2}{\Lambda_{\rm NC}^2} + x' - x''\right) \mathrm{e}^{\frac{\mathrm{i}}{2}p_1(x'+x'')} \,. \tag{B.6}$$

The desired representation of the trace of $\hat{T}(p)$ is given for the two dimensional case by

$$\begin{aligned}\operatorname{tr}\hat{T}(p) &= \frac{2\pi}{\Lambda_{\mathrm{NC}}^2}\int \mathrm{d}x\ \langle x|\,\hat{T}(p)\,|x\rangle \\ &= \frac{2\pi}{\Lambda_{\mathrm{NC}}^2}\delta\!\left(\frac{p_2}{\Lambda_{\mathrm{NC}}^2}\right)\int \mathrm{d}x\ \mathrm{e}^{\mathrm{i}\,p_1 x} = (2\pi)^2\delta^{(2)}(p_\mu)\,. \qquad \text{(B.7)}\end{aligned}$$

B.2 Time Ordering

This short section demonstrates the statement of section 2.6. Assume the time-ordered two point function with an additional phase operator coming from the Moyal-Weyl star-product

$$\begin{aligned}\langle 0|\,\mathrm{T}\left[\mathrm{e}^{\mathrm{i}\,\partial_x\wedge(\mathrm{i}\,q)}\,\phi(x)\phi(y)\right]|0\rangle &= H(x^0-y^0)\,\mathrm{e}^{\mathrm{i}\,\partial_x\wedge(\mathrm{i}\,q)}\,\langle 0|\,\phi(x)\phi(y)\,|0\rangle \\ &\quad + H(y^0-x^0)\,\mathrm{e}^{\mathrm{i}\,\partial_x\wedge(\mathrm{i}\,q)}\,\langle 0|\,\phi(y)\phi(x)\,|0\rangle\ , \qquad \text{(B.8)}\end{aligned}$$

where q is some external momenta and H is the Heaviside function. If we define the time-ordering operator by pulling out the phase operator we obtain the usual Feynman propagator times the phase factor

$$\begin{aligned}\langle 0|\,\tilde{\mathrm{T}}\left[\mathrm{e}^{\mathrm{i}\,\partial_x\wedge(\mathrm{i}\,q)}\,\phi(x)\phi(y)\right]|0\rangle &= \mathrm{e}^{\mathrm{i}\,\partial_x\wedge(\mathrm{i}\,q)}\,D_{\mathrm{F}}(x-y) \\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,\mathrm{e}^{\mathrm{i}\,p\wedge q}\,\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\,\epsilon}\,. \qquad \text{(B.9)}\end{aligned}$$

If we do *not* interchange the phase operator with the time-ordering operator in general we don't get the Feynman propagator. But let us calculate this in detail. First of all the vacuum expectation value of two scalar fields is

$$\langle 0|\,\phi(x)\phi(y)\,|0\rangle = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)}\Big|_{p^0=E_{\vec{p}}}\,. \qquad \text{(B.10)}$$

With the integral representation of the Heaviside function

$$H(x^0-y^0) = \mathrm{i}\int\frac{\mathrm{d}t}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\,t(x^0-y^0)}}{t+\mathrm{i}\,\epsilon} \qquad \text{(B.11)}$$

and $p^0 = t + E_{\vec{p}}$ the integrand of the two-point function is given by

$$\begin{aligned}&\frac{\mathrm{i}}{2E_{\vec{p}}\left(p^0-E_{\vec{p}}+\mathrm{i}\,\epsilon\right)}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,\mathrm{e}^{\mathrm{i}\,p\wedge q} \\ &\qquad + \frac{\mathrm{i}}{2E_{\vec{p}}\left(p^0-E_{\vec{p}}+\mathrm{i}\,\epsilon\right)}\,\mathrm{e}^{-\mathrm{i}\,p(y-x)}\,\mathrm{e}^{\mathrm{i}\,p\wedge q}\,. \qquad \text{(B.12)}\end{aligned}$$

Now if we exchange the momenta of the second summand by the corresponding negative momenta $p^\mu \to -p^\mu$, we get

$$\frac{\mathrm{i}}{2E_{\vec{p}}}\left(\frac{\mathrm{e}^{\mathrm{i}\,p\wedge q}}{p^0 - E_{\vec{p}} + \mathrm{i}\,\epsilon} + \frac{\mathrm{e}^{-\mathrm{i}\,p\wedge q}}{-p^0 - E_{\vec{p}} + \mathrm{i}\,\epsilon}\right)\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,, \tag{B.13}$$

so that we finally obtain for the time-ordered two point function

$$\begin{aligned}&\langle 0|\,\mathrm{T}\left[\phi(x)\phi(y)\,\mathrm{e}^{\mathrm{i}\,\frac{\partial_x+\partial_y}{2}\wedge q}\right]|0\rangle \\ &\quad = -\,\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}\,p(x-y)}\,\frac{1}{2E_{\vec{p}}}\frac{-E_{\vec{p}}\cos(p\wedge q) + \mathrm{i}\,p^0\sin(p\wedge q)}{p^2 - m^2 + \mathrm{i}\,\epsilon}\,.\end{aligned} \tag{B.14}$$

Obviously this result is not equal to (B.9). One only gets the ordinary Feynman propagator if the phase factor vanishes, i.e. $p\wedge q = 0$.

Appendix C

Seiberg-Witten maps

C.1 Some Identities

Consider the gauge field and its derivatives symmetrised in the indices with one standing out index λ. Then the identity

$$\partial_{(\mu_1} \dots \partial_{\mu_n} A_{\lambda)} = \frac{n}{n+1} \partial_\lambda \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} A_{\mu_n)} + \frac{1}{n+1} \partial_{\mu_1} \dots \partial_{\mu_n} A_\lambda \qquad \text{(C.1)}$$

holds. The proof is straightforward, namely with the definition in section A.3 we have

$$\begin{aligned}
\partial_{(\mu_1} \dots \partial_{\mu_n} A_{\lambda)} &= \frac{1}{(n+1)!} \sum_{\sigma \in S^{n+1}} \partial_{\mu_{\sigma(1)}} \dots \partial_{\mu_{\sigma(n)}} A_{\lambda_{\sigma(n+1)}} \\
= \frac{1}{(n+1)!} & \left(n \sum_{\sigma \in S^n} \partial_\lambda \partial_{\mu_{\sigma(1)}} \dots \partial_{\mu_{\sigma(n-1)}} A_{\mu_{\sigma(n)}} + \sum_{\sigma \in S^n} \partial_{\mu_{\sigma(1)}} \dots \partial_{\mu_{\sigma(n)}} A_\lambda \right) \\
&= \frac{n}{n+1} \partial_\lambda \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} A_{\mu_n)} + \frac{1}{n+1} \partial_{\mu_1} \dots \partial_{\mu_n} A_\lambda \,. \qquad \text{(C.2)}
\end{aligned}$$

In the last equation we have used that the partial derivatives commute with each other

$$\partial_{(\mu_1} \dots \partial_{\mu_n)} = \partial_{\mu_1} \dots \partial_{\mu_n} \,. \qquad \text{(C.3)}$$

A direct corollary of (C.1) is

$$\begin{aligned}
\partial_{(\mu_1} \dots \partial_{\mu_{2n}} F_{\mu)\lambda} &= \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} \partial_{\mu_n)} A_\lambda - \partial_\lambda \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} A_{\mu_n)} \\
&= \partial_{\mu_1} \dots \partial_{\mu_n} A_\lambda - \partial_\lambda \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} A_{\mu_n)} \,. \qquad \text{(C.4)}
\end{aligned}$$

With these two identities we obtain an equation, which is important for the calculations of Seiberg-Witten maps, namely

$$\partial_{\mu_1} \dots \partial_{\mu_n} A_\lambda = \partial_{(\mu_1} \dots \partial_{\mu_n} A_{\lambda)} + \frac{n}{n+1} \partial_{(\mu_1} \dots \partial_{\mu_{n-1}} F_{\mu)\lambda} \,. \qquad \text{(C.5)}$$

C.2 Sums

In this section we present some identities for series. First, let us define a general class of functions of n variables $x_1 \dots x_n$

$$Z_n\left(\pi, \{x_1, \dots, x_n\}\right) := \sum_{a_1=0}^{\infty} \cdots \sum_{a_n=0}^{\infty} \frac{x_1^{a_1}}{a_1!} \cdots \frac{x_n^{a_n}}{a_n!} \pi(a_1, \dots, a_n)\,, \tag{C.6}$$

with a parameter π depending on the summation indices $a_1 \dots a_n$. In order to transform all series which occur in our calculations into a Z-sum we need the following transformations and identities. Firstly, the binomial theorem

$$(x+y)^a = \sum_{b=0}^{a} \binom{a}{b} x^{a-b} y^b\,. \tag{C.7}$$

And for nested sums we have the identity

$$\sum_{a=0}^{\infty} \sum_{b=0}^{a} f(a,b) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} f(a+b,b)\,. \tag{C.8}$$

Finally, for sums where only monomials with an even power are summed up one can find the identity

$$\sum_{a=0}^{\infty} f(2a) = \frac{1}{2} \left(\sum_{a=0}^{\infty} f(a) + \sum_{a=0}^{\infty} (-1)^a f(a) \right)\,. \tag{C.9}$$

Derivatives of the Z-sum with respect to one variable supply a very useful identity. Let a_i be the summation index of the variable x_i. Then, for an arbitrary π,

$$\begin{aligned} &\partial_{x_i} Z_n(\pi(a_1 \dots a_n), \{x_1, \dots, x_n\}) \\ &\qquad = \frac{1}{x_i} Z_n(a_i \pi(a_1 \dots a_n), \{x_1, \dots, x_n\})\,, \end{aligned} \tag{C.10}$$

for all $i \leq n$.

Now let us express some series representations by special functions that have been studied in the literature. In the calculation of the map $\psi^{[3]}$ we need the following sums

$$\begin{aligned} Z_2\left((a,b) \mapsto \frac{1}{a+b+1}, \{\mathrm{i}\,x, \mathrm{i}\,y\} \right) &= \sum_{a,b=0}^{\infty} \frac{(\mathrm{i}\,x)^a}{a!} \frac{(\mathrm{i}\,y)^b}{b!} \frac{1}{a+b+1} \\ &\overset{(C.8)}{=} \sum_{a=0}^{\infty} \sum_{b=0}^{a} \frac{(\mathrm{i}\,x)^{a-b}}{(a-b)!} \frac{(\mathrm{i}\,y)^b}{b!} \frac{1}{a+1} \overset{(C.7)}{=} \sum_{a=0}^{\infty} \frac{(\mathrm{i}(x+y))^a}{a!} \frac{1}{a+1} \end{aligned}$$

$$\overset{a\to a+1}{=} \sum_{a=1}^{\infty} \frac{(\mathrm{i}(x+y))^{a-1}}{a!} = \frac{\mathrm{e}^{\mathrm{i}(x+y)}-1}{\mathrm{i}(x+y)} \,. \qquad \text{(C.11)}$$

$$\begin{aligned}
&Z_2\left((a,b)\mapsto \frac{1}{(a+1)(a+b+1)}, \{\mathrm{i}\,x, \mathrm{i}\,y\}\right) \\
&= \sum_{a,b=0}^{\infty} \frac{(\mathrm{i}\,x)^a}{a!}\frac{(\mathrm{i}\,y)^b}{b!}\frac{1}{(a+1)(a+b+1)} \overset{\text{(C.8)}}{=} \sum_{a=0}^{\infty}\sum_{b=0}^{a} \frac{(\mathrm{i}\,x)^{a-b}}{(a-b+1)!}\frac{(\mathrm{i}\,y)^b}{b!}\frac{1}{a+1} \\
&\qquad = \sum_{a=0}^{\infty}\sum_{b=0}^{a} \binom{a+1}{b} \frac{(\mathrm{i}\,x)^{a-b}(\mathrm{i}\,y)^b}{(a+1)!(a+1)} \\
&\overset{a\to a+1}{=} \frac{1}{\mathrm{i}\,x}\sum_{a=1}^{\infty}\sum_{b=0}^{a-1}\binom{a}{b}\frac{(\mathrm{i}\,x)^{a-b}(\mathrm{i}\,y)^b}{a!a} \overset{\text{(C.7)}}{=} \frac{1}{\mathrm{i}\,x}\sum_{a=1}^{\infty}\left[\frac{(\mathrm{i}(x+y))^a}{a!a} - \frac{(\mathrm{i}\,y)^a}{a!a}\right] \\
&\quad = \frac{1}{\mathrm{i}\,x}\sum_{a=1}^{\infty}\left[\mathrm{i}\int \mathrm{d}(x+y)\,\frac{(\mathrm{i}(x+y))^{a-1}}{a!} - \mathrm{i}\int \mathrm{d}y\,\frac{(\mathrm{i}\,y)^{a-1}}{a!}\right] \\
&\qquad\qquad = \frac{1}{x}(*_{\mathrm{ei}}(x+y) - *_{\mathrm{ei}}(y))\,, \qquad \text{(C.12)}
\end{aligned}$$

with

$$*_{\mathrm{ei}}(x) := \int \mathrm{d}x\; *_{\mathrm{e}}(x) = \int \mathrm{d}x\,\frac{\mathrm{e}^{\mathrm{i}\,x}-1}{\mathrm{i}\,x} = \sum_{a=1}^{\infty}\frac{(\mathrm{i}\,x)^a}{a!a}\,, \qquad \text{(C.13)}$$

and

$$\begin{aligned}
&Z_3\left((a,b,c)\mapsto \frac{1}{(a+b+1)(a+c+1)}, \{\mathrm{i}\,x, \mathrm{i}\,y, \mathrm{i}\,z\}\right) \\
&\qquad = \frac{\mathrm{e}^{\mathrm{i}\,\frac{yz}{x}}}{\mathrm{i}\,x}\left[*_{\mathrm{ei}}\left(\frac{(x+y)(x+z)}{x}\right) - *_{\mathrm{ei}}\left(\frac{(x+y)z}{x}\right)\right. \\
&\qquad\qquad \left. - *_{\mathrm{ei}}\left(\frac{y(x+z)}{x}\right) + *_{\mathrm{ei}}\left(\frac{yz}{x}\right)\right] =: \mathrm{Ei}^{\psi}(x,y,z)\,. \qquad \text{(C.14)}
\end{aligned}$$

One can check this identity with Mathematica or Maple.

C.3 The next to Leading Order Seiberg-Witten maps

C.3.1 The Ghost Field $C^{[3]}$

We start from the recursive solution (3.3b) where we can read off the next-to-leading order (NLO) Seiberg-Witten map of the ghost field

$$C^{[3]} = -\rho^{[0]}\left(\gamma^{[1]}C^{[2]} - \frac{\mathrm{i}}{2}\left[C \overset{*}{,} C^{[2]}\right] - \frac{\mathrm{i}}{2}\left[C^{[2]} \overset{*}{,} C\right]\right)\,. \qquad \text{(C.15)}$$

Because $\gamma^{[1]}A_\mu = 0$ and $\gamma^{[1]}C = 0$ it follows that the first term on the r.h.s. is zero, so that we get

$$\begin{aligned} C^{[3]} &= \mathrm{i}\,\rho^{[0]}\left(C * C^{[2]} + C^{[2]} * C\right) \\ &= \theta^{\mu\nu}\rho^{[0]}\left(\sin(\wedge_{12}+\wedge_{13})\frac{\sin\wedge_{23}}{\wedge_{23}} C A_\mu(\partial_\nu C)\right) . \end{aligned} \tag{C.16}$$

Now we write the sine in its series representation so that we obtain with (3.23)

$$\begin{aligned} C^{[3]} &= \theta^{\mu\nu}\rho^{[0]} \sum_{m,n=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\frac{(-1)^n}{(2n+1)!}(\wedge_{12}+\wedge_{13})^{2m+1}\wedge_{23}^{2n}\, C A_\mu(\partial_\nu C) \\ &= \frac{\theta^{\mu\nu}}{2}\rho^{[0]}\Big[\wedge_{13} Z_2(\pi_2, \{\mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}\}) \\ &\quad + \wedge_{12} Z_3(\pi_3, \{\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}\})\Big] C A_\mu(\partial_\nu C)\,, \end{aligned} \tag{C.17}$$

using the notation (C.6) with

$$\pi_2(a,b) = \frac{1}{2}\frac{(1+(-1)^a)(1+(-1)^b)}{(a+1)(b+1)}\,, \tag{C.18a}$$

$$\pi_3(a,b,c) = \frac{1}{2}\frac{(1+(-1)^{a+b})(1+(-1)^c)}{(a+1)(c+1)}\,. \tag{C.18b}$$

Before we can apply the operator $\rho^{[0]}$, we must symmetrize the above equation in the indices by using (C.5). To keep the expressions compact we use the multi-index notation (cf. section A.4). With $C^{[3]} = C^{[3]}_{Z_2} + C^{[3]}_{Z_3}$ we get

$$\begin{aligned} C^{[3]}_{Z_2} &= \rho^{[0]} \sum_{a,b=0}^{\infty} \frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\pi_2(a,b)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\mu\nu}}{2} \\ &\quad (\partial_{\boldsymbol{\alpha}[a+1]}C)\left[\partial_{(\boldsymbol{\gamma}[b]}A_{\mu)} + \frac{b}{b+1}\partial_{(\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right](\partial_{\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\delta}[b]}\partial_\nu C) \end{aligned} \tag{C.19a}$$

and

$$\begin{aligned} C^{[3]}_{Z_3} &= \rho^{[0]} \sum_{a,b,c=0}^{\infty} \frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\frac{\mathrm{i}^c}{c!}\pi_3(a,b,c)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2} \\ &\quad (\partial_{\boldsymbol{\alpha}[a+1]}\partial_{\boldsymbol{\gamma}[b]}C)\left[\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c]}A_{\mu)} + \frac{a+c+1}{a+c+2}\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c-1]}F_{\epsilon_c)\mu}\right] \\ &\quad (\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\partial_\nu C)\,. \end{aligned} \tag{C.19b}$$

Now, we let $\rho^{[0]}$ operate on $C^{[3]}_{Z_2}$ and $C^{[3]}_{Z_3}$ which leads to twice as many terms, which we again split up $C^{[3]}_{Z_2} = C^{[3]}_{Z_2,\mathfrak{A}} + C^{[3]}_{Z_2,\mathfrak{B}}$ and $C^{[3]}_{Z_3} = C^{[3]}_{Z_3,\mathfrak{A}} + C^{[3]}_{Z_3,\mathfrak{B}}$.

The $\mathfrak{B}$-type terms have a minus sign which has its origin in the interchange of the operator $\rho^{[0]}$ with the ghost field C, which anti-commutes. The four terms one obtains are

$$C^{[3]}_{Z_2,\mathfrak{A}} = \frac{1}{2}\sum_{a,b=0}^{\infty}\frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\pi_2(a,b)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{(\boldsymbol{\alpha}[a]}A_{\alpha_{a+1})})\left[\frac{1}{3}\partial_{(\boldsymbol{\gamma}[b]}A_{\mu)}+\frac{1}{2}\frac{b}{b+1}\partial_{(\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right](\partial_{\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\delta}[b]}\partial_\nu C)\,, \quad \text{(C.20a)}$$

$$C^{[3]}_{Z_3,\mathfrak{A}} = \frac{1}{2}\sum_{a,b,c=0}^{\infty}\frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\frac{\mathrm{i}^c}{c!}\pi_1(a,b,c)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{(\boldsymbol{\alpha}[a+1]}\partial_{\boldsymbol{\gamma}[b-1]}A_{\gamma_b)})\left[\frac{1}{3}\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c]}A_{\mu)}+\frac{1}{2}\frac{a+c+1}{a+c+2}\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c-1]}F_{\epsilon_c)\mu}\right]$$
$$(\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\partial_\nu C)\,, \quad \text{(C.20b)}$$

$$C^{[3]}_{Z_2,\mathfrak{B}} = -\frac{1}{2}\sum_{a,b=0}^{\infty}\frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\pi_2(a,b)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{\boldsymbol{\alpha}[a+1]}C)\left[\frac{1}{3}\partial_{(\boldsymbol{\gamma}[b]}A_{\mu)}+\frac{1}{2}\frac{b}{b+1}\partial_{(\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right](\partial_{(\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\delta}[b]}A_{\nu)})\,, \quad \text{(C.20c)}$$

$$C^{[3]}_{Z_3,\mathfrak{B}} = -\frac{1}{2}\sum_{a,b,c=0}^{\infty}\frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\frac{\mathrm{i}^c}{c!}\pi_1(a,b,c)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{\boldsymbol{\alpha}[a+1]}\partial_{\boldsymbol{\gamma}[b]}C)\left[\frac{1}{3}\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c]}A_{\mu)}+\frac{1}{2}\frac{a+c+1}{a+c+2}\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c-1]}F_{\epsilon_c)\mu}\right]$$
$$(\partial_{(\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}A_{\nu)})\,. \quad \text{(C.20d)}$$

Before we go on, let us rewrite our result. Due to the different factors which come from the t integration of the operator $\rho^{[0]}$ (3.16) we can't combine the two terms inside the parenthesis to just one term. Hence, in addition to one term which is proportional to A_μ we have all other possible terms. Our result then becomes

$$C^{[3]}_{Z_2,\mathfrak{A}} = \sum_{a,b=0}^{\infty}\frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\pi_2(a,b)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{(\boldsymbol{\alpha}[a]}A_{\alpha_{a+1})})\left[\frac{1}{6}\frac{3b+2}{b+1}\partial_{\boldsymbol{\gamma}[b]}A_\mu-\frac{1}{6}\frac{b}{b+1}\partial_\mu\partial_{(\boldsymbol{\gamma}[b-1]}A_{\gamma_b)}\right]$$
$$(\partial_{\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\delta}[b]}\partial_\nu C)\,, \quad \text{(C.21a)}$$

$$C^{[3]}_{Z_3,\mathfrak{A}} = \sum_{a,b,c=0}^{\infty} \frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\frac{\mathrm{i}^c}{c!}\pi_1(a,b,c)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{(\boldsymbol{\alpha}[a+1]}\partial_{\boldsymbol{\gamma}[b-1]}A_{\gamma_b)})\left[\frac{1}{6}\frac{3a+3c+5}{a+c+2}\partial_{\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c]}A_\mu\right.$$
$$\left.-\frac{1}{6}\frac{a+c+1}{a+c+2}\partial_\mu\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c-1]}A_{\epsilon_c)}\right](\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\partial_\nu C)\,, \quad \text{(C.21b)}$$

$$C^{[3]}_{Z_2,\mathfrak{B}} = -\sum_{a,b=0}^{\infty} \frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\pi_2(a,b)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{\boldsymbol{\alpha}[a+1]}C)\left[\frac{1}{6}\frac{3b+2}{b+1}\partial_{\boldsymbol{\gamma}[b]}A_\mu - \frac{1}{6}\frac{b}{b+1}\partial_\mu\partial_{(\boldsymbol{\gamma}[b-1]}A_{\gamma_b)}\right]$$
$$(\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\delta}[b]}A_{\nu)})\,, \quad \text{(C.21c)}$$

$$C^{[3]}_{Z_3,\mathfrak{B}} = -\sum_{a,b,c=0}^{\infty} \frac{\mathrm{i}^a}{a!}\frac{\mathrm{i}^b}{b!}\frac{\mathrm{i}^c}{c!}\pi_1(a,b,c)\frac{\theta^{\boldsymbol{\alpha}[a+1]\boldsymbol{\beta}[a+1]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{\boldsymbol{\alpha}[a+1]}\partial_{\boldsymbol{\gamma}[b]}C)\left[\frac{1}{6}\frac{3a+3c+5}{a+c+2}\partial_{\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c]}A_\mu\right.$$
$$\left.-\frac{1}{6}\frac{a+c+1}{a+c+2}\partial_\mu\partial_{(\boldsymbol{\beta}[a+1]}\partial_{\boldsymbol{\epsilon}[c-1]}A_{\epsilon_c)}\right](\partial_{(\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}A_{\nu)})\,. \quad \text{(C.21d)}$$

Note, that there are only five different combinations how two vector fields plus one indexless field and their derivatives can be arranged. So we can write the $\mathfrak{A}$-type field $C^{[3]}_{\mathfrak{A}} = C^{[3]}_{Z_3,\mathfrak{A}} + C^{[3]}_{Z_2,\mathfrak{A}}$ as

$$\begin{aligned}
C^{[3]}_{\mathfrak{A}} = {} & \frac{\theta^{\mu\nu}}{2}F^C_{\mathrm{I},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_\mu A_\nu C \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{II},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_2}A_{\mu_1})(\partial_{\nu_1}A_{\nu_2})C \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{III},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}(\partial_{\nu_1}A_{\mu_2})(\partial_{\nu_2}C) \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{IV},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_2}A_{\mu_1})A_{\nu_2}(\partial_{\nu_1}C) \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{V},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})A_{\mu_1}A_{\mu_2}(\partial_{\nu_1}\partial_{\nu_2}C)\,. \quad \text{(C.22)}
\end{aligned}$$

and the $\mathfrak{B}$-type field $C^{[3]}_{\mathfrak{B}} = C^{[3]}_{Z_3,\mathfrak{B}} + C^{[3]}_{Z_2,\mathfrak{B}}$ as

$$\begin{aligned}
C^{[3]}_{\mathfrak{B}} = {} & \frac{\theta^{\mu\nu}}{2}F^C_{\mathrm{I},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})CA_\mu A_\nu \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{II},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})C(\partial_{\mu_2}A_{\mu_1})(\partial_{\nu_1}A_{\nu_2})
\end{aligned}$$

$$+\frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{III},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}C)A_{\mu_2}(\partial_{\nu_2}A_{\nu_1})$$
$$+\frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{IV},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}C)(\partial_{\mu_2}A_{\nu_1})A_{\nu_2}$$
$$+\frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^C_{\mathrm{V},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})(\partial_{\mu_1}\partial_{\mu_2}C)A_{\nu_1}A_{\nu_2}\,, \quad \text{(C.23)}$$

respectively. To determine the functions F^C one has to go through (C.21) and collect all appropriate terms. With

$$\pi_{2,\mathfrak{A}}(a,b) = \frac{\pi_2(a,b)}{3}\,, \quad \text{(C.24a)}$$
$$\pi_{3,\mathfrak{A}}(a,b,c) = \frac{1}{6}\frac{\pi_3(a,b,c)}{(a+b+1)(a+c+2)}\,, \quad \text{(C.24b)}$$
$$\pi_{2,\mathfrak{B}}(a,b) = \frac{1}{3}\frac{\pi_2(a,b)}{a+b+2}\,, \quad \text{(C.24c)}$$
$$\pi_{3,\mathfrak{B}}(a,b,c) = \frac{1}{6}\frac{\pi_3(a,b,c)}{(b+c+1)(a+c+2)}\,, \quad \text{(C.24d)}$$

the functions F^C are given by

$$F^C_{\mathrm{I},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{12}}$$
$$\wedge_{12}\wedge_{23}Z_3(-(a+1)\pi_{3,\mathfrak{A}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\,, \quad \text{(C.25a)}$$

$$F^C_{\mathrm{II},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{12}^2}$$
$$\wedge_{12}\wedge_{23}Z_3(-a(a+1)\pi_{3,\mathfrak{A}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\,, \quad \text{(C.25b)}$$

$$F^C_{\mathrm{III},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{12}\wedge_{23}}$$
$$\wedge_{12}\wedge_{23}Z_3((a+1)(3a+2c+5)\pi_{3,\mathfrak{A}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\,, \quad \text{(C.25c)}$$

$$F^C_{\mathrm{IV},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{12}\wedge_{13}}$$
$$\wedge_{12}\wedge_{23}Z_3(-b(a+1))\pi_{3,\mathfrak{A}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\,, \quad \text{(C.25d)}$$

$$F^C_{\mathrm{V},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{13}\wedge_{23}}\Big[\wedge_{13}\wedge_{23}\,Z_2(\pi_{2,\mathfrak{A}},\{\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})$$
$$+\wedge_{12}\wedge_{23}\,Z_3(b(3a+2c+5)\pi_{3,\mathfrak{A}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\Big] \quad \text{(C.25e)}$$

and

$$F^C_{\mathrm{I},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{23}}\Big[\wedge_{13}\wedge_{23} Z_2(-\pi_{2,\mathfrak{B}},\{\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) + \wedge_{12}\wedge_{23} Z_3(-(3a+2c+5)\pi_{3,\mathfrak{B}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\Big], \quad \text{(C.26a)}$$

$$F^C_{\mathrm{II},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{23}^2}\Big[\wedge_{13}\wedge_{23} Z_2(-b\pi_{2,\mathfrak{B}},\{\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) + \wedge_{12}\wedge_{23} Z_3(-c(3a+2c+5)\pi_{3,\mathfrak{B}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\Big], \quad \text{(C.26b)}$$

$$F^C_{\mathrm{III},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{13}\wedge_{23}}\Big[\wedge_{13}\wedge_{23} Z_2(-(a+1)\pi_{2,\mathfrak{B}},\{\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) + \wedge_{12}\wedge_{23} Z_3(-b(3a+2c+5)\pi_{3,\mathfrak{B}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\Big], \quad \text{(C.26c)}$$

$$F^C_{\mathrm{IV},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{12}\wedge_{23}} \wedge_{12}\wedge_{23} Z_3((a+1)(c+1)\pi_{3,\mathfrak{B}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}), \quad \text{(C.26d)}$$

$$F^C_{\mathrm{V},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{1}{\wedge_{12}\wedge_{13}} \wedge_{12}\wedge_{23} Z_3((a+1)b\pi_{3,\mathfrak{B}},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}), \quad \text{(C.26e)}$$

respectively. The NLO Seiberg-Witten map for the ghost field is then $C^{[3]} = C^{[3]}_{\mathfrak{A}} + C^{[3]}_{\mathfrak{B}}$.

To check our result let us calculate the r.h.s. of the appropriate gauge equivalence equation (2.26). Doing so we have to calculate the commutative BRST transformation of the map $C^{[3]}$, which is with $C^{[3]} = C^{[3]}_{\mathfrak{A}} + C^{[3]}_{\mathfrak{B}}$

$$\begin{aligned}\gamma C^{[3]}_{\mathfrak{A}} = \frac{1}{2}\theta^{\mu\nu}\Big[& F^C_{\mathrm{I},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{(\partial_\mu C)A_\nu C + A_\mu(\partial_\nu C)C\} \\ & + F^C_{\mathrm{II},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{12}(\partial_\mu C)A_\nu C + \wedge_{12}A_\mu(\partial_\nu C)C\} \\ & + F^C_{\mathrm{III},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{12}CA_\mu(\partial_\nu C) + \wedge_{23}A_\mu(\partial_\nu C)C\} \\ & + F^C_{\mathrm{IV},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{13}(\partial_\mu C)A_\nu C + \wedge_{12}A_\mu C(\partial_\nu C)\} \\ & + F^C_{\mathrm{V},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{13}CA_\mu(\partial_\nu C) + \wedge_{23}A_\mu C(\partial_\nu C)\}\Big] \end{aligned} \quad \text{(C.27)}$$

and

$$\begin{aligned}\gamma C^{[3]}_{\mathfrak{B}} = -\frac{1}{2}\theta^{\mu\nu}\Big[&F^{C}_{\mathrm{I},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{C(\partial_\mu C)A_\nu + CA_\mu(\partial_\nu C)\}\\
&+ F^{C}_{\mathrm{II},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{23}C(\partial_\mu C)A_\nu + \wedge_{23}CA_\mu(\partial_\nu C)\}\\
&+ F^{C}_{\mathrm{III},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{23}(\partial_\mu C)CA_\nu + \wedge_{13}CA_\mu(\partial_\nu C)\}\\
&+ F^{C}_{\mathrm{IV},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{12}C(\partial_\mu C)A_\nu + \wedge_{23}(\partial_\mu C)A_\nu C\}\\
&+ F^{C}_{\mathrm{V},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{12}(\partial_\mu C)CA_\nu + \wedge_{13}(\partial_\mu C)A_\nu C\}\Big]\,. \qquad \text{(C.28)}\end{aligned}$$

In the next step we combine all terms which are proportional to the same fields. The subscript labels to which fields the function is proportional. We have six different combinations of fields and derivatives, namely

$$\begin{aligned}F^{C}_{(\partial_\mu C)A_\nu C}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \big[&F_{\mathrm{I},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&+ \wedge_{12}F_{\mathrm{II},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) + \wedge_{13}F_{\mathrm{IV},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&- \wedge_{23}F_{\mathrm{IV},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) - \wedge_{13}F_{\mathrm{V},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\big]\,, \qquad \text{(C.29a)}\end{aligned}$$

$$\begin{aligned}F^{C}_{A_\mu(\partial_\nu C)C}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \big[&F_{\mathrm{I},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&+ \wedge_{12}F_{\mathrm{II},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) + \wedge_{23}F_{\mathrm{III},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\big]\,, \qquad \text{(C.29b)}\end{aligned}$$

$$\begin{aligned}F^{C}_{CA_\mu(\partial_\nu C)}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \big[&\wedge_{12}F_{\mathrm{III},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&+ \wedge_{13}F_{\mathrm{V},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) - F_{\mathrm{I},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&- \wedge_{23}F_{\mathrm{II},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) - \wedge_{13}F_{\mathrm{III},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\big]\,, \qquad \text{(C.29c)}\end{aligned}$$

$$\begin{aligned}F^{C}_{A_\mu C(\partial_\nu C)}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \big[&\mathrm{i}\wedge_{12}F_{\mathrm{IV},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&+ \wedge_{23}F_{\mathrm{V},\mathfrak{A}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\big]\,, \qquad \text{(C.29d)}\end{aligned}$$

$$\begin{aligned}F^{C}_{C(\partial_\mu C)A_\nu}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \big[&-F_{\mathrm{I},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&- \wedge_{23}F_{\mathrm{II},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) - \wedge_{12}F_{\mathrm{IV},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\big]\,, \qquad \text{(C.29e)}\end{aligned}$$

$$\begin{aligned}F^{C}_{(\partial_\mu C)CA_\nu}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \big[&-\wedge_{23}\,F_{\mathrm{III},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&- \wedge_{12}F_{\mathrm{V},\mathfrak{B}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\big]\,. \qquad \text{(C.29f)}\end{aligned}$$

If we now permute these six above terms so that they become proportional to $CA_\mu(\partial_\nu C)$ we get

$$\gamma C^{[3]} = \frac{1}{2}\theta^{\mu\nu}CA_\mu(\partial_\nu C)\big[F_{(\partial_\mu C)A_\nu C}(-\mathrm{i}\wedge_{23},-\mathrm{i}\wedge_{13},-\mathrm{i}\wedge_{12})$$

$$
\begin{aligned}
&- F_{A_\mu(\partial_\nu C)C}(\mathrm{i}\wedge_{23}, -\mathrm{i}\wedge_{12}, -\mathrm{i}\wedge_{13}) + F_{CA_\mu(\partial_\nu C)}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}) \\
&+ F_{A_\mu C(\partial_\nu C)}(-\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{23}, \mathrm{i}\wedge_{13}) - F_{C(\partial_\mu C)A_\nu}(\mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{12}, -\mathrm{i}\wedge_{23}) \\
&\qquad + F_{(\partial_\mu C)CA_\nu}(-\mathrm{i}\wedge_{13}, -\mathrm{i}\wedge_{23}, \mathrm{i}\wedge_{12})\big] \,, \qquad \text{(C.30)}
\end{aligned}
$$

where the minus sign comes from either the interchange of $\mu \leftrightarrow \nu$ or from the interchange of the two ghost fields. If we perform all the sums in the above equation and simplify the expression we end up with

$$
\gamma C^{[3]} = \theta^{\mu\nu} \sin(\wedge_{12} + \wedge_{13}) \frac{\sin \wedge_{23}}{\wedge_{23}} \, CA_\mu(\partial_\nu C)\,, \qquad \text{(C.31)}
$$

which is just the l.h.s. of the gauge equivalence equation for the ghost field

$$
\begin{aligned}
\hat{\gamma}\hat{C} = \frac{\mathrm{i}}{2}[\hat{C} \stackrel{*}{,} \hat{C}] &\stackrel{\mathcal{O}(A)}{=} -2C \sin(\wedge_{12} + \wedge_{13}) C^{[2]} \\
&= \theta^{\mu\nu} \sin(\wedge_{12} + \wedge_{13}) \frac{\sin \wedge_{23}}{\wedge_{23}} CA_\mu(\partial_\nu C)\,. \qquad \text{(C.32)}
\end{aligned}
$$

C.3.2 The Matter Field $\psi^{[3]}$

As in the previous section we start from the recursive solution (3.3c) where we can read off the next to leading order Seiberg-Witten map of the matter field

$$
\psi^{[3]} = -\rho^{[0]} \left(\gamma^{[1]} \psi^{[2]} - \mathrm{i} \left[C * \psi^{[2]} + C^{[2]} * \psi \right] \right) . \qquad \text{(C.33)}
$$

With $\gamma^{[1]} A_\mu = 0$, $\gamma^{[1]}\psi = \mathrm{i}\, C\psi$ and the maps $C^{[2]}$ and $\psi^{[2]}$ we get

$$
\begin{aligned}
\psi^{[3]} = \frac{\mathrm{i}}{2} \theta^{\mu\nu} \rho^{[0]} \Bigg(&\frac{e^{\mathrm{i}(\wedge_{12}+\wedge_{13})} - 1}{\mathrm{i}\wedge_{12} + \mathrm{i}\wedge_{13}} A_\mu \partial_\nu (C\psi) \\
-e^{\mathrm{i}(\wedge_{12}+\wedge_{13})} \frac{e^{\mathrm{i}\wedge_{23}} - 1}{\mathrm{i}\wedge_{23}} CA_\mu(\partial_\nu \psi) &- \frac{\sin \wedge_{12}}{\wedge_{12}} e^{\mathrm{i}(\wedge_{13}+\wedge_{23})} A_\mu(\partial_\nu C)\psi \Bigg) . \qquad \text{(C.34)}
\end{aligned}
$$

Since the tensor structure of these three terms is different from each other we will tackle them one by one.

We begin with the first term which depends only on $\wedge_{12}$ and $\wedge_{13}$. The series representation of this term is

$$
\begin{aligned}
\psi_1^{[3]} &= \frac{\mathrm{i}}{2} \theta^{\mu\nu} \rho^{[0]} \sum_{a=0}^{\infty} \frac{1}{(a+1)!} (\mathrm{i}\wedge_{12} + \mathrm{i}\wedge_{13})^a A_\mu \partial_\nu (C\psi) \\
&= \frac{1}{2} \theta^{\mu\nu} \rho^{[0]} \sum_{a,b=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{1}{a+b+1} A_\mu \partial_\nu (C\psi)\,. \qquad \text{(C.35)}
\end{aligned}
$$

As in the case of $C^{[3]}$ we introduce the multi-index notation (A.15)

$$\psi_1^{[3]} = \mathrm{i}\,\rho^{[0]} \sum_{a,b=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{1}{a+b+1} \frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2} \frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2} \frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}A_\mu)\big[(\partial_{\boldsymbol{\beta}[a]}\partial_\nu C)(\partial_{\boldsymbol{\delta}[b]}\psi) + (\partial_{\boldsymbol{\beta}[a]}C)(\partial_{\boldsymbol{\delta}[b]}\partial_\nu\psi)\big]\,. \quad \text{(C.36)}$$

In the next step we have to symmetrise the above expression in its indices in order to perform the operator $\rho^{[0]}$

$$\psi_1^{[3]} = \mathrm{i}\,\rho^{[0]} \sum_{a,b=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{1}{a+b+1} \frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2} \frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2} \frac{\theta^{\mu\nu}}{2}$$
$$\left[\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}A_{\mu)} + \frac{a+b}{a+b+1}\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right]$$
$$\big[(\partial_{\boldsymbol{\beta}[a]}\partial_\nu C)(\partial_{\boldsymbol{\delta}[b]}\psi) + (\partial_{\boldsymbol{\beta}[a]}C)(\partial_{\boldsymbol{\delta}[b]}\partial_\nu\psi)\big]\,. \quad \text{(C.37)}$$

After we have obtained the expression symmetric in its indices we can go on and apply $\rho^{[0]}$

$$\psi_1^{[3]} = \mathrm{i} \sum_{a,b=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{1}{a+b+1} \frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2} \frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2} \frac{\theta^{\mu\nu}}{2}$$
$$\left[\frac{1}{2}\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}A_{\mu)} + \frac{a+b}{a+b+1}\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right]$$
$$\big[(\partial_{(\boldsymbol{\beta}[a]}A_{\nu)})(\partial_{\boldsymbol{\delta}[b]}\psi) + (\partial_{(\boldsymbol{\beta}[a-1]}A_{\beta_a)})(\partial_{\boldsymbol{\delta}[b]}\partial_\nu\psi)\big]\,. \quad \text{(C.38)}$$

Now we come to the second term of (C.34). The series expansion of this expression is just

$$\psi_2^{[3]} = -\frac{\mathrm{i}}{2}\theta^{\mu\nu}\rho^{[0]} \sum_{a,b,c=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{(\mathrm{i}\wedge_{23})^c}{c!} \frac{1}{c+1} C A_\mu(\partial_\nu\psi)\,. \quad \text{(C.39)}$$

With the multi-index notation we obtain for the second term

$$\psi_2^{[3]} = -\,\mathrm{i}\,\rho^{[0]} \sum_{a,b,c=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{(\mathrm{i}\wedge_{23})^c}{c!} \frac{1}{c+1}$$
$$\frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2} \frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2} \frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2} \frac{\theta^{\mu\nu}}{2}$$
$$(\partial_{\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}C)(\partial_{\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c]}A_\mu)(\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\partial_\nu\psi)\,. \quad \text{(C.40)}$$

Now we symmetrise our expression which leads to

$$\psi_2^{[3]} = -\,\mathrm{i}\,\rho^{[0]} \sum_{a,b,c=0}^{\infty} \frac{(\mathrm{i}\wedge_{12})^a}{a!} \frac{(\mathrm{i}\wedge_{13})^b}{b!} \frac{(\mathrm{i}\wedge_{23})^c}{c!} \frac{1}{c+1}$$
$$\frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2} \frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2} \frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2} \frac{\theta^{\mu\nu}}{2}(\partial_{\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}C)$$

$$\left[\partial_{(\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c]}A_{\mu)} + \frac{a+c}{a+c+1}\partial_{(\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c-1]}F_{\epsilon_c)\mu}\right](\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\partial_\nu\psi)\,. \quad (C.41)$$

After this we can apply the operator $\rho^{[0]}$

$$\begin{aligned}\psi_2^{[3]} &= -\mathrm{i}\sum_{a,b,c=0}^{\infty}\frac{(\mathrm{i}\wedge_{12})^a}{a!}\frac{(\mathrm{i}\wedge_{13})^b}{b!}\frac{(\mathrm{i}\wedge_{23})^c}{c!}\frac{1}{c+1}\\ &\qquad \frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}(\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b-1]}A_{\gamma_c)})\\ &\qquad \left[\frac{1}{2}\partial_{(\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c]}A_{\mu)} + \frac{a+c}{a+c+1}\partial_{(\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c-1]}F_{\epsilon_c)\mu}\right](\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\partial_\nu\psi)\,. \quad (C.42)\end{aligned}$$

The series expansion of the last term is

$$\begin{aligned}\psi_3^{[3]} &= -\frac{\mathrm{i}}{2}\theta^{\mu\nu}\rho^{[0]}\left(\frac{\sin\wedge_{12}}{\wedge_{12}}e^{\mathrm{i}(\wedge_{13}+\wedge_{23})}A_\mu(\partial_\nu C)\psi\right)\\ &= -\frac{\mathrm{i}}{4}\theta^{\mu\nu}\rho^{[0]}\sum_{a,b,c=0}^{\infty}\frac{(\mathrm{i}\wedge_{12})^a}{a!}\frac{(\mathrm{i}\wedge_{13})^b}{b!}\frac{(\mathrm{i}\wedge_{23})^c}{c!}\frac{1+(-1)^a}{a+1}A_\mu(\partial_\nu C)\psi\,. \quad (C.43)\end{aligned}$$

With the multi-index notation we have

$$\begin{aligned}\psi_3^{[3]} &= -\frac{\mathrm{i}}{2}\rho^{[0]}\sum_{a,b,c=0}^{\infty}\frac{(\mathrm{i}\wedge_{12})^a}{a!}\frac{(\mathrm{i}\wedge_{13})^b}{b!}\frac{(\mathrm{i}\wedge_{23})^c}{c!}\frac{1+(-1)^a}{a+1}\\ &\qquad \frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}\\ &\qquad (\partial_{\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}A_\mu)(\partial_{\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c]}\partial_\nu C)(\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\psi)\,. \quad (C.44)\end{aligned}$$

After the symmetrisation the expression is

$$\begin{aligned}\psi_3^{[3]} &= -\frac{\mathrm{i}}{2}\rho^{[0]}\sum_{a,b,c=0}^{\infty}\frac{(\mathrm{i}\wedge_{12})^a}{a!}\frac{(\mathrm{i}\wedge_{13})^b}{b!}\frac{(\mathrm{i}\wedge_{23})^c}{c!}\frac{1+(-1)^a}{a+1}\\ &\qquad \frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}\\ &\qquad \left[\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}A_{\mu)} + \frac{a+b}{a+b+1}\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right]\\ &\qquad (\partial_{\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c]}\partial_\nu C)(\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\psi)\,. \quad (C.45)\end{aligned}$$

Now applying $\rho^{[0]}$ we get

$$\psi_3^{[3]} = -\frac{\mathrm{i}}{2}\sum_{a,b,c=0}^{\infty}\frac{(\mathrm{i}\wedge_{12})^a}{a!}\frac{(\mathrm{i}\wedge_{13})^b}{b!}\frac{(\mathrm{i}\wedge_{23})^c}{c!}\frac{1+(-1)^a}{a+1}$$

$$\frac{\theta^{\boldsymbol{\alpha}[a]\boldsymbol{\beta}[a]}}{2}\frac{\theta^{\boldsymbol{\gamma}[b]\boldsymbol{\delta}[b]}}{2}\frac{\theta^{\boldsymbol{\epsilon}[c]\boldsymbol{\zeta}[c]}}{2}\frac{\theta^{\mu\nu}}{2}$$
$$\left[\frac{1}{2}\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b]}A_{\mu)} + \frac{a+b}{a+b+1}\partial_{(\boldsymbol{\alpha}[a]}\partial_{\boldsymbol{\gamma}[b-1]}F_{\gamma_b)\mu}\right]$$
$$(\partial_{(\boldsymbol{\beta}[a]}\partial_{\boldsymbol{\epsilon}[c]}A_{\nu)})(\partial_{\boldsymbol{\delta}[b]}\partial_{\boldsymbol{\zeta}[c]}\psi)\,. \quad \text{(C.46)}$$

We end up with three terms (C.38), (C.42) and (C.46) where the sum of these three expressions build the NLO Seiberg-Witten map for the matter field. As in the previous section we can write this map as

$$\begin{aligned}
\psi^{[3]}(A,\psi) = {} & \frac{\theta^{\mu\nu}}{2}F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\ A_\mu A_\nu \psi \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^{\psi}_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\ (\partial_{\mu_2}A_{\mu_1})(\partial_{\nu_1}A_{\nu_2})\psi \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^{\psi}_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\ A_{\mu_1}(\partial_{\nu_1}A_{\mu_2})(\partial_{\nu_2}\psi) \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^{\psi}_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\ (\partial_{\mu_2}A_{\mu_1})A_{\nu_2}(\partial_{\nu_1}\psi) \\
& + \frac{\theta^{\mu_1\nu_1}}{2}\frac{\theta^{\mu_2\nu_2}}{2}F^{\psi}_{\mathrm{V}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\ A_{\mu_1}A_{\mu_2}(\partial_{\nu_1}\partial_{\nu_2}\psi)\,. \quad \text{(C.47)}
\end{aligned}$$

With the parameters

$$\pi_{21}(a,b) = \frac{1}{a+b+1}\,, \quad \text{(C.48a)}$$
$$\pi_{22}(a,b) = \frac{1}{(a+1)(a+b+1)}\,, \quad \text{(C.48b)}$$
$$\pi_3(a,b,c) = \frac{1}{(a+b+1)(a+c+1)}\,, \quad \text{(C.48c)}$$

the above F^ψ functions are

$$\begin{aligned}
F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = {} & \frac{\mathrm{i}}{2}\Big[Z_3(\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) \\
& + \frac{1}{2}[Z_3(\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) + Z_3(\pi_3,\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})] \\
& - e^{\mathrm{i}\wedge_{13}}[Z_2(\pi_{22},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\}) + Z_2(\pi_{22},\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\})]\Big]\,, \quad \text{(C.49a)}
\end{aligned}$$

$$\begin{aligned}
F^{\psi}_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = {} & \frac{\mathrm{i}}{2}\frac{1}{\wedge_{12}}\Big[Z_3(a\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) \\
& + \frac{1}{2}[Z_3(a\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\}) + Z_3(a\pi_3,\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})] \\
& - e^{\mathrm{i}\wedge_{13}}[Z_2(a\pi_{22},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\}) + Z_2(a\pi_{22},\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\})]\Big]\,, \quad \text{(C.49b)}
\end{aligned}$$

$$
\begin{aligned}
&F^{\psi}_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\\
&\qquad=\frac{\mathrm{i}}{2}\frac{1}{\wedge_{12}}\left[-\frac{\mathrm{e}^{\mathrm{i}\wedge_{13}}-1}{\mathrm{i}\wedge_{13}}+\mathrm{e}^{\mathrm{i}\wedge_{23}}Z_2(\pi_{21},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13}\})\right]\\
&\quad+\frac{\mathrm{i}}{4}\frac{1}{\wedge_{13}}[Z_3(b\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})+Z_3(b\pi_3,\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})]\\
&\qquad\qquad+\frac{1}{2}\frac{\wedge_{23}}{\wedge_{12}}Z_3(\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\,,
\end{aligned}
\tag{C.49c}
$$

$$
\begin{aligned}
F^{\psi}_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})&=\frac{\mathrm{i}}{2}\frac{1}{\wedge_{23}}\Big[-2\left(\mathrm{e}^{\mathrm{i}\wedge_{23}}-1\right)Z_2(\pi_{21},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13}\})\\
&+\mathrm{e}^{\mathrm{i}\wedge_{13}}[Z_2(\pi_{21},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\})+Z_2(\pi_{21},\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\})]\\
&-2\frac{\sin(\wedge_{12})}{\wedge_{12}}\mathrm{e}^{\mathrm{i}(\wedge_{13}+\wedge_{23})}+\frac{1}{2}[Z_3(c\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\\
&+Z_3(c\pi_3,\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})]+Z_3(c\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\Big]\,,
\end{aligned}
\tag{C.49d}
$$

$$
\begin{aligned}
F^{\psi}_{\mathrm{V}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})&=\frac{\mathrm{i}}{2}\frac{\wedge_{12}}{\wedge_{13}\wedge_{23}}\Big[\frac{\sin(\wedge_{12})}{\wedge_{12}}\mathrm{e}^{\mathrm{i}(\wedge_{13}+\wedge_{23})}\\
&-\frac{1}{2}\mathrm{e}^{\mathrm{i}\wedge_{13}}[Z_2(\pi_{21},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\})+Z_2(\pi_{21},\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{23}\})]\\
&-\frac{1}{2}[Z_3(c\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})+Z_3(c\pi_3,\{-\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})]\Big]\\
&+\frac{\mathrm{e}^{\mathrm{i}\wedge_{23}}-1}{\mathrm{i}\wedge_{23}}Z_2(\pi_{21},\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13})-\frac{1}{2}Z_3(\pi_3,\{\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}\})\,.
\end{aligned}
\tag{C.49e}
$$

With the identity (C.10) and the above defined functions (C.11), (C.12) and (C.14) we can rewrite the F^{ψ} function

$$
\begin{aligned}
F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})&=-\frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}\wedge_{13}}\frac{1}{\wedge_{12}}\\
&[*_{\mathrm{ei}}(\wedge_{12}+\wedge_{23})+*_{\mathrm{ei}}(-\wedge_{12}+\wedge_{23})-2*_{\mathrm{ei}}(\wedge_{23})]\\
&+\frac{\mathrm{i}}{4}[3\,\mathrm{Ei}^{\psi}(\wedge_{12},\wedge_{13},\wedge_{23})+\mathrm{Ei}^{\psi}(-\wedge_{12},\wedge_{13},\wedge_{23})]\,,
\end{aligned}
\tag{C.50a}
$$

$$
F^{\psi}_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})=\partial_{\wedge_{12}}F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,,
\tag{C.50b}
$$

$$
\begin{aligned}
F^{\psi}_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})&=\frac{1}{2}\frac{\wedge_{23}}{\wedge_{12}}\mathrm{Ei}^{\psi}(\wedge_{12},\wedge_{13},\wedge_{23})\\
&+\frac{\mathrm{i}}{4}\partial_{\wedge_{13}}[\mathrm{Ei}^{\psi}(\wedge_{12},\wedge_{13},\wedge_{23})+\mathrm{Ei}^{\psi}(-\wedge_{12},\wedge_{13},\wedge_{23})]\\
&+\frac{\mathrm{i}}{2}\frac{1}{\wedge_{12}}\left[-*_{\mathrm{e}}(\wedge_{13})+\mathrm{e}^{\mathrm{i}\wedge_{23}}*_{\mathrm{e}}(\wedge_{12}+\wedge_{13})\right]\,,
\end{aligned}
\tag{C.50c}
$$

$$F^{\psi}_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = *_{\mathrm{e}}(\wedge_{23}) *_{\mathrm{e}} (\wedge_{12}+\wedge_{13})$$
$$+\frac{\mathrm{i}}{4}\partial_{\wedge_{23}}[3\,\mathrm{Ei}^{\psi}(\wedge_{12},\wedge_{13},\wedge_{23}) + \mathrm{Ei}^{\psi}(-\wedge_{12},\wedge_{13},\wedge_{23})]$$
$$\frac{\mathrm{i}}{2}\frac{1}{\wedge_{23}}\Big[\mathrm{e}^{\mathrm{i}\wedge_{13}}[*_{\mathrm{e}}(\wedge_{12}+\wedge_{23}) + *_{\mathrm{e}}(-\wedge_{12}+\wedge_{23})]$$
$$-2\frac{\sin(\wedge_{12})}{\wedge_{12}}\,\mathrm{e}^{\mathrm{i}(\wedge_{13}+\wedge_{23})}\Big], \quad \text{(C.50d)}$$

$$F^{\psi}_{\mathrm{V}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \frac{\mathrm{i}}{2}\frac{\wedge_{12}}{\wedge_{13}\wedge_{23}}\Big[\frac{\sin(\wedge_{12})}{\wedge_{12}}\,\mathrm{e}^{\mathrm{i}(\wedge_{13}+\wedge_{23})}$$
$$-\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\wedge_{13}}[*_{\mathrm{e}}(\wedge_{12}+\wedge_{23}) + *_{\mathrm{e}}(-\wedge_{12},\wedge_{23})]\Big]$$
$$-\frac{\mathrm{i}}{4}\frac{\wedge_{12}}{\wedge_{13}}\partial_{\wedge_{23}}[\mathrm{Ei}^{\psi}(\wedge_{12},\wedge_{13},\wedge_{23}) + \mathrm{Ei}^{\psi}(-\wedge_{12},\wedge_{13},\wedge_{23})]$$
$$+ *_{\mathrm{e}}(\wedge_{23}) *_{\mathrm{e}} (\wedge_{12}+\wedge_{13}) - \frac{1}{2}\,\mathrm{Ei}^{\psi}(\wedge_{12},\wedge_{13},\wedge_{23})\,. \quad \text{(C.50e)}$$

Again, we test our result by checking the appropriate gauge equivalence equation (2.26). The r.h.s. of this equation is

$$\gamma\psi^{[3]} = \frac{1}{2}\theta^{\mu\nu}\Big[F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{(\partial_\mu C)A_\nu\psi + A_\mu(\partial_\nu C)\psi\}$$
$$+F^{\psi}_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{12}(\partial_\mu C)A_\nu\psi + \wedge_{12}A_\mu(\partial_\nu C)\psi\}$$
$$+F^{\psi}_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{12}CA_\mu(\partial_\nu\psi) + \wedge_{23}A_\mu(\partial_\nu C)\psi\}$$
$$+F^{\psi}_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{13}(\partial_\mu C)A_\nu\psi + \wedge_{12}A_\mu C(\partial_\nu\psi)\}$$
$$+F^{\psi}_{\mathrm{V}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\,\{\wedge_{13}CA_\mu(\partial_\nu\psi) + \wedge_{23}A_\mu C(\partial_\nu\psi)\}\Big]\,. \quad \text{(C.51)}$$

We combine all terms which are proportional to the same fields. The four different types of field configurations are

$$F^{\psi}_{(\partial_\mu C)A_\nu\psi}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \Big[F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})$$
$$+\wedge_{12}F^{\psi}_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) + \wedge_{13}F^{\psi}_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\Big], \quad \text{(C.52a)}$$

$$F^{\psi}_{A_\mu(\partial_\nu C)\psi}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \Big[F^{\psi}_{\mathrm{I}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})$$
$$+\wedge_{12}F^{\psi}_{\mathrm{II}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) + \wedge_{23}F^{\psi}_{\mathrm{IV}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})\Big], \quad \text{(C.52b)}$$

$$F^{\psi}_{CA_\mu(\partial_\nu\psi)}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23}) = \Big[\wedge_{12}F^{\psi}_{\mathrm{III}}(\mathrm{i}\wedge_{12},\mathrm{i}\wedge_{13},\mathrm{i}\wedge_{23})$$

$$+ \wedge_{13} F^{\psi}_{\mathrm{V}}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23})\Big], \quad \text{(C.52c)}$$

$$F^{\psi}_{A_\mu C(\partial_\nu \psi)}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}) = \Big[\wedge_{12} F^{\psi}_{\mathrm{IV}}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}) + \wedge_{23} F^{\psi}_{\mathrm{V}}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23})\Big], \quad \text{(C.52d)}$$

where the subscription of each function labels the appropriate field configuration.

If we now appropriately combine the first and the second as well as the third and the fourth term we get

$$\begin{aligned}\gamma\psi^{[3]} = \frac{1}{2}\theta^{\mu\nu}\Big[&\big(F^{\psi}_{CA_\mu(\partial_\nu\psi)}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}) \\ &+ F^{\psi}_{A_\mu C(\partial_\nu\psi)}(-\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{23}, \mathrm{i}\wedge_{13})\big) C A_\mu(\partial_\nu\psi) \\ &+ \big(F^{\psi}_{A_\mu(\partial_\nu C)\psi}(\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{13}, \mathrm{i}\wedge_{23}) \\ &- F^{\psi}_{(\partial_\mu C)A_\nu\psi}(-\mathrm{i}\wedge_{12}, \mathrm{i}\wedge_{23}, \mathrm{i}\wedge_{13})\big) A_\mu(\partial_\nu C)\psi\Big], \end{aligned} \quad \text{(C.53)}$$

where the minus sign comes form $\theta^{\mu\nu}$ because one has to interchange $\mu \leftrightarrow \nu$. If we perform all the sums in the above equation and simplify the expression we end up with

$$\begin{aligned}\gamma\psi^{[3]} = -\frac{\mathrm{i}}{2}\theta^{\mu\nu}\Big[&\mathrm{e}^{\mathrm{i}(\wedge_{12}+\wedge_{13})}\,\frac{\mathrm{e}^{\mathrm{i}\wedge_{23}}-1}{\mathrm{i}\wedge_{23}} C A_\mu(\partial_\nu\psi) \\ &+\frac{\sin(\wedge_{12})}{\wedge_{12}}\,\mathrm{e}^{\mathrm{i}(\wedge_{13}+\wedge_{23})}\,A_\mu(\partial_\nu C)\psi\Big]. \end{aligned} \quad \text{(C.54)}$$

This is just the l.h.s. of the gauge equivalence equation of the matter field

$$\begin{aligned}\hat{\gamma}\hat{\psi} = \mathrm{i}\,\hat{C} * \hat{\psi} &\overset{\mathcal{O}(A)}{=} \mathrm{i}\,C * \psi^{[2]} + \mathrm{i}\,C^{[2]} * \psi \\ &= -\frac{\mathrm{i}}{2}\theta^{\mu\nu}\Big[\mathrm{e}^{\mathrm{i}(\wedge_{12}+\wedge_{13})}\,\frac{\mathrm{e}^{\mathrm{i}\wedge_{23}}-1}{\mathrm{i}\wedge_{23}} C A_\mu(\partial_\nu\psi) \\ &\qquad+\frac{\sin(\wedge_{12})}{\wedge_{12}}\,\mathrm{e}^{\mathrm{i}(\wedge_{13}+\wedge_{23})}\,A_\mu(\partial_\nu C)\psi\Big]. \end{aligned} \quad \text{(C.55)}$$

C.4 Proof of the Homotopy Equation

Theorem C.4.1. *Let y^α, z^α and w^α be the coordinates defined in section 3.2. Let f be a function of these variables of the form*

$$f(y,z,w) = y^{\boldsymbol{\alpha}[0,a]} z^{\boldsymbol{\beta}[1,b]} w^{\boldsymbol{\gamma}[0,c]}, \quad a \geq 0\,,\ b \geq 1\,,\ c \geq 0\,, \quad \text{(C.56)}$$

i.e. the function has to depend at least on one z. The multi-index notation is defined in section A.4. Let furthermore $\gamma^{[0]}$ be a differential of the form

$$\gamma^{[0]}y^\alpha = z^\alpha\,, \gamma^{[0]}z^\alpha = 0 \text{ and } \gamma^{[0]}w^\alpha = 0\,. \tag{C.57}$$

Then

$$\rho^{[0]}f(y,z,w) = \int_0^t \frac{\mathrm{d}t}{t}\left[y^\alpha \frac{\partial}{\partial z^\alpha}f\right](ty,tz,w) \tag{C.58}$$

is the contracting homotopy for $\gamma^{[0]}$, i.e.

$$N = \rho^{[0]}\gamma^{[0]} + \gamma^{[0]}\rho^{[0]} \tag{C.59}$$

satisfies

$$Nf = f\,. \tag{C.60}$$

Proof. First note, that $\gamma^{[0]}$, $\rho^{[0]}$ and z all have an odd Grassmann parity. So they all anticommute with each other. Now, let us derive how the differential acts on $y^{\boldsymbol{\alpha}[a]}$, namely

$$\begin{aligned}\gamma^{[0]}y^{\boldsymbol{\alpha}[1,a]} &= \sum_{i=1}^{a} y^{\alpha_1}\ldots y^{\alpha_{i-1}}z^{\alpha_i}y^{\alpha_{i+1}}\ldots y^{\alpha_a}\\ &= \sum_{i=1}^{a} y^{\boldsymbol{\alpha}[1,i-1]}z^{\alpha_i}y^{\boldsymbol{\alpha}[i+1,a]}\,.\end{aligned} \tag{C.61}$$

Up to the t integration in the definition of $\rho^{[0]}$ this operator acts on $z^{\boldsymbol{\beta}[b]}$ like

$$\left[y^\alpha\frac{\partial}{\partial z^\alpha}z^{\boldsymbol{\beta}[1,b]}\right] = \sum_{i=1}^{b}(-1)^{i+1}z^{\boldsymbol{\beta}[1,i-1]}y^{\beta_i}z^{\boldsymbol{\beta}[i+1,b]}\,. \tag{C.62}$$

The t integral is trivial, namely

$$\int_0^1 \mathrm{d}t\,\frac{t^a t^b}{t} = \frac{1}{a+b}\,. \tag{C.63}$$

Let us calculate the first term on the r.h.s. of (C.59)

$$\begin{aligned}\rho^{[0]}\gamma^{[0]}f(y,z,w) &= \rho^{[0]}\left[\sum_{i=1}^{a} y^{\boldsymbol{\alpha}[1,i-1]}z^{\alpha_i}y^{\boldsymbol{\alpha}[i+1,a]}z^{\boldsymbol{\beta}[b]}w^{\boldsymbol{\gamma}[c]}\right]\\ &= \frac{1}{a+b}\Bigg[ay^{\boldsymbol{\alpha}[1,a]}z^{\boldsymbol{\beta}[1,b]} - \sum_{i=1}^{a}\sum_{j=1}^{b} y^{\boldsymbol{\alpha}[1,i-1]}z^{\alpha_i}y^{\boldsymbol{\alpha}[i+1,a]}\\ &\qquad (-1)^{j+1}z^{\boldsymbol{\beta}[1,j-1]}y^{\beta_j}z^{\boldsymbol{\beta}[j+1,b]}\Bigg]w^{\boldsymbol{\gamma}[1,c]}\,.\end{aligned} \tag{C.64}$$

The second term is

$$\begin{aligned}\gamma^{[0]}\rho^{[0]}f(y,z,w) &= \gamma^{[0]}\frac{1}{a+b}\left[y^{\boldsymbol{\alpha}[a]}\sum_{j=1}^{b}(-1)^{j+1}z^{\boldsymbol{\beta}[1,j-1]}y^{\beta_j}z^{\boldsymbol{\beta}[j+1,b]}w^{\boldsymbol{\gamma}[c]}\right]\\ &= \frac{1}{a+b}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}y^{\boldsymbol{\alpha}[1,i-1]}z^{\alpha_i}y^{\boldsymbol{\alpha}[i+1,a]}\right.\\ &\left.(-1)^{j+1}z^{\boldsymbol{\beta}[1,j-1]}y^{\beta_j}z^{\boldsymbol{\beta}[j+1,b]}+y^{\boldsymbol{\alpha}[1,a]}bz^{\boldsymbol{\beta}[1,b]}\right]w^{\boldsymbol{\gamma}[1,c]}\,. \qquad \text{(C.65)}\end{aligned}$$

Obviously the sum $(\rho^{[0]}\gamma^{[0]}+\gamma^{[0]}\rho^{[0]})f(y,z,w)$ gives $f(y,z,w)$. □

C.5 Proofs of the Recursive Solutions

The following proofs are based on the work of Barnich, Brandt and Grigoriev [2]. Some helpful suggestions came from Rauh [32]. The first proof is worked out in more detail than the latter two, because the procedure is always the same.

Definition C.5.1. *The variables $\{y\}$ and $\{z\}$ are defined in section 3.2. The homogeneity degree $[\,\cdot\,]$ (also called just "order") of a function $f(y,z,w)$ counts the number of ys and zs. A function of homogeneity degree k will be denoted by $f^{[k]}$.*

C.5.1 Ghost Field Solution

Theorem C.5.1. *A special recursive solution*

$$\hat{C}=\sum_{k=1}^{\infty}C^{[k]}\,,\quad C^{[1]}=C\,, \qquad \text{(C.66)}$$

of the equation

$$\frac{\mathrm{i}}{2}[\hat{C}\stackrel{*}{,}\hat{C}]=\gamma\hat{C} \qquad \text{(C.67)}$$

is given by

$$C^{[k]}=-\rho^{[0]}\left(\gamma^{[1]}C^{[k-1]}-\frac{\mathrm{i}}{2}\sum_{l=1}^{k-1}\left[C^{[l]}\stackrel{*}{,}C^{[k-l]}\right]\right)\,. \qquad \text{(C.68)}$$

Proof. At first order in k, the equation (C.67) is satisfied due to $\gamma^{[0]}C=0$ and $\gamma^{[1]}C=0$

$$\frac{\mathrm{i}}{2}[C\stackrel{*}{,}C]-\gamma C=-C\sin(\wedge_{12})C=:r^{[2]}\,. \qquad \text{(C.69)}$$

Obviously, the above expression is of second order in the homogeneity. It vanishes if z, i.e. the differentiated ghost contribution, becomes zero.

Let us assume that $C^{k-1} = \sum_{l=1}^{k-1} C^{[l]}$ satisfies (C.67) up to homogeneity degree k. This means that

$$\gamma C^{k-1} - \frac{\mathrm{i}}{2}\big[C^{k-1} \overset{*}{,} C^{k-1}\big] = r^{[k]} + \sum_{m\geq 0} q^{[k+1+m]} \,. \tag{C.70}$$

If we apply $\gamma - \mathrm{i}[C^{k-1} \overset{*}{,} \cdot]$ to the l.h.s. of (C.70) we see that it vanishes identically

$$\begin{aligned}\gamma^2 C^{k-1} - \frac{\mathrm{i}}{2}\gamma\big[C^{k-1} \overset{*}{,} C^{k-1}\big] - \mathrm{i}\big[C^{k-1} \overset{*}{,} \gamma C^{k-1}\big] - \frac{1}{2}\Big[C^{k-1} \overset{*}{,} \big[C^{k-1} \overset{*}{,} C^{k-1}\big]\Big] \\ = -\frac{\mathrm{i}}{2}\big[\gamma C^{k-1} \overset{*}{,} C^{k-1}\big] + \frac{\mathrm{i}}{2}\big[C^{k-1} \overset{*}{,} \gamma C^{k-1}\big] - \mathrm{i}\big[C^{k-1} \overset{*}{,} \gamma C^{k-1}\big] = 0 \,. \end{aligned} \tag{C.71}$$

We used the nilpotency of γ and the fact that the nested commutator is always zero due to the Jacobi identity. Because the l.h.s. vanishes it follows from the r.h.s.

$$\begin{aligned}0 = \gamma r^{[k]} + \gamma \sum_{m\geq 0} q^{[k+1+m]} - \mathrm{i}\big[C^{k-1} \overset{*}{,} \cdot r^{[k]}\big] - \mathrm{i}\big[C^{k-1} \overset{*}{,} \sum_{m\geq 0} q^{[k+1+m]}\big] \\ = \gamma^{[0]} r^{[k]} + \sum_{m\geq 0} \tilde{q}^{[k+1+m]} \,. \end{aligned} \tag{C.72}$$

So in lowest order this implies that $\gamma^{[0]} r^{[k]} = 0$.

In homogeneity degree k (C.67) and (C.70) imply

$$\gamma^{[0]} C^{[k]} = -\gamma^{[1]} C^{[k-1]} + \frac{\mathrm{i}}{2} \sum_{l=1}^{k-1} \big[C^{[l]} \overset{*}{,} C^{[k-l]}\big] = r^{[k]} \,. \tag{C.73}$$

Let us assume, that $r^{[k]}(0,0,w) = 0$, or that $r^{[k]}$ depends at least on one z. Thus, one obtains with the use of theorem C.4.1

$$- r^{[k]} = -(\rho^{[0]}\gamma^{[0]} + \gamma^{[0]}\rho^{[0]}) r^{[k]} = \gamma^{[0]}\big(-\rho^{[0]} r^{[k]}\big) \,. \tag{C.74}$$

Obviously $C^{[k]} = -\rho^{[0]} r^{[k]}$ solves (C.73) as we suggested.

What we still have to show is that $r^{[k]}$ depends at least on one differentiated ghost, i.e. on one z. For $k = 2$ we have shown this explicitely by direct calculation (C.69). Suppose that no $r^{[l]}$ for $l < k$ depend on undifferentiated ghosts. Then $C^{[l]} = -\rho^{[0]} r^{[l]}$ depends only on differentiated ghosts. Thus, with (C.73), $r^{[k]}$ doesn't depend on undifferentiated ghost if

$$\gamma^{[1]} C^{[k-1]} - \mathrm{i}\big[C \overset{*}{,} C^{[k-1]}\big] \tag{C.75}$$

doesn't. And indeed, this is the case, namely the first term is zero, because $\gamma^{[1]}y = 0$ as well as $\gamma^{[1]}z = 0$. The second term is

$$-\,\mathrm{i}\big[C \stackrel{*}{,} C^{[k-1]}\big] = 2C \sin(\wedge_{12}) C^{[k-1]} \,, \tag{C.76}$$

which depends only on differentiated ghosts, because the series expansion of the sine starts with $\wedge_{12}$.

We know that $r^{[2]}$ depends on, at least, one ghost (C.69). Suppose $r^{[k-1]}$ depends on a ghost. Then $r^{[k]}$ depends on a ghost, too. Namely, the differential γ and the commutator in (C.73) increase the number of ghosts by one whereas $\rho^{[0]}$ lower the number by one. Thus, with (C.73) and $C^{[k]} = -\rho^{[0]} r^{[k]}$, one sees that $r^{[k]}$ depends, at least, on one ghost.

Hence we conclude that every term of $r^{[k]}$ depends, at least, on one differentiated ghost which means that the theorem C.4.1 is valid for the function $r^{[k]}$.

□

C.5.2 Gauge Field Solution

Theorem C.5.2. *A special recursive solution*

$$\hat{A}_\mu = \sum_{k=1}^{\infty} A_\mu^{[k]} \,, \quad A_\mu^{[1]} = A_\mu \,, \tag{C.77}$$

of the equation

$$\partial_\mu \hat{C} - \mathrm{i}\big[\hat{A}_\mu \stackrel{*}{,} C\big] = \gamma \hat{A}_\mu \tag{C.78}$$

is given by

$$A_\mu^{[k]} = -\rho^{[0]} \left(\gamma^{[1]} A_\mu^{[k-1]} - \partial_\mu C^{[k]} + \mathrm{i} \sum_{l=1}^{k-1} \left[A_\mu^{[l]} \stackrel{*}{,} C^{[k-l]} \right] \right) . \tag{C.79}$$

Proof. At first order in k, the equation (C.78) is satisfied due to $\gamma^{[0]} A_\mu = \partial_\mu C$ and $\gamma^{[1]} A_\mu = 0$

$$\partial_\mu C + \partial_\mu C^{[2]} - \mathrm{i}\big[A_\mu \stackrel{*}{,} C\big] - \gamma A_\mu = \partial_\mu C^{[2]} - \mathrm{i}\big[A_\mu \stackrel{*}{,} C\big] = t_\mu^{[2]} \,. \tag{C.80}$$

Obviously, the expression $t_\mu^{[2]}$ of second order in the homogeneity depends only on differentiated ghosts.

Let us assume that $A_\mu^{k-1} = \sum_{l=1}^{k-1} A_\mu^{[l]}$ satisfies (C.78) up to order k. This means that

$$\gamma A_\mu^{k-1} - \partial_\mu \hat{C} + \mathrm{i}\big[A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big] = t_\mu^{[k]} \sum_{m \geq 0} v_\mu^{[k+1+m]} \,, \tag{C.81}$$

whereas $t_\mu^{[k]}$ depends only on differentiated ghosts. Now if we apply $\gamma - \mathrm{i}[\hat{C} \stackrel{*}{,} \cdot\,]$ to the l.h.s. of (C.70) we see that it vanishes identically

$$\begin{aligned}
\gamma^2 A_\mu^{k-1} - \partial_\mu \gamma \hat{C} + \mathrm{i}\,\gamma\big[A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big] \\
- \mathrm{i}\big[\hat{C} \stackrel{*}{,} \gamma A_\mu^{k-1}\big] + \mathrm{i}\big[\hat{C} \stackrel{*}{,} \partial_\mu \hat{C}\big] + \Big[\hat{C} \stackrel{*}{,} \big[A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big]\Big] \\
= -\,\mathrm{i}\,\partial_\mu(\hat{C} * \hat{C}) + \mathrm{i}\big[\gamma A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big] + \mathrm{i}\big[A_\mu^{k-1} \stackrel{*}{,} \gamma \hat{C}\big] \\
- \mathrm{i}\big[\gamma A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big] + \mathrm{i}\,\partial_\mu(\hat{C} * \hat{C}) + \Big[\hat{C} \stackrel{*}{,} \big[A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big]\Big] \\
= \frac{1}{2}\bigg(\Big[A_\mu^{k-1} \stackrel{*}{,} \big[\hat{C} \stackrel{*}{,} \hat{C}\big]\Big] + \Big[\hat{C} \stackrel{*}{,} \big[A_\mu^{k-1} \stackrel{*}{,} \hat{C}\big]\Big] + \Big[\hat{C} \stackrel{*}{,} \big[\hat{C} \stackrel{*}{,} A_\mu^{k-1}\big]\Big]\bigg) = 0\,. \qquad \text{(C.82)}
\end{aligned}$$

The last equality is satisfied due to the Jacobi identity. In homogeneity degree k (C.78) and (C.81) imply

$$\gamma^{[0]} A_\mu^{[k]} = -\gamma^{[1]} A_\mu^{[k-1]} + \partial_\mu \hat{C} + \mathrm{i} \sum_{l=1}^{k-1} \big[A_\mu^{[l]} \stackrel{*}{,} C^{[k-l]}\big] = -t_\mu^{[k]}\,. \qquad \text{(C.83)}$$

Under the assumption that $t_\mu^{[k]}(0,0,w) = 0$ the theorem C.4.1 is valid. With $\gamma^{[0]} t_\mu^{[k]} = 0$ one finds, that $A_\mu^{[k]} = -\rho^{[0]} t_\mu^{[k]}$ is a solution to (C.83), as suggested.

What we sill have to show is, that $t_\mu^{[k]}(0,0,w)$ is indeed zero. Thus we have to show, that $r^{[k]}$ depends only on differentiated ghosts.

For $k = 2$ we have shown by a direct calculation (C.80) that $t_\mu^{[2]}(0,0,w) = 0$. For $k > 2$ the only dependence on undifferentiated ghosts can come from the terms

$$\gamma^{[1]} A_\mu^{[k-1]} - \mathrm{i}\big[A_\mu^{[k]} \stackrel{*}{,} C\big] = 2 A_\mu^{[k]} \sin(\wedge_{12}) C\,. \qquad \text{(C.84)}$$

which depend only on differentiated ghosts, because the series expansion of the sine starts with $\wedge_{12}$. □

C.5.3 Matter Field Solution

Theorem C.5.3. *A special recursive solution*

$$\hat{\psi} = \sum_{k=1}^{\infty} \psi^{[k]}\,, \quad \psi^{[1]} = \psi\,, \qquad \text{(C.85)}$$

of the equation

$$\mathrm{i}\,\hat{C} * \hat{\psi} = \gamma \hat{\psi} \qquad \text{(C.86)}$$

is given by

$$\psi^{[k]} = -\rho^{[0]} \left(\gamma^{[1]} \psi^{[k-1]} - \mathrm{i} \sum_{l=1}^{k-1} C^{[l]} * \psi^{[k-l]} \right)\,. \qquad \text{(C.87)}$$

Proof. At first order in k, the equation (C.86) is satisfied due to $\gamma^{[0]}\psi = 0$ and $\gamma^{[1]}C = \mathrm{i}\,C\psi$

$$\mathrm{i}\,C * \psi - \gamma\psi = \mathrm{i}(C * \psi - C\psi) =: s^{[2]}\,. \tag{C.88}$$

Obviously, the expression $s^{[2]}$ of second order in the homogeneity depends only on differentiated ghosts. The zeroth order of the star-product cancels $-C\psi$. So it vanishes if z becomes zero.

Let us assume that $\psi^{k-1} = \sum_{l=1}^{k-1} \psi^{[l]}$ satisfies (C.86) up to order k. This means that

$$\gamma\psi^{k-1} - \mathrm{i}\,\hat{C} \stackrel{*}{,} \psi^{k-1} = s^{[k]} + \sum_{m\geq 0} u^{[k+1+m]}\,, \tag{C.89}$$

whereas $s^{[k]}$ depends only on differentiated ghosts. Now if we apply $\gamma - \mathrm{i}\,\hat{C}*$ to the l.h.s. of (C.89) we see that all terms up to order $k+1$ vanish

$$\begin{aligned}\gamma^2\psi^{k-1} - \mathrm{i}\,\gamma\big(\hat{C} * \psi^{k-1}\big) - \mathrm{i}\,\hat{C} * \big(\gamma\psi^{k-1}\big) - \hat{C} * \big(\hat{C} * \psi^{k-1}\big)& \\ = \frac{1}{2}\big[\hat{C}, \hat{C}\big] * \psi^{k-1} + \mathrm{i}\,\hat{C} * \big(\gamma\psi^{k-1}\big) - \mathrm{i}\,\hat{C} * \big(\gamma\psi^{k-1}\big)& \\ - \hat{C} * \big(\hat{C} * \psi^{k-1}\big) = -\hat{C} * \hat{C} * \psi^{k-1}\,.&\end{aligned} \tag{C.90}$$

So in homogeniety degree k this implies that $\gamma^{[0]}s^{[k]} = 0$.

In order k (C.86) and (C.89) imply

$$\gamma^{[0]}\psi^{[k]} = -\gamma^{[1]}\psi^{[k-1]} + \mathrm{i}\sum_{l=1}^{k-1} C^{[l]} \stackrel{*}{,} \psi^{[k-l]}\big] = s^{[k]}\,. \tag{C.91}$$

Under the assumption that $s^{[k]}(0,0,w) = 0$ the theorem C.4.1 is valid. With $\gamma^{[0]}s^{[k]} = 0$ one finds, that $\psi^{[k]} = -\rho^{[0]}s^{[k]}$ is a solution to (C.91), as suggested.

What we still have to show is, that $s^{[k]}(0,0,w)$ is indeed zero. Thus we have to show, that $s^{[k]}$ depends only on differentiated ghosts.

For $k = 2$ we have shown by a direct calculation (C.88) that $s^{[2]}(0,0,w) = 0$. For $k > 2$ the only dependence on undifferentiated ghosts can come from the terms

$$\gamma^{[1]}\psi^{[k-1]} - \mathrm{i}\,C * \psi^{[k-1]} = \mathrm{i}\,C\psi^{[k-1]} - \mathrm{i}\,C * \psi^{[k-1]}\,, \tag{C.92}$$

which depend only on differentiated ghost, because the zeroth order of the star-product cancels $\mathrm{i}\,C\psi^{[k-1]}$. □

Appendix D

Alternative Ansatz

We want show by an explicit calculation that the "differential evolution equations" from [34] don't result in a valid Seiberg-Witten map for the ghost field in next-to-leading order.

Based on the "differential evolution equations"

$$\dot{C}(t) = \frac{1}{4}\theta^{\mu\nu}\Big[\partial_\mu C \overset{*(t)}{,} A_\nu\Big]_+ , \tag{D.1a}$$

$$\dot{A}_\lambda(t) = -\frac{1}{4}\theta^{\mu\nu}\Big[A_\mu \overset{*(t)}{,} \partial_\nu A_\lambda + F_{\nu\lambda}\Big]_+ , \tag{D.1b}$$

we want calculate $\tilde{C}^{[3]}$, which is composed of one ghost field and two gauge fields. The function $*(t)$ is defined in (4.1). In order to be able to get the desired map we want first calculate the n-th derivative of $\dot{C}(t)$ with respect to t

$$\frac{\partial^n \dot{C}(t)}{\partial t^n} = \frac{1}{4}\theta^{\mu\nu}\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}\left[\partial_\mu C^{(n-i)}(t) \overset{*^{(i-j)}(t)}{,} A_\nu^{(j)}(t)\right]_+ . \tag{D.2}$$

In the general case this equation would become very complex because one recursively has to replace the derivative of the fields by (D.1). But because we only want to calculate $\tilde{C}^{[3]}$, we could stop the recursive replacement after the first step. So for that case we obtain for the n-th derivative

$$\begin{aligned}\frac{\partial^n \dot{C}_3(t)}{\partial t^n} = \frac{1}{16}\theta^{\mu\nu}\theta^{\rho\sigma}&\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}\\ &\bigg(\Big[\partial_\mu\big[\partial_\rho C \overset{*^{(n-i-1)}(t)}{,} A_\sigma\big]_+ \overset{*^{(i-j)}(t)}{,} A_\nu^{(j)}(t)\delta_{j0}\Big]_+\\ -\Big[\partial_\mu C^{(n-i)}(t)\delta_{in} &\overset{*^{(i-j)}(t)}{,} \big[A_\rho \overset{*^{(j-1)}(t)}{,} \partial_\sigma A_\nu + F_{\sigma\nu}\big]_+\Big]_+\bigg) ,\end{aligned} \tag{D.3}$$

where δ_{ij} is the usual Kronecker delta. Our desired Seiberg-Witten map $\tilde{C}^{[3]}$ is according to (4.5) the sum over all n for $t = 0$

$$\tilde{C}^{[3]} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left. \frac{\mathrm{d}^n \dot{C}_3(t)}{\mathrm{d}t^n} \right|_{t=0} = \frac{1}{16}\theta^{\mu\nu}\theta^{\rho\sigma} \sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{i}{j}$$
$$\Bigg(\big((\mathrm{i}\wedge_{12})^{n-i-1} + (-\mathrm{i}\wedge_{12})^{n-i-1} \big)$$
$$\big((\mathrm{i}(\wedge_{13}+\wedge_{23}))^{i-j} + (-\mathrm{i}(\wedge_{13}+\wedge_{23}))^{i-j} \big)\, \delta_{j\,0} \partial_\mu \big((\partial_\rho C) A_\sigma \big) A_\nu$$
$$- \big((\mathrm{i}(\wedge_{12}+\wedge_{13}))^{i-j} + (-\mathrm{i}(\wedge_{12}+\wedge_{13}))^{i-j} \big) \big((\mathrm{i}\wedge_{23})^{j-1} + (-\mathrm{i}\wedge_{23})^{j-1} \big)$$
$$\delta_{i\,n} (\partial_\mu C) A_\rho \big(\partial_\sigma A_\nu + F_{\sigma\nu} \big) \Bigg). \quad \text{(D.4)}$$

If we perform the sums we obtain

$$\tilde{C}^{[3]} = \frac{1}{16}\theta^{\mu\nu}\theta^{\rho\sigma}$$
$$\Bigg(\left(\frac{\mathrm{e}^{\mathrm{i}(\wedge_{12}+\wedge_{13}+\wedge_{23})} - 1}{\mathrm{i}\wedge_{12}\,\mathrm{i}(\wedge_{12}+\wedge_{13}+\wedge_{23})} + \frac{\mathrm{e}^{\mathrm{i}(-\wedge_{12}+\wedge_{13}+\wedge_{23})} - 1}{\mathrm{i}(-\wedge_{12})\,\mathrm{i}(-\wedge_{12}+\wedge_{13}+\wedge_{23})} \right)$$
$$\left(\frac{\mathrm{e}^{\mathrm{i}(\wedge_{12}-\wedge_{13}-\wedge_{23})} - 1}{\mathrm{i}\wedge_{12}\,\mathrm{i}(\wedge_{12}-\wedge_{13}-\wedge_{23})} + \frac{\mathrm{e}^{\mathrm{i}(-\wedge_{12}-\wedge_{13}-\wedge_{23})} - 1}{\mathrm{i}(-\wedge_{12})\,\mathrm{i}(-\wedge_{12}-\wedge_{13}-\wedge_{23})} \right)$$
$$\partial_\mu \big((\partial_\rho C) A_\sigma \big) A_\nu$$
$$- \left(\frac{\mathrm{e}^{\mathrm{i}(\wedge_{12}+\wedge_{13}+\wedge_{23})} - 1}{\mathrm{i}\wedge_{23}\,\mathrm{i}(\wedge_{12}+\wedge_{13}+\wedge_{23})} + \frac{\mathrm{e}^{\mathrm{i}(\wedge_{12}+\wedge_{13}-\wedge_{23})} - 1}{\mathrm{i}(-\wedge_{23})\,\mathrm{i}(\wedge_{12}+\wedge_{13}-\wedge_{23})} \right)$$
$$\left(\frac{\mathrm{e}^{\mathrm{i}(-\wedge_{12}-\wedge_{13}+\wedge_{23})} - 1}{\mathrm{i}\wedge_{23}\,\mathrm{i}(-\wedge_{12}-\wedge_{13}+\wedge_{23})} + \frac{\mathrm{e}^{\mathrm{i}(-\wedge_{12}-\wedge_{13}-\wedge_{23})} - 1}{\mathrm{i}(-\wedge_{23})\,\mathrm{i}(-\wedge_{12}-\wedge_{13}-\wedge_{23})} \right)$$
$$(\partial_\mu C) A_\rho \big(\partial_\sigma A_\nu + F_{\sigma\nu} \big) \Bigg). \quad \text{(D.5)}$$

Now let us rewrite the above expression in trigonometric functions, which leads to

$$\tilde{C}^{[3]} = -\frac{1}{8}\theta^{\mu\nu}\theta^{\rho\sigma}$$
$$\Bigg(\left(\frac{\cos(\wedge_{12}+\wedge_{13}+\wedge_{23}) - 1}{\mathrm{i}\wedge_{12}\,\mathrm{i}(\wedge_{12}+\wedge_{13}+\wedge_{23})} + \frac{\cos(\wedge_{12}-\wedge_{13}-\wedge_{23}) - 1}{\mathrm{i}\wedge_{12}\,\mathrm{i}(\wedge_{12}-\wedge_{13}-\wedge_{23})} \right)$$
$$\partial_\mu \big((\partial_\rho C) A_\sigma \big) A_\nu$$
$$- \left(\frac{\cos(\wedge_{12}+\wedge_{13}+\wedge_{23}) - 1}{\mathrm{i}\wedge_{23}\,\mathrm{i}(\wedge_{12}+\wedge_{13}+\wedge_{23})} + \frac{\cos(-\wedge_{12}-\wedge_{13}+\wedge_{23}) - 1}{\mathrm{i}\wedge_{23}\,\mathrm{i}(-\wedge_{12}-\wedge_{13}+\wedge_{23})} \right)$$
$$(\partial_\mu C) A_\rho \big(\partial_\sigma A_\nu + F_{\sigma\nu} \big) \Bigg). \quad \text{(D.6)}$$

Let

$$F(\mathrm{i}\,x, \mathrm{i}\,y) := \frac{\cos(x+y) - 1}{\mathrm{i}\,x\,\mathrm{i}(x+y)} + \frac{\cos(x-y) - 1}{\mathrm{i}\,x\,\mathrm{i}(x-y)}\,, \tag{D.7}$$

which has the property that it is even in both of its arguments

$$F(\mathrm{i}\,x, \mathrm{i}\,y) = F(\mathrm{i}\,x, -\mathrm{i}\,y) = F(-\mathrm{i}\,x, \mathrm{i}\,y) = F(-\mathrm{i}\,x, -\mathrm{i}\,y)\,. \tag{D.8}$$

With this function we can rewrite (D.6), namely

$$\begin{aligned}\tilde{C}^{[3]} = -\frac{1}{8}\theta^{\mu\nu}\theta^{\rho\sigma}\Big(&F(\mathrm{i}\wedge_{12}, \mathrm{i}(\wedge_{13} + \wedge_{23}))\partial_\mu\big((\partial_\rho C)A_\sigma\big)A_\nu \\ &- F(\mathrm{i}\wedge_{23}, \mathrm{i}(\wedge_{12} + \wedge_{13}))(\partial_\mu C)A_\rho\big(\partial_\sigma A_\nu + F_{\sigma\nu}\big)\Big)\,.\end{aligned} \tag{D.9}$$

Now let us check if $\tilde{C}^{[3]}$ satisfies the gauge equivalence equation (2.26). The r.h.s. of that equation is

$$\begin{aligned}\gamma\tilde{C}^{[3]} = -\frac{1}{8}\theta^{\mu\nu}\theta^{\rho\sigma}\Big(&F(\mathrm{i}\wedge_{12}, \mathrm{i}(\wedge_{13} + \wedge_{23}))\big[-(\partial_\mu\partial_\rho C)(\partial_\sigma C)A_\nu \\ &- (\partial_\rho C)(\partial_\mu\partial_\sigma C)A_\nu - (\partial_\mu\partial_\rho C)A_\sigma(\partial_\nu C) - (\partial_\rho C)(\partial_\mu A_\sigma)(\partial_\nu C)\big] \\ &- F(\mathrm{i}\wedge_{23}, \mathrm{i}(\wedge_{12} + \wedge_{13}))\big[-2(\partial_\mu C)(\partial_\rho C)(\partial_\sigma A_\nu) + (\partial_\mu C)(\partial_\rho C)(\partial_\nu A_\sigma) \\ &- (\partial_\mu C)A_\rho(\partial_\nu\partial_\sigma C)\big]\Big)\,.\end{aligned} \tag{D.10}$$

After exchanging the fields in such a way that we obtain one field configuration we get

$$\begin{aligned}\gamma\tilde{C}^{[3]} = -\frac{1}{2}\theta^{\mu\nu}\Big[&\wedge_{13}F(\mathrm{i}\wedge_{13}, \mathrm{i}(\wedge_{12} - \wedge_{23})) \\ &- \wedge_{12}F(\mathrm{i}\wedge_{12}, \mathrm{i}(\wedge_{13} + \wedge_{23}))\Big]CA_\mu(\partial_\nu C)\,.\end{aligned} \tag{D.11}$$

If we compare this result with (C.32) we find that, actually

$$\begin{aligned}&\frac{1}{2}\Big[-\wedge_{13}\,F(\mathrm{i}\wedge_{13}, \mathrm{i}(\wedge_{12} - \wedge_{23})) + \wedge_{12}F(\mathrm{i}\wedge_{12}, \mathrm{i}(\wedge_{13} + \wedge_{23}))\Big] \\ &\qquad\neq \sin(\wedge_{12} + \wedge_{13})\frac{\sin\wedge_{23}}{\wedge_{23}}\,.\end{aligned} \tag{D.12}$$

One can see that the left and right hand side is unequal if one expands the two sides in $\wedge_{12}$, $\wedge_{13}$ and $\wedge_{23}$. The zeroth and first order are equal, but even the second order in each argument is different. So the solution of the "differential evolution equations" doesn't satisfy the gauge equivalence equations.

Appendix E

Ghost Vertex

As we know from ordinary QCD, the existence of a 3-gauge-boson vertex leads to contributions in perturbation theory from intermediate states with unphysical longitudinal and scalar modes which do not cancel. To get rid of these unwanted polarisation states one introduces a pair of massless scalar fermions which are called Faddeev-Popov ghosts. Because we have a 3-gauge-boson vertex in abelian noncommutative theories, one would assume that we have to consider these Faddeev-Popov ghosts, too. This is actually the case if we base our calculation on the action (2.30) where we first did the gauge fixing and afterwards replaced the fields by the appropriate Seiberg-Witten maps.

But we know that the noncommutative gauge transformation is equivalent to the commutative one by construction. This ensures us the gauge equivalence equations or the Seiberg-Witten maps. So instead of using the noncommutative gauge transformation we can also use the commutative one in which, in the abelian case, the ghosts decouple from the photons. This leads to the action (2.31). But if we don't have Faddeev-Popov ghosts the unphysical photon polarisations obtained by the 3-photon vertex have to cancel in a somehow different manner. Namely this job was done by the contact vertex $\mathrm{f\bar{f}gg}$.

As we stated in section 2.4.4 we chose the action (2.31) which is independent of the ghost field. Although we don't need the photon-ghost vertex we will derive it from the action (2.30) in this appendix for completeness. First at all we combine the terms with the same number of gauge fields

$$\mathrm{i}\int \mathrm{d}^4x \left[\tilde{\bar{C}} * \partial^\mu\partial_\mu \tilde{C} - \mathrm{i}\ \mathrm{tr}\left[\tilde{\bar{C}} * \partial_\mu[\tilde{A}^\mu \stackrel{*}{,} \tilde{C}]\right]\right]$$
$$= \mathrm{i}\int \mathrm{d}^4x\ [\mathcal{L}_{\mathrm{c\bar{c}}} + \mathcal{L}_{\mathrm{c\bar{c}g}}] + \mathcal{O}(A^2)\,, \quad \text{(E.1)}$$

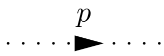
where the free and the interaction parts are

$$\mathcal{L}_{\mathrm{c\bar{c}}} = \bar{C} * \partial^\mu\partial_\mu C\,, \quad \text{(E.2a)}$$

$$\mathcal{L}_{\mathrm{c\bar{c}g}} = -\,\mathrm{i}\,\bar{C} * \partial_\mu[A^\mu \stackrel{*}{,} C] + \bar{C} * \partial^\mu \partial_\mu C^{[2]} + \bar{C}^{[2]} * \partial^\mu \partial_\mu C\,. \tag{E.2b}$$

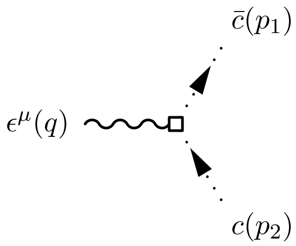
As in $\mathcal{L}_{\mathrm{f\bar{f}g}}$ we have one term in the photon-ghost Lagrangian which doesn't depend on Seiberg-Witten maps. The other two terms have their origin in the free part where one replaces one ghost field by the appropriate Seiberg-Witten map.

The ghost propagator remains the same one as, for example, in QCD, namely



$$= \frac{\mathrm{i}}{p^2 + \mathrm{i}\,\epsilon}\,.$$

Now we come to the vertex coupling a photon to two ghosts and which is pictured below.



$$= V^\mu_{\mathrm{c\bar{c}g}}(p_1, p_2, q)$$

The relevant Lagrangian for this vertex is given by $\mathcal{L}_{\mathrm{c\bar{c}g}}$ (E.2b). The calculation is straightforward. One obtains a sine in the first term due to the star commutator and in the last two terms one has a $*_{\mathrm{s}}$-function coming from the $C^{[2]}$ map. Thus, the $\mathrm{c\bar{c}g}$ vertex is

$$\begin{aligned} V^\mu_{\mathrm{c\bar{c}g}}(p_1, p_2, q) = \mathrm{i}\,g \Big[& 2\sin(p_1 \wedge q) p_1^\mu \\ & + \frac{1}{2} *_{\mathrm{s}} (p_1 \wedge q) \left(p_1^2 (\theta p_2)^\mu + p_2^2 (p_1 \theta)^\mu \right) \Big]. \end{aligned} \tag{E.3}$$

As we see from the above result, the last two terms will vanish if the ghosts are on the mass shell, i.e. $p_i^2 = 0$. In this case the only term which survives is the first term which is exactly the ghost vertex in the noncommutative QED without Seiberg-Witten maps.

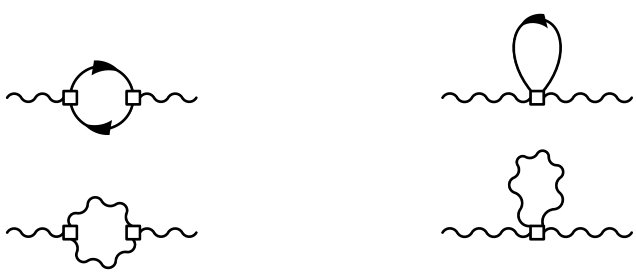
Appendix F

Loop Calculation

After we showed in chapter 7 that the scattering process $e^+e^- \rightarrow \gamma\gamma$ is unitary in the sense of tree-level-unitarity, we now want to do a step towards the calculation of the photon self-energy. By calculating the photon self-energy one can check the optical theorem by direct calculations. In our case the optical theorem states that two times the imaginary part of the photon self energy has to be equal to the square of the absolute value of the appropriate vertices.

It is known [26] that the optical theorem is violated in the case of NCQED without Seiberg-Witten maps. Thus, in order to get additional contributions to the optical theorem from the photon self-energy there has to exist new imaginary terms coming from the Seiberg-Witten maps.

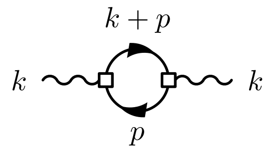
The one loop photon self-energy contains four diagrams.



With the Feynman rules we calculated in chapter 5 to investigate tree level unitarity on the basis of the electron positron pair annihilation process we are able to calculate three of the four diagrams. The lower right diagram we can't calculate with the Feynman rules derived in the main part of this thesis because it contains the 4-photon vertex. But nevertheless it is reasonable to calculate the two fermion diagrams. Namely, these two fermion diagrams are separately gauge invariant because they are proportional to the fermion number.

F.1 Fermion Loop Contribution

The first step towards the calculation of the photon self-energy is to calculate the fermion loop which is already present in ordinary QED.



$$= \mathrm{i}\,\mathcal{M}^{\mu\nu}_{\mathrm{L1}}(k)$$

In the case of our model we have the noncommutative vertex (5.7) which has not only an additional noncommutative phase factor but also new terms coming from the Seiberg-Witten maps. The amplitude of this loop is obtained straightforwardly, namely

$$\mathcal{M}^{\mu\nu}_{\mathrm{L1}}(k) = (-1)\,\mathrm{i} \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{tr}\left[V^{\mu}_{\bar{\mathrm{f}}\mathrm{fg}}(-k-p,p,k)\frac{\mathrm{i}(\not{k}+\not{p}+m)}{(k+p)^2-m^2+\mathrm{i}\,\epsilon}\right.$$
$$\left. V^{\nu}_{\bar{\mathrm{f}}\mathrm{fg}}(-p,k+p,-k)\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\,\epsilon}\right], \quad \text{(F.1)}$$

where $V_{\bar{\mathrm{f}}\mathrm{fg}}$ is the noncommutative fermion gauge boson vertex (5.7). The minus sign in front of the integral is the fermion loop factor. After a lengthy but not difficult calculation using dimensional regularization one ends up with

$$\mathrm{i}\,\mathcal{M}^{\mu\nu}_{\mathrm{L1}} = \mathrm{i}\,\mathcal{M}^{\mu\nu}_{\mathrm{QED}} - 2\,\mathrm{i}\,g^2\tilde{k}^{\mu}\tilde{k}^{\nu}I^{(1)}_{\mathrm{L1}}(\tilde{k}) - 2\,\mathrm{i}\,g^2\theta^{\mu\rho}\theta^{\nu\sigma}I^{(2)}_{\mathrm{L1},\rho\sigma}(\tilde{k})\,, \qquad \text{(F.2)}$$

in the massless limit $m \to 0$, with $\tilde{k}^{\mu} := (k\theta)^{\mu}$. The first summand is just the fermion loop of the ordinary QED. The existence of this factor is easy to see because the noncommutative phases in the first summand of the $\bar{\mathrm{f}}\mathrm{fg}$-vertex cancel each other. Thus, only $\mathrm{i}\,g\gamma^{\mu}$ survives. The result for $\mathrm{i}\,\mathcal{M}^{\mu\nu}_{\mathrm{QED}}$ in dimensional regularization is well known (cf. [37])

$$\mathrm{i}\,\mathcal{M}^{\mu\nu}_{\mathrm{QED}} = -\,\mathrm{i}\,\frac{g^2}{2\pi^2}(k^2 g^{\mu\nu} - k^{\mu}k^{\nu})$$
$$\int\limits_0^1 \mathrm{d}x\; x(1-x)\left(\frac{2}{\epsilon} - \log(-x(1-x)k^2) - \gamma + \log(4\pi)\right), \quad \text{(F.3)}$$

with the Euler constant γ and the dimension $D = 4-\epsilon$.

Now we come to the latter two integrals which are new in NCQED with Seiberg-Witten maps

$$I^{(1)}_{\mathrm{L1}}(n) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1-\cos(np)}{(np)^2}\,, \qquad \text{(F.4a)}$$

$$I^{(2)}_{\mathrm{L1},\mu\nu}(n) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{p_\mu p_\nu}{p^2 + \mathrm{i}\,\epsilon} \frac{1 - \cos(np)}{(np)^2} \,. \tag{F.4b}$$

At first sight one would assume that $I^{(1)}_{\mathrm{L1}}$ is easier to calculate than $I^{(2)}_{\mathrm{L1},\mu\nu}$. But this isn't the case. Actually we don't need to calculate the first integral, because if one contracts the indices of the second integral with $g^{\mu\nu}$ one obtains the first one

$$I^{(1)}_{\mathrm{L1}}(n) = g^{\mu\nu} I^{(2)}_{\mathrm{L1},\mu\nu}(n) \,. \tag{F.5}$$

So we have to calculate only $I^{(2)}_{\mathrm{L1},\mu\nu}$. Due to the cosine the integral doesn't appear in standard perturbation theory, so that we have to calculate it from scratch. The detailed calculation can be found in section F.4. Here we only give the result, namely in the sense of dimensional regularization the second integral is after subtracting the divergences

$$I^{(2),\mu\nu}_{\mathrm{L1,\ conv.}}(n) = \mathrm{i}\, \frac{1}{(4\pi)^2} \frac{1}{(n^2)^2} \frac{4}{3} \left[-g^{\mu\nu} + \frac{4}{n^2} n^\mu n^\nu \right] \,, \tag{F.6}$$

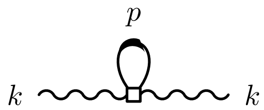
with $D = 4$. Let us discuss this result. Firstly, we see that this integral is purely imaginary. This means that its contribution to $\mathrm{i}\,\mathcal{M}^{\mu\nu}_{\mathrm{L1}}$ (F.2) is purely real. As we stated in the introduction of this chapter this means that it doesn't lead to a new contribution to the optical theorem.

Secondly, the above integral as it is present in (F.2) depends on $\tilde{k}$, the external photon momentum times the noncommutative parameter. In the commutative limit $|\theta^{\mu\nu}| \to 0$ the fermion loop contribution $-2g^2 \theta^{\mu\rho}\theta^{\nu\sigma} I^{(2)}_{\mathrm{L1},\alpha\beta}(\tilde{k})$ diverges due to the factor $\tilde{k}^{-4}$. Thus, there doesn't exist a limit to the ordinary QED anymore. The same holds of course if the photon momenta becomes very small, i.e. the integral is infrared (IR) divergent. This phenomena is known as UV/IR-mixing [14].

Thirdly, if we multiply the integral with $g^{\alpha\beta}$ it becomes zero for $D = 4$. Thus the integral $I^{(1)}_{\mathrm{L1}}$ is zero, i.e. the middle term in (F.2) vanishes.

F.2 Fermion Tadpole Contribution

Due to the existence of a contact vertex which couples two fermions to two gauge bosons we have also a tadpole diagram which contributes to the photon self energy.



$$= \mathcal{M}^{\mu\nu}_{\mathrm{L2}}(k)$$

Basically the amplitude for this tadpole is the contact vertex times the fermion propagator

$$\mathcal{M}^{\mu\nu}_{\mathrm{L2}}(k) = (-1) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \,\mathrm{tr} \left[V^{\mu\nu}_{\mathrm{\bar{f}fgg}}(-p, p, -k, k) \frac{\not{p} + m}{p^2 - m^2\, \mathrm{i}\,\epsilon} \right] \,, \tag{F.7}$$

where the minus sign in front of the integral is the fermion loop factor. Because we have only one photon and one fermion the two boson and two fermion momenta of the vertex are equal up to a sign. So all wedge-products (2.4) between the boson momenta or the fermion momenta are zero, due to the antisymmetry of $\theta^{\mu\nu}$. Especially, the argument $k_1 \wedge k_2$ of the $*_{\mathrm{f}}$- and the $*_{\mathrm{s}}$-functions as well as the sine is zero. Obviously, the first two functions became unity and the sine becomes zero. So we can record, that the above vertex is independent of the concrete choice of the function $*_{\mathrm{f}}$.

Let us ask whether the terms with the functions F^{ψ}, namely (5.16) and (5.17), do contribute to this diagram. Remember, in the electron positron pair annihilation process discussed in chapter 6 the terms depending on the function F^{ψ} became zero because they were proportional to the equation of motion. But now, the fermion is off the mass shell and thus the terms don't vanish.

Let us look somewhat closer at the part V_{IV} (5.16) and V_{V} (5.17) of the contact vertex which depend on the F^{ψ}s. Because the factors $-\not{p}_1 - m = \not{p}_2 - m$ are equal due to $-p_1 = p_2 = p$ the two parts of the vertex are equal up to a sign. The terms proportional to F^{ψ}_{I}, F^{ψ}_{II} and F^{ψ}_{V} are equal, hence they become doubled. The terms proportional to F^{ψ}_{III} and F^{ψ}_{IV} have the opposite sign, hence they cancel.

Next, the function F^{ϕ}_{V} vanishes if the first argument becomes zero. So at the end we obtain

$$\begin{aligned} &V^{\mu\nu}_{\mathrm{IV}}(-p,p,-k,k) + V^{\mu\nu}_{\mathrm{V}}(-p,p,-k,k) + (k,\mu) \leftrightarrow (-k,\nu) \\ &\quad = (\not{p} - m)\Bigg[\frac{1}{2}\theta^{\mu\nu}\Big(F^{\phi}_{\mathrm{I}}(0,\mathrm{i}\,\tilde{k}p,-\mathrm{i}\,\tilde{k}p) - F^{\phi}_{\mathrm{I}}(0,-\mathrm{i}\,\tilde{k}p,\mathrm{i}\,\tilde{k}p)\Big) \\ &\qquad - \tilde{k}^{\mu}\tilde{k}^{\nu}\Big(F^{\phi}_{\mathrm{II}}(0,\mathrm{i}\,\tilde{k}p,-\mathrm{i}\,\tilde{k}p) + F^{\phi}_{\mathrm{II}}(0,-\mathrm{i}\,\tilde{k}p,\mathrm{i}\,\tilde{k}p)\Big)\Bigg]. \end{aligned} \tag{F.8}$$

The difference of the two F_{I}s is

$$F^{\phi}_{\mathrm{I}}(0,\mathrm{i}\,\tilde{k}p,-\mathrm{i}\,\tilde{k}p) - F^{\phi}_{\mathrm{I}}(0,-\mathrm{i}\,\tilde{k}p,\mathrm{i}\,\tilde{k}p) = \frac{\cos(\tilde{k}p) - 1}{\mathrm{i}\,\tilde{k}p} := T^{(1)}(\tilde{k}p). \tag{F.9}$$

And the sum of the two F_{II}s is given by

$$\begin{aligned} &F^{\phi}_{\mathrm{II}}(0,\mathrm{i}\,\tilde{k}p,-\mathrm{i}\,\tilde{k}p) + F^{\phi}_{\mathrm{II}}(0,-\mathrm{i}\,\tilde{k}p,\mathrm{i}\,\tilde{k}p) \\ &\qquad = \frac{2 + (\tilde{k}p)^2 - 2\cos(\tilde{k}p) - 2(\tilde{k}p)\sin(\tilde{k}p)}{(\tilde{k}p)^4} \\ &\qquad - \frac{-\cos(\tilde{k}p) + \cos(2\tilde{k}p) + (\tilde{k}p)\sin(2\tilde{k}p)}{(\tilde{k}p)^2} := T^{(2)}(\tilde{k}p). \end{aligned} \tag{F.10}$$

Important is, that the above functions depend only on $\tilde{k}p$, i.e. they depend only on one variable.

Now we come to the remaining parts of the contact vertex. One observes that the sum of the second and third part of the vertex cancel each other

$$V_{\mathrm{II}}^{\mu\nu}(-p,p,-k,k) + V_{\mathrm{III}}^{\mu\nu}(-p,p,-k,k) + (k,\mu) \leftrightarrow (-k,\nu) = 0\,.$$

Thus, one only has one contribution from the part V_{I}

$$\begin{aligned} &V_{\mathrm{I}}^{\mu\nu}(-p,p,-k,k) + V_{\mathrm{I}}^{\nu\mu}(-p,p,k,-k) \\ &\qquad = \mathrm{i}\big(2\tilde{k}^{\nu} g^{\mu\rho} - 2\tilde{k}^{\mu} g^{\nu\rho} - \theta^{\mu\nu} k^{\rho}\big)\gamma_{\rho} \end{aligned} \tag{F.11}$$

and another one from the part V_{VI}

$$\begin{aligned} &V_{\mathrm{VI}}^{\mu\nu}(-p,p,-k,k) + V_{\mathrm{VI}}^{\nu\mu}(-p,p,k,-k) \\ &\qquad = 4\frac{\cos(\tilde{k}p) - 1}{(\tilde{k}p)^2}\tilde{p}^{\mu}\tilde{p}^{\nu}(\not{p} - m) = 4T^{(3)}(\tilde{k}p)\tilde{p}^{\mu}\tilde{p}^{\nu}(\not{p} - m)\,, \end{aligned} \tag{F.12}$$

where $\tilde{p}^{\mu} = (p\theta)^{\mu}$.

If we take all the above statements into account, we can rewrite the tadpole integral

$$\begin{aligned} \mathcal{M}_{\mathrm{L2}}^{\mu\nu}(k) = (-1)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\bigg[&\frac{1}{2}\theta^{\mu\nu} T^{(1)}(\tilde{k}p) - \tilde{k}^{\mu}\tilde{k}^{\nu} T^{(2)}(\tilde{k}p) \\ &+ 4\tilde{p}^{\mu}\tilde{p}^{\nu} T^{(3)}(\tilde{k}p) + 4\,\mathrm{i}\big(2\tilde{k}^{\nu} g^{\mu\rho} - 2\tilde{k}^{\mu} g^{\nu\rho} - \theta^{\mu\nu} k^{\rho}\big)\frac{p_{\rho}}{p^2 - m^2 + \mathrm{i}\,\epsilon}\bigg]\,, \end{aligned} \tag{F.13}$$

where we already performed the trace. As one can see, the pole in the first three terms doesn't exist anymore. It cancels because these three terms are proportional to $\not{p} - m$.

Now, what can we say about the four integrals? Firstly, the last integrand is odd in p^{μ}. Thus the integral over the whole energy-momentum space vanishes. The same is also valid for the first integral if one substitutes the momenta p by $\tilde{k}p$.

By substituting the integration variable of the second and third integral one can transfer the momentum integrals into scale-less integrals. The open problem of the regularization still poses conceptual and technical problems due to UV/IR-mixing. Hence, it is not settled if one is allowed to simply set them to zero. It makes sense that this tadpole diagram vanishes in dimensional regularization, but we can't give a final answer yet.

F.3 Discussion

One can ask whether the other two photon loop diagrams could lead to a contribution to the imaginary part of the photon self-energy. Based on the ordinary QCD one would assume that the photon tadpole diagram as well as the fermion tadpole diagram vanish. If this is the case the only additional imaginary contribution could come from the photon loop diagram. It is hard to conceive how this loop could contain an imaginary part whereas the fermion loop doesn't. Nevertheless one has to perform the calculations which has to be done in further studies.

F.4 The Integral $I^{(2)}_{\mathbf{L1}}$

The considerations from section F.1 led to the integral

$$I^{(2),\mu_1\mu_2}_{\mathrm{L1}}(n) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{p^{\mu_1} p^{\mu_2}}{p^2 + \mathrm{i}\,\epsilon} \frac{\cos(p\cdot n) - 1}{(p\cdot n)^2}\,, \tag{F.14}$$

which we now want to calculate in dimensional regularization. First of all we "Wick-rotate" the momenta p and n and write the denominator $p^2 + \mathrm{i}\,\epsilon$ as an integral over the Schwinger parameter α

$$I^{(2),\mu_1\mu_2}_{\mathrm{L1}}(n_{\mathrm{E}}) = -\,\mathrm{i} \int\limits_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^D p_{\mathrm{E}}}{(2\pi)^D} \eta^{\mu_1}_{\lambda_1} p^{\lambda_1}_{\mathrm{E}} \eta^{\mu_2}_{\lambda_2} p^{\lambda_2}_{\mathrm{E}} \frac{\cos p_{\mathrm{E}}\cdot n_{\mathrm{E}} - 1}{(p_{\mathrm{E}}\cdot n_{\mathrm{E}})^2} e^{-\alpha p^2_{\mathrm{E}}}\,, \tag{F.15}$$

with $p^\mu_{\mathrm{E}} := (-\,\mathrm{i}\,p^0, p^1, p^2, p^3)\,,\; n^\mu_{\mathrm{E}} := (-\,\mathrm{i}\,n^0, n^1, n^2, n^3)$ and

$$\eta^\mu_\nu := \begin{pmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{F.16}$$

In order to calculate the integral over p_{E} we expand the remaining fraction in its series representation

$$I^{(2),\mu_1\mu_2}_{\mathrm{L1}}(n_{\mathrm{E}}) = -\,\mathrm{i} \int\limits_0^\infty \mathrm{d}\alpha \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l)!} \eta^{\mu_1}_{\lambda_1} \eta^{\mu_2}_{\lambda_2} n_{\mathrm{E}\nu_1} \dots n_{\mathrm{E}\nu_{2l-2}} \int \frac{\mathrm{d}^D p_{\mathrm{E}}}{(2\pi)^D} p^{\lambda_1}_{\mathrm{E}} p^{\lambda_2}_{\mathrm{E}} p^{\nu_1}_{\mathrm{E}} \dots p^{\nu_{2l-2}}_{\mathrm{E}} e^{-\alpha p^2_{\mathrm{E}}}\,. \tag{F.17}$$

We can perform the momentum space integral, namely

$$\int \frac{\mathrm{d}^D p_{\mathrm{E}}}{(2\pi)^D} p^{\rho_1}_{\mathrm{E}} \dots p^{\rho_{2l}}_{\mathrm{E}} e^{-\alpha p^2_{\mathrm{E}}} = \frac{1}{2^l \alpha^l} \frac{1}{(4\pi\alpha)^{D/2}} \mathcal{P}(\delta^{\rho_1\rho_2} \dots \delta^{\rho_{2l-1}\rho_{2l}})\,, \tag{F.18}$$

with $\mathcal{P}$ meaning all different combinations of products of Kronecker-Deltas. The double factorial is defined as $n!! = n(n-2)(n-4)\dots$. Note, that there are $(2l-1)!!$ different combinations and not $(2l)!$ because of the equality of $\delta^{\mu_i\mu_j}$ and $\delta^{\mu_j\mu_i}$.

If we insert this integral in (F.17), we obtain

$$I_{\mathrm{L1}}^{(2),\mu_1\mu_2}(n_{\mathrm{E}}) = -\frac{\mathrm{i}}{n_{\mathrm{E}}^2}\int_0^\infty \mathrm{d}\alpha\,\frac{1}{(4\pi\alpha)^{D/2}}\sum_{l=1}^{\infty}\frac{(-1)^l}{(2l)!}\frac{1}{(2\alpha)^l}(n_{\mathrm{E}}^2)^l$$
$$\left[(2l-3)!!\,\eta^{\mu_1}_{\lambda_1}\delta^{\lambda_1\lambda_2}\eta^{\mu_2}_{\lambda_2} + \frac{\eta^{\mu_1}_{\lambda_1}n_{\mathrm{E}}^{\lambda_1}\eta^{\mu_2}_{\lambda_2}n_{\mathrm{E}}^{\lambda_2}}{n_{\mathrm{E}}^2}\big((2l-1)!!-(2l-3)!!\big)\right]. \quad (\text{F.19})$$

With the identities

$$(2l-3)!! = \frac{(2l-1)!!}{2l-1}\,, \qquad \frac{(2l-1)!!}{(2l)!} = \frac{1}{(2l)!!} = \frac{1}{2^l l!}\,, \quad (\text{F.20})$$

we can perform the sums

$$\sum_{l=1}^{\infty}\frac{(-1)^l(2l-3)!!}{(2l)!}\left(\frac{n_{\mathrm{E}}^2}{2\alpha}\right)^l = \sum_{l=1}^{\infty}\frac{1}{l!(2l-1)}\left(-\frac{n_{\mathrm{E}}^2}{4\alpha}\right)^l$$
$$= -\sqrt{\frac{n_{\mathrm{E}}^2}{4\alpha}}\,\sqrt{\pi}\ \mathrm{erf}\left(\sqrt{\frac{n_{\mathrm{E}}^2}{4\alpha}}\right) - \left(e^{-\frac{n_{\mathrm{E}}^2}{4\alpha}}-1\right) \quad (\text{F.21})$$

and

$$\sum_{l=1}^{\infty}\frac{(-1)^l(2l-1)!!}{(2l)!}\left(\frac{n_{\mathrm{E}}^2}{2\alpha}\right)^l = \sum_{l=1}^{\infty}\frac{1}{l!}\left(-\frac{n_{\mathrm{E}}^2}{4\alpha}\right)^l = e^{-\frac{n_{\mathrm{E}}^2}{4\alpha}}-1\,. \quad (\text{F.22})$$

Hence, with the two one-dimensional integrals over the Schwinger parameter α

$$I_{2a}(n_{\mathrm{E}}) := \frac{1}{(4\pi)^{D/2}}\int_0^\infty \mathrm{d}\alpha\,\alpha^{-(D+1)/2}\,\mathrm{erf}\left(\sqrt{\frac{n_{\mathrm{E}}^2}{4\alpha}}\right), \quad (\text{F.23})$$

$$I_{2b}(n_{\mathrm{E}}) := \frac{1}{(4\pi)^{D/2}}\int_0^\infty \mathrm{d}\alpha\,\alpha^{-D/2}\left(e^{-\frac{n_{\mathrm{E}}^2}{4\alpha}}-1\right). \quad (\text{F.24})$$

we can write $I_{\mathrm{L1}}^{(2),\mu_1\mu_2}$ as

$$I_{\mathrm{L1}}^{(2),\mu_1\mu_2}(n_{\mathrm{E}}) = -\frac{\mathrm{i}}{n_{\mathrm{E}}^2}\left[-\left(\frac{\sqrt{\pi n_{\mathrm{E}}^2}}{2}I_{2a}(n_{\mathrm{E}}) + I_{2b}(n_{\mathrm{E}})\right)\eta^{\mu_1}_{\lambda_1}\delta^{\lambda_1\lambda_2}\eta^{\mu_2}_{\lambda_2}\right.$$
$$\left. + \left(\frac{\sqrt{\pi n_{\mathrm{E}}^2}}{2}I_{2a}(n_{\mathrm{E}}) + 2I_{2b}(n_{\mathrm{E}})\right)\frac{\eta^{\mu_1}_{\lambda_1}n_{\mathrm{E}}^{\lambda_1}\eta^{\mu_2}_{\lambda_2}n_{\mathrm{E}}^{\lambda_2}}{n_{\mathrm{E}}^2}\right]. \quad (\text{F.25})$$

If we substitute in the first and second integral

$$\beta_a := \sqrt{\frac{n_{\rm E}^2}{4\alpha}} \qquad \text{and} \qquad \beta_b := \frac{n_{\rm E}^2}{4\alpha}\,, \tag{F.26}$$

respectively, we obtain

$$I_{2a}(n_{\rm E}) = 2\frac{1}{(4\pi)^{D/2}}\left(\frac{4}{n_{\rm E}^2}\right)^{(D-1)/2}\int_0^\infty \mathrm{d}\beta_a\,\beta_a^{D-2}\ \mathrm{erf}(\beta_a)\,, \tag{F.27a}$$

$$I_{2b}(n_{\rm E}) = \frac{1}{(4\pi)^{D/2}}\left(\frac{4}{n_{\rm E}^2}\right)^{(D-2)/2}\int_0^\infty \mathrm{d}\beta_b\,\beta_b^{D/2-2}\left(e^{-\beta_b}-1\right)\,. \tag{F.27b}$$

Let us discuss the two integrals. It is important to determine for which dimension D the integrals diverges at the lower and at the upper limit. The lower limit is equivalent to small loop momenta, i.e. the IR region. The upper limit is equivalent to high loop momenta, i.e. the UV region.

At the upper limit the integral I_{2a} is convergent for $D < 1$ (note, that the error-function becomes unity for $\beta_a \to \infty$), whereas the whole integral I_{2b} is convergent for $D < 2$. Thus, in the UV both integrals converge for $D < 1$.
If we look at the lower limit of the integral I_{2a} we find that it is convergent for $D > 0$, whereas the integral I_{2a} is convergent for $D > 2$. Thus, in the IR both integrals converges for $D > 2$.
This means that for $D = 4$ both integrals diverge in the UV region. To determine the power of divergence let us derive the indefinite integrals

$$\int \mathrm{d}\beta_a\,\beta_a^{D-2}\ \mathrm{erf}(\beta_a) = \frac{1}{D-1}\left(\beta_a^{D-1}\,\mathrm{erf}(\beta_a) + \frac{1}{\sqrt{\pi}}\Gamma\big(D/2,\beta_a^2\big)\right)\,,$$

$$\int \mathrm{d}\beta_b\,\beta_b^{D/2-2}\left(e^{-\beta_b}-1\right) = -2\frac{\beta_b^{D/2-1}}{D-2} - \Gamma\big(D/2-1,\beta_b\big)\,,$$

with the incomplete Γ-function

$$\Gamma(n,x) := \int_x^\infty \mathrm{d}t\,t^{n-1}\,\mathrm{e}^{-t}\ . \tag{F.28}$$

As mentioned, the IR region, $\beta_i \to 0$, is convergent for $D = 4$, namely

$$I_{2a}^{\rm IR}(n_{\rm E}) = -\frac{2}{3\sqrt{\pi}(4\pi)^2}\left(\frac{4}{n_{\rm E}^2}\right)^{3/2}\,, \tag{F.29a}$$

$$I_{2b}^{\rm IR}(n_{\rm E}) = \frac{1}{(4\pi)^2}\frac{4}{n_{\rm E}^2}\,. \tag{F.29b}$$

One can see in the definition of the incomplete Γ-function (F.28) that for all a $\lim_{x\to\infty}\Gamma(a,x) = 0$. Thus, in the UV region we are left with two terms which diverge for $D = 4$, namely

$$I^{\mathrm{UV}}_{2a}(n_{\mathrm{E}}) = 2\frac{1}{(4\pi)^2}\left(\frac{4}{n_{\mathrm{E}}^2}\right)^{3/2}\frac{1}{3}\lim_{\beta_a\to\infty}\beta_a^3\,\mathrm{erf}(\beta_a)$$

$$\stackrel{\delta\to 0}{=} \frac{2}{3}\frac{1}{(4\pi)^2}\left(\frac{4}{n_{\mathrm{E}}^2}\right)^{3/2}\frac{1}{\delta^3}\,, \qquad \text{(F.30a)}$$

$$I^{\mathrm{UV}}_{2b}(n_{\mathrm{E}}) = -\frac{1}{(4\pi)^2}\left(\frac{4}{n_{\mathrm{E}}^2}\right)\lim_{\beta_a\to\infty}\beta_b \stackrel{\delta\to 0}{=} -\frac{1}{(4\pi)^2}\left(\frac{4}{n_{\mathrm{E}}^2}\right)\frac{1}{\delta}\,. \qquad \text{(F.30b)}$$

We see that in the UV region the integrals are purely divergent. But the corresponding counterterms are non-local due to n_{E}^{-3} or n_{E}^{-2}, respectively. Thus, in the sense of dimensional regularization we can subtract them, but different from the ordinary QED we have to accept a non-local counterterm as has been proposed in [15].

If we combine the convergent parts (F.29) we obtain

$$I^{(2),\mu_1\mu_2}_{\mathrm{L1,\,conv.}}(n_{\mathrm{E}}) = \int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{p^{\mu_1}p^{\mu_2}}{p^2+\mathrm{i}\,\epsilon}\frac{\cos(p\cdot n)-1}{(p\cdot n)^2}$$

$$= \mathrm{i}\,\frac{1}{(4\pi)^2}\frac{1}{(n_{\mathrm{E}}^2)^2}\frac{4}{3}\left[-g^{\mu_1\mu_2} - \frac{4}{n_{\mathrm{E}}^2}\eta^{\mu_1}_{\lambda_1}n_{\mathrm{E}}^{\lambda_1}\eta^{\mu_2}_{\lambda_2}n_{\mathrm{E}}^{\lambda_2}\right]\,. \qquad \text{(F.31)}$$

Going back to Minkowski spacetime we finally get

$$I^{(2),\mu_1\mu_2}_{\mathrm{L1,\,conv.}}(n) = \int\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{p^{\mu_1}p^{\mu_2}}{p^2+\mathrm{i}\,\epsilon}\frac{\cos(p\cdot n)-1}{(p\cdot n)^2}$$

$$= \mathrm{i}\,\frac{1}{(4\pi)^2}\frac{1}{(n^2)^2}\frac{4}{3}\left[-g^{\mu_1\mu_2} + \frac{4}{n^2}n^{\mu_1}n^{\mu_2}\right]\,. \qquad \text{(F.32)}$$

Bibliography

[1] Nathan Seiberg and Edward Witten. String theory and noncommutative geometry. *JHEP* **09**, 032 (1999).

[2] Glenn Barnich, Friedemann Brandt, and Maxim Grigoriev. Local BRST cohomology and Seiberg-Witten maps in noncommutative Yang-Mills theory. *Nuclear Physics B* **677**, 503 (2004).

[3] Sergio Doplicher, Klaus Fredenhagen, and John E. Roberts. The Quantum structure of space-time at the Planck scale and quantum fields. *Commun. Math. Phys.* **172**, 187–220 (1995).

[4] Karl von Meyenn, editor. "Wolfgang Pauli, Scientific Correspondence", volume II. Springer Verlag (1985).

[5] H. S. Snyder. Quantized spacetime. *Phys. Rev.* **71**, 38 (1947).

[6] C. N. Yang. On quantized spacetime. *Phys. Rev.* **72**, 874 (1947).

[7] Sergio Doplicher, Klaus Fredenhagen, and John E. Roberts. Space-time quantization induced by classical gravity. *Phys. Lett.* **B331**, 39–44 (1994).

[8] Thomas Filk. Divergences in a field theory on quantum space. *Phys. Lett.* **B376**, 53–58 (1996).

[9] O. Ogievetsky, W. B. Schmidke, J. Wess, and B. Zumino. q deformed Poincare algebra. *Commun. Math. Phys.* **150**, 495–518 (1992).

[10] I. M. Gel'fand and M. A. Naimark. On the embedding of normal linear rings into the ring of operators in Hilbert space. *Mat. Sbornik* **12**, 197 (1943).

[11] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu. On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT. *Phys. Lett.* **B603**, 98–102 (2004).

[12] M. Hayakawa. Perturbative analysis on infrared aspects of noncommutative QED on R**4. *Physics Letters B* **478**, 394 (2000).

[13] M. Hayakawa. Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R**4. (1999).

[14] Alec Matusis, Leonard Susskind, and Nicolaos Toumbas. The IR/UV Connection in the Non-Commutative Gauge Theories. *JHEP* **0012**, 002 (2000).

[15] V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar. Renormalization of noncommutative phi 4-theory by multi-scale analysis. *Communications in Mathematical Physics* **262**, 565 (2006).

[16] X. Calmet, B. Jurco, P. Schupp, J. Wess, and M. Wohlgenannt. The standard model on non-commutative space-time. *Eur. Phys. J.* **C23**, 363–376 (2002).

[17] The OPAL collaboration. Test of non-commutative QED in the process e+e- -> gamma gamma at LEP. *Physics Letters B* **568**, 181 (2003).

[18] Ana Alboteanu, Thorsten Ohl, and Reinhold Rückl. Probing the noncommutative standard model at hadron colliders. *Phys. Rev. D* **74**, 096004 (2006).

[19] A. Gerhold et al.. The energy-momentum tensor on noncommutative spaces: Some pedagogical comments. (2000).

[20] Carsten Grosse-Knetter. Effective Lagrangians with higher derivatives and equations of motion. *Phys. Rev. D* **49**, 6709–6719 (1994).

[21] C. Becchi, A. Rouet, and R. Stora. Renormalization of Gauge Theories. *Annals Phys* **98**, 287–321 (1976).

[22] I. V. Tyutin. Gauge Invariance In Field Theory And Statistical Physics In Operator Formalism. *LEBEDEV-75* **39**, 62pp (1975).

[23] L. Alvarez-Gaume and M. A. Vazquez-Mozo. General properties of noncommutative field theories. *Nuclear Physics B* **668**, 293 (2003).

[24] Jochen Zahn. “Dispersion relations in quantum electrodynamics on the noncommutative Minkowski space”. PhD thesis, (2007).

[25] C. Itzykson and J. B. Zuber. Quantum Field Theory. *ISBN-13* **978-0486445687**.

[26] Jaume Gomis and Thomas Mehen. Space-time noncommutative field theories and unitarity. *Nucl. Phys.* **B591**, 265–276 (2000).

[27] Yi Liao and Klaus Sibold. Time-ordered perturbation theory on noncommutative spacetime: Basic rules. *Eur. Phys. J.* **C25**, 469–477 (2002).

[28] Thorsten Ohl, Reinhold Rückl, and Jörg Zeiner. Unitarity of time-like noncommutative gauge theories: The violation of Ward identities in time-ordered perturbation theory. *Nucl. Phys.* **B676**, 229–242 (2004).

[29] H. Bozkaya et al.. Space/time noncommutative field theories and causality. *Eur. Phys. J.* **C29**, 133–141 (2003).

[30] Kazuo Fujikawa. Quantization of space-time noncommutative theory. (2004).

[31] Friedemann Brandt. Local BRST cohomology in the antifield formalism. *Lectures at Institut für Theoretische Physik, Universität Hannover* (WS 2000/2001).

[32] Johannes Rauh. Die Produktion schwerer Quarks an Hadron-Collider in nichtkommutativer QCD. Master's thesis, (2006).

[33] B. L. Cerchiai, A. F. Pasqua, and B. Zumino. The Seiberg-Witten Map for Noncommutative Gauge Theories. (2002).

[34] D. Brace, B. L. Cerchiai, A. F. Pasqua, U. Varadarajan, and B. Zumino. A Cohomological Approach to the Non-Abelian Seiberg-Witten Map. *JHEP* **0106**, 047 (2001).

[35] Marcel Froissart. Asymptotic Behavior and Subtractions in the Mandelstam Representation. *Phys. Rev.* **123**(3), 1053–1057 (Aug 1961).

[36] A. Martin. Unitarity and High-Energy Behavior of Scattering Amplitudes. *Phys. Rev.* **129**(3), 1432–1436 (Feb 1963).

[37] Michael E. Peskin. "An Introduction to Quantum Field Theory". HarperCollins Publishers (1995).

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