Landau level quantization on the sphere

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It is well established that the Hilbert space for charged particles in a plane subject to a uniform magnetic field can be described by two mutually commuting ladder algebras. We propose a similar formalism for Landau level quantization on a sphere involving two mutually commuting SU(2) algebras.

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I. INTRODUCTION

The formalism for Landau level quantization in a spherical geometry, i.e., for the dynamics of a charged particle on the surface of a sphere with radius R, in a magnetic (monopole) field, was pioneered by Haldane[1] in 1983 as an alternative geometry for the formulation of fractionally quantized Hall states.[2] In comparison to the disk geometry used by Laughlin[3] when he originally proposed the Jastrow-type wave functions for the ground state, the sphere has the advantage that it does not have a boundary. At the same time, it does not display the topological degeneracies associated with the torus geometry (i.e., a plane with periodic boundary conditions), which has genus one.[4] These two properties make the spherical geometry particularly suited for numerical work on bulk properties of quantized Hall states.

Haldane,[1] however, worked out the formalism only for the lowest Landau level, and never generalized it to higher Landau levels, even though these became more and more important as time passed. The presently most vividly discussed quantum Hall state, the Pfaffian state[5–15] at Landau level filling fraction $\nu = 5/2$, is observed only in the second Landau level.

In this paper, we first review Haldane's formalism for the lowest Landau level,[1,16] and then generalize it to the full Hilbert space, which includes higher Landau levels as well. The key insight permitting this generalization is that there is not one but there are two mutually commuting SU(2) algebras with spin s, one for the cyclotron variables and one for the guiding center variables. The formalism we develop will prove useful for numerical studies of fractionally quantized Hall states involving higher Landau levels. In particular, it will instruct us how to calculate pseudopotentials[1] for higher Landau levels on the sphere, which we will discuss as well. Finally, we will present a convenient way to write the wave function for M-filled Landau levels on the sphere.

II. HALDANE'S FORMALISM

Following Haldane,[1] we assume a radial magnetic field of strength

$$B = \frac{\hbar c s_0}{eR^2} \quad (e > 0). \tag{1}$$

The number of magnetic Dirac flux quanta through the surface of the sphere is

$$\frac{\Phi_{\text{tot}}}{\Phi_0} = \frac{4\pi R^2 B}{2\pi\hbar c/e} = 2s_0, \tag{2}$$

which must be an integer due to Dirac's monopole quantization condition.[17] In the following, we take $\hbar = c = 1$.

The Hamiltonian is given by

$$H = \frac{\Lambda^2}{2MR^2} = \frac{\omega_c}{2s_0}\Lambda^2, \tag{3}$$

where $\omega_c = eB/M$ is the cyclotron frequency,

$$\Lambda = r \times [-i\nabla + eA(r)] \tag{4}$$

is the dynamical angular momentum, $r = Re_r$, and $\nabla \times A = Be_r$. With (A4)–(A6) from the Appendix, we obtain

$$\Lambda = -i\left(e_\varphi \frac{\partial}{\partial\theta} - e_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + eR[e_r \times A(r)]. \tag{5}$$

Note that

$$e_r\Lambda = \Lambda e_r = 0, \tag{6}$$

as one can easily verify with (A5). The commutators of the Cartesian components of Λ with themselves and with e_r can easily be evaluated using (5) and (A3)–(A5). This yields

$$[\Lambda^i, \Lambda^j] = i\epsilon^{ijk}\left(\Lambda^k - s_0 e_r^k\right), \tag{7}$$

$$\left[\Lambda^i, e_r^j\right] = i\epsilon^{ijk}e_r^k, \tag{8}$$

where $i,j,k = x,y,$ or z, and e_r^k is the kth Cartesian coordinate of e_r. From (6)–(8), we see that that the operator

$$L = \Lambda + s_0 e_r \tag{9}$$

is the generator of rotations around the origin,

$$[L^i, X^j] = i\epsilon^{ijk}X^k \quad \text{with} \quad X = \Lambda, e_r, \quad \text{or} \quad L, \tag{10}$$

and hence the angular momentum. As it satisfies the angular-momentum algebra, it can be quantized accordingly. Note that L has a component in the e_r direction

$$Le_r = e_r L = s_0. \tag{11}$$

If we take the eigenvalue of L^2 to be $s(s+1)$, this implies $s = s_0 + n$, where $n = 0,1,2,\ldots$ is a non-negative integer (while s and s_0 can be an integer or half integer, according to the number of Dirac flux quanta through the sphere).

With (9) and (6), we obtain

$$\Lambda^2 = L^2 - s_0^2. \tag{12}$$

The energy eigenvalues of (3) are hence

$$E_n = \frac{\omega_c}{2s_0}\left[s(s+1)-s_0^2\right] = \frac{\omega_c}{2s_0}\left[(2n+1)s_0 + n(n+1)\right] = \omega_c\left[\left(n+\frac{1}{2}\right)+\frac{n(n+1)}{2s_0}\right]. \quad (13)$$

The index n hence labels the Landau levels.

To obtain the eigenstates of (3), we have to choose a gauge and then explicitly solve the eigenvalue equation. We choose the latitudinal gauge

$$A = -e_\varphi \frac{s_0}{eR}\cot\theta. \quad (14)$$

The singularities of $B = \nabla \times A$ at the poles are without physical significance. They describe infinitely thin solenoids admitting flux $s_0\Phi_0$ each and reflect our inability to formulate a true magnetic monopole.

The dynamical angular momentum (5) becomes

$$\Lambda = -i\left[e_\varphi \frac{\partial}{\partial\theta} - e_\theta \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\varphi} - i s_0\cos\theta\right)\right]. \quad (15)$$

With (A5), we obtain

$$\Lambda^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\varphi} - i s_0\cos\theta\right)^2. \quad (16)$$

To formulate the eigenstates, Haldane[1] introduced spinor coordinates for the particle position

$$u = \cos\frac{\theta}{2}\exp\left(\frac{i\varphi}{2}\right), \quad v = \sin\frac{\theta}{2}\exp\left(-\frac{i\varphi}{2}\right), \quad (17)$$

such that

$$e_r = \Omega(u,v) \equiv (u,v)\sigma\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}, \quad (18)$$

where $\sigma = (\sigma_x,\sigma_y,\sigma_z)$ is the vector consisting of the three Pauli matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \quad (19)$$

In terms of these, a complete, orthogonal basis of the states spanning the lowest Landau level ($n = 0$, $s = s_0$) is given by

$$\psi^s_{m,0}(u,v) = u^{s+m}v^{s-m}, \quad (20)$$

with $m = -s, s+1, \ldots, s$. For these states,

$$L^z\psi^s_{m,0} = m\psi^s_{m,0}, \quad H\psi^s_{m,0} = \tfrac{1}{2}\omega_c\psi^s_{m,0}. \quad (21)$$

To verify (21), we consider the action of (16) on the more general basis states

$$\phi^s_{m,p}(u,v) = \left(\cos\frac{\theta}{2}\right)^{s+m}\left(\sin\frac{\theta}{2}\right)^{s-m} e^{i(m-p)\varphi} = \begin{cases}\bar{v}^{-p}u^{s+m}v^{s-m+p} & \text{for } p<0,\\ \bar{u}^{p}u^{s+m-p}v^{s-m} & \text{for } p\geqslant 0.\end{cases} \quad (22)$$

This yields

$$\Lambda^2\phi^s_{m,p} = \left[s - \left(\frac{s\cos\theta - m}{\sin\theta}\right)^2 + \left(\frac{s_0\cos\theta - m + p}{\sin\theta}\right)^2\right]\phi^s_{m,p} = \left[s + \frac{2(s\cos\theta - m + p)(p - n\cos\theta) - (p^2 - n^2\cos^2\theta)}{\sin^2\theta}\right]\phi^s_{m,p}. \quad (23)$$

For $p = n = 0$, this clearly reduces to $\Lambda^2\psi^s_{m,0} = s\psi^s_{m,0}$, and hence (21). The normalization of (20) can easily be obtained with the integral

$$\frac{1}{4\pi}\int d\Omega\,\bar{u}^{s'+m'}\bar{v}^{s'-m'}u^{s+m}v^{s-m} = \frac{(s+m)!(s-m)!}{(2s+1)!}\delta_{mm'}\delta_{ss'}, \quad (24)$$

where $d\Omega = \sin\theta d\theta d\phi$.

To describe particles in the lowest Landau level, which are localized at a point $\Omega(\alpha,\beta)$ with spinor coordinates (α,β),

$$\Omega(\alpha,\beta) = (\alpha,\beta)\sigma\begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}, \quad (25)$$

Haldane[1] introduced "coherent states" defined by

$$\{\Omega(\alpha,\beta)L\}\psi^s_{(\alpha,\beta),0}(u,v) = s\psi^s_{(\alpha,\beta),0}(u,v). \quad (26)$$

In the lowest Landau level, the angular momentum L can be written as

$$L = \frac{1}{2}(u,v)\sigma\begin{pmatrix}\frac{\partial}{\partial u}\\ \frac{\partial}{\partial v}\end{pmatrix}. \quad (27)$$

Note that u,v may be viewed as Schwinger boson creation, and $\frac{\partial}{\partial u},\frac{\partial}{\partial v}$ as the corresponding annihilation operators.[18] The solutions of (26) are given by

$$\psi^s_{(\alpha,\beta),0}(u,v) = (\bar{\alpha}u + \bar{\beta}v)^{2s}, \quad (28)$$

as one can verify easily with the identity

$$(\underline{a}\sigma\underline{b})(\underline{c}\sigma\underline{d}) = 2(\underline{a}\underline{d})(\underline{c}\underline{b}) - (\underline{a}\underline{b})(\underline{c}\underline{d}). \quad (29)$$

where $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ are two-component spinors.

III. GENERALIZATION TO HIGHER LANDAU LEVELS

We will first present the formalism we developed and then motivate it. In analogy to the two mutually commuting ladder algebras $a,a^\dagger$ and $b,b^\dagger$ in the plane,[19–23] we describe the Hilbert space of a charged particle on a sphere with a magnetic monopole in the center by two mutually commuting SU(2) angular-momentum algebras. The first algebra for the cyclotron momentum S consists of operators that allow us to raise or lower eigenstates from one Landau to the next (as $a,a^\dagger$ do in the plane). The second algebra for the guiding center momentum L consists of operators that rotate the eigenstates on the sphere while preserving the Landau level index (as $b,b^\dagger$ do in the plane).

The reason that this structure was not discovered long ago may be that it is possible to obtain the spectrum without introducing S, as the eigenvalue of both S^2 and L^2 is $s(s+1)$. The necessity to introduce S is therefore not obvious.

We have already seen above that the spinor coordinates u,v and the derivatives $\frac{\partial}{\partial u},\frac{\partial}{\partial v}$ may be viewed as Schwinger boson creation and annihilation operators, respectively. A complete basis for the eigenstates of H in the lowest Landau level is given by $u^{m+s}v^{m-s}$, i.e., it could be expressed in terms of u and v. For higher Landau levels, the analogy to the plane suggests that we will need $\bar{u},\bar{v}$ as well. With the derivatives $\frac{\partial}{\partial \bar{u}},\frac{\partial}{\partial \bar{v}}$, we have a total of four Schwinger boson creation and annihilation operators. This suggests that we span two mutually commuting SU(2) algebras with them.

We will motivate below that the appropriate combinations are

$$\begin{aligned} S^x + iS^y = S^+ &= u\frac{\partial}{\partial \bar{v}} - v\frac{\partial}{\partial \bar{u}},\\ S^x - iS^y = S^- &= \bar{v}\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial v},\\ S^z &= \frac{1}{2}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - \bar{u}\frac{\partial}{\partial \bar{u}} - \bar{v}\frac{\partial}{\partial \bar{v}}\right) \end{aligned} \tag{30}$$

for the cyclotron momentum, and

$$\begin{aligned} L^x + iL^y = L^+ &= u\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{u}},\\ L^x - iL^y = L^- &= v\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{v}},\\ L^z &= \frac{1}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - \bar{u}\frac{\partial}{\partial \bar{u}} + \bar{v}\frac{\partial}{\partial \bar{v}}\right) \end{aligned} \tag{31}$$

for the guiding center momentum. We can write these more compactly as

$$S = \frac{1}{2}(u,\bar{v})\sigma\begin{pmatrix}\frac{\partial}{\partial u}\\ \frac{\partial}{\partial \bar{v}}\end{pmatrix} - \frac{1}{2}(\bar{u},v)\sigma^T\begin{pmatrix}\frac{\partial}{\partial \bar{u}}\\ \frac{\partial}{\partial v}\end{pmatrix}, \tag{32}$$

$$L = \frac{1}{2}(u,v)\sigma\begin{pmatrix}\frac{\partial}{\partial u}\\ \frac{\partial}{\partial v}\end{pmatrix} - \frac{1}{2}(\bar{u},\bar{v})\sigma^T\begin{pmatrix}\frac{\partial}{\partial \bar{u}}\\ \frac{\partial}{\partial \bar{v}}\end{pmatrix}, \tag{33}$$

where $\sigma^T = (\sigma_x,-\sigma_y,\sigma_z)$ is the vector consisting of the three transposed Pauli matrices.

From (32) and (33), we see that both S and L obey the SU(2) angular-momentum algebras

$$[S^i,S^j] = i\epsilon^{ijk}S^k, \quad [L^i,L^j] = i\epsilon^{ijk}L^k. \tag{34}$$

With (30) and (31), it is easy to show that the two algebras are mutually commutative,

$$[S^i,L^j] = 0 \quad \text{for all } i,j. \tag{35}$$

For S^2 and L^2, we find

$$L^2 = S^2 = S(S+1) \tag{36}$$

with

$$S = \frac{1}{2}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + \bar{u}\frac{\partial}{\partial \bar{u}} + \bar{v}\frac{\partial}{\partial \bar{v}}\right). \tag{37}$$

The component of L normal to the surface of the sphere is

$$e_r L = \Omega(u,v)L = S^z, \tag{38}$$

which is easily verified with (18), (29), (33), and

$$\Omega(u,v) = (\bar{u},\bar{v})\sigma^T\begin{pmatrix}u\\ v\end{pmatrix}, \tag{39}$$

$$(\underline{a}\sigma^T\underline{b})(\underline{c}\sigma^T\underline{d}) = 2(\underline{ad})(\underline{cb}) - (\underline{ab})(\underline{cd}). \tag{40}$$

It implies that the physical Hilbert space is limited to states with S^z eigenvalue s_0, i.e.,

$$S^z\psi = s_0\psi \quad \text{for all eigenstates } \psi. \tag{41}$$

With (36)–(38), we write

$$\begin{aligned} H &= \frac{\omega_c}{2s_0}[L^2 - (e_r L)^2]\\ &= \frac{\omega_c}{2s_0}[S^2 - (S^z)^2]\\ &= \frac{\omega_c}{4s_0}(S^+S^- + S^-S^+). \end{aligned} \tag{42}$$

With $[S^+,S^-] = 2S^z$ and (41), we obtain

$$H = \omega_c\left(\frac{1}{2s_0}S^-S^+ + \frac{1}{2}\right). \tag{43}$$

This is our main result. The operators S^- and S^+ hence play the role of Landau level raising and lowering operators, respectively, as $a^\dagger$ and a do in the plane.[23] At the same time, the raising operator S^- lowers the eigenvalue of S^z (i.e., s_0) by one, as

$$[S^z,S^-] = -S^z. \tag{44}$$

This has to be taken into account when constructing the Hilbert space.

The guiding center momentum L generates rotations of the states within each Landau level around the sphere, while leaving the Landau level structure unaltered. Note that the seemingly unrelated forms (33) and (9) of L describe the same operator, as both generate identical rotations around the sphere.

The basis states (20) are obviously eigenstates of (43) with energy $\frac{1}{2}\omega_c$. To lift them into the $(n+1)$th Landau level, we only have to increase the flux from s_0 to $s = s_0 + n$, and then apply $(S^-)^n$:

$$\psi^s_{m,n}(u,v) = (S^-)^n\psi^s_{m,0}(u,v), \tag{45}$$

where $s = s_0 + n$ and $m = -s,\ldots,s$. The states $\psi^s_{m,n}(u,v)$ constitute a complete, orthogonal basis for the $(n+1)$th

Landau level on a sphere in a monopole field with $2s_0$ Dirac flux quanta through its surface.

We will now show that the states (45) are indeed eigenstates of (3) with energy (13). Note first that since

$$\psi^s_{m,0}(u,v) = \frac{(2s-m)!}{(2s)!}(L^-)^m \psi^s_{S,0}(u,v)$$

and $[S^-,L^-] = [H,L^-] = 0$, it is sufficient to show that

$$\psi^s_{0,n}(u,v) = (S^-)^n \underbrace{\psi^s_{s,0}(u,v)}_{=u^{2s}} = \frac{(2s)!}{(2s-n)!}\bar{v}^n u^{2s-n}$$

is an eigenstate. This is (up to a normalization) equal to our earlier basis state $\phi^s_{m,p}$ [see (22)] with $m = s - n$, $p = -n$. With (23), we find

$$\begin{aligned}\Lambda^2 \psi^s_{0,n} &= \left[s + \frac{2(s\cos\theta - s)(-n - n\cos\theta) - (n^2 - n^2\cos^2\theta)}{\sin^2\theta}\right]\psi^s_{0,n} \\ &= [(2n+1)s - n^2]\psi^s_{0,n} \\ &= [(2n+1)s_0 + n(n+1)]\psi^s_{0,n},\end{aligned} \tag{46}$$

which completes the proof.

Note that since S commutes with L, Haldane's coherent states remain coherent as we elevate them into higher Landau levels. In particular, the state

$$\psi^s_{(\alpha,\beta),n}(u,v) = (S^-)^n(\bar{\alpha}u + \bar{\beta}v)^{2s} \tag{47}$$

in the $(n+1)$th Landau level still satisfies (26).

IV. PSEUDOPOTENTIALS

Haldane[1] also introduced two-particle coherent lowest Landau level states defined by

$$\{\Omega(\alpha,\beta)(L_1+L_2)\}\psi^{s,j}_{(\alpha,\beta),0}[u,v] = j\psi^{s,j}_{(\alpha,\beta),0}[u,v], \tag{48}$$

where $[u,v] := (u_1,u_2,v_1,v_2)$ and j is the total angular momentum

$$(L_1+L_2)^2\psi^{s,j}_{(\alpha,\beta),0}[u,v] = j(j+1)\psi^{s,j}_{(\alpha,\beta),0}[u,v]. \tag{49}$$

The solution of (48) is given by

$$\psi^{s,j}_{(\alpha,\beta),0}[u,v] = (u_1v_2 - u_2v_1)^{2s-j}\prod_{i=1,2}(\bar{\alpha}u_i + \bar{\beta}v_i)^j. \tag{50}$$

It describes two particles with relative momentum $2s - j$ precessing about their common center of mass at $\Omega(\alpha,\beta)$. It is straightforward to elevate this state into the $(n+1)$th Landau level

$$\psi^{s,j}_{(\alpha,\beta),n}[u,v] = \prod_{i=1,2}(S^-)^n\psi^{s,j}_{(\alpha,\beta),0}[u,v]. \tag{51}$$

Note that (51) still satisfies (48) and (49).

Since $0 \leqslant j \leqslant 2s$, the relative momentum quantum number $l = 2s - j$ has to be a non-negative integer. For bosons or fermions, l has to be even or odd, respectively. This implies that the projection Π_n onto the $(n+1)$th Landau level of any translationally invariant (i.e., rotationally invariant on the sphere) operator $V(\Omega_1\cdot\Omega_2)$, such as two-particle interaction potentials, can be expanded as

$$\Pi_n V(\Omega_1\cdot\Omega_2)\Pi_n = \sum_l^{2s} V^n_l P_{2s-l}(L_1+L_2), \tag{52}$$

where the sum over l is restricted to even (odd) integers for bosons (fermions), $P_j(L)$ is the projection operator on states with total momentum $L^2 = j(j+1)$, and the V^n_l are pseudopotential coefficients.

The pseudopotential V^n_l denotes the potential energy cost of $V(\Omega_1\cdot\Omega_2)$ for two particles with relative angular momentum l in the $(n+1)$th Landau level. We can use the coherent states (51) to evaluate them. As the result will not depend on the center of rotation, we can take $(\alpha,\beta) = (1,0)$, i.e., work with the coherent states

$$\psi^{s,j}_{(1,0),n}[u,v] = (S_1^-)^n(S_2^-)^n(u_1v_2 - u_2v_1)^{2s-j}u_1^j u_2^j. \tag{53}$$

This yields

$$V^n_{2s-j} = \frac{\left\langle \psi^{s,j}_{(1,0),n}\middle| V(\Omega_1\cdot\Omega_2)\middle|\psi^{s,j}_{(1,0),n}\right\rangle}{\left\langle \psi^{s,j}_{(1,0),n}\middle|\psi^{s,j}_{(1,0),n}\right\rangle} \tag{54}$$

for the pseudopotentials. Since the chord distance between two points on the unit sphere is given by

$$|\Omega_1 - \Omega_2| = 2\,|u_1v_2 - u_2v_1|, \tag{55}$$

a $1/r$ or Coulomb interaction on the sphere is given by

$$V(\Omega_1\cdot\Omega_2) = \frac{1}{2\,|u_1v_2 - u_2v_1|}. \tag{56}$$

Fano, Ortolani, and Colombo[16] evaluated the pseudopotential coefficients for Coulomb interactions in the lowest Landau level by explicit integration, and found

$$V^0_l = \frac{\binom{2l}{l}\binom{8s+2-2l}{4s+1-l}}{\binom{4s+2}{2s+1}^2}. \tag{57}$$

The potential interaction Hamiltonian acting on many-particle states expanded in a basis of L^z eigenstates (20) or (45) is given by

$$\begin{aligned}H^{(n)}_{\text{int}} = \sum_{m_1=-s}^{s}\sum_{m_2=-s}^{s}\sum_{m_3=-s}^{s}\sum_{m_4=-s}^{s} a^\dagger_{m_1,n}a^\dagger_{m_2,n}a_{m_3,n}a_{m_4,n} \\ \cdot\,\delta_{m_1+m_2,m_3+m_4}\sum_{l=0}^{2s}\langle s,m_1;s,m_2|2s-l,m_1+m_2\rangle\, V^n_l \\ \langle 2s-l,m_3+m_4|s,m_3;s,m_4\rangle\,,\end{aligned} \tag{58}$$

where $a_{m,n}$ annihilates a boson or fermion in the properly normalized single-particle state

$$\psi^{s}_{m,n}(u,v) = C_{m,n}(S^{-})^{n} u^{s+m} v^{s-m} \tag{59}$$

with

$$C_{m,n} = \sqrt{\frac{(2s-n)!}{(2s)!n!}}\sqrt{\frac{(2s+1)!}{4\pi(s+m)!(s-m)!}}. \tag{60}$$

In (58), we take two particles with L_z eigenvalues m_3 and m_4, use the Clebsch-Gordan coefficients[24] $\langle 2s-l, m_3+m_4|s,m_3;s,m_4\rangle$ to change the basis into one where m_3+m_4 and the total two-particle momentum $2s-l$ are replacing the quantum numbers m_3 and m_4, multiply each amplitude by V_l^n, and convert the two-particle states back into a basis of L_z eigenvalues m_1 and m_2.

Note that since this basis transformation commutes with $\boldsymbol{S}_i$ for all i, (58) depends on the Landau level index n only through the pseudopotentials. This means that if we write out the potential interaction term (58) in a higher Landau level, the matrix we obtain is exactly as in the lowest Landau level for the same value of s, except that we have to use the pseudopotential V_l^n for the $(n+1)$-th Landau level instead of V_l^0. Note further that the normalization $C_{m,n}$ for the basis states factorizes into a term which depends only on n and a term which depends only on the L^z eigenvalue m. This follows again from the commutativity of $\boldsymbol{S}$ and $\boldsymbol{L}$. It is hence sufficient to write out the wave function of a quantized Hall state in the lowest Landau level using the basis states (59) for $n=0$, and use the Hamiltonian matrix (58) with the $(n+1)$-th Landau level pseudopotentials V_l^n to evaluate the interaction energy this state would have if we were to elevate it into the $(n+1)$-th level with $\prod_i (S_i^-)^n$. In other words, the only difference between an exact diagonalization study in a higher Landau as compared to the lowest Landau level is that we have to use V_l^n instead of V_l^0.

The generalization of the pseudopotentials for three- and more-particle interactions[25] to higher Landau levels proceeds without incident.

V. FILLED LANDAU LEVELS

The wave function for a filled $(n+1)$th Landau level for $N=2s+1$ particles with $s_0 = s-n$ is given by

$$\psi^{s}_{n}[u,v,\bar{u},\bar{v}] = \prod_{i=1}^{N}(S_i^{-})^{n}\prod_{i<j}^{N}(u_i v_j - u_j v_i). \tag{61}$$

Except for $n=0$, this does not reduce to any particularly simple form when we write out all the terms.

We have found, however, a convenient way to write the wave function for M filled Landau levels with index $n=0,\ldots,M-1$ (i.e., from the first to the Mth Landau level). We assume a total of LM particles labeled by two integers $l=1,\ldots,L$ and $m=1,\ldots,M$, with spinor coordinates $(u_{lm},v_{lm},\bar{u}_{lm},\bar{v}_{lm})$. The LM particle wave function for a sphere with $2s_0 = L-M>0$ flux quanta is then given by

$$\psi^{s_0}[u,v,\bar{u},\bar{v}] = \mathcal{A}\left\{\prod_{m=1}^{M}\prod_{l<l'}^{L}(u_{lm}v_{l'm} - u_{l'm}v_{lm}) \times \prod_{l=1}^{L}\prod_{m<m'}^{M}(\bar{u}_{lm}\bar{v}_{lm'} - \bar{u}_{lm'}\bar{v}_{lm})\right\}, \tag{62}$$

where $\mathcal{A}$ denotes antisymmetrization. To verify (62), multiply the wave functions (61) for each Landau level n with

$$\prod_{i=1}^{N}(u_i\bar{u}_i + v_i\bar{v}_i)^{M-1-n},$$

which is equal to 1 and commutes with both S_i and L_i for all i, and then antisymmetrize over all the single-particle states in the Landau levels with index $n=0,\ldots,M-1$.

The formulation (62) may be useful in the construction of composite fermion states[26] for hierarchical filling fractions, and in particular as a starting point for obtaining such states from several filled Landau levels through a process of adiabatic localization of magnetic flux onto the particles.[27]

VI. CONCLUSION

We have developed a formalism to describe the Hilbert space of charged particles on a sphere subject to a magnetic monopole field, using two mutually commuting SU(2) algebras for cyclotron and guiding center momenta. As the previously developed formalism for the lowest Landau level has been highly important for numerical studies of fractionally quantized Hall states, we expect our generalization to higher Landau levels to be of similar significance.

APPENDIX: SPHERICAL COORDINATES

The formalism requires vector analysis in spherical coordinates. In this appendix, we will briefly review the conventions.

Vectors and vector fields are given by

$$\boldsymbol{r} = r\boldsymbol{e}_r, \tag{A1}$$

$$\boldsymbol{v}(\boldsymbol{r}) = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_\varphi\boldsymbol{e}_\varphi, \tag{A2}$$

with

$$\boldsymbol{e}_r = \begin{pmatrix}\cos\varphi\sin\theta\\ \sin\varphi\sin\theta\\ \cos\theta\end{pmatrix},\quad \boldsymbol{e}_\theta = \begin{pmatrix}\cos\varphi\cos\theta\\ \sin\varphi\cos\theta\\ -\sin\theta\end{pmatrix},$$

$$\boldsymbol{e}_\varphi = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \tag{A3}$$

where $\varphi \in [0,2\pi[$ and $\theta\in[0,\pi]$. This implies

$$\boldsymbol{e}_r\times\boldsymbol{e}_\theta = \boldsymbol{e}_\varphi,\quad \boldsymbol{e}_\theta\times\boldsymbol{e}_\varphi = \boldsymbol{e}_r,\quad \boldsymbol{e}_\varphi\times\boldsymbol{e}_r = \boldsymbol{e}_\theta, \tag{A4}$$

and

$$\frac{\boldsymbol{e}_r}{\theta} = \boldsymbol{e}_\theta,\quad \frac{\boldsymbol{e}_\theta}{\theta} = -\boldsymbol{e}_r,\quad \frac{\boldsymbol{e}_\varphi}{\theta} = 0,\quad \frac{\boldsymbol{e}_r}{\varphi} = \sin\theta\,\boldsymbol{e}_\varphi,$$

$$\frac{\boldsymbol{e}_\theta}{\varphi} = \cos\theta\,\boldsymbol{e}_\varphi,\quad \frac{\boldsymbol{e}_\varphi}{\varphi} = -\sin\theta\,\boldsymbol{e}_r - \cos\theta\,\boldsymbol{e}_\theta. \tag{A5}$$

With the nabla operator

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}, \tag{A6}$$

we obtain

$$\nabla v = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}, \tag{A7}$$

$$\begin{aligned}\nabla\times v = {} & e_r\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, v_\varphi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\right)\\ & + e_\theta\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r}\right)\\ & + e_\varphi\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right),\end{aligned} \tag{A8}$$

$$\begin{aligned}\nabla^2 = {} & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\\ & + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.\end{aligned} \tag{A9}$$

Comparing (16) with (A9), we see that

$$\Lambda^2\big|_{s_0=0} = \nabla^2\big|_{r\equiv 1}, \tag{A10}$$

as expected.

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