Two-leg *t-J* ladder: A spin liquid generated by Gutzwiller projection of magnetic bands

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The ground state of the two-leg Heisenberg ladder is identified as a resonating-valence-band-RVB type spin liquid, which is generated by Gutzwiller projection of tight-binding bands with flux π per plaquet. Explicit trial wave functions for the magnon and hole excitations are formulated in terms of spinons and holons. The spinon-holon bound state is shown to violate P and T. This is interpreted as a manifestation of fractional statistics.

Itinerant antiferromagnets confined to coupled chains, or t-J ladders, have enjoyed enormous popularity over the past few years.[1–8] They provide the simplest example of a generic spin liquid in dimensions greater than one, and the only example thereof which is presently fully amenable to numerical methods. (It is furthermore widely believed that they constitute the first step towards understanding the two-dimensional t-J model starting from one dimension, but I rather believe the two-leg ladder to be just a special case.) These models are approximately realized in $(VO)_2P_2O_7$, $SrCu_2O_3$, and $Sr_{14-x}Ca_xCu_{24}O_{41}$, and hence accessible to experiment.

The t-J Hamiltonian for the ladder is given by

$$H_{t-J} = -\sum_{\langle ij \rangle \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \qquad (1)$$

where $(t_{ij}, J_{ij}) = (t, J)$ if i and j are nearest neighbors along one of the chains, and $(t_\perp, J_\perp)$ if they are nearest neighbors across the rungs; each pair $\langle ij \rangle$ is summed over twice and no doubly occupied sites are allowed.

One of the most striking features of the undoped two-leg t-J ladder is the persistence of a spin gap $\Delta \approx J_\perp/2$ in the weak coupling limit $J_\perp \ll J$. (For sufficiently strong couplings $J_\perp > J$, the system can be described by a perturbative expansion around the strong coupling limit consisting of singlets across the rungs,[9] which yields a spin gap $\Delta \approx J_\perp - J + \frac{1}{2}J^2/J_\perp$; a weak coupling expansion starting from decoupled chains, however, is not possible, as the individual spin chains are quantum critical in the sense that the tiniest perturbation can change the universality class.) In this paper, we will formulate a microscopic theory of the two-leg t-J ladder, which is universally valid at *all ratios* $J_\perp/J$, in terms of explicit spin liquid trial wave functions for the ground state, magnon (spinon-spinon bound state), and the hole (holon-spinon bound state) excitations.

The trial wave function for the ground state of the Heisenberg ladder, the t-J ladder without any holes, is constructed as follows. Consider a tight binding ladder with flux π per plaquet, hopping terms of magnitude $\tilde{t}$ along the chains, and $\tilde{t}_\perp$ across the rungs. In the gauge depicted in Fig. 1, we write the single particle Bloch states $\psi_q(j) = e^{iq \cdot R} u_q(j)$, where the $u_q(j)$ are strictly periodic in both real and momentum space and obey

$$\tilde{H}_q \begin{pmatrix} u_q(1) \\ u_q(2) \end{pmatrix} = \tilde{E}_q \begin{pmatrix} u_q(1) \\ u_q(2) \end{pmatrix}, \qquad (2)$$

where

$$\tilde{H}_q = 2\tilde{t} \begin{pmatrix} \cos q_x & m e^{-iq_y} \\ m e^{iq_y} & -\cos q_x \end{pmatrix} \qquad (3)$$

with $m \equiv \tilde{t}_\perp / 2\tilde{t}$. Since Eq. (2) is a two-dimensional Dirac equation, i.e., $\tilde{H}_q^2$ is diagonal, we immediately obtain the eigenvalues $\tilde{E}_q = \pm 2\tilde{t} \sqrt{\cos^2 q_x + m^2}$. The coupling between the tight-binding chains hence induces an energy gap of magnitude $2\tilde{t}_\perp$. We now fill the lower band twice, once with up-spin electrons, and once with down-spin electrons; the resulting Slater determinant $|\psi_{\text{SD}}\rangle$ is obviously a spin singlet. The spin liquid trial wave function for the Heisenberg ladder with $J_\perp/J = \tilde{t}_\perp/\tilde{t}$ is obtained by eliminating all the doubly occupied sites via Gutzwiller projection

$$|\psi_{\text{trial}}\rangle = \mathcal{P}_G |\psi_{\text{SD}}\rangle. \qquad (4)$$

Since the Gutzwiller projector $\mathcal{P}_G$ commutes with the total spin operator, $|\psi_{\text{trial}}\rangle$ is also a singlet. This trial wave function is as accurate an approximation as the Haldane-Shastry state[10] for the one-dimensional Heisenberg chain in the weak coupling limit $J_\perp/J = 0$, and exact in the strong coupling limit $J_\perp/J \to \infty$; the approximation has its worst point at isotropic coupling (see Table I).[11,12]

There are essentially two ways of constructing spinon and holon excitations for spin liquids (they are obtained from each other by annihilating or creating an electron on the spinon or holon site). The first one is Anderson's projection technique:[13] inhomogenities in both spin and charge created
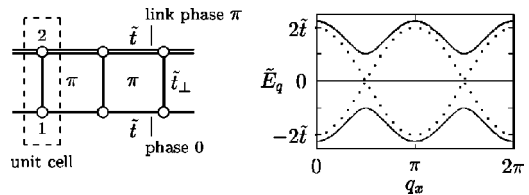

FIG. 1. Flux band structure of a tight binding ladder with flux π per plaquet for $\tilde{t}_\perp/\tilde{t} = 0$ (dotted lines) and $\tilde{t}_\perp/\tilde{t} = 1$ (solid lines). The energy gap is given by $2\tilde{t}_\perp$.

TABLE I. Energy expectation values and nearest neighbor spin correlations for the spin liquid trial wave functions in comparison with the exact ground states of a 2×10 Heisenberg ladder with periodic boundary conditions, as well as overlaps between trial wave functions and exact ground states. Throughout this article, all energies quoted are in units of $\max(J_\perp,J)$. The boundary phase before Gutzwiller projection has been 0.

| $J_\perp/J$ | E_{tot} | | % over- | | $\langle\vec{S}_i\vec{S}_j\rangle_\parallel$ | | $\langle\vec{S}_i\vec{S}_j\rangle_\perp$ | |
	exact	trial	off	lap	exact	trial	exact	trial
0	-9.031	-9.015	0.2	0.997	-0.452	-0.451	0.000	0.000
0.1	-9.062	-9.024	0.4	0.986	-0.450	-0.451	-0.062	-0.011
0.2	-9.155	-9.073	0.9	0.969	-0.445	-0.449	-0.123	-0.045
0.5	-9.755	-9.568	1.9	0.952	-0.420	-0.413	-0.269	-0.263
1	-11.577	-11.346	2.0	0.941	-0.354	-0.302	-0.450	-0.530
2	-8.594	-8.444	1.8	0.957	-0.222	-0.143	-0.638	-0.702
5	-7.664	-7.594	0.9	0.981	-0.085	-0.029	-0.732	-0.748
10	-7.539	-7.513	0.3	0.993	-0.040	-0.007	-0.746	-0.750
∞	-7.500	-7.500	0.0	1.000	0.000	0.000	-0.750	-0.750

before Gutzwiller projection yield inhomogenities in spin only after projection. Anderson writes a state with two spinons localized at sites i and j

$$|\psi_{i\uparrow,j\downarrow}\rangle = \mathcal{P}_G c^\dagger_{i\uparrow}c_{j\uparrow}|\psi_{SD}\rangle. \tag{5}$$

In the case of the ladder, however, the spinons are not free particles, but bound into pairs by a linear confinement force.[14] To obtain the magnon trial wave function

$$|\psi_{\text{magnon}}(k)\rangle = \sum_{i,j}\phi_{i,j}(k)|\psi_{i\uparrow,j\uparrow}\rangle, \tag{6}$$

a hence nontrivial internal wave function $\phi_{i,j}(k)$ for the spinon-spinon bound state is required.

It is therefore expedient to use the second method, which has been successful in describing the fractionally charged
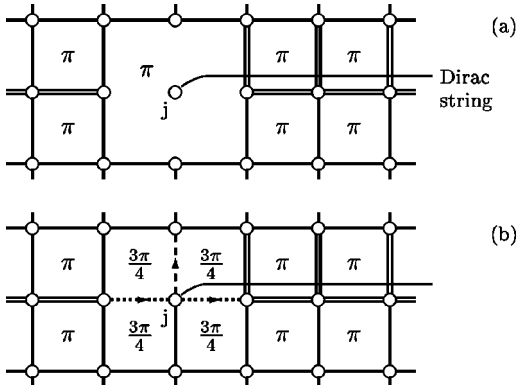
FIG. 2. Magnetic tight-binding configuration for a holon of the chiral spin liquid (a) as proposed by Rokhsar (cutting all the links to a given site and adjusting the flux according to Rokhsar's loop rules generates a stationary holon or s holon) or (b) by combining this flux adjustment with Anderson's projection technique (only the phases of the hopping parameters around the holon site are adjusted).

TABLE II. Energy expectation values including individual contributions to the kinetic energy from chains and rungs and overlaps for the trial wave functions describing holon-spinon bound states in comparison with the exact one hole eigenstates for a periodic 2×8 ladder with $t_\perp/J_\perp=t/J=1$ for three ratios $J_\perp/J$. The energies are in units of $\max(J_\perp,J)$; the transverse momentum is always $k_y=0$, as a $k_y=\pi$ state corresponds to a hole (holon-spinon bound state) plus a magnon (spinon-spinon bound state). The trial wave functions are given by $|\psi_{\text{trial}}\rangle = \mathcal{N}(|\psi_+\rangle + a|\psi_-\rangle)$, where $|\psi_+\rangle$ is the trial wave function constructed with the flux configuration shown in Fig. 3(b) and $|\psi_-\rangle$ its P or T conjugate. The boundary phase for the flux band structure before projection has been 0 for the chain containing the holon, and π for the other chain.

| $J_\perp/J$ | k_x/π | E_{tot} | | % over- | | $E_{t_\parallel}$ | | $E_{t_\perp}$ | |
		exact	trial	off	lap	exact	trial	exact	trial
	0	-7.40	-6.88	7.1	0.560	-1.51	-0.45	0.04	0.06
	1/4	-7.92	-7.83	1.1	0.975	-1.63	-1.62	0.01	0.04
0.2	1/2	-8.17	-8.07	1.2	0.952	-1.89	-1.87	-0.02	0.04
	3/4	-7.59	-7.22	4.9	0.540	-1.49	-1.01	-0.10	0.04
	1	-7.21	-5.98	17	0.000	-1.55	0.45	-0.09	0.06
	0	-9.04	-8.36	7.6	0.776	-0.88	0.42	-0.38	-0.89
	1/4	-9.46	-8.91	5.8	0.849	-1.21	-0.47	-0.45	-0.67
1	1/2	-9.89	-9.52	3.7	0.894	-1.54	-1.23	-0.59	-0.57
	3/4	-9.84	-9.47	3.8	0.917	-1.35	-0.99	-0.81	-0.70
	1	-9.67	-9.19	5.0	0.917	-1.07	-0.42	-0.98	-0.89
	0	-6.20	-6.12	1.3	0.981	0.13	0.19	-0.96	-1.00
	1/4	-6.25	-6.18	1.1	0.984	0.08	0.11	-0.97	-0.99
5	1/2	-6.38	-6.33	0.7	0.987	-0.05	-0.04	-0.98	-0.99
	3/4	-6.50	-6.46	0.7	0.988	-0.17	-0.16	-0.99	-0.99
	1	-6.55	-6.50	0.7	0.988	-0.21	-0.19	-1.00	-1.00

solitons in polyacytelene:[15] Rokhsar[16] constructs elementary excitations of spin liquids via localized midgap states, which are either occupied by a single electron (spinon) or left unoccupied (holon). The topology of the ladder dictates that midgap states can only be created in pairs, which implies that we automatically obtain spinon-spinon (or holon) bound states rather then isolated spinons (or holons). This general observation, however, leaves us still with a large number of possible choices for the midgap states; most constructions
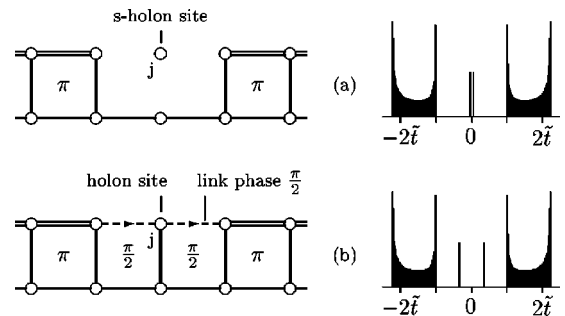
FIG. 3. Magnetic tight-binding configurations and the corresponding density of states for the ladder with a spinon bound to (a) a stationary s holon or (b) a mobile holon before Gutzwiller projection. Only the latter flux configuration violates P and T.

TABLE III. As in Table II, but now for spinon-spinon bound states (magnons) with $k_y = \pi$ obtained by creation of an electron at the holon site. For $k_x = 0$, spinetto-spinon bound states yield better trial wave functions (the energy is only 0.4% off at $J_\perp/J = 0.2$, and 2.7% off at $J_\perp/J = 1$).

$J_\perp/J$ k_x/π	E_{tot} exact	trial	% off	over-lap	$E_{J_\parallel}$ exact	trial	$E_{J_\perp}$ exact	trial
0	-6.31	-5.86	7.2	0.844	-6.23	-5.70	-0.09	-0.16
1/4	-6.16	-6.13	0.4	0.994	-6.08	-6.08	-0.08	-0.05
0.2 1/2	-5.60	-5.56	0.7	0.983	-5.44	-5.47	-0.16	-0.09
3/4	-5.96	-5.89	1.1	0.961	-5.70	-5.73	-0.26	-0.16
1	-6.98	-6.94	0.6	0.990	-6.73	-6.76	-0.25	-0.18
0	-7.33	-6.99	4.6	0.885	-5.11	-3.79	-2.22	-3.20
1/4	-7.24	-7.01	3.1	0.927	-4.78	-3.74	-2.46	-3.27
1 1/2	-7.28	-7.13	2.0	0.961	-4.11	-3.44	-3.18	-3.69
3/4	-7.96	-7.87	1.1	0.981	-4.56	-4.22	-3.39	-3.65
1	-8.73	-8.63	1.1	0.982	-5.68	-5.36	-3.04	-3.27
0	-4.91	-4.86	1.0	0.986	-0.04	0.12	-4.88	-4.99
1/4	-4.96	-4.92	0.9	0.988	-0.07	0.07	-4.89	-4.98
5 1/2	-5.09	-5.06	0.6	0.991	-0.18	-0.08	-4.91	-4.97
3/4	-5.24	-5.20	0.7	0.991	-0.34	-0.23	-4.90	-4.98
1	-5.31	-5.26	0.9	0.987	-0.43	-0.28	-4.88	-4.99

yield satisfactory magnon, but only very few acceptable hole trial wave functions. To identify those, let us step back and take a broader view.

The spin liquid proposed above is, in fact, a special case of the Kalmeyer-Laughlin chiral spin liquid,[17] obtained by imposing a periodic boundary condition with a periodicity of only two lattice spacings in the y direction. This chiral spin liquid may be generated from a tight-binding lattice with flux π per plaquette and hopping magnitudes $\tilde{t}$ and $\frac{1}{2}\tilde{t}_\perp$ in x and y directions, respectively; the P and T violating diagonal hopping elements, which are otherwise required to open an energy gap, cancel due to the boundary condition.

Spinons and holons for the chiral spin liquid may be constructed via Anderson's method or via midgap states; Rokhsar creates a midgap state in the flux band structure before projection by cutting all the links to a given site and adjusting the flux according to his loop rules, which require that the kinetic energy on the loops around each plaquette is minimal [see Fig. 2(a)]. The resulting holon is not nearly as mobile as Anderson's, but optimal with regard to the magnetic energy; it adequately describes stationary charge excitations. We call it a stationary holon or "s holon." To obtain the generic and mobile holon, we create a midgap state by adjusting the flux according to Rokhsar's procedure (i.e., we create a defect of flux π around the holon site) without cutting any links (i.e., we adjust the hopping phases without adjusting the magnitudes), and then project such that this site is unoccupied [see Fig. 2(b)]. This holon is equivalent to Anderson's in the case of the chiral spin liquid, but more generally applicable.

The flux configurations used to construct s-holon-spinon and holon-spinon bound states for the ladder are shown in

TABLE IV. As in Table II, but now with $J_\perp/J = t_\perp/t = 1$ for different ratios t/J. The momentum is $(k_x, k_y) = (\pi/2, 0)$, which corresponds to the ground state of the 2×8 ladder with periodic boundary conditions. For $t/J \leq 0.5$, data for s-holon-spinon bound states are shown as well, marked with an asterisk; these adequately describe stationary holes ($t = 0$).

t/J	E_{tot} exact	trial	% off	over-lap	$E_{t_\parallel}$ exact	trial	$E_{t_\perp}$ exact	trial
0	-8.18	-7.88	3.7	0.888	0.00	0.00	0.00	0.00
*		-8.00	2.2	0.938		0.00		0.00
0.2	-8.37	-8.14	2.8	0.927	-0.17	-0.14	-0.10	-0.16
*		-8.12	2.9	0.923		0.00		-0.13
0.5	-8.87	-8.63	2.7	0.929	-0.65	-0.53	-0.28	-0.33
*		-8.32	6.3	0.862		0.00		-0.32
1	-9.89	-9.52	3.7	0.894	-1.54	-1.23	-0.59	-0.57
2	-12.1	-11.3	6.5	0.797	-3.36	-2.63	-1.31	-1.02
5	-19.5	-16.8	14	0.585	-8.86	-6.84	-3.76	-2.33

Figs. 3(a) and 3(b). In the case of the s holon, we cut all the links to a given site; as the topology of the ladder does not provide a context for a flux adjustment around this site, we obtain a second midgap state, and hence a spinon, localized nearby. This trial wave function describes a stationary hole. To construct a mobile hole, we create the midgap states by only adjusting the flux, and project such that the holon site is unoccupied; as we are creating two rather than one midgap state, we remove flux $\pi/2$ from each neighboring plaquet.[18] The flux configuration now violates P and T, and the holon-spinon bound state carries a chirality quantum number, which is $+$ for the configuration shown in Fig. 3(b), and $-$ for its complex conjugate; states of opposite chiralities map into each other under P or T. The final trial wave functions for the hole is a linear superposition of the holon-spinon bound states of both chiralities at each momentum
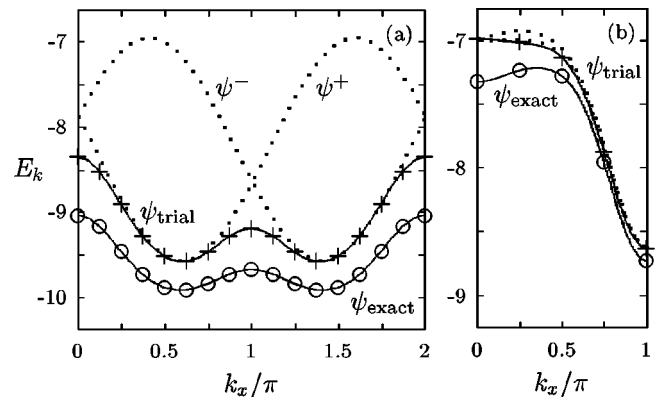

FIG. 4. Dispersions for a single (a) hole (holon-spinon bound state) and (b) magnon (spinon-spinon bound state) as predicted by the spin liquid proposed here in comparison with the exact dispersions for a 2×8 ladder with $J_\perp = J = t_\perp = t = 1$ (see Tables II and III) and periodic (or antiperiodic) boundary conditions. The dotted lines correspond to the individual $+$ and $-$ chirality trial wave functions [generated from the tight-binding configuration shown in Fig. 3(b) and its P or T conjugate, respectively].

$$|\psi_{\text{hole}}(k)\rangle = \mathcal{N}\sum_j e^{ir_jk}(|\psi_j^+\rangle + a(k)|\psi_j^-\rangle), \qquad (7)$$

where r_j is the holon cordinate and $a(k)$ is a variational parameter. Magnons or spinon-spinon bound states are obtained from the holon-spinon bound states by creating an electron at the holon site, which forms a spin triplet with the spinon bound to it.

Numerical comparisons of the trial wave functions for holes and magnons with the exact eigenstates are presented in Fig. 4 and in Tables II–IV. The hole trial wave functions for $t_\perp/J_\perp = t/J = 1$ (see Table II) are excellent at stong coupling, and less accurate at isotropic coupling; for weak coupling, they are excellent only at momenta close to the one-hole ground states, as there is a large amplitude to find a hole and a magnon rather than just a hole at other momenta. The holon-spinon bound state wave functions adequately describe the hole when t and J are comparable (see Table IV); for $t \ll J$, the s-holon-spinon bound state is more appropriate, while holes with sufficiently large t are detrimental to antiferromagnetic correlations and eventually destabilize the spin liquid.[19] The trial wave functions for the magnons are generally satisfactory (see Table III).

The P and T violation of the localized holon-spinon bound states, or the appearance of a chirality quantum number, is a physical property of the system; any real trial wave function for the localized hole would yield a dispersion $E_k \propto \cos(k_x)$, and thus be inconsistent with the dispersion obtained by exact diagonalization [bottom curve in Fig. 4(a)].[20] The chirality quantum number is a manifestation of the fractional statistics[21] of spinons and holons[17,22] in dimensions greater than one; it determines the sign of the statistical phases acquired as they encircle each other.

A state with two holons is a superposition of two holon-spinon bound states and a holon-holon bound state, which are connected through virtual processes of annihilation or creation of spinon pairs in singlet configurations. These processes mediate an effective pairing force between holes.

In conclusion, we have shown that the Heisenberg or lightly doped t-J ladder can be adequately described by a Gutzwiller-type spin liquid generated from magnetic bands. This enabled us to construct explicit wave functions for spinons and holons, which are confined in pairs to form magnons or holes. Possibly the most striking result is the P and T violation of the hole, which indicates that spinons and holons carry fractional statistics.

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[11] For a finite ladder with a periodic boundary condition, the trial wave function depends slightly on the boundary phase of the flux ladder before Gutzwiller projection; it appears that the best trial wave function is obtained when the kinetic energy before projection is minimal.

[12] The trial wave function can be improved by optimizing $\tilde{t}_\perp/\tilde{t}$ for each value $J_\perp/J$; at isotropic coupling $J_\perp/J = 1$, for example, the energy is only 1.35% off if one assumes $\tilde{t}_\perp/\tilde{t} = 0.84$.

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