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The 1D Ising model and the topological phase of the Kitaev chain

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ABSTRACT

It has been noted that the Kitaev chain, a p-wave superconductor with nearest-neighbor pairing amplitude equal to the hopping term $\Delta = t$, and chemical potential $\mu = 0$, can be mapped into a nearest neighbor Ising model via a Jordan–Wigner transformation. Starting from the explicit eigenstates of the open Kitaev chain in terms of the original fermion operators, we elaborate that despite this formal equivalence the models are physically inequivalent, and show how the topological phase in the Kitaev chain maps into conventional order in the Ising model.

1. Introduction

A few years ago, in the lovely town of Trieste, one of us engaged in a bet with a highly esteemed colleague. The issue was whether fermions were physically distinguishable from hard-core bosons in one dimension (1D), or whether they would only be different descriptions of the same particles which could be obtained from each other through gauge transformations. That they are distinguishable was settled with the example of two particles on a ring, where fermions with periodic boundary conditions (PBCs) are equivalent to hard-core bosons with anti-periodic boundary conditions (anti-PBCs) and vice versa. Delivery of the espresso at stake was promised thereafter.

In this paper, we provide a much more compelling example of the difference between fermions and hard-core bosons in 1D. We will investigate two simple Hamiltonians, one formulated in terms of fermions, the other in terms of hard-core bosons realized through spin-flip operators acting on a

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Hilbert space with spin $s = \frac{1}{2}$. Written in a basis of the appropriate operators, the entire spectrum of eigenstates including their energy eigenvalues is equivalent for both models. There is, however, a key difference. The states in the fermionic model are topologically ordered [1–11], while the spin model is conventionally ordered in the sense of a spontaneously broken symmetry.

To be more precise, we investigate the eigenstates of the Kitaev chain [2,6], a one-dimensional p-wave superconductor with nearest-neighbor pairing amplitude equal to the hopping term $\Delta = t$, and chemical potential $\mu = 0$, with open boundary conditions (OBCs). While those are well known in terms of the Majorana fermion [12,13] operators introduced by Kitaev, we show that they take a very simple yet somewhat surprising form in terms of the fermion operators which span the Hilbert space of the model. We find that both the states and the Hamiltonian equivalent to those of an Ising model, with just one crucial difference: The spinless fermion creation and annihilation operators in the Kitaev model are replaced by bosonic spin flip operators.

This result can of course be anticipated from the widely appreciated mapping of the Kitaev model onto the Ising model via a simple Jordan–Wigner transformation [14–16]. The non-trivial part of our finding, however, stems from the non-locality of the Jordan–Wigner transformation, which triggers a dichotomy of a formal equivalence and a physical inequivalence. The ground state of both models is two-fold degenerate, but the physics of the order displayed could hardly be more different. In the Ising model, the $\mathbb{Z}_2$ spin reflection symmetry is spontaneously broken, while the degeneracy in the Kitaev chain stems from the Majorana zero mode (i.e., the isolated Majorana fermions at the ends of the chain) characteristic of the symmetry-protected topological (SPT) phase.

2. The Kitaev chain

Kitaev [2] studied a lattice model of a p-wave superconductor in 1D,

$$H = -\mu \sum_x c_x^\dagger c_x - \sum_x (t c_x^\dagger c_{x+1} + \Delta e^{\mathrm{i}\phi} c_x c_{x+1} + \text{H.c.}), \tag{1}$$

where μ is the chemical potential, $t \geq 0$ the nearest-neighbor hopping, and $\Delta \geq 0$ the p-wave pairing amplitude. Since the model is particle hole symmetric, we may restrict our attention to the case $\mu \leq 0$; since the order parameter phase ϕ can be absorbed into the definition of c_x and $c_x^\dagger$, we may set $\phi = 0$. Kitaev showed that this model has two phases: a topologically trivial strong-coupling phase for $\mu < -2t$, and a topologically non-trivial weak-coupling phase for $\mu > -2t$. To understand this, consider first PBCs and diagonalize (1) in k-space with a standard Bogoliubov transformation [17]. This yields the quasiparticle spectrum $\epsilon_k = \sqrt{\xi_k^2 + \Delta_k^2}$, where $\xi_k = -2t\cos k - \mu$, $\Delta_k = 2\Delta \sin k$, and we have set the lattice constant to unity. The topological universality class can change only where the gap closes, which is for $\mu = -t$ at $k = 0$. To illustrate the two topologically distinct phases, Kitaev turned to a chain with OBCs, and rewrote the fermion operators in terms of Majorana fermion operators,

$$\gamma_{A,x} = -\mathrm{i}c_x + \mathrm{i}c_x^\dagger, \qquad \gamma_{B,x} = c_x + c_x^\dagger. \tag{2}$$

This yields

$$\begin{aligned} H = &-\frac{\mu}{2}\sum_{x=1}^{N}(1 + \mathrm{i}\gamma_{B,x}\gamma_{A,x}) \\ &-\frac{\mathrm{i}}{2}\sum_{x=1}^{N-1}(\Delta + t)\gamma_{B,x}\gamma_{A,x+1} + (\Delta - t)\gamma_{A,x}\gamma_{B,x+1}). \end{aligned} \tag{3}$$

The trivial phase is illustrated by the case $t = \Delta = 0$, $\mu < 0$, in which Majorana fermions are paired on the same site, and all the sites are unoccupied. The topologically non-trivial phase is illustrated by the case $\mu = 0$, $t = \Delta > 0$, in which Majorana fermions are paired on neighboring sites. This yields an unpaired Majorana fermion at each end, or a Majorana zero mode formed by combining these two into a fermion state, which can be occupied or unoccupied. For OBCs, the Majorana fermions on the boundaries are a characteristic feature of the topologically non-trivial phase. (For PBCs, a characteristic feature is the fermion parity of the ground state, which is even (i.e., the state consists only of terms with an even numbers of fermions) in the trivial phase, but odd in the topologically non-trivial phase. This simple observation seems to have been overlooked in some of the literature reviewed by Alicea [6].)

In this paper, we further investigate the case $\mu = 0$, $t = \Delta = 1$, a model we refer to as the Kitaev chain. The Hamiltonian may be written

$$H_{\text{Kitaev}} = -\sum_{x=1}^{N-1}(c_{x+1}^\dagger - c_{x+1})(c_x^\dagger + c_x) \tag{4}$$

$$= -\mathrm{i}\sum_{x=1}^{N-1}\gamma_{B,x}\gamma_{A,x+1} = \sum_{x=1}^{N-1}(2d_x^\dagger d_x - 1) \tag{5}$$

where

$$2d_x^\dagger = \gamma_{B,x} + \mathrm{i}\gamma_{A,x+1} = c_{x+1} - c_{x+1}^\dagger + c_x + c_x^\dagger. \tag{6}$$

Closing the OBCs would add another term $(2d_0^\dagger d_0 - 1)$ to (5), where

$$2d_0^\dagger = \gamma_{B,N} + \mathrm{i}\gamma_{A,1} = c_1 - c_1^\dagger + c_N + c_N^\dagger. \tag{7}$$

One ground state of (5) is obviously given by the vacuum defined by the operators d_x, $x = 0, 1, 2, \ldots, N-1$, and the other is obtained by acting with $d_0^\dagger$ on this vacuum state. All the other eigenstates are trivially obtained by creation of various $d_x^\dagger$ excitations.

3. Eigenstates in terms of local fermion operators

It is not obvious, however, how the eigenstates look like in terms of the original, local fermion operators c_x and $c_x^\dagger$. A conceptually straightforward way to obtain them is to choose two seed states, one with even and one with odd fermion parity, like $|0\rangle$ and $c_1^\dagger\,|0\rangle$ (where $c_x\,|0\rangle = 0\ \forall x$), and project them with

$$\mathcal{P} \equiv \prod_{x=1}^{N-1} d_x d_x^\dagger \tag{8}$$

onto ground states of (5). Note that since

$$2d_x d_x^\dagger = (c_{x+1}^\dagger - c_{x+1})(c_x^\dagger + c_x) + 1 \tag{9}$$

preserves fermion parity, the projected eigenstates inherit the fermion parity of the seed states. For the (unnormalized) ground states we find (by building up the states site by site and carrying out the algebra)

$$\left|\psi_0^{\substack{\text{even}\\\text{odd}}}\right\rangle = \prod_{x=1}^{N}(1 + c_x^\dagger)\bigg|_{M\,\substack{\text{even}\\\text{odd}}} |0\rangle\,, \tag{10}$$

where M denotes the number of fermion operators in the preceding product, which we project onto even or odd numbers. We choose a convention where products acting on kets are build up from right to left,

$$\prod_{x=1}^{N}(1 + c_x^\dagger) \equiv (1 + c_N^\dagger)\cdot\cdots\cdot(1 + c_2^\dagger)(1 + c_1^\dagger). \tag{11}$$

For our purposes, it is convenient to introduce an alternative basis for the two degenerate ground states,

$$\left|\psi_0^{\pm}\right\rangle = \prod_{x=1}^{N}(1 \pm c_x^\dagger) = \left|\psi_0^{\text{even}}\right\rangle \pm \left|\psi_0^{\text{odd}}\right\rangle. \tag{12}$$

We obtain the excited states

$$d_x^\dagger \left|\psi_0^\pm\right\rangle = (d_x^\dagger + d_x)\left|\psi_0^\pm\right\rangle = (c_x + c_x^\dagger)\left|\psi_0^\pm\right\rangle$$
$$= \pm \prod_{y=x+1}^{N}(1 \mp c_y^\dagger)\prod_{y=1}^{x}(1 \pm c_y^\dagger)\left|0\right\rangle . \tag{13}$$

These are just domain walls between the two ground states $\left|\psi_0^+\right\rangle$ and $\left|\psi_0^-\right\rangle$. Trivially, we could have obtained this result also with

$$d_x^\dagger \left|\psi_0^\pm\right\rangle = (d_x^\dagger - d_x)\left|\psi_0^\pm\right\rangle = (c_{x+1} - c_{x+1}^\dagger)\left|\psi_0^\pm\right\rangle . \tag{14}$$

The terms we sum over in the Hamiltonian (4) hence first create a domain wall between sites x and $x+1$ from one side, and then annihilate it from the other side.

4. Correspondence with the 1D Ising model

Since the operators $d_x^\dagger$ commute for different sites x, we can immediately write down all the eigenstates of (4),

$$\left|\sigma_1\sigma_2\ldots\sigma_N\right\rangle \equiv \prod_{x=1}^{N}(1+\sigma_x c_x^\dagger)\left|0\right\rangle , \tag{15}$$

where $\sigma_x = \pm 1$. The corresponding energy eigenvalues, defined by

$$H\left|\sigma_1\sigma_2\ldots\sigma_N\right\rangle = E_{\sigma_1\sigma_2\ldots\sigma_N}\left|\sigma_1\sigma_2\ldots\sigma_N\right\rangle , \tag{16}$$

are given by

$$E_{\sigma_1\sigma_2\ldots\sigma_N} = -\sum_{x=1}^{N-1}\sigma_x\sigma_{x+1}. \tag{17}$$

The last two equation describe an Ising model in 1D. We have hence shown that there is a formal equivalence between the eigenstates and energy eigenvalues of the Kitaev model and the Ising model.

We can make the correspondence more explicit by choosing the Ising spins in the x-direction, while the quantization axis remains the z-axis. Then the Ising model eigenstates corresponding to (15) are given by

$$\left|\sigma_1\sigma_2\ldots\sigma_N\right\rangle \equiv \prod_{x=1}^{N}(1+\sigma_x S_x^+)\left|\downarrow\right\rangle^{\otimes N} , \tag{18}$$

where $\left|\downarrow\right\rangle^{\otimes N}$ denotes a state with all spins $\downarrow$, and S_x^+ flips a spin at site x, $S_x^+\left|\downarrow\right\rangle = \left|\uparrow\right\rangle$. The corresponding Ising Hamiltonian is

$$H_{\text{Ising}} = -4\sum_{x=1}^{N-1}S_{x+1}^{\mathrm{x}}S_x^{\mathrm{x}}$$
$$= -\sum_{x=1}^{N-1}(S_{x+1}^+ + S_{x+1}^-)(S_x^+ + S_x^-). \tag{19}$$

Note that as compared to (4), the sign in the first factor in (19) is reversed. This is simply a consequence of having substituted the fermion operators $c^\dagger$ and c by the (hard-core) boson operators S^+ and S^-. If the site $x+1$ is occupied in the fermionic model, commuting the factor $(c_x^\dagger + c_x)$ through it in the state vector we act on will give us an extra minus sign, which is not present in the bosonic model.

5. Conventional order vs. topological phase

Irrespective of the formal equivalence of the two models in the sense elaborated above, the physical order displayed by them is highly distinct. The Ising model displays conventional order, and the $\mathbb{Z}_2$ spin reflection symmetry $S^x \to -S^x$ is spontaneously broken. There are no local matrix elements between the two ground states, as one would have to flip all the spins on the entire chain to transform one state into the other. The Kitaev model displays an SPT phase, and the two-fold ground state degeneracy is due the Majorana zero-mode, i.e., the mode described by the fermion d_0, $d_0^\dagger$, which consists of the two Majorana fermions $\gamma_{A,0}$ and $\gamma_{B,N}$ at the end of the chain. In equations,

$$\begin{aligned}(d_0^\dagger - d_0)\left|\psi_0^\pm\right\rangle &= (c_1 - c_1^\dagger)\left|\psi_0^\pm\right\rangle = \pm\left|\psi_0^\mp\right\rangle,\\ (d_0^\dagger + d_0)\left|\psi_0^\pm\right\rangle &= (c_N + c_N^\dagger)\left|\psi_0^\pm\right\rangle = \pm\left|\psi_0^\pm\right\rangle,\end{aligned} \tag{20}$$

and hence

$$d_0\left|\psi_0^{\text{odd}}\right\rangle = 0, \qquad d_0^\dagger\left|\psi_0^{\text{odd}}\right\rangle = \left|\psi_0^{\text{even}}\right\rangle. \tag{21}$$

The only physical difference between $\left|\psi_0^{\text{odd}}\right\rangle$ and $\left|\psi_0^{\text{even}}\right\rangle$ is the occupation of the Majorana-zero mode, which can easily be altered by creation and annihilation of fermions at the boundaries. These two ground states differ in their fermion parity, which is only a global, but not a local property.

Interestingly, if we diagonalize both models numerically, and set up Hilbert space conventions in which at each site x for the Kitaev model empty (i.e., $|0\rangle$) and occupied (i.e., $c_x^\dagger\,|0\rangle$), and for the Ising model $\downarrow$-spin (i.e., $|\downarrow\rangle$) and $\uparrow$-spin (i.e., $S_x^+\,|\downarrow\rangle$), by 0 and 1, the eigenstates of (4) and (19) would be identical.

This is not to say that the correlations of both models are identical, or even related. A correlation function is, like an order parameter, an expectation value of an operator (or product of operators) in a ground state. While we can easily measure the Ising spin $2S_x^x = S_x^+ + S_x^-$ on any site x in an eigenstate of (19),

$$\langle\sigma_1\sigma_2\ldots\sigma_N|\,S_x^+ + S_x^-\,|\sigma_1\sigma_2\ldots\sigma_N\rangle = \sigma_x, \tag{22}$$

there is no corresponding, local operator to measure σ_x in an eigenstate of the Kitaev model (4). In particular,

$$\langle\sigma_1\sigma_2\ldots\sigma_N|\,c_x^\dagger + c_x\,|\sigma_1\sigma_2\ldots\sigma_N\rangle = 0 \quad \forall\, x < N. \tag{23}$$

It is worth pointing out, however, that the entanglement spectrum [18,19], is identical for the ground states of both models. The comparison illustrates that not only the nature of the cut itself, but also the (non-)locality of the basis (i.e., fermions vs. bosonic spin flips operators) in which the reduced density matrix is formulated, must be taken into account when interpreting the entanglement spectrum.

6. Reconciliation with the BCS pairing wave function

We now wish to reconcile our ground state wave function (10) for the Kitaev's p-wave superconductor (4) with the conventional form of a BCS wave function in position space. To begin with, let us take another look at our wave function. As we close the OBCs by adding a term $(2d_0^\dagger d_0 - 1)$ to (5), the ground state becomes non-degenerate and is given by $\left|\psi_0^{\text{odd}}\right\rangle$ (see (21)). Note that if we reinstate the phase ϕ in (1) which we absorbed into the definition of $c_x^\dagger$ and c_x, we may write the ground state as

$$\left|\psi_0^{\text{odd}}(\phi)\right\rangle = \prod_{x=1}^{N}(1 + e^{-\frac{i}{2}\phi}c_x^\dagger)\Big|_{M\text{ odd}}|0\rangle\,, \tag{24}$$

$$= \pm\prod_{x=1}^{N}(1 \pm e^{-\frac{i}{2}\phi}c_x^\dagger)\Big|_{M\text{ odd}}|0\rangle\,. \tag{25}$$

At first sight, this may look like a BCS wave function for the condensation of single fermions rather than Cooper pairs. This is of course misguided, as there is no order parameter associated with the phase between the two terms in (24). At the same time, it does not look much like the wave function of a superconductor, and does not allow us to read off the Cooper pair wave function directly. (On a side note, (24) shows that a rotation of the superconducting order parameter phase in (1) maps onto a rotation of the Ising spin axis in the xy-plane in (19).)

To obtain the Cooper pair wave function, we go back to the Kitaev Hamiltonian (4), and solve it via a standard Bogoliubov transformation in momentum space. This yields

$$|\psi_0\rangle = \prod_{0<k<\pi} (u_k + v_k c_k^\dagger c_{-k}^\dagger) \cdot c_{k=0}^\dagger |0\rangle \,, \tag{26}$$

where the product extends over all discrete $k = \frac{2\pi}{N} n$ (with n integer) in the specified interval, $u_k = \sin\frac{k}{2}$, and $v_k = -\mathrm{i}\cos\frac{k}{2}$. Leaving aside the overall normalization, we may rewrite (26) as (see e.g. [20], App. A)

$$|\psi_0\rangle = \exp(b^\dagger) \cdot c_{k=0}^\dagger |0\rangle \,, \tag{27}$$

where

$$b^\dagger = \sum_{0<k<\pi} \frac{v_k}{u_k} c_k^\dagger c_{-k}^\dagger \tag{28}$$

creates a Cooper pair. Transforming this into position space, we obtain

$$b^\dagger = \sum_{x>x'} \varphi_{x-x'} c_x^\dagger c_{x'}^\dagger \tag{29}$$

with

$$\varphi_{x-x'} = \frac{1}{N} \sum_{k\neq 0} \frac{v_k}{u_k} e^{\mathrm{i}k(x-x')} = 1 - \frac{2(x-x')}{N}, \tag{30}$$

where we have evaluated the sum for $0 < x - x' < N$ using (see e.g. [21], App. B)

$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^n}{\eta_\alpha - 1} = \frac{N+1}{2} - n, \quad \eta_\alpha \equiv e^{\mathrm{i}\frac{2\pi}{N}\alpha}, \tag{31}$$

which holds for $1 \le n \le N$.

The analysis presented so far implies that (24) (with $\phi = 0$) and (27) with (29) and (30) are equivalent. As this is not obvious to the eye, we now show it explicitly by comparing terms with the same number of fermions M in

$$\exp(b^\dagger) \cdot c_{k=0}^\dagger |0\rangle \quad \text{and} \quad \prod_{x=1}^{N} (1 + c_x^\dagger)\Big|_{M\ \text{odd}} |0\rangle \,.$$

Since

$$\prod_{x=1}^{N} (1 + c_x^\dagger)\Big|_{M} |0\rangle = \sum_{y_M > \cdots > y_2 > y_1} c_{y_M}^\dagger \ldots c_{y_2}^\dagger c_{y_1}^\dagger |0\rangle \,, \tag{32}$$

it is sufficient to show that

$$\langle 0| c_{y_1} c_{y_2} \ldots c_{y_M} \left(b^\dagger\right)^m \sum_{x_1} c_{x_1}^\dagger |0\rangle = m!, \tag{33}$$

where $m = (M-1)/2$ is the number of Cooper pairs, and $y_1 < y_2 < \cdots < y_M$. As (33) holds trivially for $M = 1$, all we have to show to complete the proof inductively is that

$$\begin{aligned} &\langle 0| c_{y_1} \ldots c_{y_M}\, b^\dagger \sum_{x_{M-2} > \cdots > x_1} c_{x_{M-2}}^\dagger \ldots c_{x_1}^\dagger |0\rangle \\ &\quad = \langle 0| c_{y_1} \ldots c_{y_M} \sum_{x_{M-2} > \cdots > x_1} c_{x_{M-2}}^\dagger \ldots c_{x_1}^\dagger\, b^\dagger |0\rangle = m \end{aligned} \tag{34}$$

holds for $M \geq 3$, $y_j < y_{j+1}$, and $b^\dagger$ given by (29) and (30). In evaluating (34), we first consider the contribution of the second term in (30). When we order all the site indices $x', x, x_1, \ldots, x_{M-2}$ in ascending order, let x' be number i' and x number i in the list. For a given y_j to contribute $-\frac{2}{N}y_j$ in (34), either x or x' has to be equal to y_j. For $x = y_j$, x' has to be equal to a smaller y, and hence all values $i' \in [1, j-1]$ will contribute with sign $(-1)^{j+i'+1}$. Similarly, for $x' = y_j$, all values $i \in [j+1, M]$ will contribute with sign $(-1)^{i+j+1}$. The overall contribution $\propto y_j$ is hence

$$-\frac{2}{N}y_j\left\{\sum_{i'=1}^{j-1}(-1)^{j+i'+1} - \sum_{i=j+1}^{M}(-1)^{i+j+1}\right\} = 0. \tag{35}$$

This leaves us with the first term in (30), which by a similar argument yields

$$\sum_{i>i'}^{M}(-1)^{i+i'+1} = m. \tag{36}$$

This completes the proof.

7. Implications of physical inequivalence

As mentioned above, it has been appreciated previously that the models (4) and (19) can be transformed into each other via a Jordan–Wigner-transformation [14–16]. The dichotomy of the formal equivalence and the physical inequivalence elaborated in this paper, however, has not been universally appreciated. For example, in the context of quantum spin chains or optical cavities designed to simulate them, it has been proposed to investigate Majorana fermions in bosonic chains [15,16]. Relying on the equivalence of (4) and (19) via Jordan–Wigner transformations, these studies implicitly assume that the Ising model is topologically non-trivial as well, and that a Majorana fermion zero-mode can hence be observed in bosonic models. We have shown here that this is not possible.

8. Conclusion

We have analyzed how a fermion model with non-trivial topological properties, the 1D p-wave superconductor studied by Kitaev, can (as far as eigenstates and their energies are concerned) be mapped into a boson model with conventional order, the 1D Ising model. This suggests that other models with topological order, such as Kitaev's toric code or honeycomb model in 2D [22,4], might have simpler, bosonic cousins with conventional order [23]. Inversely, reformulating certain bosonic models with conventional order due to a broken discrete symmetry, in terms of fermion operators, may provide a route to novel models with topological order.

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