ELSEVIER

15 September 1994

PHYSICS LETTERS B

Physics Letters B 336 (1994) 48–53

Microscopic formulation of the hierarchy of quantized Hall states

Martin Greiter [1,2]

School of Natural Sciences Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Received 12 May 1994
Editor: R. Gatto

Abstract

Explicit wave functions for the hierarchy of fractionally quantized Hall states are proposed, and a method for integrating out the quasiparticle coordinates in the spherical geometry is developed. Their energies and overlaps with the exact ground states for small numbers of particles with Coulomb interactions are found to be excellent. We then generalize the adiabatic transport argument of Arovas, Schrieffer, and Wilczek to evaluate quasiparticle charges and statistics, and show that none of the proposed states is the exact ground state of any model Hamiltonian with two-body interactions only.

1. Introduction

Most of our understanding of the fractionally quantized Hall effect is based on a highly original trial wave function [1] for the ground state at filling fractions $1/m$, where m is an odd integer. It describes an incompressible quantum fluid with fractionally charged excitations, and has been shown [2] to capture the correct universality classes at those fillings. The hierarchy [3-5] is an extension of this picture to other rational filling fractions, based on the idea that the quasiparticle excitations themselves condense into a Laughlin-Jastrow type fluid. This procedure can be iterated, and all odd denominator filling fractions appear within this picture, at various levels of the hierarchy.

The problem with the existing formulations of the hierarchy is that they only predict the corresponding universality classes, specified by certain quantum numbers, but do not provide us with explicit wave functions. It is the aim of the present letter to fill this gap. In the following chapter, we will propose such wave functions, develop a method for integrating out the quasiparticle coordinates, and present the results of extensive numerical studies, which provide strong evidence in support of our proposal. In the chapter thereafter, we obtain general formulas for the quasiparticle charges and statistics, and present preliminary results for energy gaps. We then present a simple proof that none of the proposed hierarchy states is the exact ground of any model Hamiltonian

[1] Research supported in parts by DOE grant DE-FG02-90ER40542.

[2] Present address: Theory Division, CERN, CH 1211 Geneva 23; greiter@cernvm.cern.ch.

SSDI 0370-2693(94)00957-0

containing two body interactions only, and conclude with comments on Jain's construction, double layer electron systems and paired Hall states.

2. Microscopic formulation

The general principles are most easily demonstrated by considering first a specific example. The simplest hierarchy state we can write down is a m parent state with a p daughter of quasi-holes, denoted by $[m,+p]$, where the + sign indicates that we have quasi-holes rather then quasi-electrons. For convenience, we work in the spherical geometry [3]. The wave function we wish to propose for this state is

$$\Psi_{[m,+p]}[u,v] = \int D[a,b] \prod_{k<l}^{N_1} (\bar{a}_k\bar{b}_l - \bar{a}_l\bar{b}_k)^p \prod_{k=1}^{N_1}\prod_{i=1}^{N} (a_k u_i - b_k v_i) \prod_{i<j}^{N} (u_i v_j - u_j v_i)^m \tag{1}$$

where (u_i, v_i) with $i = 1, \dots N$ are the spinor coordinates of the electrons on the sphere, (a_k, b_k) with $k = 1, \dots N_1$ are those of the quasiparticles, and $(\bar{a}_k, \bar{b}_k)$ their complex conjugates. The interpretation of (1) is straight forward: we take a Laughlin $1/m$-state for N electrons, create N_1 quasi-holes, weight the different quasi-hole configurations by a Jastrow factor taken to an even power p, and integrate the quasi-hole coordinates over the unit sphere.

To obtain a $[m,-p]$ state, we just need to replace the quasi-holes in (1) by quasi-electrons, that is,

$$\prod_{k=1}^{N_1}\prod_{i=1}^{N} (a_k u_i - b_k v_i) \to \prod_{k=1}^{N_1}\prod_{i=1}^{N} (\bar{b}_k \frac{\partial}{\partial u_i} - \bar{a}_k \frac{\partial}{\partial v_i}). \tag{2}$$

This wave function would, of course, not be of much practical use if we had no method to perform the quasiparticle integration. Fortunately, this integration is rather straight forward. Expressing the spinor coordinates for a given particle in terms of the polar and azimuthal angles ϕ and θ on the sphere, $a = \cos\frac{\theta}{2}\, e^{+\frac{i}{2}\phi}$ and $b = \sin\frac{\theta}{2}\, e^{-\frac{i}{2}\phi}$, and substituting $\int D(a,b) \equiv \frac{1}{4\pi}\int d\Omega$, we find

$$\int D(a,b)\, \bar{a}^{n'}\, \bar{b}^{2S'-n'} a^n\, b^{2S-n} = \frac{n!\,(2S-n)!}{(2S+1)!}\, \delta_{nn'}\, \delta_{2S,2S'}. \tag{3}$$

In any given polynomials in the present context, the integer $2S$ is given by the number of flux quanta through the sphere; it is the same for all terms in an expansion, and we may absorb the $(2S+1)!$ on the right of (3) in the overall normalization of the wave function. Thus we may, instead of performing the integration, replace $\bar{a} \to \partial_a$, $\bar{b} \to \partial_b$ and then take $a, b \to 0$. This leaves us with a large polynomial with derivatives, which is in principle not significantly more difficult to expand then say a Laughlin state.

For our wave function to be non-zero the total degree $2S$ of the polynomials in $(\bar{a}_k, \bar{b}_k)$ and (a_k, b_k) has to be the same; this leads to the constraint

$$p\,(N_1 - 1) = N.$$

The total number of Dirac flux quanta seen by the physical electrons is therefore

$$2S = m\,(N-1) \pm N_1 = \left(m \pm \frac{1}{p}\right) N - m \pm 1\,,$$

where the + sign corresponds to quasi-holes, and the − sign to quasi-electrons. The inverse filling fraction and the flux shift on the sphere, defined via

$$2S = \frac{1}{\nu}\, N - N_{\text{shift}}\,, \tag{4}$$

are thus as predicted by Haldane [3], and consisted with the result of exact diagonalization studies for small numbers of particles.

Before proceeding to the general case, we wish to refine our construction. If we take a close look at the trial wave function involving quasi-electrons, we note that we cannot obtain a $[5,-2]$ state ($\nu = 2/9$) from a $[3,-2]$ state ($\nu = 2/5$) by just multiplying the latter with a Jastrow factor squared, as we would expect from a very general argument based on an adiabatic localization of magnetic flux [6]. To circumvent this problem, we are led to consider an alternative trial wave function for the quasi-electron,

$$\Psi^{+}_{m}[u,v] = \left[\prod_{i=1}^{N} (\bar{b}\frac{\partial}{\partial u_i} - \bar{a}\frac{\partial}{\partial v_i}) \prod_{i<j}^{N} (u_i v_j - u_j v_i)^2 \right] \prod_{i<j}^{N} (u_i v_j - u_j v_i)^{m-2}, \tag{5}$$

where the derivatives act only on the term in the square brackets. This trial wave function turns out to be energetically favorable over Laughlins—which is not surprising, as it preserves the maximal possible number of Jastrow factors. Indeed, when calculating the overlap between a $2/5$ state and the exact ground state for Coulomb interactions for $N = 6$, we find 0.9985 if we use Laughlin's quasi-particle and 0.9995 if we use (5).

Extending our construction to hierarchy wave functions in general, we write the recursion relations

$$\Psi_{[m,+p_1,\dots\alpha_n p_n]}[u,v] = \Psi_{[p_1,\dots\alpha_n p_n]}[\partial_a,\partial_b] \prod_{k=1}^{N_1}\prod_{i=1}^{N}(a_k u_i - b_k v_i) \prod_{i<j}^{N}(u_i v_j - u_j v_i)^m$$

$$\Psi_{[m,-p_1,\dots\alpha_n p_n]}[u,v] = \prod_{i<j}^{N}(u_i v_j - u_j v_i)^{m-2} \Psi_{[p_1,\dots\alpha_n p_n]}[\partial_{\bar{a}},\partial_{\bar{b}}] \prod_{k=1}^{N_1}\prod_{i=1}^{N}(\bar{b}_k \partial_{u_i} - \bar{a}_k \partial_{v_i}) \prod_{i<j}^{N}(u_i v_j - u_j v_i)^2. \tag{6}$$

Note that the quasiparticle wave function $\Psi_{[p_1,\dots\alpha_n p_n]}[a,b]$ is always symmetric, as all the p's are even; $\Psi_{[m,\alpha_1 p_1,\dots\alpha_n p_n]}[u,v]$ is symmetric if m is even, or antisymmetric if m is odd, as appropriate for an electron wave function [7].

These wave functions lead to the iterative equations

$$\frac{1}{\nu} \equiv [m,\alpha_1 p_1,\dots,\alpha_n p_n] = m + \frac{\alpha_1}{[p_1,\alpha_2 p_2,\dots,\alpha_n p_n]}$$

$$N_{\text{shift}} \equiv \{m,\alpha_1 p_1,\dots,\alpha_n p_n\} = m - \alpha_1 \frac{\{p_1,\alpha_2 p_2,\dots,\alpha_n p_n\}}{[p_1,\alpha_2 p_2,\dots,\alpha_n p_n]} \tag{7}$$

for inverse filling fractions and flux shifts on the sphere, and thus reproduce the universality classes predicted by Haldane [3].

For some of the most important hierarchy states, we have numerically evaluated overlaps with the exact ground state and energy expectation values for small numbers of particles with Coulomb interactions. The results confirm the validity of our trial wave functions; they are shown in Table 1. Note in particular, that the hierarchy states at $\nu = 2/5$ and $\nu = 3/7$ rate considerably better then Laughlin's original $1/3$ state, which has been included for comparison. It should also be noted that a hierarchy $2/3$ state is energetically less favorable then the particle hole conjugate of a Laughlin $1/3$ state, while a hierarchy $2/7$ state is better then a $2/7$ state constructed by multiplying this particle hole conjugate by a Jastrow factor squared.

3. Quasiparticle charge and statistics

The elementary excitations of the general hierarchy states (6) are just conventional quasiparticles (quasi-holes or quasi-electrons) in the last daughter fluid $\Psi_{[p_n]}$. To determine their charge, we just have to observe how the flux–particle number relationship is altered by the presence of a quasiparticle; this yields

Table 1
Overlaps and ground state energies per particle for some of the most important hierarchy states.

ν	$[m, \alpha_1 p_1 ...]$	N	$2S+1$	$\langle\Psi_{trial}\vert\Psi_{ex}\rangle$	E_{trial}	E_{ex}
1/3	[3]	6	16	0.9964	−0.44995	−0.45017
2/5	[3, −2]	6	12	0.9995	−0.50036	−0.50040
		8	17	0.9988	−0.48017	−0.48024
3/7	[3, −2, −2]	6	10	0.9981	−0.53950	−0.53964
2/3	[1, +2]	8	13	0.9939	−0.53372	−0.53415
	—			0.9990 [a]	−0.53408 [a]	
2/7	[3, +2]	6	20	0.9906	−0.40415	−0.40436
	—			0.9856 [b]	−0.40405 [b]	
2/9	[5, −2]	6	22	0.9915	−0.38695	−0.38714

[a] Particle hole conjugate of Laughlin 1/3 state.
[b] Particle hole conjugate of Laughlin 1/3 times Jastrow factor squared.

$$e^* \equiv (m, \alpha_1 p_1, ..., \alpha_n p_n) = -\alpha_1 \frac{(p_1, ..., \alpha_n p_n)}{[m, \alpha_1 p_1, ..., \alpha_n p_n]} = \frac{(-1)^n \alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n}{\text{denominator of filling fraction } \nu}, \tag{8}$$

where a negative sign is meant to indicate that a quasi-hole in the last daughter fluid corresponds to a quasi-electron in the electron fluid and vice versa. For example, the charge of a quasiparticle in a 2/5 state is $e^* = 1/5$.

Turning to the quasiparticle statistics, let us first recall the adiabatic transport argument of Arovas, Schrieffer, and Wilczek [8] for Laughlin-Jastrow type states. The quasiparticle charge e^* is obtained by equating the geometric phase γ acquired by the wave function as a quasi-hole is adiabatically carried around a closed loop with the corresponding Aharonov-Bohm phase,

$$i \oint \langle \psi_m^- | \frac{d}{dt} | \psi_m^- \rangle = -2\pi e^* \phi,$$

where ϕ is the flux through the loop measured in Dirac quanta. Evaluation of the geometric phase yields $\gamma = -2\pi \langle N \rangle$, where $\langle N \rangle$ is the average number of particles inside the loop. Thus $e^* = 1/m$. If there is a second quasi-hole in the loop, there will be a deficit of $1/m$ in $\langle N \rangle$, and we obtain an additional phase $\Delta\gamma = 2\pi/m$. This additional phase is identified as quasiparticle statistics $\theta = 1/m$.

This argument can be generalized to the hierarchy; the only difference is that the geometric phase depends now on the average number $\langle N_n \rangle$ of those quasiparticles in the loop which condense as the last daughter fluid,

$$\gamma = -2\pi(-1)^n \alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_n \langle N_n \rangle .$$

The statistics depends thus on how $\langle N_n \rangle$ is altered by the presence of a quasiparticle excitation in the loop; that is to say, on the amount we have to shift N_n on the sphere to accommodate a quasiparticle while keeping $2S$ fixed. This yields

$$\theta = \frac{(m, \alpha_1 p_1, ..., \alpha_n p_n)}{|(m, \alpha_1 p_1, ..., \alpha_{n-1} p_{n-1})|}, \tag{9}$$

where $(m, \alpha_1 p_1 ...)$ are the quasiparticle charges given by (8). This result is consistent with Halperin's predictions [5] and with conclusions drawn from the effective field theory [9]. To give an example, the mutual statistics of the elementary quasiparticle excitations in a 2/5 state is $\theta = 3/5$. This might look implausible at first sight. To make it plausible, let us consider pairs of these quasiparticles. They will have charge 2/5 and statistics $4 \times 3/5 = 12/5 \rightarrow 2/5$, since each particle in a pair will pick up a statistical phase from each of the particles in any other pair. The statistics of this excitation is thus consistent with the statistics of the charge 2/5 quasiparticle we would obtain by directly inserting a Laughlin quasiparticle in the parent fluid. This consistency, which carries through for quasiparticles constructed by inserting flux tubes at any level of a general hierarchy state, is most essential to the consistency of the whole theory. For otherwise we could combine quasi-electrons and quasi-holes at different levels of the hierarchy to form new excitations which carry statistics but no charge, and our charged quasiparticles might no longer be the elementary excitations.

The explicit wave functions proposed above can further be used to calculate quasiparticle energies, and in particular energy gaps. One way to estimate the gap for systems with small numbers of particles is to insert a quasi-electron at the north and a quasi-hole at the south pole of the sphere; for our 2/5 state, we find energy gaps of 0.07569 and 0.06970 for $N = 6$ and 8, respectively, which ought to be compared with 0.07505 and 0.6809 obtained by exact diagonalization of the corresponding Hamiltonians.

4. Model Hamiltonian

The question we wish to address next is whether we can construct a model Hamiltonian involving two body potentials only which singles out any of our hierarchy states as unique and exact ground state. Let us assume such a Hamiltonian would exist, and write $H|\psi\rangle = 0$. In a finite geometry, any two body interaction between particles in the lowest Landau level can be parametrized by a finite number of pseudopotentials v_l, which just denote the potential energy cost of having relative angular momenta l between two particles. Thus we may write

$$H|\psi\rangle = \Big(\sum_{l=0}^{2S} v_l H_l\Big)|\psi\rangle = \sum_{l=0}^{2S} v_l \left(H_l|\psi\rangle\right) = 0. \tag{10}$$

For the explicit wave functions proposed above we find numerically that there is only one nontrivial solution to (10):

$$v_l = (2S - l)(2S - l + 1) + \text{const.} \tag{11}$$

This solution, however, corresponds to a Hamiltonian equal to the square of the total angular momentum of the many body wave function on the sphere, $H = L_{\text{tot}}^2$, and can therefore not be used to distinguish one trial wave function from another. Thus we have shown that there is no model Hamiltonian involving two body potentials only which singles out any of our hierarchy states as the unique ground state. The same is true for the trial wave functions proposed by Halperin [10] and Jain [11].

In this context it is perhaps propitious to note that interesting new possibilities arise as one considers many body interaction potentials [12]. For example, one can identify the exact ground state of a model Hamiltonian which demands that the wave function vanishes as the sixth power as three particles approach each other; this state has filling fraction 2/5.

5. Comments

We wish to conclude with a few comments:

a) Comparing ground state energies and overlaps of our hierarchy wave functions with the trial wave functions proposed by Jain [11], we find that Jain's 2/5 state is slightly better than ours. We conjecture that the reason for this is as follows: The improved trial wave function for the quasi-electron (5) is still not the best in town; the pair wave function proposed by Halperin [10] and the trial wave function proposed by Jain [11] are even better. In fact, we conjecture that these two are identical, as indicated by expansions of the wave functions for $\nu = 1/3$ on the sphere up to $N = 6$. Now suppose we wish to construct a hierarchy 2/5 state along the lines of

(6), but with Jain's trial wave function for the quasi-electron instead of (5). For a 2/5 state, we need half as many quasiparticles as there are electrons in the liquid — but this is to say, the starting point for this state is two filled Landau levels! In this context, it is of course not clear what one could mean by a condensation of the quasi-particles into a $p = 2$ daughter fluid, as they are already condensed in the sense that they occupy all the states in the second Landau level. It is clear, however, that the trial wave function one obtains in the end will be identical to the one proposed by Jain. We are thus led to conjecture that Jain's trial wave function for the 2/5 state is identical to a hierarchy state constructed with Halperin's pair wave function for the quasi-electron.

b) The generalization of our wave functions to the hierarchy of double layer electron systems [13] follows without incident.

c) The methods developed above can be used to construct a trial wave functions for other universality classes as well, most noteworthy among them paired Hall states [10,14]:

$$\Psi^m_{\rm phs}[u,v] = \mathcal{A}\int D[a,b]\prod_{i<j}^{n}(a_i b_j - a_i b_j)^m$$
$$\times\prod_{i=1}^{n}\left((u_i\bar{a}_i - v_i\bar{b}_i)(u_{i+n}\bar{a}_i - v_{i+n}\bar{b}_i)\right)^{\frac{m(n-1)}{2}}\prod_{i=1}^{n}(u_i v_{i+n} - u_{i+n}v_i)\,, \qquad (12)$$

where $\mathcal{A}$ denotes antisymmetrization, n is the number of electron pairs with center-of-mass coordinates (a_i, b_i), and m is an even integer related to the filling fraction via $\nu = 4/m$.

Acknowledgements

Much of my knowledge of the subject was acquired in numerous discussions with F.D.M. Haldane, B.I. Halperin, X.G. Wen and F. Wilczek; my deepest gratitude goes to all of them. I further wish to thank I.A. McDonald for his critical reading of the manuscript, and the Laboratoires de Physique Statistique and Theorique at the Ecole Normale Superieure in Paris for hospitality.

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