Exact solutions and the adiabatic heuristic for quantum Hall states

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An operator formalism is developed for an exactly soluble model of fractional statistics, and used to show that a heuristic principle suggested earlier is rigorously valid in one particular case. For a class of model hamiltonians, Laughlin's Jastrow-type wave functions are obtained explicitly from a filled Landau level by smooth extrapolation in quantum statistics. The gap is shown not to close, which allows us to infer the incompressibility of the final states. The analysis is further extended to paired Hall states at even-denominator fillings, which arise adiabatically from an exact but unnormalizable model of superconductivity. Finally, we generalize the model to the torus geometry, and show that theorems restricting the possibilities of quantum statistics on closed surfaces are circumvented in the presence of a magnetic field.

1. Introduction

Recently we proposed [1] an approach to understanding the fractional quantum Hall effect [2,3] by relating it continuously to the integer effect (adiabatic heuristic.) The connection is made by the process of trading uniform magnetic flux for flux localized on the particles. In this way, incompressible many-body fermions systems at different filling fractions are continuously connected through intermediate incompressible many-body anyon systems. Our earlier argument was frankly heuristic.

In the meantime Girvin and collaborators [4], and others [5,6], have provided a simple recipe for constructing model hamiltonians whose ground states can be identified exactly, and that is very well suited to implementing our adiabatic heuristic. In this paper we shall exploit the ideas of their construction to give a concrete realization of our approach in a context where it is very close to being demonstrably correct. We shall also show how the discussion can be extended to
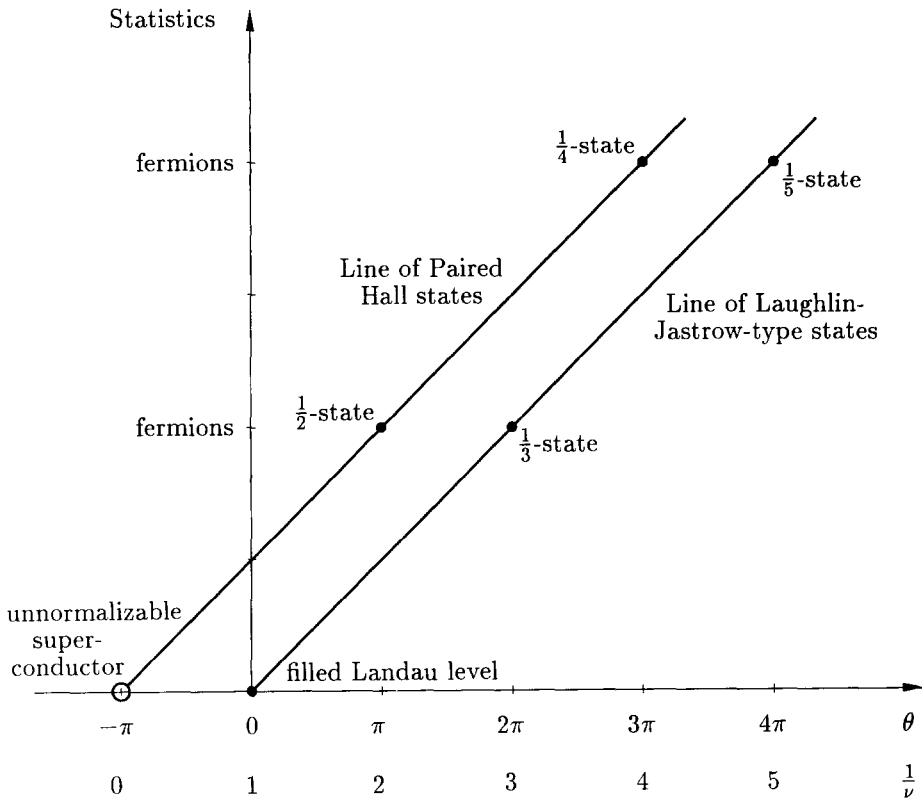
Fig. 1. The adiabatic heuristic, which is shown in this paper to apply rigorously for certain model hamiltonians, connects incompressible ground states along lines of unit slope in the statistics–inverse filling fraction plane. One such line contains the single filled Landau level and the Laughlin–Jastrow-type states at odd-denominator fillings $\nu = 1/m$ for fermions, together with similar boson states at even denominator fillings. Another such line contains incompressible states with pairing correlations, including a boson state at filling fraction $\nu = 1$, a fermion state at $\nu = 1/2$, and (as a limiting case, for $\theta \rightarrow -\pi$) a singular p-wave BCS superfluid in zero magnetic field.

accommodate the paired Hall state at half filling [7], and to periodic boundary conditions.

Many of our results are aptly summarized in fig. 1. The adiabatic heuristic, which is exactly and explicitly implemented for the ground states of the models discussed in this paper, trades uniform flux for particle statistics. It therefore relates states along parallel lines of constant slope in the statistics-inverse filling fraction plane. Starting from a ground state separated by a gap in the spectrum, that is an incompressible state, the adiabatic theorem guarantees that slow changes in the hamiltonian will evolve this state continuously into the ground state of the altered hamiltonian, so long as the gap does not close. Along the line containing a single filled Landau level of fermions one also finds the Laughlin odd-denominator

$\nu = 1/m$ states for fermions, similar bosonic even-denominator states including (as a limiting case, for $m \to 0$) the ideal Bose-condensed superfluid, and other anyon fractional quantized Hall states at appropriate filling fractions. Another line contains incompressible states with pairing correlations, including a boson state at filling fraction $\nu = 1$, a fermion state at $\nu = 1/2$, and (as a limiting case) a singular p-wave BCS superfluid in zero magnetic field.

Before beginning the technical discussion, it may be appropriate to mention briefly some quasi-philosophical points, which were important in the motivation for this work.

There can be little doubt that the leading idea of the current theory of the fractional quantum Hall effect, namely that it is due to the formation of an incompressible quantum liquid at suitable filling fractions, is essentially correct. However, in our opinion the theoretical *understanding* of this effect leaves much to be desired. The main foundation of the existing theory is Laughlin's inspired guess of a trial wave function for $\nu = 1/m$, m odd. One then argues that it represents an incompressible state, basically as follows. First, this wave function is so beautiful, and does such a good job of keeping the particles apart while treating them all symmetrically, that it clearly is very advantageous energetically. Second, wave functions of this type can only be constructed at special, discretely different filling fractions. Thus the liquid has preferred densities, and accommodates small density perturbations by producing localized quasiparticles while keeping the favored density in bulk. This argumentation has been buttressed by numerical simulations [8], which show both that there are incompressible ground states (for realistic interactions) at the predicted filling fraction and that these states have large overlaps with the Laughlin states.

We feel that the case for incompressibility of the $1/m$ states, though quite persuasive – especially in the light of laboratory and numerical experiments – is not entirely crisp or transparent. The situation is much worse for the other odd-denominator fractions.

We believe the adiabatic approach suggested previously and developed here does go a long way toward providing a crisp, transparent understanding of some key aspects of the fractional quantized Hall effect for the $1/m$ states. It is possible that some generalization of these methods will also help elucidate more general fractions. (The adiabatic heuristic may be used – at a heuristic level – to motivate either the hierarchical approach to other fractions, or some constructions using higher Landau levels, in a spirit similar to Jain's [9]. However we have not yet managed to find a rigorous version in the more elaborate cases.) In any case, they have provided the impetus leading to some results of independent interest: quantized Hall states for anyons, smooth connection of anyon superfluidity to fractional quantized Hall states, $\nu = 1/2$ states for fermions smoothly connected to BCS theory, and new possibilities for fractional statistics on torus, as we now shall discuss.

2. Exactly soluble model of fractional statistics

The analysis presented in the following sections is largely based on a model of fractional statistics, closely related to the one which has been introduced recently by Girvin and his collaborators [4]. This model is exactly soluble for the ground state, and some excited states are easily identified. In this section we introduce an attractive operator formalism to review and extend the relevant aspects of that model.

(1) Consider the following N-particle hamiltonian in $2 + 1$ dimensions:

$$H = \frac{1}{2m} \sum_{i=1}^{N} \left[\left(-i\nabla_i + eA_i \right)^2 - eB_i \right], \tag{2.1}$$

where

$$B_i = \nabla_i \times A_i = -\frac{2\theta}{e} \sum_{j \neq i} \delta^2(r_i - r_j) + B \tag{2.2}$$

and $\hbar = c = 1$. It describes a gas of fractional statistics particles, interacting via a delta-function potential, in the presence of an external magnetic field. If we impose the gauge condition $\nabla_i A_i = 0$, we can solve eq. (2.2) for the vector potential and obtain

$$A_i = -\frac{\theta}{\pi e} \sum_{j \neq i} \frac{(r_i - r_j) \times \hat{z}}{|r_i - r_j|^2} + \tfrac{1}{2} B(r_i \times \hat{z}). \tag{2.3}$$

The first term in (2.3) corresponds to magnetic flux tubes attached to the particles and implements fractional statistics with parameter θ, while the second term merely represents a uniform magnetic background field, of magnitude B and oriented in the negative $\hat{z}$-direction.

This model can be solved exactly for the ground state. To make it evident, we introduce the creation- and annihilation-like operators

$$a_i = e^{-S_a} \left(+\sqrt{2} \, \frac{\partial}{\partial \bar{z}_i} \right) e^{+S_a},$$

$$a_i^\dagger = e^{+S_a} \left(-\sqrt{2} \, \frac{\partial}{\partial z_i} \right) e^{-S_a}. \tag{2.4}$$

Here $z \equiv x + iy$, and S_a is defined by

$$S_a \equiv -\frac{\theta}{\pi} \sum_{i<j} \ln|z_i - z_j| + \tfrac{1}{4} eB \sum_i |z_i|^2. \tag{2.5}$$

Now the hamiltonian can be rewritten as

$$H = \frac{1}{m} \sum_i a_i^\dagger a_i. \tag{2.6}$$

In this form, the strong resemblance of our problem to Landau level quantization is evident. It follows immediately from (2.6) that those states annihilated by the a_i-operators are exact eigenstates of zero energy. The vastly degenerate manifold of ground states is therefore described by

$$\psi_0[z] = f[z] \, e^{-S_a} \tag{2.7}$$

where $f[z]$ is an arbitrary entire function of the complex particle positions. These states are, of course, not always normalizable.

(2) It is instructive to consider the commutation relations of the fundamental operators a_i and $a_i^\dagger$. While these are rather clumsy in general, there is one notable exception:

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0,$$

$$\left[a_i, \sum_j a_j^\dagger\right] = \left[\sum_i a_i, a_j^\dagger\right] = eB. \tag{2.8}$$

Now we can see the precise sense in which a_i and $a_i^\dagger$ resemble creation and annihilation operators. (We have refrained from normalizing them, to avoid problems near $B = 0$.) Using (2.8), we obtain the higher energy eigenstates [5]

$$\psi_n[z] = \left(\sum_i a_i^\dagger\right)^n f[z] \, e^{-S_a}$$

$$= e^{+S_a} \left(-\sqrt{2} \sum_i \frac{\partial}{\partial z_i}\right)^n f[z] \, e^{-2S_a}. \tag{2.9}$$

with energy eigenvalues $E_n = n\omega_c$, where n is a nonnegative integer, and $\omega_c = eB/m$ the cyclotron frequency. ψ_n is not the most general eigenstate of (2.1) for general values of θ.

Let us now digress briefly on the physical interpretation of these excitations. ψ_1 describes a state with all but one of the particles in the first Landau level, and one in the second Landau level – or, to be more precise, a linear combination of all N possible choices for this particular particle. In other words, we can create excitations into higher Landau levels, but we are unable to provide single-particle labels for them. This peculiarity becomes more significant as we turn to higher-order excitations. The state ψ_2, for example, contains a large amplitude for finding two

particles in the second Landau level, but also a small amplitude for having only one particle excited into the third Landau level.

(3) As with the standard Landau level problem [10], it proves convenient to introduce a second set of fundamental operators, which can be used to classify the vastly degenerate energy eigenstates. They are given by

$$b_i = e^{+S_b} \left(+ \sqrt{2} \, \frac{\partial}{\partial z_i} \right) e^{-S_b},$$

$$b_i^\dagger = e^{-S_b} \left(- \sqrt{2} \, \frac{\partial}{\partial \bar{z}_i} \right) e^{+S_b}, \tag{2.10}$$

where S_b is defined

$$S_b \equiv - \frac{\theta}{\pi} \sum_{i<j} \ln |z_i - z_j| - \tfrac{1}{4} eB \sum_i |z_i|^2. \tag{2.11}$$

S_b differs from S_a only in that the sign of B is reversed; the corresponding operators are therefore related by

$$b_i(\theta, +B) = -a_i^\dagger(\theta, -B), \qquad b_i^\dagger(\theta, +B) = -a_i(\theta, -B). \tag{2.12}$$

The commutation relations of the new operators satisfy

$$[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0,$$

$$\left[b_i, \sum_j b_j^\dagger \right] = \left[\sum_i b_i, b_j^\dagger \right] = eB, \tag{2.13}$$

in close analogy to (2.8), while operators belonging to different ladders obey

$$[a_i, b_j^\dagger] = [a_i^\dagger, b_j] = 0,$$

$$\left[a_i, \sum_j b_j \right] = \left[\sum_i a_i, b_j \right] = \left[a_i^\dagger, \sum_j b_j^\dagger \right] = \left[\sum_i a_i^\dagger, b_j^\dagger \right] = 0. \tag{2.14}$$

The energy eigenstates (2.9) may now be written in terms of fundamental operators only:

$$\psi_n[z] = \left(\sum_i a_i^\dagger \right)^n f[b^\dagger] \, e^{-S_a}. \tag{2.15}$$

(4) It is also possible to express the total canonical angular momentum

$$L \equiv \sum_{i=1}^{N} \left[-ir_i \times \nabla_i \right] \qquad (2.16)$$

in terms of ladder-like operators:

$$L = \frac{1}{2eB} \sum_j \left[\left(b_j^\dagger b_j + b_j b_j^\dagger \right) - \left(a_j^\dagger a_j + a_j a_j^\dagger \right) \right] - \frac{\theta}{\pi} \frac{N(N-1)}{2}. \qquad (2.17)$$

The last term reflects, in some sense, the fact that we consider the canonical rather than the kinetical angular momentum. Even though there is, once again, a strong similarity to the familiar case of Landau level quantization, (2.17) cannot be verified as easily as any of the other formulas stated above. However, using the identities (2.12), we can rewrite (2.17) as

$$L = \frac{1}{\omega_c} \left[H(\theta, -B) - H(\theta, B) \right] - N - \frac{\theta}{\pi} \frac{N(N-1)}{2}. \qquad (2.18)$$

To conclude the proof, we only have to substitute (2.1) and carry through a few steps of algebra.

The total angular momentum commutes with the hamiltonian,

$$[L, H] = 0, \qquad (2.19)$$

consequently, the angular momentum eigenvalues can be used to classify the degenerate energy eigenstates (2.9), even though such classification proves naturally incomplete. Note that the fiducial state, obtained by setting $f \equiv 0$ in (2.7), is an angular momentum eigenstate with eigenvalue zero:

$$L \, e^{-S_a} = 0. \qquad (2.20)$$

It is not possible to provide a similar framework for the angular momentum of the individual particles — reflecting the fact that not even the fiducial state of the interacting many-particle model is an eigenstate of the angular momentum operator for any individual particle.

(5) A similar situation arises if one attempts to construct a magnetic translation operator for an individual particle — such an operator would neither commute with the hamiltonian, nor with its equivalent for any other particle. However, one can construct an operator which translates all the particles simultaneously,

$$t(\xi) = \exp\left[\frac{1}{\sqrt{2}} \sum_i \left(\xi b_i - \bar{\xi} b_i^\dagger \right) \right]. \qquad (2.21)$$

It commutes trivially with the hamiltonian,

$$[t(\xi), H] = 0. \tag{2.22}$$

Of course, two successive translations do not commute in general, as can be seen from

$$t(\xi)t(\eta) = t(\xi + \eta) \exp\left[\tfrac{1}{4}NeB(\overline{\eta}\xi - \eta\overline{\xi})\right]. \tag{2.23}$$

However, if the parallelogram spanned by ξ and η includes n/N magnetic flux quanta, i.e.

$$\tfrac{1}{2}NeB(\overline{\eta}\xi - \eta\overline{\xi}) = 2\pi in \tag{2.24}$$

where n is an integer, $t(\xi)$ and $t(\eta)$ commute.

The proper generalization of the model to incorporate periodic boundary conditions is based on quite different considerations, and is provided in sect. 5.

3. The adiabatic heuristic and odd-denominator Hall states

We now construct, along the lines just sketched, models which allow the philosophy of the adiabatic heuristic to be implemented explicitly, so that integer to fractional quantized Hall states are connected continuously.

(1) The adiabatic heuristic is as follows. Quantized Hall states – that is incompressible quantum liquids – are related to other quantized Hall states through adiabatic localization of magnetic flux: uniform flux is traded for an equal amount of fictitious flux localized on the particles. The latter implements a change in quantum statistics, related to the change in the filling fraction by

$$\Delta\frac{\theta}{\pi} = \Delta\frac{1}{\nu}. \tag{3.1}$$

For suitable repulsive interactions, the gap in the excitation spectrum of the initial state is likely to carry through as we evolve along this line in the statistics-magnetic field plane, and new incompressible states of fermions are obtained as the statistical parameter θ has increased or decreased by integer multiples of 2π.

The most fundamental quantized Hall states are, of course, integer fillings of Landau levels. They are ideal points of departure, not only because their explicit wave functions are known, but also because their incompressibility is established on rigorous grounds. As explained elsewhere [11], explicit wave functions for other odd-denominator Hall states can be motivated via the heuristic prescription from filled Landau levels, together with particle–hole conjugation. In this section,

however, we wish to focus on the simplest case only. We start with a single filled Landau level, and obtain, according to (3.1), incompressible fermion states at $\nu = 1/m$, with m odd integer. These are, as we shall show below, the Laughlin–Jastrow-type states.

(2) We now return to the formal world of the exact solutions. Suppose the bare particles are fermions, as is appropriate for electron states. Then the most natural choice for the characteristic function $f[z]$ of the ground state (2.7) is the Vandermonde determinant

$$f[z] = \prod_{i<j} (z_i - z_j). \qquad (3.2)$$

The corresponding ground-state wave function is

$$\psi[z] = \prod_{i<j} (z_i - z_j) \prod_{i<j} |z_i - z_j|^{\theta/\pi} \prod_i \exp\left(-\tfrac{1}{4}eB|z_i|^2\right). \qquad (3.3)$$

For $\theta = 0$, $\psi[z]$ describes a single filled Landau level – the initial state for the adiabatic process. Now as we gradually evolve the statistics from fermions to super-fermions, and take θ from 0 to $(m-1)\pi$, we obtain states which obviously resemble Laughlin's trial wave functions. In fact, their physical consequences are identical. The flux tubes attached to the particles, each of them carrying $m-1$ Dirac flux quanta, are no longer of physical significance. They can be removed by the singular gauge transformation

$$A_i \rightarrow A_i + \nabla_i \Lambda_i \quad \text{for all } j,$$

$$\psi[z] \rightarrow \psi[z] \prod_i \exp(-ie\Lambda_i), \qquad (3.4)$$

with

$$\Lambda_i = -\frac{m-1}{2e} \sum_{i\neq j} \arg(z_i - z_j). \qquad (3.5)$$

This gauge transformation is called singular, because some of the Λ_i are ill defined if two particles sit on top of each other. However a centrifugal barrier excludes this possibility, and no restrictions result.

The removal of the flux tubes affects the ground state only by a phase, and yields the familiar wave functions

$$\psi_m[z] = \prod_{i<j} (z_i - z_j)^m \prod_i \exp\left(-\tfrac{1}{4}eB|z_i|^2\right). \qquad (3.6)$$

The only change left in the hamiltonian, after adiabatic evolution and subsequent gauge transformation, is the coefficient of the δ-function potential – which is to

say, no change at all, because this potential is zero when acting on any non-singular fermion wave function.

Strictly to comply with the heuristic prescription we have to trade real magnetic flux into ficticious flux, instead of merely attaching the latter to the particles. We therefore adjust the magnetic field strength according to

$$B = \left(1 + \frac{\theta}{\pi}\right)B_0. \tag{3.7}$$

Then the size of the circular droplet described by (3.3) remains constant during the evolution. This condition will prove essential for generalizations of the analysis to closed surfaces.

(3) The object of the exercise is of course not so much to provide an elegant path to Laughlin's wave functions, as to supply an argument for their incompressibility. The construction is robust, if the gap in the excitation spectrum of the initial state – then a consequence of Landau-level quantization – carries through as we travel along in the statistics-magnetic field plane. For then – and only then – will the adiabatic theorem of quantum mechanics [12] guarantee that the initial ground state will indeed evolve into the final one.

This is not at all a formality. Indeed the model hamiltonian we have been considering so far basically collapses to the hamiltonian for free fermions in a magnetic field at fractional filling fractions, and that problem certainly does not have an isolated incompressible ground state.

The energy gap in our model will persist only if we refine the construction in two respects. One of the problems is essentially trivial: there is no gap associated with quasihole excitations, and not even the initial state is strictly incompressible. This is of no real physical significance however: we should really consider whether there is a gap for creation of quasi-particle–quasi-hole paris. (In a more proper treatment, where we paid more attention to boundary conditions, the single quasihole state would be in a different Hilbert space. This emerges clearly if the problem is formulated on a closed surface.) We can cure the formal problem with quasi-holes by adding a pressure or chemical potential, or, even simpler, an energy term proportional to the total angular momentum. These methods all produce, at least for the initial state, the desired cusp in the energy versus the filling fraction.

More work is required to remove the other inadequacy, which really does concern essential physics. The repulsive interactions essential to producing a gap in the fractional quantized Hall states have not been included. A vast degeneracy emerges for the final states (and also for intermediate ones; see below). The challenge is therefore to find a local interaction potential, which singles out the wave function (3.3) as the exact and unique ground state for each value of θ during the evolution.

For the Laughlin $1/m$ states, the standard prescription [13] is as follows. Since these wave functions vanish like $|z_i - z_j|^m$ as two particles approach one another, they are manifestly annihilated by the potential

$$V^{(m)}[z] \propto \sum_{i \neq j} \left(\nabla_i^2\right)^{(m-1)/2} \delta^2(z_i - z_j), \qquad (3.8)$$

and of course also by all other potentials of this form with the Laplace operator taken to any smaller non-negative power. (The δ-function is meant to be taken over real and imaginary part separately.) Thus the wave function is an exact eigenstate of the hamiltonian with short-range interparticle repulsions of this form and the usual kinetic term for charged particles in an external magnetic field. Note that the expectation value of the potential energy is positive for the sign corresponding to repulsion. While a pure δ-function repulsion is negligible for any non-singular fermion wave function due to the antisymmetry of the wave function (for spinless fermions, as we consider here exclusively), the vanishing of these derivative interactions is a highly non-trivial property of the Laughlin wave functions. In fact it is easy to see that the polynomial factor in the $1/m$ wave function is the polynomial of least degree that is annihilated by $V^{(m)}$. Modulo subtleties of the kind alluded to above, which could be removed by adding to the hamiltonian a piece proportional to the total angular momentum, this makes it quite plausible that the Laughlin wave function is incompressible. The only thing that keeps it short of a full proof is the difficulty of verifying that the energy gap remains finite in the thermodynamic limit. (Though this is not implausible, given that the basic underlying interactions and correlations are short ranged.)

The relevant uniqueness property of the states (3.3) during the adiabatic evolution is clearly that they vanish like

$$|z_i - z_j|^{1+\theta/\pi}$$

as particles approach each other. This feature suggests, as a first thought, that in interpolating between the states we ought to consider fractional powers of the derivative operators in interaction potentials of the general form (3.8). However fractional derivatives are non-local operators, and a hamiltonian including them could well be considered unphysical.

On reflection, one comes to realize that a much better solution is possible. To appreciate this – one of our major points in the present paper – let us step back a moment to take a broader view of the problem.

In the course of our adiabatic procedure, we might move, for example, between fermions at filling fractions $\nu = 1$ and $\nu = 1/3$ through intermediate anyons at intermediate filling fractions. At the start, there is a natural isolated state – the full Landau level. At the end, we have arrived for all intents and purposes back at

the same (kinetic) hamiltonian, but as a result of the change in filling fraction there is no longer a natural isolated state. As we have seen, the "natural" continuation of the full Landau level, according to (3.2) and (3.3), leads to the Laughlin state. Where do the additional states, degenerate with it, come from?

In fact they are not difficult to trace. They come from holomorphic functions $f(z)$ in (2.7) that include *inverse powers* of difference $z_i - z_j$. Although at the starting point these are excluded, because they led to non-normalizable states, as soon as we depart from this point they must be included. The normalization integral

$$\int d^2z \, \frac{1}{|z|^2} \, |z|^{2\theta/\pi}$$

is finite for any positive θ, though very large for small θ. Starting after $\theta/\pi = 2$ we must also include inverse third powers, and so forth. All these states are exact eigenstates of the model hamiltonian (2.1).

The distinguishing characteristic of the additional states is to have very large amplitudes at small particle separations. Thus we may suppress them with a repulsive potential emphasizing short distances – which we obtain by adequate generalization of (3.8).

Let us first recall the precise role of the Laplace operators in $V^{(m)}$. Its significance becomes transparent only when we integrate in position space, as we do to evaluate matrix elements: each laplacian peels off a factor $|z_i - z_j|^2$ from the wave function, through integration by parts. The wave function is annihilated by the potential, if there is at least one power of $|z_i - z_j|$ left once all derivatives are stripped off the δ-function. Similarly, we can peel off a fractional power $|z_i - z_j|^\epsilon$ from the wave function if we replace the laplacian in a general potential (3.8) by the operator

$$|z_i - z_j|^{2p-\epsilon} (\nabla_i^2)^p, \tag{3.9}$$

where p is an integer such that $2p - \epsilon \geqslant 0$. We are therefore led to consider

$$V[z] \propto \sum_{i \neq j} |z_i - z_j|^{2p-\theta/\pi} (\nabla_i^2)^p \delta^2(z_i - z_j). \tag{3.10}$$

A local interaction potential of this form – in conjunction with a suitable energy term proportional to the total angular momentum, as explained above – singles out the wave functions (3.3) as unique and exact ground states during the entire evolution.

It is worth noting that the desired result may also be accomplished with the inverse power potential

$$V'[z] \propto \sum_{i<j} \left| \frac{1}{z_i - z_j} \right|^{\theta/\pi} \delta^2(z_i - z_j) \tag{3.11}$$

instead of $V[z]$. Of course the product of an inverse power with a δ-function is not an entirely reputable mathematical object. However, this difficulty can be circumvented by a suitable limiting procedure. For example, we may replace the δ-function in (3.11) with one of its standard realizations,

$$\delta^2(z) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon} \exp\left(-\frac{1}{\epsilon} |z|^2 \right),$$

and multiply it by a factor $\epsilon^{-\epsilon/|z|}$, to regulate the singularity at the origin. (Since the important values of z inside the gaussian representing the δ-function are of order $|z| \sim \sqrt{\epsilon}$, this regulator does not significantly distort it.)

(4) Thus we have explained how to derive the final wave function, and shown why we expect them to be incompressible. It is our believe that the existence and incompressibility of fractional quantized Hall states for screened Coulomb interactions, as they appear in realistic samples, can be understood along exactly this line of reasoning. The analysis presented here provides evidence that the adiabatic localization of flux onto particles (the adiabatic heuristic) can in fact be regarded as the underlying physical principle of the fractional quantum Hall effect. It traces the incompressibility of the fractional states to the incompressibility of a single full Landau level in an explicit and elementary way, using the adiabatic theorem of quantum mechanics. As a bonus, it predicts the existence of a line of anyon quantized Hall states, with the statistical parameter of the anyons related to the filling fraction according to (3.1).

4. Paired Hall states at even denominator fillings

Additional applications of the adiabatic heuristic arise in connection with a generically new phase of matter, the paired Hall state, as we shall discuss now.

(1) Assume that the bare particles described by the ground state (2.7) of the Girvin model are bosons. Then the most obvious choice for the characteristic function would be $f[z] \equiv 1$. Unfortunately, no new states are obtained by starting there: as we evolve into the statistics-magnetic field plane, we just reproduce the line of Laughlin–Jastrow states discussed above. It is of some interest to note, however, that this construction relates the full Landau level of fermions, and other quantized Hall states in a magnetic field, to the very simplest superfluid: the Bose

condensed ground state in zero magnetic field, with all particles at zero momentum.

Another interesting choice, which gives something essentially new, is to start once more with exactly one filled Landau level. For bosons, this is implemented by taking $f[z]$ a product of a pfaffian and a Vandermonde determinant,

$$f[z] = \text{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j),\tag{4.1}$$

where

$$\text{Pf}\left(\frac{1}{z_i - z_j}\right) \equiv \mathscr{A}\left\{\prod_{i \text{ even}}\frac{1}{z_{i-1} - z_i}\right\}.\tag{4.2}$$

The operator $\mathscr{A}$ indicates antisymmetrization over all $(N-1)!!$ different possible choices of breaking an even number of particles N up into pairs. $f[z]$ is a product of two antisymmetric entities, and thus symmetric, as required for boson statistics. Note also that it is an entire function of the complex particle positions z_i. The filling factor in the thermodynamic (large N) limit is insensitive to the pfaffian, as can be seen from a simple angular momentum argument.

The analysis below is very similar to the one presented in sect. 3. The full wave function during the adiabatic evolution is given by

$$\psi[z] = \text{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)\prod_{i<j}|z_i - z_j|^{\theta/\pi}\prod_i \exp\left(-\tfrac{1}{4}eB|z_i|^2\right),\tag{4.3}$$

where the external magnetic field is adjusted such that the total amount of flux per unit area (i.e. the sum of real and ficticious magnetic flux) remains constant,

$$B = \left(1 + \frac{\theta}{\pi}\right)B_0.\tag{4.4}$$

The focus of our attention below is on those final points where Fermi statistics is recovered, at $\theta = (2n-1)\pi$, with n integer ($n \geqslant 0$ since $B \geqslant 0$).

(2) The most peculiar of these fermion states is the very first one: $n = 0$, or $\theta = -\pi$. It is obtained from the initial state (the Landau level filled with bosons) by evolving backwards in statistics – all the way until the point where the entire magnetic field is converted into fictitious flux tubes. The latter are removed via a singular gauge transformation, in close analogy to (3.4). The new ground state is described by the pfaffian factor alone,

$$\psi_{\text{sc}}[z] = \text{Pf}\left(\frac{1}{z_i - z_j}\right).\tag{4.5}$$

It is an exact zero energy eigenstate of the transformed hamiltonian

$$H_{sc} = \frac{1}{2m} \sum_i (-i\nabla_i)^2 - \frac{\pi}{m} \sum_{i \neq j} \delta^2(r_i - r_j). \tag{4.6}$$

In fact, we have obtained an exact model of BCS pairing (which may not look familiar, though, because it is formulated in position rather than momentum space). The pfaffian ground state can be obtained from a BCS product wave function by projecting out a definite number of particles, as pointed out by Dyson [14] a long time ago. Note that the interaction is now attractive, and that the δ-function in position space corresponds to a constant potential in momentum space.

The exactness of the solution may also be verified directly with the identity

$$\nabla^2 \frac{1}{z} = -2\pi \frac{\delta^2(z)}{z}$$

(which is to be interpreted as a prescription for integrating smooth functions which vanish at the origin, slightly generalizing the usual definition of distributions).

Unfortunately, ψ_{sc} is not normalizable and therefore without evident physical meaning by itself. Formally, it attempts to describe a BCS superconductor with a short-range pairing potential so strong that the potential energy gained is large enough to compensate entirely for the total kinetic energy of the system.

(3) The real virtue of this solution is its heuristic connection with paired Hall states at filling fractions $\nu = 1/2n$, into which it can be continuously deformed. We obtain such states for $\theta = (2n - 1)\pi$, one for each positive value of the integer n,

$$\psi_n[z] = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^{2n} \prod_j \exp\left(-\tfrac{1}{4}eB|z_i|^2\right). \tag{4.7}$$

They are exact eigenstates of

$$H_n = \frac{1}{2m} \sum_i (-i\nabla_i - eA_i)^2 + \frac{(2n-1)\pi}{m} \sum_{i \neq j} \delta^2(r_i - r_j) \tag{4.8}$$

with

$$A_i = \tfrac{1}{2}B(r_i \times \hat{z}).$$

Physically, these wave functions describe Laughlin states modulated by a strong attractive pairing correlation – which is to say, the very strong anti-correlation implicit in Laughlin's states are partially ameliorated.

Pairing is conventionally associated with an attractive interaction. Our exact solution, however, makes it very clear that this association is not inevitable. As we travel through statistics-magnetic field space, the coefficient of the delta-function potential in the hamiltonian changes continuously with θ. It is attractive for the unnormalizable superconductor, vanishes as we pass through boson statistics, and remains repulsive as we reach the Hall states. The pairing in the Hall states evidently does *not* require an attractive interaction; rather, it arises indirectly as a necessary accessory of Jastrow–Laughlin correlations at even-denominator filling fractions.

(4) Just as in the case of the $1/m$ states discussed in sect. 3, the construction here is robust only if a gap in the excitation spectrum is maintained during the adiabatic evolution. This is accomplished by adding an energy term proportional to the total angular momentum in conjunction with a suitable local interaction potential.

The uniqueness property we shall exploit in constructing the latter is as follows: the amplitudes for the paired Hall states at $\nu = 1/2n$ vanish at least like

$$|z_i - z_j|^{2n} \cdot |z_i - z_k|^{2n-1}$$

or like

$$|z_i - z_j|^{2n-1} \cdot |z_i - z_k|^{2n}$$

as two arbitrary particles (we label them j and k) approach a third particle (labeled by i). Thus wave functions are annihilated by the three-body potential

$$V^{(n)}[z] \propto \sum_{\text{triples}} \left(\nabla_i^2\right)^{2n-1}\left(\delta^2(z_i - z_j)\delta^2(z_i - z_k)\right), \qquad (4.9)$$

and of course also by all other potentials of this form with the Laplace operator taken to any smaller non-negative power. In fact, this potential may be considered as a generalization of (3.8), for it is equally effective in isolating an incompressible ground state from the remaining parts of the spectrum.

It is now rather straightforward to generalize $V^{(n)}[z]$ to the intermediate anyon states – following the prescription developed in sect. 3. For non-negative values of θ, the paired Hall states (4.3) are singled out uniquely by

$$V[z] \propto \sum_{\text{triples}} |(z_i - z_j)(z_i - z_k)|^{p-\theta/\pi}\left(\nabla_i^2\right)^p\left(\delta^2(z_i - z_j)\delta^2(z_i - z_k)\right), \quad (4.10)$$

where p is an integer such that $2p - \theta/\pi \geqslant 0$ during the entire evolution. (No additional interaction is required for negative values of θ.)

(5) Thus the construction is robust everywhere except at the point $\theta = -\pi$ (the superconductor), when the ground state (4.3) ceases to be normalizable. The

question of what happens near this particular point can be addressed from a different point of view: one starts with a free gas of fermions, and treats the onset of the adiabatic process as a perturbation. Careful analysis of the residual interactions reveals then that they do indeed trigger a p-wave pairing instability, and that the strength of the pairing increases with the induced change in quantum statistics $\Delta\theta$.

5. Generalization to periodic boundary conditions

Finally, we will generalize the exactly soluble anyon model and the applications discussed in sects. 3 and 4 to the torus geometry (related studies have been carried out by Iengo and Lechner [15]).

(1) The starting point for our discussion is the construction of Laughlin–Jastrow states on the torus, as first obtained by Haldane and Rezayi [16]. For the sake of consistency with the previous sections, we use symmetric rather then Landau gauge:

$$A(r) = \tfrac{1}{2}B(r \times \hat{z}).$$ (5.1)

The standard formalism for Landau-level quantization, as we employ it in this number, is contained in the more general formalism developed in sect. 2 for the exact model: it is the special case when no flux is attached to the particles, obtained by setting $\theta \equiv 0$. The operators a_i and $a_i^\dagger$, and b_i and $b_i^\dagger$ now obey the proper commutation relations for independent creation and annihilation operators.

Periodic boundary conditions are imposed by

$$t_i(\xi_\alpha)\,\psi[z] = e^{i\phi_\alpha}\psi[z] \quad \text{for all } i; \quad \alpha = 1, 2,$$ (5.2)

where ξ_1 and ξ_2 are two nonparallel displacements in the complex plane, ϕ_1 and ϕ_2 are boundary phases, and $t_i(\xi)$ is the magnetic translation operator for the ith particle,

$$t_i(\xi) = \exp\left[\frac{1}{\sqrt{2}}\left(\xi b_i - \bar{\xi} b_i^\dagger\right)\right].$$ (5.3)

Note that $t_i(\xi)$ commutes trivially with the hamiltonian. The periodic boundary conditions require further that $t_i(\xi_1)$ and $t_i(\xi_2)$ commute, i.e. the parallelogram spanned by ξ_1 and ξ_2 must contain an integer number of magnetic flux quanta, which we call N_ϕ.

The wave functions subject to (5.2) are, because of the magnetic field, not strictly periodic, but only quasiperiodic,

$$\psi(z_1, \ldots, z_i + \xi_\alpha, \ldots, z_N) = \exp\left[-\tfrac{1}{4}eB\left(\xi_\alpha \bar{z}_i - \bar{\xi}_\alpha z_i\right)\right] e^{i\phi_\alpha}\psi(z_1, \ldots, z_i, \ldots, z_N).$$

(5.4)

For convenience, we set the principal displacements $\xi_1 \equiv 1$ and $\xi_2 \equiv \tau$, with $\mathrm{Im}(\tau) > 0$, and call the area bounded by the four points $z = \tfrac{1}{2}(\pm 1 \pm \tau)$ the principal region. The magnetic field strength is now given by

$$eB = \frac{2\pi N_\phi}{\mathrm{Im}(\tau)}.$$

(5.5)

Note that N_ϕ is also equal to the number of states in the first Landau level.

Following ref. [16], we obtain for Laughlin's $1/m$ states for N particles (with $N_\phi = mN$)

$$\psi_m[z] = f[z] \prod_{i=1}^{N} \exp\left(-\tfrac{1}{4}eB \,|z_i|^2\right)$$

(5.6)

with

$$f[z] = \exp(iKZ) \prod_{\nu=1}^{m} \vartheta_{\frac{1}{2},\frac{1}{2}}(Z - Z_\nu \,|\, \tau) \prod_{i<j}^{N} \vartheta_{\frac{1}{2},\frac{1}{2}}(z_i - z_j \,|\, \tau)^m \prod_{i=1}^{N} \exp\left(+\tfrac{1}{4}eBz_i^2\right),$$

(5.7)

where $\vartheta_{\frac{1}{2},\frac{1}{2}}(z \,|\, \tau)$ is the odd Jacobi theta function [17]. The theta functions are defined in general by

$$\vartheta_{a,b}(z \,|\, \tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i(n+a)^2 \tau}\, e^{2\pi i(n+a)(z+b)}$$

(5.8)

and satisfy the quasiperiodicity relations

$$\vartheta_{a,b}(z + 1 \,|\, \tau) = e^{2\pi ia}\vartheta_{a,b}(z \,|\, \tau),$$

$$\vartheta_{a,b}(z + \tau \,|\, \tau) = e^{-\pi i\tau}\, e^{-2\pi i(z+b)}\vartheta_{a,b}(z \,|\, \tau).$$

(5.9)

Z denotes the center-of-mass coordinate

$$Z \equiv \sum_i z_i.$$

All the center-of-mass zeros Z_ν are located in the principal region. The sum of the center-of-mass zeros and the real parameter K are subject to the boundary conditions

$$(-1)^{N_\phi} \exp(iK) = e^{i\phi_1},$$

$$(-1)^{N_\phi} \exp(iK\tau) \exp\left(2\pi i \sum_\nu Z_\nu\right) = e^{i\phi_2}. \tag{5.10}$$

For fixed values of the boundary phases, the equations (5.10) possess a total of m^2 solutions for K and $\sum_\nu Z_\nu$; no further restrictions for the allowed choices of the individual Z_ν result. However, all these distinct solutions yield only m linearly independent states, as can be seen from a very general argument.

Indeed, the abstract properties we require of the center-of-mass part $F(Z)$ of the wave function, viz. the product of the first two factors in (5.7), is that it is entire and that it has exactly m zeroes in the principal region. Given one such solution F, its ratio $\tilde F/F$ with any other solution $\tilde F$ is a meromorphic truly periodic function on the torus, with at most simple poles at m prescribed points (namely the zeroes of F). It is a standard theorem in complex function theory – a very special case of the Riemann–Roch theorem – that the space of such functions is m dimensional, including the constant function (see e.g. ref. [18]). Consequently, a Laughlin $1/m$ state subject to periodic boundary conditions is m-fold degenerate.

We have gone into some detail regarding this apparently esoteric subject of degeneracies, both because it is important in the higher theory of the Hall states, and because some special features of the adiabatic procedure and the $\nu = 1/2$ state are tied up with it, as we shall soon see.

(2) The generalization of the exact model to the torus geometry is based on the observation that a wave function with the key properties of the Laughlin $1/m$ state may be obtained on any geometry, including the torus, from the wave function of a filled Landau level by raising the latter to the mth power,

$$\psi_m[z] = (\psi_1[z])^m. \tag{5.11}$$

Indeed, this wave function is entire, satisfies the appropriate boundary conditions, and has zeroes of order m as particles approach. This method does not display the full scope of possible choices for the center-of-mass zeroes, and consequently does not reveal the degeneracy of the states obtained. Nevertheless, it leads to fully consistent wave functions. The boundary phases are mapped according to

$$\phi_\alpha \to m\phi_\alpha. \tag{5.12}$$

Note also that the magnetic field for ψ_m is m times stronger then for ψ_1.

An observation similar to (5.11) can also be made for the exact model: the ground state for arbitrary values of the statistical parameter θ is obtained by taking

the product of the fiducial state (the state for $\theta = 0$) with the absolute value of the wave function for a single filled Landau level raised to a fractional power:

$$\psi[z] = \psi_{\theta=0}[z] \cdot |\psi_1[z]|^{\theta/\pi}. \tag{5.13}$$

Again, this relation is not specific to any geometry. Of course, any generalization following this method requires knowledge of how to generalize the fiducial state

$$\psi_{\theta=0}[z] = f[z] \prod_i \exp\left(-\tfrac{1}{4}eB_0 |z_i|^2\right) \tag{5.14}$$

(or more precisely the characteristic function f) to the desired geometry, here the torus. Note that the boundary conditions of ψ, as specified by the quasiperiodicity relation (5.4), are identical to those of $\psi_{\theta=0}$.

With these simple observations, it becomes conceptually straightforward to obtain the generalized hamiltonian for the anyon model. From (5.13), the torus ground state ψ retains the general form

$$\psi[z] = f[z] \, e^{-S_a}, \tag{5.15}$$

but S_a is now consistent with periodic boundary conditions,

$$S_a = -\frac{\theta}{\pi} \ln |\psi_1[z]| + \tfrac{1}{4}eB_0 \sum_i |z_i|^2$$

$$= -\frac{\theta}{\pi} \left(\ln |\vartheta_{\frac{1}{2},\frac{1}{2}}(Z - Z_0|\tau)| + \sum_{i<j} \ln |\vartheta_{\frac{1}{2},\frac{1}{2}}(z_i - z_j|\tau)| \right) + \tfrac{1}{4}eB \sum_i |z_i|^2, \tag{5.16}$$

where

$$eB = \frac{2(\pi + \theta)N}{\mathrm{Im}(\tau)} = \left(1 + \frac{\theta}{\pi}\right)B_0. \tag{5.17}$$

From S_a, we obtain the operators

$$a_i = e^{-S_a} \left(+\sqrt{2}\, \frac{\partial}{\partial \bar{z}_i} \right) e^{+S_a},$$

$$a_i^\dagger = e^{+S_a} \left(-\sqrt{2}\, \frac{\partial}{\partial z_i} \right) e^{-S_a}. \tag{5.18}$$

and thus the hamiltonian

$$H = \frac{1}{m} \sum_i a_i^\dagger a_i. \tag{5.19}$$

Note that a_i and $a_i^\dagger$ do not obey the commutation relations (2.8) any more. It is instructive to rewrite (5.19) in the physically more transparent form

$$H = \frac{1}{2m} \sum_i \left[(-i\, \nabla_i + eA_i)^2 - eB_i \right], \tag{5.20}$$

where

$$B_i = \nabla_i \times A_i = -\frac{2\theta}{e} \sum_{n,m=-\infty}^{+\infty} \left(\delta^2(Z - Z_0 + n + m\tau) \right.$$

$$\left. + \sum_{j \neq i} \delta^2(z_i - z_j + n + m\tau) \right) + B. \tag{5.21}$$

The hamiltonian on the torus is therefore almost what one would have anticipated naively from (2.1). The only peculiarity is its explicit dependence on the center-of-mass zero Z_0. On reflection, however, one comes to realize that this dependence is inevitable. For otherwise the ground-state degeneracy would vary with θ, and one could interpolate continuously between states having different degeneracies, which is absurd.

(3) Now let us reconsider the particular applications elaborated in the previous two sections – the lines of the Laughlin–Jastrow-type states and of the paired Hall states, for the toroidal geometry. By virtue of the formalism just developed, the problem of generalizing them to the torus geometry reduces to the problem of adapting the corresponding fiducial states.

This is trivial in the first of the two cases, for the initial state there is just a filled Landau level. The intermediate anyon states are therefore given by (5.15) with

$$f[z] = \exp(iKZ)\,\vartheta_{\frac{1}{2},\frac{1}{2}}(Z - Z_1 | \tau) \prod_{i<j} \vartheta_{\frac{1}{2},\frac{1}{2}}(z_i - z_j | \tau) \prod_i \exp\left(+\tfrac{1}{4}eBz_i^2 \right). \tag{5.22}$$

Note that the boundary phases ϕ_α are specified by K and Z_1 only, through

$$(-1)^N \exp(iK) = e^{i\phi_1},$$

$$(-1)^N \exp(iK\tau) \exp(2\pi i Z_1) = e^{i\phi_2}, \tag{5.23}$$

and that it is B_0 rather than B which enters the quasiperiodicity relation (5.4) for $\psi[z]$. The conventional boundary conditions for Laughlin's $1/m$ states, as stated above, may of course by recovered at those fillings by removing the flux tubes via a singular gauge transformation.

The second case (the paired Hall state) is much more interesting, because it requires the pfaffian to be adapted to periodic boundary conditions. This is most elegantly done by the replacement

$$\frac{1}{z_i - z_j} \rightarrow \frac{\vartheta_{a,b}(z_i - z_j \mid \tau)}{\vartheta_{\frac{1}{2},\frac{1}{2}}(z_i - z_j \mid \tau)},$$

where $\vartheta_{a,b}(z_i - z_j \mid \tau)$ is any one of the three even theta functions, i.e. a and b may take the values 0 and 0, $\frac{1}{2}$ and 0, or 0 and $\frac{1}{2}$. These different choices yield linearly independent states – which is to say, the pairing causes an additional three-fold degeneracy. Explicitly, we obtain for the characteristic functions describing paired Hall states

$$f_{a,b}[z] = \exp(iKZ)\vartheta_{\frac{1}{2},\frac{1}{2}}(Z - Z_1 \mid \tau)$$

$$\times \operatorname{Pf}\left(\frac{\vartheta_{a,b}(z_i - z_j \mid \tau)}{\vartheta_{\frac{1}{2},\frac{1}{2}}(z_i - z_j \mid \tau)}\right) \prod_{i<j} \vartheta_{\frac{1}{2},\frac{1}{2}}(z_i - z_j \mid \tau) \prod_i \exp\left(+\tfrac{1}{4}eBz_i^2\right), \quad (5.24)$$

where K, Z_1, a and b are subject to the boundary conditions

$$(-1)^N (-1)^{2a-1} \exp(iK) = e^{i\phi_1},$$

$$(-1)^N (-1)^{2b-1} \exp(iK\tau) \exp(2\pi i Z_1) = e^{i\phi_2}. \quad (5.25)$$

Note that (5.25) reduces to (5.23) in the (illegitimate) case $a = b = \frac{1}{2}$.

The factor three in the degeneracy is quite unusual. It may be an indication of non-abelian statistics [19] for the quasiparticles in this state. We hope to have more to say on this subject soon.

(4) Our constructions supply us with anyon fractional quantized Hall states for arbitrary statistics on a torus. Learned readers may find this troubling, because there are general theorems [20,21] claiming to forbid the existence of anyons on a torus, except for statistics $\theta = \pi/m$.

However there is no contradiction, when one carefully examines the premises of these theorems. For they all apply *in the absence of a magnetic field*.

To appreciate the difference a magnetic field can make, consider the simpler case of N anyons with statistical parameter θ on a sphere. If we transport one of these around the equator, the wave function must acquire a phase $e^{i2\theta n}$, where n is the number of particles in the northern hemisphere, since that many anyons are

enclosed by the loop. On the other hand it must also acquire a phase $e^{-2i\theta(N-n-1)}$, counting particles in the southern hemisphere. For consistency, we evidently must require

$$e^{i2\theta n} = e^{-i2\theta(N-n-1)} \tag{5.26}$$

or

$$\frac{\theta}{\pi} = \frac{k}{N-1} \tag{5.27}$$

where k is an integer.

When there is a magnetic field present, however, the condition is changed. The two hemispheric integrals differ by a multiplicative factor $e^{i\Phi}$, where Φ is the magnetic flux through the sphere. Thus the consistency condition is modified to read

$$\frac{\theta}{\pi} = \frac{k + \Phi/2\pi}{N-1}. \tag{5.28}$$

Clearly (5.28), as opposed to (5.27), allows arbitrary values of the statistics.

By the way, (5.28) indicates that in applying the adiabatic heuristic on a sphere containing few particles one should supply slightly less external flux as one changes the statistics (by a factor $N/(N-1)$). This can be interpreted as a manifestation of a spin-statistics connection (!). Indeed, because of the curvature of the sphere, in general the parallel transport of a particle around the equator will also involve a purely geometrical phase factor $e^{\pm i2\pi s}$, with phase proportional to the (two-dimensional) spin s. If the change in statistics involves a simultaneous change in spin – as one expects from the spin-statistics connection – then (5.28) is modified to read

$$e^{i2\theta n} = e^{-i2\theta(N-n-1)} e^{-i4\pi s} e^{i\Phi} \tag{5.29}$$

or

$$2\theta(N-1) + 4\pi s = \Phi \bmod 2\pi. \tag{5.30}$$

Thus if s changes by $\theta/2\pi$ as θ varies, a modified but basically straightforward version of the heuristic principle applies.

Similar arguments apply on a torus. Since the torus is geometrically flat, there should be absolutely no discrepancy with the heuristic principle. In fact there is not. Although the argument leading to (5.28) would suggest an $N-1$ where N should appear, that argument was not fully general. It did not allow for extra phases in addition to those associated with particles and with external magnetic field, that arise instead from more complicated multi-particle correlations in the wave function. Now we can understand on a profound level why special points, the center-of-mass zeroes, had to be introduced. They represent complicated many-

particle correlations, reflecting charge density oscillations at some essentially arbitrary fixed in space. Passage around these special points generates additional phases, and (it can be shown) modifies (5.28) to agree exactly with the heuristic principle.

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