

Spin Hamiltonian for which the Chiral Spin Liquid is the Exact Ground State

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We construct a Hamiltonian that singles out the chiral spin liquid on a square lattice with periodic boundary conditions as the exact and, apart from the twofold topological degeneracy, unique ground state.

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Introduction.—The fractionalization of quantum numbers, in which the excitations of a strongly correlated system carry only a fraction of the quantum numbers of the constituents, is currently of great interest in condensed matter physics and a significant body of recent work has focused on finding solvable theoretical models in which the phenomenon occurs [1–5]. In addition to its intrinsic interest, the phenomenon of fractionalization may well have a bearing on one of the most vexing problems in condensed matter theory, should the long-standing suggestion of a link between fractionalization and high- T_C superconductivity [6,7] be established; recently, it has been shown [8] that the topological degeneracy in these systems might be used to protect quantum bits and be applicable to the emerging field of quantum computing.

Fractional statistics, as a generalization of the idea of quantum statistics based on Berry's phase [9], is a sensible idea only in one or two dimensions, where one can define a winding number. In 1D, the behavior is known to occur in spin-1/2 antiferromagnets [10,11], where exactly solvable models exhibiting this behavior exist [12–16]. Fractionalization of statistics is also known to occur in 2D in the presence of a magnetic field that violates the discrete symmetries of parity (P) and time reversal (T); this situation is realized in the fractional quantum Hall effect [17–22] (FQHE). Very recently, the fractional statistics of the quasiparticle excitations in the FQHE has been observed experimentally [21,22]. In contrast to the one-dimensional case, however, there has been no definite evidence as to whether fractional statistics occurs in the absence of an external field breaking these symmetries.

In this Letter, we present a spin Hamiltonian for which the chiral spin liquid [23,24] (CSL) is the exact ground state. The CSL, one of the paradigmatic systems to introduce the concept of fractional statistics in 2D spin systems, is constructed to spontaneously violate the symmetries P and T ; this violation is generally associated with fractional statistics. The excitations of the liquid—spinons, which carry spin 1/2 but no charge, and holons, which carry charge but no spin—obey fractional statistics. In addition, the spinons exhibit quantum-number fractionalization and carry only half the spin of the excitations in conventional

magnetically ordered systems, which are spin-1. In many respects, the Hamiltonian we present is a generalization of the Haldane-Shastry model [12,13] (HSM) to 2D, and it provides an exact spin model in which fractional quantization can be studied. A spin Hamiltonian for a 2D system where the ground state is a chiral spin state, but not a liquid, has been discovered by Wen *et al.* [25]. These authors additionally argue for the plausibility of a CSL ground state in a Heisenberg-like model including six-site interactions; the model presented in this Letter is precisely of that form.

The proof presented below shows that the model has an exact twofold topologically degenerate ground state for any number of lattice sites N . This is in contrast to models, such as the Rokhsar-Kivelson dimer model [7] (RKM), where the topological degeneracy is only realized in the thermodynamic limit [26]. The exact topological degeneracy supports the view that the model will increase the accessibility of studying aspects of fractional statistics in 2D on an analytical and exact footing. At present, we have numerically verified the results in this Letter by exact diagonalization of the Hamiltonian on a 4×4 lattice. The numerical work confirms that the Hamiltonian has two zero-energy ground states and that these are the two topologically degenerate CSL ground states introduced below. A detailed discussion of the numerics will be presented elsewhere [27]. In the following, we briefly review the CSL ground state, present the exact parent-Hamiltonian for the state, and show analytically that our positive semidefinite Hamiltonian annihilates the CSL ground states.

Ground state.—The CSL was originally conceived by D. H. Lee as a spin liquid constructed by condensing the bosonic spin flip operators on a 2D lattice into a FQH liquid at Landau level filling factor $\nu = 1/2$. The ground state wave function for a circular droplet with open boundary conditions, on a square lattice with lattice constant of length one, is given by [23,24]

$$\langle z_1 \cdots z_M | \psi \rangle = \prod_{j < k}^M (z_j - z_k)^2 \prod_{j=1}^M G(z_j) e^{-(\pi/2)|z_j|^2}, \quad (1)$$

where $|\psi\rangle$ is always referred to as the CSL state. The z 's in

the above expression are the complex positions of the up-spins on the lattice: $z = x + iy$, with x and y integer. $G(z) = (-1)^{(x+1)(y+1)}$ is a gauge factor, which ensures that (1) is a spin singlet. Lattice sites not occupied by z 's correspond to down-spins.

For our purposes, it is propitious to choose periodic boundary conditions (PBCs) with equal periods $L_1 = L_2 = L$, L even and with $N = L^2$ sites. Following Haldane and Rezayi [28], the wave function for the CSL then takes the form

$$\langle z_1 \cdots z_M | \psi \rangle = \prod_{\nu=1}^2 \vartheta_1 \left(\frac{\pi}{L} [Z - Z_\nu] \right) \prod_{j < k}^M \vartheta_1 \left(\frac{\pi}{L} [z_j - z_k] \right)^2 \cdot \prod_{j=1}^M G(z_j) e^{(\pi/2)(z_j^2 - |z_j|^2)}, \quad (2)$$

where $M = N/2$ and $\vartheta_1(w) = -\vartheta_1(-w) \equiv \vartheta_1(w|e^{-\pi})$ is the odd Jacobi theta function [29]. The zeros for the center-of-mass coordinate $Z = \sum_j z_j$ must lie in the principal region $0 \leq \text{Re}(Z_1) < L$, $0 \leq \text{Im}(Z_1) < L$ and satisfy $Z_1 + Z_2 = L + iL$; the freedom to choose Z_1 reflects the topological degeneracy and yields two linearly independent ground states for the CSL. These states are spin singlets, are invariant under lattice translations, and are strictly periodic with regard to the PBCs.

Hamiltonian.—The Hamiltonian for which the CSL is the exact ground state is defined in terms of vector operators Ω_j as

$$H = \sum_{\langle ij \rangle} (\Omega_i - \Omega_j)^\dagger \cdot (\Omega_i - \Omega_j), \quad (3)$$

where the sum extends over all nearest-neighbor pairs on the square lattice. The vector operators contain one-through-three-site interactions and, in terms of two sets of coefficients K_{ijk} and U_{ij} , are defined as

$$\Omega_j = \sum'_{i,k \neq j} K_{ijk} \left[\frac{1}{2i} (\mathbf{S}_j \times \mathbf{S}_k) + \frac{4}{5} (\mathbf{S}_j \cdot \mathbf{S}_k) \mathbf{S}_i - \frac{1}{5} (\mathbf{S}_k \cdot \mathbf{S}_i) \mathbf{S}_j - \frac{1}{5} (\mathbf{S}_i \cdot \mathbf{S}_j) \mathbf{S}_k \right] + \sum_{i \neq j} U_{ij} \mathbf{S}_i, \quad (4)$$

where the prime on the sum indicates $i \neq k$. The coefficients $K_{ijk} = K(z_k - z_j, z_i - z_j)$ in the first term of (4) are given by

$$K(x, y) = \lim_{R \rightarrow \infty} \sum_{0 \leq |z_0 - x| \leq R} \frac{1}{x - z_0} \frac{P(x - z_0, y)}{N/2 - 1}, \quad (5)$$

where the sum over all full lattice translations $z_0 = (\ell + im)L$ guarantees periodicity in the first argument of K . The function $P(x, y)$ is given by

$$P(x, y) = \sum_{0 \leq |z_0 - y| \leq R} \frac{\text{Co} \left(\frac{\pi}{2L} [z_0 - y] \right)}{\text{Co} \left(\frac{\pi}{2L} [x - (y - z_0)] \right)} \frac{e^{-(\pi/L^2)|z_0 - y|^2}}{n(y)}, \quad (6)$$

where $\text{Co}(x) = \cos x + \cosh x$ and where $n(y)$ is a normalization factor:

$$n(y) = \lim_{R \rightarrow \infty} \sum_{0 \leq |z_0 - y| \leq R} e^{-(\pi/L^2)|z_0 - y|^2}, \quad (7)$$

chosen such that $P(0, y) = 1$. The sums in (6) and (7) enforce the periodicity of K in its second argument.

The coefficients in the second term of (4) are given by $U_{ij} = \frac{\pi}{L} U \left(\frac{\pi}{L} [z_j - z_i] \right)$, where

$$\frac{\pi}{L} U \left(\frac{\pi}{L} z \right) = \frac{\pi}{L} W \left(\frac{\pi}{L} z \right) + \frac{1}{N - 2} \cdot \left[\frac{d}{dx} P(x, -z) \Big|_0 + \lim_{R \rightarrow \infty} \sum_{0 < |z_0| \leq R} \frac{P(z_0, -z)}{z_0} \right]. \quad (8)$$

In this expression, the function $W(z)$ is the periodic extension of $1/z$ to the torus:

$$\frac{\pi}{L} W \left(\frac{\pi}{L} z \right) = \lim_{R \rightarrow \infty} \sum_{0 \leq |z_0| \leq R} \frac{1}{z - z_0}. \quad (9)$$

The Hamiltonian (3) is constructed to be positive semi-definite. Therefore, if $\Omega_i - \Omega_j$ annihilates the CSL states (2), these states will be zero-energy ground states of (3).

Proof.—In order to prove that the vector operator $\Omega_i - \Omega_j$ annihilates the CSL ground state, we first demonstrate that the related tensor operator $\omega_i - \omega_j$ annihilates it. Here, ω_i is a reducible tensor, i.e., a composition of tensor components of different ranks that may be decomposed into irreducible spherical first-rank (vector) and third-rank tensors; the operator Ω_i is the vector component of ω_i . The operator $\omega_i = \omega_i^+ - \omega_i^-$, where ω_i^\pm are related by a π rotation about the x axis, will be discussed in detail below after constructing the portion of the proof that does not depend on its precise form; it is later defined as $\omega_i^+ = T_i + V_i$ with the two operators T_i and V_i given in (13) and (14) below. The Wigner-Eckart theorem, in conjunction with the fact that the ground state defined in (2) is a spin singlet, guarantees that if $\omega_i - \omega_j$ is a destruction operator for the state, then each of its irreducible tensor components are also destruction operators. Therefore, given that the operator $\omega_i - \omega_j$ destroys the ground state, it follows that the vector operator $\Omega_i - \Omega_j$ does as well.

In order to show that the operator $\omega_i - \omega_j$ is a destruction operator for the ground state, we first demonstrate the following property:

$$\frac{\langle z_1 \cdots z_M | \omega_j | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = f(Z). \quad (10)$$

The fact that the function on the right-hand side of (10) is independent of the site-index j ensures that the difference of any two operators $\omega_i - \omega_j$ is a destruction operator. We consider only nearest-neighbor pairs of operators in constructing the Hamiltonian in (3), as this is the simplest and most local operator that is also translationally invariant. Other choices, however, are possible.

The reducible tensor operators ω_j can be decomposed into two operators as $\omega_j = \omega_j^+ - \omega_j^-$, where ω_j^+ and ω_j^- are related to each other through a π rotation about the x axis that maps S_z and S_y into $-S_z$ and $-S_y$. The operator ω_j^+ will be further decomposed as $\omega_j^+ = T_j + V_j$ with the explicit forms for these operators given in (13) and (14) below. In order to prove (10), we will first demonstrate that

$$\frac{\langle z_1 \cdots z_M | \omega_j^+ | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = f(Z) \begin{cases} 1 & z_j \in \{z_1 \cdots z_M\} \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where $f(Z)$ is an odd, periodic function of the center-of-mass coordinate Z . Using the relation between ω_j^\pm and the invariance of the CSL ground state under such a rotation, one can show, without specific knowledge of the function $f(Z)$, that

$$\frac{\langle z_1 \cdots z_M | \omega_j^- | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = f(\mathcal{W}) \begin{cases} 0 & z_j \in \{z_1 \cdots z_M\} \\ 1 & \text{otherwise.} \end{cases} \quad (12)$$

In the above expression, $\mathcal{W} = \sum w_i$ is the center of mass of the down-spins on the lattice, such that $\{w_i\}$ is the complement of $\{z_i\}$. It is straightforward to show, regardless of the chosen origin, that the sum of the two center of mass terms is a full lattice translation: $Z + \mathcal{W} = (\ell + im)L$. This means that $f(\mathcal{W}) = -f(Z)$ and, given the definition of ω_j above, (10) follows from (11).

Having developed the remainder of the argument, it remains only to demonstrate (11) for the operator ω_j^+ to prove that (3) is the exact parent Hamiltonian for the CSL; this last step is the heart of the proof. The operator ω_j^+ is defined in terms of off-diagonal and diagonal contributions as $\omega_j^+ = T_j + V_j$ where

$$T_j = \frac{1}{2} \sum_{i,k \neq j} K_{ijk} S_j^+ S_k^- (\frac{1}{2} + S_i^z) \quad (13)$$

$$V_j = \sum_{i \neq j} U_{ij} (\frac{1}{2} + S_i^z) (\frac{1}{2} + S_j^z), \quad (14)$$

with the coefficients defined in terms of the functions in (5), (7), and (8) above. Considering first the off-diagonal term, its action on the CSL ground state may be expressed as

$$\langle z_1 \cdots z_M | T_j | \psi \rangle = \frac{1}{2} \sum_{i,k \neq j} K_{ijk} \langle z_1 \cdots z_M | S_j^+ S_k^- (\frac{1}{2} + S_i^z) | \psi \rangle. \quad (15)$$

This is clearly zero if $z_j \notin \{z_1 \cdots z_M\}$ giving half of the equality in (11). Otherwise, acting onto the bra with the operator removes the site z_j and replaces it with the site z_k . In addition, the matrix element vanishes if $z_i \notin \{z_1 \cdots z_M\}$. Dividing by the wave function yields

$$\frac{\langle z_1 \cdots z_M | T_j | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = \frac{1}{2} \sum_{i,k \neq j} K_{ijk} \frac{\langle z_1 \cdots z_i \cdots z_k \cdots z_M | \psi \rangle}{\langle z_1 \cdots z_i \cdots z_j \cdots z_M | \psi \rangle}. \quad (16)$$

Using the definition in (5) above, this may be written as

$$\frac{\langle z_1 \cdots z_M | T_j | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = \frac{1}{N-2} \sum_{i \neq j} \sum_{z \neq 0} \lim_{R \rightarrow \infty} \sum_{0 \leq |z_0 - z| < R} \frac{P(z - z_0, z_i - z_j)}{z - z_0} \frac{\langle z_1 \cdots z_i \cdots z_j + z \cdots z_M | \psi \rangle}{\langle z_1 \cdots z_i \cdots z_j \cdots z_M | \psi \rangle}, \quad (17)$$

where the sum over k has been replaced by a sum over $z = z_k - z_j$. As the wave function is periodic in all of its M coordinates, the sums on z_0 and z may be replaced with a sum on $x = z - z_0$ that runs over the entire complex plane. Additionally, the ratio of wave function coefficients appearing in (16) has the form

$$\frac{\langle z_1 \cdots z_i \cdots z_j + x \cdots z_M | \psi \rangle}{\langle z_1 \cdots z_i \cdots z_j \cdots z_M | \psi \rangle} = -G(x)F(x)e^{-(\pi/2)|x|^2} \quad (18)$$

$$F(x) = e^{\pi(z_j - z_j^*)x} e^{-\pi x^2/2} \prod_{i=1}^2 \frac{\vartheta_1(\frac{\pi}{L}[Z - Z_i + x])}{\vartheta_1(\frac{\pi}{L}[Z - Z_i])} \times \prod_{k \neq i} \frac{\vartheta_1^2(\frac{\pi}{L}[z_j - z_k + x])}{\vartheta_1^2(\frac{\pi}{L}[z_j - z_k])}, \quad (19)$$

where $F(x)$ is an analytic function of x . Being careful to pick up the points excluded by the sum in (17) and using the above definition of $F(x)$, the action of T_j on the CSL ground state may be written as

$$\frac{\langle z_1 \cdots z_M | T_j | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = -\frac{1}{N-2} \sum_{i \neq j} \left[\sum_{x \neq 0} \frac{P(x, z_i - z_j)}{x} \times F(x)G(x)e^{-(\pi/2)|x|^2} + \sum_{0 < z_0} \frac{P(z_0, z_i - z_j)}{z_0} \right]. \quad (20)$$

This is the major step in the proof since the first term may be evaluated with the singlet sum-rule [30]. This sum meets the requirements for convergence [31] that were not satisfied in the original work, due to the exponential falloff of $P(x, y)$ with increasing x . This gives

$$\begin{aligned} & \sum_{x \neq 0} \frac{P(x, z_i - z_j)}{x} F(x) G(x) e^{-(\pi/2)|x|^2} \\ &= - \frac{d}{dx} \left[\frac{P(x, z_i - z_j)}{x} F(x) \right] \Big|_{x=0}. \end{aligned} \quad (21)$$

Combining this with the second term in Eq. (20) gives

$$\frac{\langle z_1 \cdots z_M | T_j | \psi \rangle}{\langle z_1 \cdots z_M | \psi \rangle} = f(Z) - \sum_{i \neq j} \frac{\pi}{L} U\left(\frac{\pi}{L} [z_j - z_i]\right), \quad (22)$$

where the function of the center-of-mass coordinate $f(Z)$, which first appears in (10), may now be written down explicitly:

$$f(Z) = - \sum_{i=1}^2 \frac{\pi}{2L} W\left(\frac{\pi}{L} [Z - Z_i]\right). \quad (23)$$

The U function appearing in (22) is the one introduced in (8) when defining the Hamiltonian. The equality here is a result of the fact that the W function, introduced in (9), is related to the logarithmic derivative of the odd Jacobi theta functions:

$$W(z) = \frac{d}{dz} \ln \vartheta_1(z) + \frac{z - z^*}{\pi}. \quad (24)$$

The operator V_j introduced in (14), which only generates diagonal terms, is chosen to exactly cancel the second term in (22). This proves the identity in (11) and from here, the arguments at the beginning of the section may be traced backwards to show that $\Omega_i - \Omega_j$ annihilates the CSL state (2), and hence that the CSL is an exact ground state of (3). The fact that the topological degeneracy is exact for any number of lattice sites N in this model is due to the fact that $\omega_i - \omega_j$ destroys the state regardless of the choice of the location of the center-of-mass zeros Z_i .

Conclusion.—We have constructed a Hamiltonian that singles out the chiral spin liquid state as the exact and, apart from the topological twofold degeneracy for PBCs, unique zero-energy ground state. The proof has been numerically verified on a 4×4 lattice. In analogy to the HSM in one dimension, this model provides a framework to study spinon excitations and their interactions in a two-dimensional spin liquid. For example, one may investigate whether the spinons in this model are similarly free in the sense that they only interact through their fractional statistics, and, if so, whether the many spinon states can be classified in similar terms [16]. In any event, we have promoted the CSL from an intriguing trial wave function to the exact ground state of a spin Hamiltonian, and we have hence accomplished something analogous to the promotion of Gutzwiller's wave function [32] to an exact solution by Haldane and Shastry [12,13].

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