

$S = 1$ Spin Liquids: Broken Discrete Symmetries Restored*

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We introduce a novel kind of spin liquid, the spin 1 chirality liquid, which provides a generic paradigm for a disordered spin 1 antiferromagnet in two dimensions. It supports spinon and holon excitations, which carry a chirality quantum number. These excitations obey half-fermi statistics. The sign of the statistical parameter is determined by the chirality. The spin 1 chirality liquid is the first example of a two dimensional quantum state which supports excitations with fractional statistics but does not violate P or T .

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1. INTRODUCTION

The key thought of this article is easily expressed. Whenever we have a spin liquid (or valence bond solid) state with spin- s on a lattice, we can obtain another spin liquid with spin- ms , where m is an integer, by combining m copies of such a state and projecting out the spin- ms representation contained in

$$\underbrace{\mathbf{s} \otimes \mathbf{s} \otimes \dots \otimes \mathbf{s}}_m = \mathbf{ms} \oplus m(\mathbf{m}-1)\mathbf{s} \oplus \dots \quad (1)$$

at each site. This is particularly useful if the spin- s state (spontaneously) violates one or many discrete symmetries, which we desire not to be violated for the spin- ms state we construct. We then only have to combine one copy of each of the “degenerate” spin- s states. If there are now m of them, we will obtain a state with spin- ms which does not violate any of these symmetries.

*Dedicated to Peter Wölfle on the occasion of his 60th birthday.

As we shall see below, the projection can be formulated very elegantly using Schwinger bosons¹.

Affleck, Kennedy, Lieb and Tasaki (AKLT)² have constructed a series of states by availing themselves of this principle for valence bond solids, which discretely violate various lattice symmetries depending on lattice type and dimension. In this article, I advocate that the principle may have other interesting applications, and introduce a particular and generically new example of a two dimensional spin 1 liquid, which I call the *spin 1 chirality liquid*. This liquid supports charge neutral spin- $\frac{1}{2}$ spinon (and spinless charge 1 holon) excitations, which carry a chirality quantum number. This number can be $\chi = +1$ or -1 . The excitations obey half-fermi statistics, both in the sense of Haldane’s generalized exclusion principle³ and in the sense of the Berry’s phases encountered by the state as particles adiabatically encircle each other⁴. The chirality quantum number determines the sign of this phase. This liquid is the first example of a state with fractional statistics excitations in more than one space dimension which does not violate the discrete symmetries parity (P) and time reversal (T).

The spin 1 chirality liquid is, at least at the time of Peter Wölfle’s 60th birthday, mainly of conceptual importance. As alluded to above, it is rather straightforward to formulate it. What is not so easily explained is the conceptual importance, the motivation. Why are we interested in spin liquids in dimensions larger than one? Why should we believe that they generically support spinon excitations, which carry spin $\frac{1}{2}$ and no charge? Why do I put forward the hypothesis that spinons in any dimension must obey half-fermi statistics?⁵

If the reader agrees with my suggested answers to these questions, he or she will be quick to realize that spin liquids in two space dimensions must either violate P and T (as it has been advocated by R.B. Laughlin⁶) or support spinon (and holon) excitations, which carry a chirality quantum number. (The individual excitations then still violate P and T , but this is not any more significant than the P and T violation of right and left moving fermions in one dimension: one can always use the complex eigenfunctions of any supposed real Hamiltonian H to construct real ones, and hence combine them into simultaneous eigenfunctions of H , P , and T .)

In this article, I present the first spin liquid of this second kind. It is a spin liquid with spin $s = 1$. By now, I have also succeeded in constructing a P and T invariant spin liquid for $s = \frac{1}{2}$, which I believe to be relevant to the problem of CuO-superconductivity¹⁰. The construction of a generically invariant spin liquid is, however, only simple in the $s = 1$ case discussed here. It is one of the more inspired applications of the simple projection method outlined above, and at the same time the simplest paradigm for a sophisti-

cated principle—the principle of fractional quantization and emerging particles without P or T violation in two-dimensional antiferromagnets—known today.

In the following three sections, I will illustrate the general projection principle with the simplest example, the spin 1 AKLT chain, which is obtained by combining two dimer or Majumdar-Gosh chains¹¹. In sections 5. and 6., I will introduce the concepts of Jastrow-type wave functions and spinon excitations in antiferromagnetically correlated spin liquids in a brief review of the Haldane-Shastry model¹², and illustrate why I believe that spinons must obey a fractional exclusion principle or fractional statistics, regardless of the model specifics including the dimension. In section 7., I will review the chiral spin liquid state^{6,7,9}, which supports spinon excitations with half-fermi statistics, but also violates P and T. I will introduce the spin 1 chirality liquid, which does not violate any symmetry, in section 8. In section 9., I will present a modest amount of numerical work on spin 1 systems. Finally, I will address the question of spinon confinement in the spin 1 chirality liquid in section 10.

2. THE MAJUMDAR-GOSH MODEL

Majumdar and Gosh¹¹ noticed in 1967 that on a linear spin $S = \frac{1}{2}$ chain with an even number of sites, the two valence bond solid or dimer states

$$\begin{aligned} |\psi_{\text{even}}^{\text{MG}}\rangle &= \prod_{\substack{i \text{ even} \\ (i \text{ odd})}} (c_{i\uparrow}^\dagger c_{i+1\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{i+1\uparrow}^\dagger) |0\rangle = \\ &= \begin{cases} | \text{---} \text{---} \text{---} \text{---} \text{---} \rangle & \text{“even”} \\ | \text{---} \text{---} \text{---} \text{---} \text{---} \rangle & \text{“odd”} \end{cases} \end{aligned} \quad (2)$$

where the product runs over all even sites i for one state and over all odd sites for the other, are exact zero energy ground states of the parent Hamiltonian

$$H^{\text{MG}} = \sum_i \left(\mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{2} \mathbf{S}_i \mathbf{S}_{i+2} + \frac{3}{8} \right). \quad (3)$$

The proof is exceedingly simple. We rewrite

$$H^{\text{MG}} = \frac{1}{4} \sum_i H_i \quad \text{with} \quad H_i = (\mathbf{S}_i + \mathbf{S}_{i+1} + \mathbf{S}_{i+2})^2 - \frac{3}{4}. \quad (4)$$

Clearly, any state in which the total spin of three neighboring spins is $\frac{1}{2}$ will be annihilated by H_i . (The total spin can only be $\frac{3}{2}$ or $\frac{1}{2}$, as $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} =$

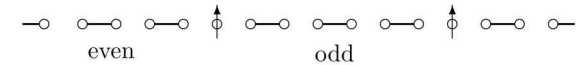
$\frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$.) In the dimer states above, this is always the case as two of the three neighboring spins are in a singlet configuration, and $0 \otimes \frac{1}{2} = \frac{1}{2}$. Graphically, we may express this as

$$H_i | \text{---} \text{---} \text{---} \rangle = H_i | \text{---} \text{---} \text{---} \rangle = 0. \quad (5)$$

As H_i is positive definite, the two zero energy eigenstates of H_{MG} are also ground states¹³.

Is the Majumdar-Gosh or dimer state in the universality class generic to one-dimensional spin- $\frac{1}{2}$ liquids, and hence a useful paradigm to understand, say, the nearest-neighbor Heisenberg chain? The answer is clearly no, as the dimer states (2) violate translational symmetry modulo translations by two lattice spacings, while the generic liquid is invariant. A useful paradigm for the generic liquid is provided by the Haldane-Shastry model, which we will review in section 5.

Nonetheless, the dimer chain shares some important properties of this generic liquid. First, the spinon excitations¹⁶—here domain walls between “even” and “odd” ground states—are free (rather than confined). Second, there are (modulo the overall two-fold degeneracy) only $M+1$ orbitals available for the spinons if $2M$ spins are condensed into dimers or valence bond singlets. This is to say, if there are only a few spinons in a long chain, the number of orbitals available to them is roughly half the number of sites. This can easily be seen graphically:



If we start with an even ground state on the left, the spinon to its right must occupy an even lattice site and vice versa. The resulting state counting is precisely what we will find in the Haldane-Shastry model, where it is directly linked to the half-fermi statistics of the spinons³.

The dimer chain is further meaningful as a piece of a general paradigm. The two degenerate dimer states (2) can be combined into an $s = 1$ chain, the AKLT chain, which serves as a generic paradigm for $s = 1$ chains which exhibit the Haldane gap^{17,18}. To formulate the AKLT model, we will avail ourselves of Schwinger bosons, which we review first.

3. SCHWINGER BOSONS

Schwinger bosons^{1,14} constitute a way to formulate spin- s representations of an $SU(2)$ algebra¹⁹. The spin operators

$$\begin{aligned} S^x + iS^y &= S^+ = a^\dagger b \\ S^x - iS^y &= S^- = b^\dagger a \\ S^z &= \frac{1}{2}(a^\dagger a - b^\dagger b) \end{aligned} \quad (6)$$

are given in terms of boson creation and annihilation operators which obey the usual commutation relations

$$\begin{aligned} [a, a^\dagger] &= [b, b^\dagger] = 1 \\ [a, b] &= [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0. \end{aligned} \quad (7)$$

It is readily verified with (7) that

$$[S^i, S^j] = i\epsilon^{ijk} S^k \quad \text{where } i, j, k = x, y, \text{ or } z. \quad (8)$$

The spin quantum number s is given by half the number of bosons,

$$2s = a^\dagger a + b^\dagger b, \quad (9)$$

and the usual spin states (simultaneous eigenstates of \mathbf{S}^2 and S^z) are given by

$$|s, m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} |0\rangle. \quad (10)$$

In particular, the spin- $\frac{1}{2}$ states are given by

$$|\uparrow\rangle = a^\dagger |0\rangle = c_\uparrow^\dagger |0\rangle \quad |\downarrow\rangle = b^\dagger |0\rangle = c_\downarrow^\dagger |0\rangle, \quad (11)$$

i.e., a^\dagger and b^\dagger act just like the fermion creation operators c_\uparrow^\dagger and c_\downarrow^\dagger in this case. The difference shows up only when two (or more) creation operators act on the same site or orbital. The fermion operators create an antisymmetric or singlet configuration (in accordance with the Pauli principle),

$$|0, 0\rangle = c_\uparrow^\dagger c_\downarrow^\dagger |0\rangle, \quad (12)$$

while the Schwinger bosons create a totally symmetric or triplet (or higher spin if we create more than two bosons) configuration,

$$\begin{aligned} |1, 1\rangle &= \frac{1}{\sqrt{2}}(a^\dagger)^2 |0\rangle \\ |1, 0\rangle &= a^\dagger b^\dagger |0\rangle \\ |1, -1\rangle &= \frac{1}{\sqrt{2}}(b^\dagger)^2 |0\rangle. \end{aligned} \quad (13)$$

Accordingly, may rewrite the Majumdar-Gosh states as

$$|\psi_{\text{even}}^{\text{MG}}\rangle = \prod_{\substack{i \text{ even} \\ (i \text{ odd})}} (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle \quad (14)$$

$$\equiv \underbrace{\Psi_{\text{even}}^{\text{MG}} [a^\dagger, b^\dagger]}_{(i \text{ odd})}$$

This formulation readily suggests a generalization to higher spin.

4. THE AKLT CHAIN

Affleck, Kennedy, Lieb and Tasaki² noticed that the valance bond solid state

$$\begin{aligned} |\psi^{\text{AKLT}}\rangle &= \prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle = \\ &= \Psi_{\text{even}}^{\text{MG}} [a^\dagger, b^\dagger] \cdot \Psi_{\text{odd}}^{\text{MG}} [a^\dagger, b^\dagger] |0\rangle = \\ &= | \text{---} \bigcirc \text{---} \rangle \end{aligned} \quad (15)$$

projections onto spin $s = 1$

is the exact zero energy ground state of the spin 1 extended Heisenberg Hamiltonian

$$H^{\text{AKLT}} = \sum_i \left(\mathbf{S}_i \mathbf{S}_{i+1} + \frac{1}{3} (\mathbf{S}_i \mathbf{S}_{i+1})^2 + \frac{2}{3} \right) \quad (16)$$

with periodic boundary conditions. Each term in the sum (16) projects onto the subspace in which the total spin of a pair of neighboring sites is $s = 2$. The Hamiltonian (16) thereby lifts all states except (15) to higher energies²⁰. The valance bond solid state (15) is a generic paradigm as it shares all the symmetries, but in particular the Haldane spin gap^{17,18}, of the spin-1 Heisenberg chain. It even offers a particularly simple understanding of this gap, or of the linear confinement force between spinons responsible for it, as illustrated by a cartoon:

$$\text{---} \text{---} \text{---} \uparrow = = \uparrow \text{---} \text{---} \text{---} \text{---}$$

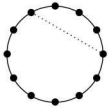
energy cost \propto distance

We have reviewed the AKLT model here as the simplest and the only generally known application of the general projection principle described in

the introduction. The Haldane gap illustrates that not only the symmetries, but also the physical properties, and in particular the low energy properties, may be very different for the combined spin- ms states as for the original spin- s state.

5. THE HALDANE-SHASTRY MODEL

The Haldane-Shastry model^{12,21,32} is the most important paradigm for a generic spin- $\frac{1}{2}$ liquid. Consider a spin- $\frac{1}{2}$ chain with periodic boundary conditions and an even number of sites N on a unit circle embedded in the complex plane:



N sites with spin $\frac{1}{2}$ on unit circle:

$$\eta_\alpha = e^{i\frac{2\pi}{N}\alpha} \quad \text{with } \alpha = 0, 1, \dots, N-1$$

The $1/r^2$ -Hamiltonian

$$H^{\text{HS}} = \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha < \beta} \frac{\mathbf{S}_\alpha \mathbf{S}_\beta}{|\eta_\alpha - \eta_\beta|^2}, \quad (17)$$

where $|\eta_\alpha - \eta_\beta|$ is the chord distance between the sites α and β , has the exact ground state

$$|\psi_0^{\text{HS}}\rangle = \sum_{\{z_1, z_2, \dots, z_M\}} \psi_0^{\text{HS}}(z_1, z_2, \dots, z_M) S_{z_1}^+ \dots S_{z_M}^+ \underbrace{|\downarrow \downarrow \dots \downarrow\rangle}_{\text{all } N \text{ spins } \downarrow} \quad (18)$$

where the sum extends over all possible ways to distribute the $M = \frac{N}{2}$ \uparrow -spin coordinates z_i and

$$\psi_0^{\text{HS}}(z_1, z_2, \dots, z_M) = \prod_{j < k}^M (z_j - z_k)^2 \prod_{j=1}^M z_j. \quad (19)$$

This state is real, a spin singlet, and has ground state energy $-\frac{\pi^2}{24} (N - \frac{5}{N})$.

The proof of solution is rather lengthy^{12,32}. We will content ourselves here to showing that (19) is real, and that it is a spin singlet. As for the former, we use

$$(z_j - z_k)^2 = -z_j z_k |z_j - z_k|^2, \quad (20)$$

to write

$$\begin{aligned} \psi_0^{\text{HS}}(z_1, z_2, \dots, z_M) &= \pm \prod_{j < k}^M |z_j - z_k|^2 \prod_{j < k}^M z_j z_k \prod_{j=1}^M z_j = \\ &= \pm \prod_{j < k}^M |z_j - z_k|^2 \prod_{j=1}^M G(z_j) \end{aligned} \quad (21)$$

where

$$G(\eta_\alpha) = (\eta_\alpha)^{\frac{N}{2}} = \begin{cases} +1 & \alpha \text{ even} \\ -1 & \alpha \text{ odd.} \end{cases} \quad (22)$$

The gauge factor $G(z_j)$ effects that the Marshall sign criteria¹⁵ is fulfilled.

We now proof that (19) is a singlet. Since $S_{\text{tot}}^z |\psi_0^{\text{HS}}\rangle = 0$, it suffices to show that $|\psi_0^{\text{HS}}\rangle$ is annihilated by S_{tot}^- :

$$\begin{aligned} S_{\text{tot}}^- |\psi_0^{\text{HS}}\rangle &= \sum_{\alpha=1}^N S_\alpha^- \sum_{\{z_1, \dots, z_M\}} \psi_0^{\text{HS}}(z_1, z_2, \dots, z_M) S_{z_1}^+ \dots S_{z_M}^+ |\downarrow \dots \downarrow\rangle = \\ &= \sum_{\{z_2, \dots, z_M\}} \underbrace{\sum_{\alpha=1}^N \psi_0^{\text{HS}}(\eta_\alpha, z_2, \dots, z_M) S_{z_2}^+ \dots S_{z_M}^+ |\downarrow \dots \downarrow\rangle}_{=0} \end{aligned} \quad (23)$$

since $\psi_0^{\text{HS}}(\eta_\alpha, z_2, \dots, z_M)$ contains powers $\eta_\alpha^1, \eta_\alpha^2, \dots, \eta_\alpha^{N-1}$ and

$$\sum_{\alpha=1}^N \eta_\alpha^n = N \delta_{n,0} \pmod{N}. \quad (24)$$

The elementary excitations for this model are deconfined and only weakly interacting spinon excitations, which carry spin- $\frac{1}{2}$ and no charge. They constitute an instance of fractional quantization, similar (both conceptually and mathematically) to the fractional quantization of charge in the fractional quantum Hall effect²³. Their fractional quantum number is the spin, which takes the value $\frac{1}{2}$ in a Hilbert space (18) made out of spin flips S^+ , which carry spin 1.

To write down the wave for a \downarrow spin spinon localized at site η_α , consider a chain with an odd number of sites N and let $M = \frac{N-1}{2}$ be the number of \uparrow or \downarrow spins condensed in the uniform liquid. The spinon is then given by

$$\psi_{\alpha\downarrow}^{\text{HS}}(z_1, z_2, \dots, z_M) = \prod_{j=1}^M (\eta_\alpha - z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_{j=1}^M z_j, \quad (25)$$

which we again understand substituted into (18). It is easy to verify that $S_{\text{tot}}^z \psi_{\alpha\downarrow}^{\text{HS}} = -\frac{1}{2} \psi_{\alpha\downarrow}^{\text{HS}}$ and that $S_{\text{tot}}^- \psi_{\alpha\downarrow}^{\text{HS}} = 0$, which shows that the spinon transforms as a spinor under rotations.

The localized spinon (25) is not an eigenstate of the Haldane-Shastry Hamiltonian (17). To obtain exact eigenstates, we construct momentum eigenstates according to

$$\psi_{\alpha\downarrow}^{\text{HS}}(z_1, z_2, \dots, z_M) = \sum_{\alpha=1}^N \bar{\eta}_{\alpha}^m \psi_{\alpha\downarrow}^{\text{HS}}(z_1, z_2, \dots, z_M). \quad (26)$$

where the integer m corresponds to a momentum quantum number. Since $\psi_{\alpha\downarrow}^{\text{HS}}(z_1, z_2, \dots, z_M)$ contains only powers $\eta_{\alpha}^0, \eta_{\alpha}^1, \dots, \eta_{\alpha}^M$ and

$$\sum_{\alpha=1}^N \bar{\eta}_{\alpha}^m \eta_{\alpha}^n = \delta_{mn} \mod N, \quad (27)$$

$\psi_{\alpha\downarrow}^{\text{HS}}(z_1, z_2, \dots, z_M)$ will vanish unless $m = 0, 1, \dots, M$. There are only roughly half as many spinon orbitals as sites. Spinons on neighboring sites hence cannot be orthogonal.

To make a correspondence between m and the usual wave number q , we translate (26) counterclockwise by one lattice spacing around the unit circle

$$\mathbf{T} |\psi_{\alpha\downarrow}^{\text{HS}}\rangle = e^{iq} |\psi_{\alpha\downarrow}^{\text{HS}}\rangle, \quad (28)$$

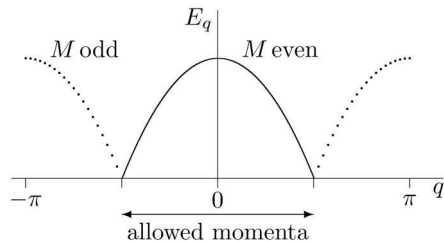
and find

$$q = \pi M + \pi \frac{M - 2m}{2M + 1}. \quad (29)$$

The spinon dispersion is given by

$$E_q = \frac{1}{2} \left[\left(\frac{\pi}{2} \right)^2 - q^2 \right] \mod \pi, \quad (30)$$

as depicted below.



The construction (25,26) can be generalized to many spinon states, which weakly attract each other. Then the individual spinon momenta labeled by m_l ($l = 1, 2, \dots, L$, where L is the number of spinons) are no longer good quantum numbers, as they have a channel to decay into. The construction can, however, still be used to count the number of available orbitals for spinons, which leads us to the subject of fractional statistics.

6. FRACTIONAL STATISTICS IN SPIN LIQUIDS

Consider a Haldane-Shastry chain with an even number of sites N and an even number of spinons L . According to (25),(26) and (27), the number of orbitals available to each spinon is given by $M + 1$, where $M = \frac{N-L}{2}$ is the number of \uparrow or \downarrow spins in the remaining uniform liquid. (In this representation, the spinon wave functions are symmetric; two or more spinons can have the same value for m .) The creation of *two* spinons reduces the number of available orbitals hence by *one*. They obey half-fermi statistics in the sense of Haldane's exclusion principle³. (For fermions, the creation of two particles would decrease the number of available orbitals by two, while this number would not change for bosons.)

It is very instructive to use this fractional exclusion principle to count the dimension of the Hilbert space spanned by the ground state plus all possible spinon configurations³. In general, the number of ways to place x bosons (as the spinon wave functions are symmetric—the spinons are half-fermions formulated in a bosonic representation) into y orbitals is given by

$$\frac{(x + y - 1)!}{x! (y - 1)!}. \quad (31)$$

For a configuration with L spinons, we have $x = L$ and $y = 2(M + 1) = N - L + 2$ orbitals, where the factor 2 stems from the internal spin degeneracy of each spinon orbital. The total number of states is

$$\sum_{\substack{L=0 \\ \text{even}}}^N \frac{(N + 1)!}{L! (N - L + 1)!} = 2^N \quad (32)$$

as it should be for a spin $\frac{1}{2}$ system with N sites.

We may interpret the Haldane-Shastry model as a reparameterization of a Hilbert space spanned by spin flips (18) into a basis which consists of the Haldane-Shastry ground state plus all possible spinon excitations. The reward for such a reparameterization is that a highly non-trivial Hamiltonian in the original basis may be approximately or exactly diagonal in

the new basis. This reparameterization is particularly useful as it usually instructs us directly about the quantum numbers of the elementary excitations. In the fractional quantum Hall effect, we have learned from such a reparameterization—or trial wave function—that the quasiparticles carry fractional charge. For the spin $\frac{1}{2}$ chain, we can learn that the elementary excitations are spinons, which carry spin $\frac{1}{2}$.

Of course, it is a condition that the Hilbert space dimensions before and after the reparameterization match. It is not very difficult to convince oneself that the fractional state counting, and hence the fractional or half fermi statistics, is a necessary condition to accomplish this for a spin $\frac{1}{2}$ system with spin $\frac{1}{2}$ spinon excitations. This leads me to the following hypothesis:

Spinons must carry fractional or half-fermion⁵ statistics, regardless of the dimension, in order for the states to count up consistently.

In one space dimension, fractional statistics can only be defined through a fractional exclusion principle, which does not yield any symmetry violations. In two space dimensions, however, we can define statistics alternatively through the Berry's phase acquired by the state as we adiabatically exchange particles by moving them counterclockwise around each other⁴. This phase is given by $e^{i\theta}$, where θ is the statistical parameter.



$$|\psi\rangle \rightarrow e^{i\theta} |\psi\rangle$$

For $\theta = \pi$ we have fermions, for $\theta = 0$ bosons, and for both $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ half-fermions. The choice of sign for θ is physically meaningful, as the allowed values for the relative angular momentum l depend on it:

$$l = \frac{\theta}{\pi} + 2n \quad \text{where } n \text{ integer.} \quad (33)$$

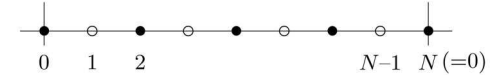
Clearly either choice of sign for half-fermions violates parity (P) and time reversal (T).

If we now assume that both definitions of statistics match for spinons in two dimensional spin liquids, as they do for the quasiparticles in the fractionally quantized Hall effect, the hypothesis of half fermi statistics leaves us with only two choices. The first is that the spin liquid ground state violates P and T spontaneously and thereby fixes the sign for the statistical parameter for the spinons. This possibility has been advocated by R.B. Laughlin. The chiral spin liquid^{6,7,9}, which we will review in the following section, provides us with a paradigm for this situation.

The second possibility, which I would like to advocate, is that the spinons carry a chirality quantum number, which determines the sign of the statistical parameter θ . Then there is no need for the ground state to violate any symmetries. Spinons of different chiralities map into each other under P and T. The situation is somewhat analogous to fermions in one dimension, where we have right and left moving excitations, which are P and T conjugates of each other. I will construct the simplest paradigm for such a liquid, the spin 1 chirality liquid, in section 8.

7. THE CHIRAL SPIN LIQUID

The chiral spin liquid⁶⁻⁹ may be viewed as a brute-force generalization of the Haldane-Shastry wave function to two space dimensions²⁴. Consider a periodic one-dimensional lattice on the real axis of a complex plane, with lattice points at integer values:

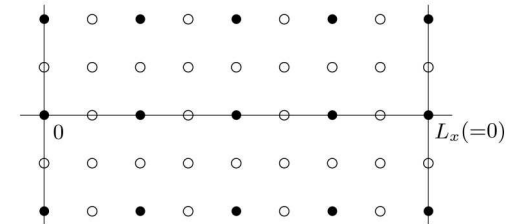


Filled and empty circles represent even and odd integers, with gauge factors $G(z) = -1$ and $G(z) = +1$, respectively. The Haldane-Shastry wave functions (21) then becomes

$$\psi_0^{\text{HS}}(z_1, \dots, z_M) = \prod_{j=1}^M G(z_j) \prod_{j < k}^M \sin\left(\frac{\pi}{N}(z_j - z_k)\right)^2, \quad (34)$$

where we took advantage of the fact that now the coordinates z_j are real.

The chiral spin liquid is obtained by extending the lattice from a circle to a cylinder, or from a segment of the real axis to a strip in the two-dimensional complex plane:



where $G(z) = (-1)^{(x+1)(y+1)}$ for lattice site $z = x + iy$. The wave function for a chiral spin liquid on this cylinder is given by (34) multiplied by an exponential factor, which effects that the density of spin flips is $\frac{1}{2}$ for $-L_y/2 < y < L_y/2$, i.e., up to the boundaries of the liquid:

$$\psi_+^{\text{CSL}}(z_1, \dots, z_M) = \prod_{j=1}^M G(z_j) \prod_{j < k}^M \sin\left(\frac{\pi}{L_x}(z_j - z_k)\right)^2 \prod_{j=1}^M e^{-\pi|y_j|^2}. \quad (35)$$

The chiral spin liquid takes a more familiar form if we consider open boundary conditions. The wave function for a circular droplet of fluid is simply

$$\psi_+^{\text{CSL}}(z_1, \dots, z_M) = \prod_{j=1}^M G(z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_{j=1}^M e^{-\frac{\pi}{2}|z_j|^2}. \quad (36)$$

Note that the exponential in (35) or (36) corresponds to a (fictitious) magnetic field of strength $2\pi/\text{plaquet}$. Apart from the gauge factors $G(z_j)$, (36) is formally equivalent to the wave function for a fractionally quantized Hall liquid for bosons at filling fraction $\nu = \frac{1}{2}$, which corresponds to half a particle (or spin flip) per plaquet. ψ_+^{CSL} is a spin singlet.

The chiral spin liquid is in general complex as the z 's are complex. There are two “degenerate” states, ψ_+^{CSL} and its complex conjugate ψ_-^{CSL} , which are P and T conjugates of each other. (I have written “degenerate” in quotation marks as no-one has ever been able to construct a local parent Hamiltonian for this liquid.) Since the chiral spin liquid violates the discrete symmetries P and T, it is not in the same universality class as the generic disordered spin $\frac{1}{2}$ antiferromagnet in two space dimensions, which I presume to be stabilized for some lattice type or upon doping.

It is rather easy to formulate spinon excitations for the chiral spin liquid. In analogy to both the quasiholes in fractionally quantized Hall liquids and the spinons (25) for the Haldane-Shastry model, we write the wave function for a spinon localized at η

$$\psi_{+\eta\downarrow}^{\text{CSL}}(z_1, \dots, z_M) = \prod_{j=1}^M (\eta_\alpha - z_j) \prod_{j=1}^M G(z_j) \prod_{j < k}^M (z_j - z_k)^2 \prod_{j=1}^M e^{-\frac{\pi}{2}|z_j|^2}. \quad (37)$$

The spinons obey half-fermi statistics, both in the sense of Haldane's exclusion principle as well as in the sense of the Berry's phases encountered by the state as we exchange particles by winding them counterclockwise around each other. We can evaluate this phase using the adiabatic transport argument of Arovas, Schrieffer, and Wilczek²⁵. We obtain $\pi/2$ for (37) and $-\pi/2$ for its complex conjugate.

The chiral spin liquid is a wonderful paradigm, with only one imperfection: the violation of P and T. There is no indication that these symmetries are spontaneously broken in any two-dimensional spin system.

Like in the case of the Majumdar-Gosh state discussed in section 2., the chiral spin liquid can be used to construct a symmetry invariant spin liquid with $s = 1$, the spin 1 chirality liquid. For this construction, it is propitious to formulate the chiral spin liquid in a basis which constitutes of electron creation operators rather than spin flips. To begin with, we use $M = N/2$ to rewrite (36) as

$$\psi_+^{\text{CSL}}(z_1, \dots, z_M) = \prod_{j=1}^M \left(G(z_j) e^{-\frac{M\pi}{N}|z_j|^2} \prod_{\substack{k=1 \\ (k \neq j)}}^M (z_j - z_k) \right). \quad (38)$$

Let w_l with $l = 1, 2, \dots, N - M$ be the lattice sites not occupied by the z_j 's. Then the octopus theorem⁶

$$G(z_j) e^{-\frac{\pi}{2}|z_j|^2} \prod_{\substack{\alpha=1 \\ (\eta_\alpha \neq z_j)}}^N (z_j - \eta_\alpha) = \prod_{\alpha=1}^N e^{+\frac{\pi}{2N}|\eta_\alpha|^2} \cdot \text{const.}, \quad (39)$$

where the first product runs over all lattice sites except z_j , implies

$$\psi_+^{\text{CSL}}(z_1, \dots, z_M) = \prod_{j=1}^M \left(\prod_{l=1}^M \frac{1}{z_j - w_l} \right) \prod_{j=1}^M e^{+\frac{(N-M)\pi}{2N}|z_j|^2} \prod_{l=1}^M e^{+\frac{M\pi}{2N}|w_l|^2}. \quad (40)$$

Let

$$\mathcal{S}[z, w] \equiv \langle 0 | c_{z_1\downarrow} \dots c_{z_M\downarrow} c_{w_1\downarrow} \dots c_{w_{N-M}\downarrow} \underbrace{|\downarrow\downarrow \dots \downarrow\rangle}_{\text{all } N \text{ spins } \downarrow} \rangle, \quad (41)$$

be the sign associated with ordering the z 's and w 's according to their lattice positions. Then we can use

$$\begin{aligned} & \prod_{j < k}^M (z_j - z_k) \prod_{j=1}^M \prod_{l=1}^{N-M} (z_j - w_l) \prod_{l < m}^{N-M} (w_l - w_m) \cdot \\ & \cdot \prod_{j=1}^M e^{-\frac{\pi}{2}|z_j|^2} \prod_{l=1}^{N-M} e^{-\frac{\pi}{2}|w_l|^2} = \mathcal{S}[z, w] \cdot \text{const.} \end{aligned} \quad (42)$$

to rewrite (40) as

$$\psi_+^{\text{CSL}}(z_1, \dots, z_M) = \mathcal{S}[z, w] \phi(z_1, \dots, z_M) \phi(w_1, \dots, w_{N-M}) \quad (43)$$

where

$$\phi(z_1, \dots, z_M) = \prod_{j < k}^M (z_j - z_k) \prod_{j=1}^M e^{-\frac{M\pi}{2N} |z_j|^2} \quad (44)$$

is simply the wave function for a filled Landau level in a (fictitious) magnetic field with flux $\frac{2\pi M}{N}$ /plaquet. If we rewrite (36) in terms of (44) and compare it to (43), we obtain the lattice particle-hole symmetry

$$\prod_{j=1}^M G(z_j) \phi(z_1, \dots, z_M) = \mathcal{S}[z, w] \phi(w_1, \dots, w_{N-M}) \cdot \text{const.}, \quad (45)$$

which holds for any M . The chiral spin liquid ground state, where $M = N/2$, is according to (43) simply given by

$$|\psi_+^{\text{CSL}}\rangle = \sum_{\{z_1, \dots, z_M; w_1, \dots, w_M\}} \phi(z_1, \dots, z_M) \phi(w_1, \dots, w_M) \cdot c_{z_1 \uparrow}^\dagger \dots c_{z_M \uparrow}^\dagger c_{w_1 \downarrow}^\dagger \dots c_{w_M \downarrow}^\dagger |0\rangle, \quad (46)$$

where the sum extends over all possible ways to distribute the coordinates z_j and w_l on mutually distinct lattice sites. It is often convenient to write (46) as

$$|\psi_+^{\text{CSL}}\rangle = P_{\text{GW}} |\psi_{\text{SD}}^N\rangle \quad (47)$$

where the Gutzwiller projector

$$P_{\text{GW}} \equiv \prod_{i=1}^N (1 - c_{i \uparrow}^\dagger c_{i \uparrow} c_{i \downarrow}^\dagger c_{i \downarrow}) \quad (48)$$

eliminates doubly occupied sites and $|\psi_{\text{SD}}^N\rangle$ is the Slater determinant wave function for the lowest Landau level filled once with $M = \frac{N}{2}$ \uparrow -spin and once with M \downarrow -spin electrons.

The spin singlet property of the chiral spin liquid mentioned above is now evident. Clearly the Slater determinant $|\psi_{\text{SD}}^N\rangle$ is a singlet. Since the Gutzwiller projector commutes with the spin operator on each site,

$$[P_{\text{GW}}, \mathbf{S}_i] = 0, \quad (49)$$

$|\psi_+^{\text{CSL}}\rangle$ must be a singlet as well.

The Gutzwiller form (47) also allows a rather elegant formulation of the spinon excitations. For example, a state with L \downarrow -spin spinons at sites η_1, \dots, η_L is given by

$$|\psi_{+\eta_1 \dots \eta_L \downarrow}^{\text{CSL}}\rangle = P_{\text{GW}} c_{\eta_1 \uparrow} \dots c_{\eta_L \uparrow} |\psi_{\text{SD}}^{N+L}\rangle, \quad (50)$$

where $N + L = 2M$ must be an even integer. The equivalence of (50) to (37) is readily seen with (45). This form nicely illustrates the (fractional) spin $\frac{1}{2}$ of the spinon. The electron annihilation operators create inhomogeneities in spin and charge before projection. The projector enforces one particle per site and hence restores the homogeneity in the charge distribution, but commutes with spin. We are left with a neutral object of spin $\frac{1}{2}$.

Note that the spinon coordinates η in either (37) or (50) do not have to coincide with lattice points. This allows us to define a winding path and thus a statistical parameter via a Berry's phase encountered upon adiabatic interchange.

Holon excitations, which carry a positive unit charge and no spin, are constructed from spinon excitations by annihilation of an electron at the spinon site, which must now coincide with a lattice point. For example, a chiral spin liquid with L holons at sites η_1, \dots, η_L is given by

$$|\psi_{+\eta_1 \dots \eta_L \circ}^{\text{CSL}}\rangle = c_{\eta_1 \downarrow} \dots c_{\eta_L \downarrow} P_{\text{GW}} c_{\eta_1 \uparrow} \dots c_{\eta_L \uparrow} |\psi_{\text{SD}}^{N+L}\rangle, \quad (51)$$

where $N + L = 2M$ must again be an even integer.

The chiral spin liquid, when formulated as a Gutzwiller projection of filled Landau levels, is readily generalized to different lattice types.

8. THE SPIN 1 CHIRALITY LIQUID

The spin 1 chirality liquid is obtained by combining ψ_+^{CSL} and its complex conjugate ψ_-^{CSL} via a projection to spin 1 on each site, which can be accomplished elegantly via Schwinger bosons. Let

$$\begin{aligned} |\psi_+^{\text{CSL}}\rangle &= \sum_{\{z_1, \dots, z_M; w_1, \dots, w_M\}} \phi(z_1, \dots, z_M) \phi(w_1, \dots, w_M) \cdot a_{z_1}^\dagger \dots a_{z_M}^\dagger b_{w_1}^\dagger \dots b_{w_M}^\dagger |0\rangle = \\ &\equiv \Psi_+^{\text{CSL}}[a^\dagger, b^\dagger] |0\rangle. \end{aligned} \quad (52)$$

be a chiral spin liquid (46) written in terms of Schwinger bosons. The spin 1 chirality liquid is then given by

$$|\psi^{\text{S1CL}}\rangle = \Psi_+^{\text{CSL}}[a^\dagger, b^\dagger] \Psi_-^{\text{CSL}}[a^\dagger, b^\dagger] |0\rangle. \quad (53)$$

This is the central proposal of this article.

The spinon or holon excitations for the spin 1 chirality liquid are just the spinon or holon excitations of the individual chiral spin liquids. As the spinons for the chiral spin liquid are massive, we expect an energy gap for

the spin 1 liquid as well. The spinons in the latter carry a chirality quantum number χ , which is $+1$ if they are constructed as excitations of ψ_+^{CSL} and -1 if they are constructed as excitations of ψ_-^{CSL} .

Let us first consider the case where we only have spinons of chirality $+$. They span a Hilbert space of dimension 2^N , just as in the case of the chiral spin liquid or Haldane-Shastry chain. The spinons obey a fractional exclusion principle according to half fermi statistics. The adiabatic transport argument of Arovas, Schrieffer, and Wilczek²⁵ remains unaffected by the projection as well. We can evaluate the phase acquired by $\Psi_+^{\text{CSL}}[a^\dagger, b^\dagger]$ and hence $|\psi^{\text{S1CL}}\rangle$ as we exchange two $+$ chirality spinons by winding them counterclockwise around each other, and obtain $\pi/2$. If we had only spinons of chirality $-$, we would obtain $-\pi/2$. In this sense, spinons of either chirality obey half-fermi statistics, with statistical parameter

$$\theta = \chi \frac{\pi}{2}. \quad (54)$$

Non-trivialities arise, however, as we consider states with spinons of both chiralities. If spinons of different chiralities were independent of each other, the total Hilbert space dimension of the ground state plus all possible spinon excitations would be $2^N \cdot 2^N = 4^N$. The Hilbert space dimension for a spin 1 system with N sites, however, is only 3^N . The spinon basis is consequently overcomplete, and spinons of different chiralities cannot always be orthogonal. They have a significant overlap when they are spatially close to each other.

What does this mean? One possibility is that a parameterization of the spin 1 Hilbert space in terms of a ground state plus spinons excitations is not sensible. This would be the case if there is no energy scale at which spinon degrees of freedom offer an adequate description of the system. I do not believe in this possibility but cannot rule it out at this stage.

The other possibility is that there is a way to transform the 4^N non-orthogonal orbitals into 3^N orthogonal orbitals, which then provide the desired Hilbert space basis. Note that a similar situation had occurred in the Haldane-Shastry model, when we first constructed spinon excitations in real space and ended up with a vastly overcomplete basis. The problem was resolved by a transformation to momentum space. Of course, the possibility of such a transformation confronts us with a variety of unresolved questions. The most pressing one, in my opinion, concerns the statistics. Since we do not expect the transformation to dissolve the concept of spinons, it will be linear and hence preserve the fractional exclusion principle. It is not clear, however, what happens to chirality quantum numbers, winding phases, and relative angular momenta.

Fortunately, the issue of overcompleteness only matters when we have

spinons of different chiralities nearby. For the low energy dynamics of a spin 1 chirality liquid another question is more immediate: are the spinons free, as they are in spin $\frac{1}{2}$ chains or in chiral spin liquids, or are they confined, as they are in spin 1 chains? Even though I will not be able to give a definite answer, I will present a modest amount of evidence for a conjecture in section 10. First, however, I will interpret a few numerical studies on the the spin 1 chirality liquid ground state in the following section.

9. FINITE SIZE STUDIES

In this section, we report on a few numerical studies of the spin 1 chirality liquid (S1CL) for square, triangular and kagomé lattices, as well as the chiral spin liquids (CSLs) used to construct them. The calculations are performed on finite size systems with periodic boundary conditions (PBC). To construct the spin liquid states for this geometry, we have to generalize the wave function (44) for a filled Landau level to PBCs, which is in essence done by replacing the factors $(z_j - z_k)$ by odd Jacobi theta functions $\vartheta_{\frac{1}{2}, \frac{1}{2}}(z_j - z_k, \tau)$.²⁶

The most severe limitation for exact diagonalization studies is the size of the Hilbert space. For a spin 1 system, the dimension of the subspace with $S_{\text{tot}}^z = 0$ is given by

$$\sum_{n=0}^{N/2} \frac{N!}{n! n! (N - 2n)!} \quad (55)$$

which amounts to 5,196,627 for $N=16$. This limits the system size to an order of 16 sites, which is far too small to address most questions of interest. For example, it is well established that all the spin 1 Heisenberg antiferromagnets we study possess long range order. This feature, however, is hardly manifest in the systems we consider here. The comparisons of the spin liquid trial wave functions with exact ground states of Heisenberg models reported in Table 1 merely indicate to which extent the energetically relevant nearest-neighbor correlations of the models are captured by the liquids. If these correlations are sufficiently close, we may conjecture that the spin liquid is stabilized if we disorder the system by doping.

Let us now take a look at Table 1. First of all, the results confirm that the P and T violating CSLs are inadequate for all three lattice types. The energies always differ by an amount of order 10 % or larger, which is enormous in light of the fact that the energy of the classical ground state is only about 30 % higher than the energy of the quantum model. Let us now turn to the S1CL for the square lattice. For a Heisenberg antiferromagnet

Lattice type		square		triangular	kagomé
		$J'=0$	$J'=\frac{1}{2}J$		
Number of sites N		16		16	12
Principal displacement \mathbf{L}_1		(4,0)		(4,0)	(4,0)
\mathbf{L}_2		(0,4)		(2,2 $\sqrt{3}$)	(2,2 $\sqrt{3}$)
LL bound. phases φ_1, φ_2		π, π		$\frac{2\pi}{3}, \frac{4\pi}{3}$	$\frac{\pi}{4}, \frac{3\pi}{4}$
$S = \frac{1}{2}$	E_{exact}	-11.2285	-8.4579	-8.5555	-5.4448
	E_{CSL}	-8.2572	-7.6435	-7.8251	-4.9756
	$\langle \psi^{\text{CSL}} \psi^{\text{exact}} \rangle$.6727	.8626	.0000	.0000
$S = 1$	E_{exact}	-38.3047	-25.0858	-25.3699	-17.6208
	E_{S1CL}	-30.7987	-24.4782	-24.1504	-15.7025
	$\Delta E/E_{\text{exact}}$	19.6 %	2.42 %	4.80 %	10.89 %
	$\langle \psi^{\text{S1CL}} \psi^{\text{exact}} \rangle$.6019	.9484	.7948	.5527
	$E_{\text{S1F1/2}}$	-38.1286	-24.8957	-25.0184	-15.9249
	$\Delta E/E_{\text{exact}}$	0.46 %	0.76 %	1.39 %	9.62 %
	$\langle \psi^{\text{S1F1/2}} \psi^{\text{exact}} \rangle$.9927	.9784	.8962	.02954

Table 1. Energy expectation values and overlaps for the chiral spin liquid (CSL), the spin 1 chirality liquid (S1CL), and a spin 1 liquid constructed by combining two identical copies of the exact spin $\frac{1}{2}$ ground states according to the Schwinger boson projection principle discussed in this article (S1F1/2), for finite lattices of various types with PBC. The trial states are compared to exact ground states of nearest-neighbor Heisenberg models with $J = 1$ for all three lattice types, plus a model augmented by a next-nearest-neighbor coupling J' on the square lattice. The displacement vectors \mathbf{L}_1 , \mathbf{L}_2 span the principal region for the PBCs in units of lattice constants. The boundary phases φ_1 and φ_2 are a property of the Landau level wave functions adapted to PBCs²⁶ and projected onto the lattice, which are used to construct the chiral spin liquids: φ_1 and φ_2 are the phases acquired by $\phi_{\mathbf{L}_1, \mathbf{L}_2}(z_1, \dots, z_M)$ as one of the coordinates (which must coincide with a lattice point) is translated by \mathbf{L}_1 or \mathbf{L}_2 , respectively, plus the phase acquired by a unit charge under this translation through the coupling to the vector potential generating the magnetic field.²⁷ These phases have been chosen to minimize the energies of the corresponding spin liquids, which slightly depend on it.

with only a nearest-neighbor coupling J , the S1CL is way off as well, as shown in the first column of Table 1. If we use, however, the S1CL as a trial wave function for a Heisenberg model with a next-nearest-neighbor coupling $J' = J/2$, we find an energy difference of 2.42 % and an overlap of 95 %, which is reasonable. There is a very realistic chance that the S1CL is realized in this antiferromagnet if it is disordered by additional frustration or disorder. It may further be a candidate for a disordered state on the triangular lattice but appears inadequate for the kagomé lattice.

In the last three lines on the bottom of Table 1, I have evaluated energies and overlaps of yet another trial wave function for $s = 1$ antiferromagnets, which is simply obtained by combining two identical copies of the exact ground states for the corresponding $s = \frac{1}{2}$ Heisenberg model

$$|\psi^{\text{exact}}\rangle = \Psi^{1/2}[a^\dagger, b^\dagger] |0\rangle \quad (56)$$

by Schwinger boson projection:

$$|\psi^{\text{S1F1/2}}\rangle = \Psi^{1/2}[a^\dagger, b^\dagger] \Psi^{1/2}[a^\dagger, b^\dagger] |0\rangle. \quad (57)$$

I call this trial state “spin 1 from 1/2” (S1F1/2). Comparisons of this state with the exact ground states of the spin 1 antiferromagnets reveal information about how different the correlations in the spin 1 and spin $\frac{1}{2}$ models are. For the nearest-neighbor Heisenberg model on the square lattice, the energy of these states differs only by 0.46 %, with an overlap of 99 %. I attribute this to the fact that both states share the same type of long range order. For the triangular lattice, this overlap is already reduced to 90 %. I interpret this as an indication that the order is much more pronounced in the triangular spin 1 system. For the kagomé lattice, the overlap is virtually zero, which confirms that spin $\frac{1}{2}$ and spin 1 models possess very different correlations. The spin $\frac{1}{2}$ system is an intrinsically disordered spin liquid²⁸, while the spin 1 model on the kagomé lattice possesses order. Most likely, there is a disordered state for the spin 1 kagomé system which can be stabilized by doping, but it is apparently not obtained by Schwinger boson projection of two identical spin $\frac{1}{2}$ liquids. I conjecture that nature avails herself of the possibility that the individual spin $\frac{1}{2}$ states used in this construction may violate a discrete symmetry, which is not P or T, but symmetry under rotations by $\pi/3$ modulo $2\pi/3$. This is, however, merely a speculation.

10. SPINON CONFINEMENT

In section 4., we illustrated with a little cartoon how a linear confinement force between spinons arises for the AKLT chain, even though spinons

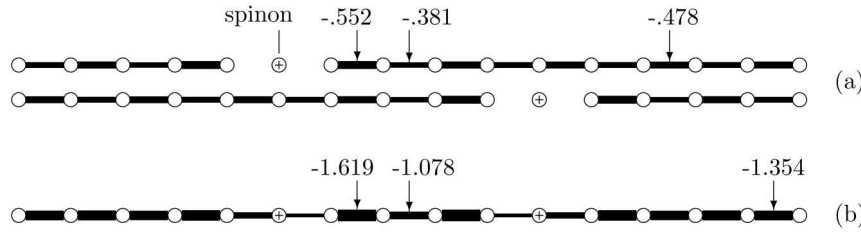


Fig. 1. Spin-spin correlations in two spin $\frac{1}{2}$ Haldane-Shastry chains with 15 sites and one spinon each (a) and a spin 1 Gutzwiller chain (b) constructed via Schwinger boson projection from two Haldane-Shastry chains with spinons at different sites. The thickness of the lines reflects the magnitudes of the antiferromagnetic correlations $\langle \mathbf{S}_i \mathbf{S}_{i+1} \rangle$ on nearest neighbor sites.

are free for the individual Majumdar-Gosh chains. This example might suggest that if spinons are confined, this property can always be understood through such a simple picture. Let me first demonstrate that this is not the case, and then turn to the subtle question of whether spinons are confined in the spin 1 chirality liquid.

Consider a spin 1 Gutzwiller chain (S1GW), that is, a spin 1 chain obtained by Schwinger boson projection of two identical Haldane-Shastry chains:

$$|\psi^{\text{S1GW}}\rangle = \Psi_0^{\text{HS}}[a^\dagger, b^\dagger] \Psi_0^{\text{HS}}[a^\dagger, b^\dagger] |0\rangle. \quad (58)$$

As the individual Haldane-Shastry chains are completely disordered, there is no indication of confining forces on the level of the cartoon mentioned above. Nonetheless, the S1GW chain does not support free spinons. There are several ways to see this. The first is that the S1GW chain is in the same universality class as the AKLT or spin 1 Heisenberg chain, which is known to possess an energy gap²⁹. Since the spinons for the Haldane-Shastry are (in contrast to the Majumdar-Gosh chain or the chiral spin liquid) massless, the energy gap must stem from spinon-spinon interactions, or confining forces. This is in accordance with Haldane's analysis^{17,18} of the one-dimensional nonlinear σ model, which shows that a topological term and hence deconfined spinons exist only for half-integer spin chains.

The second way to see that spinons in the S1GW chain are confined is by direct computation of the confining potential, as shown in Fig. 1. The numerically accessible system size is again too small for a full analysis, as we must consider periodic boundary conditions³⁰ and are hence limited to a few lattice spacings when we pull two spinons apart. Nonetheless, Fig. 1 clearly

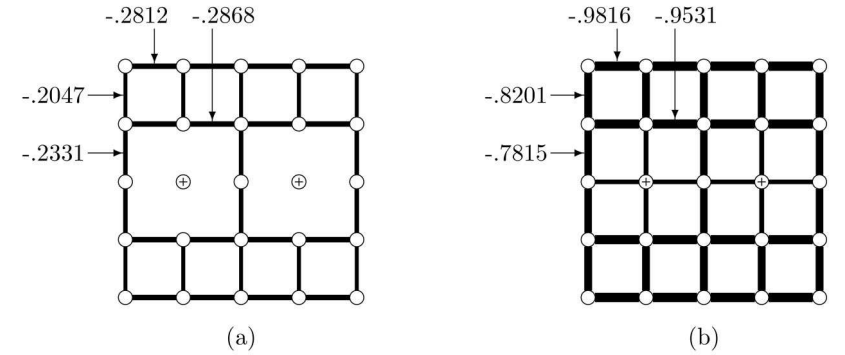


Fig. 2. Spin-spin correlations in a chiral spin liquid (a) and a spin 1 chirality liquid (b) constructed with it for a 16 site square lattice with periodic boundary conditions in the presence of two spinons at the positions indicated.

shows a dimerization of the spin correlations between the spinons. The dimerized chain segment compares unfavorably in energy to the uniformly correlated chain far away. I should mention here that when I say spinons are confined, I do not necessarily mean that there is a string force between them, as it is the case for spinons in AKLT chains or two-leg Heisenberg ladders³¹. I merely mean that it costs less energy to create another spinon pair than to completely separate two spinons. In the S1GW chain, the confining force approaches a constant for large spinon separations, as a single localized spinon in a Haldane-Shastry chain dimerizes the spin correlations over the entire chain. The strength of the dimerization and hence the force between widely separated spinons, however, decreases with the system size, and vanishes in the thermodynamic limit.³²

Let us now turn to the question of whether there is spinon confinement in the spin 1 chirality liquid. First of all, from an analysis of the nonlinear σ model in two dimensions, there is no reason to expect deconfined spinons for either half-integer or integer spins.³³ The fact that the chiral spin liquid supports deconfined spinons may be interpreted as yet another indication that it does not describe the universality class of the generic spin $\frac{1}{2}$ liquid in two dimensions, which is believed to be of central importance to the problem of CuO superconductivity.

Evidence for confining forces in the spin 1 chirality liquid is presented in Fig. 2. In a CSL with two spinons (a), the energetically relevant nearest-neighbor spin correlations are slightly enhanced near the spinons. This enhancement is due to the fact that the sites around the spinons effectively have fewer neighbors, as there is no need to correlate the links to the spinon

sites. Combining this + chirality CSL with a – chirality CSL ground state (which has spatially uniform correlations) yields a S1CL with two + chirality spinons (b). The spin correlations for this state are slightly depressed in the immediate vicinity of the spinons, in sharp contrast to what one would expect from the correlations in the CSLs. I interpret this depression as an indication of confining forces between the spinons. I should emphasize, however, that the system size is too small to allow for a conclusion.

If one believes that there is a confining force between spinons in the S1CL, one is led to wonder where it is coming from. In the spin 1 chain, we have been able to trace it to a dimerization in the spin correlations in the chain segment between the spinons. The spin correlations provided information about both the existence and the origin of the force. In the S1CL, however, Fig. 2 only suggests that a force exists, but does not reveal its origin. It seems to come from no-where. The origin of the force, I believe, only reveals itself if one considers chirality correlations, that is, expectation values of the chirality operator⁷

$$\chi \equiv \mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k), \quad (59)$$

where i, j , and k are three lattice sites on the same plaquet. This operator has been used as an order parameter for the CSL, as it effectively provides a local measure of the P and T violation in a spin liquid. The expectation value is trivially zero in the S1CL ground state. As the individual spinons in the S1CL carry chirality and hence violate P and T, we expect a cloud of chirality, that is, non-zero expectation values of (59) on triangles, around an isolated spinon. This chirality upsets the spin correlations, and costs energy. The chiral disturbance can be repaired by a second spinon nearby, which according to my intuition could be of either chirality. The confining force can hence be traced back to the appearance of chirality in the liquid as one pulls spinons apart. I conjecture that similar forces play a central role in the pairing of charge carriers in CuO superconductors.¹⁰

11. CONCLUSION

There are many interesting and unresolved questions regarding the generic spin 1 liquid state in two dimensions. The spin 1 chirality liquid introduced here offers a paradigm for this state and hence provides a framework to formulate, and possibly resolve, some of these.

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13. Strictly speaking, this proof, which is in principle due Majumdar and Gosh¹¹ (see also ref. 14), only shows that the states (2) are ground states, but not that they are the only ground states of (3). It is, however, rather simple to complete the proof. Marshall¹⁵ has shown in his classic treatment of antiferromagnetism that any singlet configuration of a spin- $\frac{1}{2}$ system can be written as a sum over dimer configurations, where the dimers are taken over all possible pairs of spins. (The space of all possible pairings is much larger than the space of singlets, so that the mapping is not unique.) It is easy to see that none of these dimer configurations except for those given in (2) are simultaneously annihilated by H_i for all i . The argument is easily generalized to triplet and higher total spin states.
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18. For a review see I. Affleck, *J. Phys. Cond. Mat.* **1**, 3047 (1989), and references therein.
19. In principle, SU(N).
20. For details, see references 2 or 14.
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22. R.B. Laughlin, D. Giuliano, R. Caracciolo, and O. White in G. Morandi et al. (eds), *Field Theories for Low-Dimensional Condensed Matter Systems* (Springer, 2000).
23. R.B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
24. This view does not reflect the history adequately. D.H. Lee thought of using the quantized Hall wave function for bosons with $m = 2$ to formulate a spin liquid, and Kalmeyer and Laughlin⁶ showed that it is a singlet when multiplied by a gauge factor $G(z_j)$. The Haldane-Shastry model was discovered only in the following year.
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26. F.D.M. Haldane and E.H. Rezayi, *Phys. Rev. B* **31**, 2529 (1985). A brief summary can be found on pages 593-595 of M. Greiter and F. Wilczek, *Nucl. Phys.* **B370**, 577 (1992).
27. For the kagomé lattice, these phases are not (and cannot be) equal for all lattice sites. The chiral spin liquid on the kagomé lattice specified in Table 1 is not invariant under rotations.
28. For references see, for example, P. Azaria *et al.*, *Phys. Rev. Lett.* **81**, 1694 (1998).
29. Numerically, the S1GW chain is an even better approximation to the spin 1 nearest-neighbor Heisenberg chain than the AKLT chain. For a 10 site chain with PBCs, the energies are 2.60 % and 5.39 % off, respectively. The overlaps with the exact ground states are 0.9450 and 0.8638.
30. The problem with open boundary conditions is that the generic spin 1 chain has spinons at the ends. This can easily be seen from an AKLT cartoon.
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32. B.A. Bernevig, D. Giuliano, R.B. Laughlin, *Phys. Rev. Lett.* **86**, 3392 (2001); *Phys. Rev. B* **64**, 24425 (2001).
33. X.G. Wen and A. Zee, *Phys. Rev. Lett.* **61**, 1025 (1988); F.D.M. Haldane, *ibid.* 1029 (1988); E. Fradkin and M. Stone, *Phys. Rev. B* **38**, 7215 (1988).