Dynamics of charged particles in magnetic fields

SI units

Lorentz Force $\vec{F}_L = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$

$\searrow \dot{\vec{p}}$

assume $\vec{E} = 0 \rightarrow \frac{\partial \vec{B}}{\partial t} = 0$ due to $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\frac{1}{2}\frac{d\vec{p}^2}{dt} = \vec{p} \cdot \dot{\vec{p}} = q(\vec{v} \times \vec{B}) \cdot \vec{p} = 0$$

$E = E(p) \Rightarrow \frac{dE}{dt} = 0$ energy conserved

1st integral of motion, $\dot{\gamma} = 0$

$$\gamma m \frac{d\vec{v}}{dt} = q \cdot \vec{v} \times \vec{B} \Rightarrow$$

eq. of motion in static magnetic field

Special Relativity

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$ Lorentz-Factor

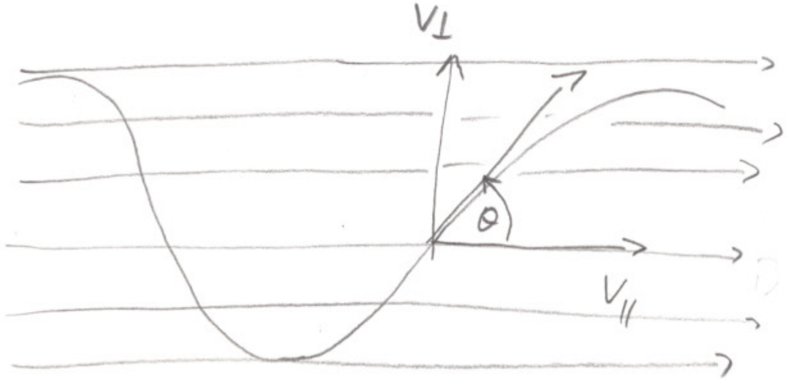
$U = \sqrt{p^2c^2 + m^2c^4} = \gamma mc^2$ total energy

$E = U - mc^2$ kinetic energy

m = rest mass, mc^2 rest energy

$$1\,eV = 1.602 \cdot 10^{-19}\,J$$

www.astro.uni-wuerzburg.de/den/spacephysics/index.html homegeneous magnetic field



$\vec{B} = B\vec{e}_z$

$\mu = \cos\theta$ pitch angle $\sphericalangle(\vec{v}, \vec{B})$

$$\theta = \arctan \frac{v_\perp}{v_\parallel}$$

define $\vec{\Omega} = \dfrac{q\vec{B}}{\gamma m} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

relativistic gyro frequency

n.r. $\Omega_e = -\dfrac{|e|B}{m_e} \Rightarrow \nu_e = \dfrac{\Omega_e}{2\pi} = 28\,\mathrm{GHz}/\mathrm{T}$

$= 2.8\,\mathrm{MHz}/\mathrm{G}$

$1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$

$= 28\,\mathrm{Hz} \cdot \left(\dfrac{B}{\mathrm{nT}}\right)$

protons $\nu = \dfrac{\Omega_p}{2\pi} = 0.0153 \cdot \left(\dfrac{B}{\mathrm{nT}}\right)\,\mathrm{Hz}$

$$\dot{\vec{v}} = \vec{v} \times \vec{\Omega}$$

$$\dot{v}_x = v_y \Omega - \cancel{v_z \Omega_y}$$

$$\dot{v}_y = \cancel{v_z \Omega_x} - v_x \Omega$$

$$\dot{v}_z = 0 \longrightarrow v_z = v_\parallel = \mathrm{const}$$

$$z(t) = v_\parallel \cdot t + z_0$$

$$\dot{v}_x = \Omega v_y$$

$$i\dot{v}_y = -i\Omega v_x \qquad i = +\sqrt{-1}$$

$$\dot{v}_x + i\dot{v}_y = -i\Omega v_x + \Omega v_y$$

$$= -i\Omega(v_x + i v_y)$$

define $v_+ = v_x + i v_y \Rightarrow \dot{v}_+ = -i\Omega v_+$

solution $\boxed{v_+(t) = A e^{-i(\Omega t + \alpha)}}$ $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$v_x = \frac{v_+ + v_+^*}{2} = \frac{A}{2}\left(e^{-i()} + e^{+i()}\right) = A\cos(\Omega t + \alpha)$$

$$v_y = \frac{v_+ - v_+^*}{2i} = \frac{A}{2i}\left(e^{-i()} - e^{+i()}\right) = -A\sin(\Omega t + \alpha)$$

$$v_\perp^2 = v_x^2 + v_y^2 \Rightarrow A = v_\perp$$

integrate $\rightarrow$

$$\boxed{\begin{aligned} x(t) &= x_0 + \frac{v_\perp}{\Omega}\sin(\Omega t + \alpha) \\ y(t) &= y_0 + \frac{v_\perp}{\Omega}\cos(\Omega t + \alpha) \\ z(t) &= z_0 + v_\parallel t \end{aligned}}$$

particle trajectory is helix with radius $R_L = \left|\frac{v_\perp}{\Omega}\right|$ (Larmor radius) and inclination angle

$$\theta = \arctan\frac{v_\perp}{v_\parallel}$$

$$v_+ = v_\perp + i v_y$$
$$v_+^* = v_\perp - i v_y$$
$$\frac{v_+ - v_+^*}{2} = i v_y$$

$$i \cdot i = -1$$
$$i = -\frac{1}{i}$$

$$R_L = \frac{v_\perp}{\Omega} = \frac{\gamma m v_\perp}{qB} = \frac{\gamma m v}{ze} \cdot \frac{\sin\theta}{B} = \underbrace{\left(\frac{pc}{ze}\right)}_{P\ [V]} \frac{\sin\theta}{Bc}$$

Rigidity

particles of the same rigidity follow the same path in a static magnetic field

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$$R_L = \frac{p}{eB} = 3300 \left(\frac{p_\perp}{\text{MeV/c}}\right) \cdot \left(\frac{B}{\text{nT}}\right)^{-1} \text{ km}$$

$R_L \ll$

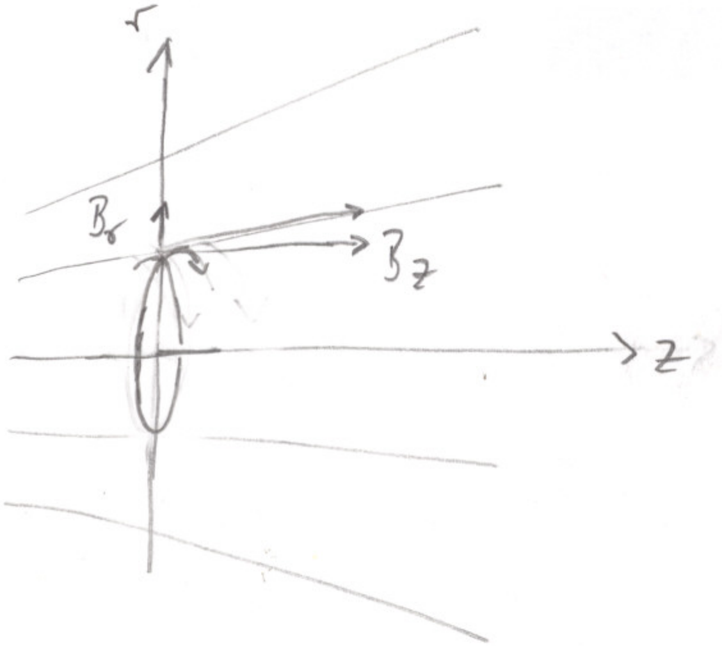
solar wind: $B \approx 5\,\text{nT}$, $p_\perp = 1\,\text{GeV/c}$ $\Rightarrow R_L \approx \frac{3300 \cdot 1000}{5}$

$= 660\,000\,\text{km} \ 2 \times R_{E,M}$

galaxy: $B \approx 0.3\,\text{nT}$, $p = 10^{20}\,\text{eV/c} = 10^{14}\,\text{MeV/c}$

$R_L = \frac{3300 \cdot 10^{14}}{0.3} \approx 1.1 \cdot 10^{18}$ $\rightarrow$ size of galaxy!

ii) inhomogeneous magnetic field



cylindrical symmetry and coordinates

$$\vec{\nabla}\cdot\vec{B} = 0 = \frac{\partial B_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r B_r)$$

$$\frac{\partial}{\partial r}(r B_r) = - r \frac{\partial B_z}{\partial z}$$

= const

for $B_z \gg B_r$ $\frac{\partial B_z}{\partial z}$ does not depend on r, approximately constant during one gyration $\rightarrow B_z \approx B$

integrate $r B_r = -\frac{r^2}{2}\frac{\partial B_z}{\partial z}$

$$\boxed{B_r = -\frac{r}{2}\frac{\partial B}{\partial z}}$$

Lorentz-Force along z $\boxed{F_z = (q \cdot \vec{v} \times \vec{B})_z}$

$$\frac{dp_\parallel}{dt} = q v_\perp B_r \overset{r = R_L}{=} -\frac{q v_\perp R_L}{2}\frac{\partial B}{\partial z} = -\frac{\gamma m v_\perp^2}{2B}\frac{\partial B}{\partial z}$$

$$\boxed{\frac{dp_\parallel}{dt} = -\frac{p_\perp^2}{2\gamma m B}\frac{\partial B}{\partial z}}$$

only magnetic forces: $\frac{d}{dt}\left(p_\perp^2 + p_\parallel^2\right) = 0$

or $\frac{dp_\parallel^2}{dt} = -\frac{dp_\perp^2}{dt}$

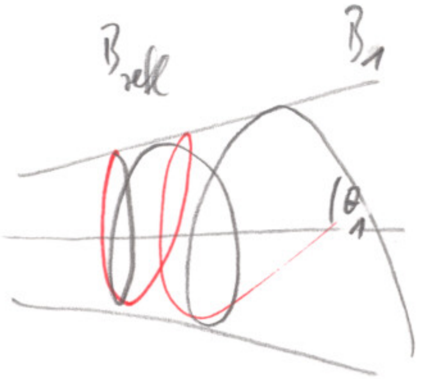
$$\frac{dp_\parallel^2}{dt} = 2 p_\parallel \frac{dp_\parallel}{dt} = 2\gamma m \underbrace{\frac{dz}{dt}}_{v_\parallel} \frac{dp_\parallel}{dt} = -\frac{p_\perp^2}{B} \underbrace{\frac{dz}{dt}\frac{\partial B}{\partial z}}_{\frac{dB(z(t))}{dt}}$$

outer, inner

$$\frac{1}{p_\perp^2}\frac{dp_\perp^2}{dt} = \frac{1}{B}\frac{dB}{dt} \Rightarrow \boxed{\frac{d}{dt}\left(\frac{p_\perp^2}{B}\right) = 0}$$

1st adiabatic invariant

with $p_\perp = p \sin\theta \Rightarrow \frac{\sin^2\theta}{B} = \text{const}$



$$\frac{\sin^2\theta_1}{B_1} = \frac{\sin^2 90^\circ}{B_{refl}}$$

$$\uparrow$$

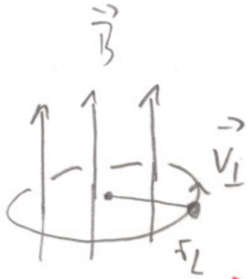
$$p = p_\perp$$

flip

magnetic mirror

$$\vec{\mu} = \int_S d\vec{S} \cdot I \quad \text{magnetic moment}$$

$d\vec{S}$: area element



$\vec{\mu} = \frac{q}{2} \vec{r}_L \times \vec{v}_\perp \rightarrow$ electron

$$|\vec{\mu}| = \mu = I \cdot S = I \pi R_L^2 = \pi \cdot I \cdot \frac{v_\perp^2}{\Omega^2}$$

$$= \frac{q\Omega}{2} \frac{v_\perp^2}{\Omega^2} = \frac{q}{2} \frac{v_\perp^2}{\Omega} = \frac{v_\perp^2 \gamma m}{2}$$

$$I = \frac{q}{T} = \frac{q\Omega}{2\pi}$$

$$= \frac{p_\perp^2}{2\gamma m B} \Rightarrow \boxed{\gamma \cdot |\vec{\mu}| = \text{const}}$$

$$\Omega = \frac{qB}{\gamma m}$$

$$\text{const} = \frac{p_\perp^2}{B} = \frac{\gamma^2 m^2 v_\perp^2}{B} = \frac{\gamma^2 m^2 R_L^2 \Omega^2}{B}$$

$$= \frac{\gamma^2 m^2 R_L^2 q^2 B^2}{B \gamma^2 m^2} \Rightarrow \boxed{B \cdot R_L^2 = \text{const}}$$

magnetic flux through Larmorradius conserved