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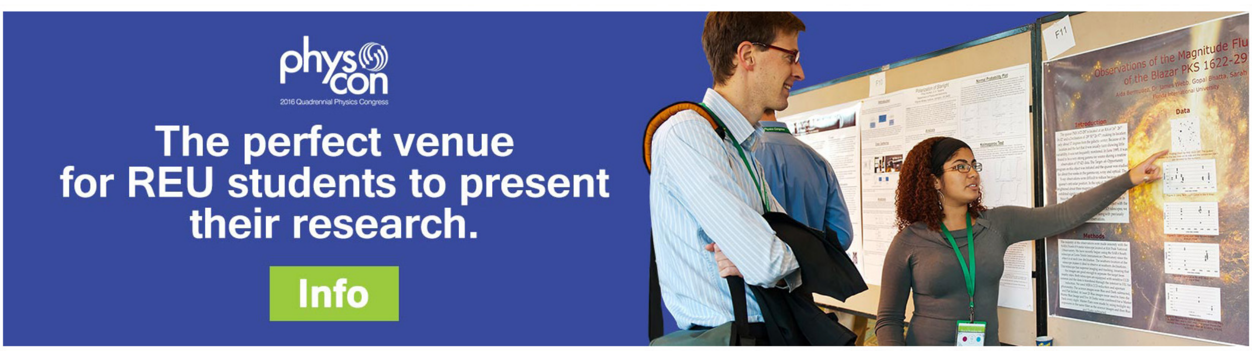

likely to play an important role at low speeds. Furthermore, our aim here is to examine Maxwell theory in its most "classical" interpretation.

[8]One can show furthermore that $d\mathbf{A}_1/dt = 0$, and then by (13) and (8) it follows, since $\mathbf{A}_1$ is equal up to second order in β to $\mathbf{A}_0(\mathbf{x} - \mathbf{v}t)$, that $\mathbf{F}(\mathbf{I}')$, is equal in the same range of approximation to $q(\mathbf{v}\cdot\text{grad})\ \mathbf{A}_0$, which can be compared with (9).

[9]Of course by making a suitable gauge transformation one could write the force only as $-q\partial\mathbf{A}^0/\partial t$ for a new choice of the vector potential $\mathbf{A}^0$. Our question may be rephrased by asking whether this choice of $\mathbf{A}^0$ can be vindicated within Maxwell theory.

[10]In the translation contained in A. I. Miller, Ref. 4, p. 392.

[11]Of course Einstein was aware that if we consider phenomena in *the same* inertial frame, we see symmetric effects only "if we neglect the terms multiplied by the second and higher powers of v/c" (A. I. Miller, Ref. 4, p. 406). Lorentz invariance means symmetry in another sense, involving a comparison between *different* inertial frames. A detailed conceptual analysis, with explicit computations in classical and relativistic electrodynamics, is contained in U. Bartocci and M. Mamone Capria, "Symmetries and asymmetries in classical and relativistic electrodynamics," Found. Phys. **21**, 787–801 (1991).

[12]Cf. Bartocci and Mamone Capria, Ref. 11.

In defense of relativistic mass

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The concept of relativistic mass brings a consistency and simplicity to the teaching of special relativity to introductory students. For example, $E = mc^2$ then expresses the beautifully simplifying equivalence of mass and energy. Those who claim not to use relativistic mass actually do so—if not by name—when considering systems of particles or photons. Relativistic mass does not depend on the angle between force and velocity—this supposed dependence results from incorrect use of Newton's second law of motion.

I. INTRODUCTION

The concept of relativistic mass has come under attack recently.[1–4] The standard definition of the relativistic mass m of a particle is

$$m = m_0/(1 - v^2/c^2)^{1/2} = \gamma m_0, \tag{1}$$

where m_0 is the rest mass of the particle and v is its speed relative to the observer. An equivalent definition of the relativistic mass is

$$m = E/c^2, \tag{2}$$

with

$$E = K + E_0 = K + m_0c^2, \tag{3}$$

where E is the total energy, K is the kinetic energy, and $E_0 = m_0c^2$ is the rest energy. It may also be defined through

$$\mathbf{p} = m\mathbf{v}, \tag{4}$$

where $\mathbf{p}$ and $\mathbf{v}$ are the particle's momentum and velocity relative to the observer. I join in the traditional use of relativistic mass[5] because, in my experience, it's the most consistent and understandable way to introduce many concepts of special relativity to students.

II. THE PRO- AND ANTI- VIEWS

The one equation nearly every student brings to an introductory treatment of relativity is $E = mc^2$. From the pro-relativistic mass point of view (the *pro- view*), this famous equation states with elegant simplicity that energy and mass are equivalent: If you increase one you increase the other proportionally (c^2 is merely the conversion factor from mass units to energy units). Thus the increase in translational kinetic energy causes the relativistic increase in mass with speed: $K/c^2 = m - m_0$.

The anti-relativistic mass point of view (the *anti- view*) does not have this simple beauty. In the anti- view, $E = mc^2$ supposedly refers only to rest mass. Thus, if a single particle is given a speed v, its mass remains m_0 while its energy increases to $E_0 + K$. However, consider a system composed of more than one particle, such as a monatomic ideal gas in a container at rest in an inertial reference frame. Then the anti- view states with seeming inconsistency that, although the mass of each atom remains m_{0i}, the rest mass of the system is not Σm_{0i}, but is $\Sigma(m_{0i} + K_i/c^2)$. We see that, in either view, we obtain this rest mass of the system from what the pro- view simply calls the sum of the constituent relativistic masses. Suppose we change the temperature of the gas by changing its internal energy by ΔU. Both views agree that the mass of the gas increases by $\Delta U/c^2$, but the anti- view says there is no change in mass for any of the atoms. The pro- view disagrees, giving an average mass change to the N ideal gas atoms of $\Delta U/(c^2N)$.

III. MORE EXAMPLES OF THE TWO VIEWS

To contrast these points of view numerically, let's consider a simple, but expensive, two-particle system: Two protons, each with a kinetic energy of 20 000 GeV, moving head-on toward one another in the SSC. (A proton has a rest mass and rest energy of about 1 GeV/c^2 and 1 Gev, respectively. The electric potential energy of the two protons is negligible here.) In the pro- view, proton 1 and proton 2 each have an energy of 20 001 GeV and a mass of 20 001 GeV/c^2 relative to the lab. Therefore, simple addition tells us that the system composed of proton 1 and proton 2 has a total energy of 40 002 GeV and a total mass of 40 002 GeV/c^2.

The anti- view agrees with the energy values, but states that each proton mass remains at about 1 GeV/c^2. However, the anti- view does not add 1 and 1 to arrive at a total mass of 2 GeV/c^2, but calculates a total mass of 40 002 GeV/c^2. Which of these points of view will make more sense to an introductory student?

We can also compare the views of the SSC collision in the frame of reference of one of the protons. Both views give the moving proton about 8×10^8 GeV and the fixed proton 1 GeV of energy, for a total energy of about 8×10^8 GeV. The pro- view says the moving proton has about 8×10^8 GeV/c^2 and the fixed proton 1 GeV/c^2 of mass, for a total mass of about 8×10^8 GeV/c^2. This total mass is relativistic mass—the invariant rest mass of the system remains 40 002 GeV/c^2. In contrast, the anti- view says that the moving proton and the fixed proton each have 1 GeV/c^2 of mass and the total mass of the system is 40 002 GeV/c^2.

All energy and corresponding mass values are numerically equal in the pro- view because that view holds that $E = mc^2$ is a general relation between energy and mass. The energy calculations and the mass calculations are quite different in the anti- view because that view holds that $E = mc^2$ is valid only for rest energy and rest mass. In this frame of reference, part of the calculation of the 40 002 GeV/c^2 invariant mass in both views involves adding about 8×10^8 GeV/c^2 to 1 GeV/c^2. From the pro- view, this addition is a sum of a relativistic mass and a rest mass. From the anti- view, it is the sum of two E_i/c^2 terms that have units of mass: The term for the proton at rest is mass, but the term for the proton in motion is mostly not mass.

For another example, let's consider a system of particles that only interact through contact forces. Told by an anti- view professor that the only mass a particle has is its rest mass, wouldn't a logical introductory student infer the position and velocity of the center of mass to be $\mathbf{r}_{\text{c.m.}} = \Sigma m_{0i}\mathbf{r}_i/\Sigma m_{0i}$ and $\mathbf{v}_{\text{c.m.}} = \Sigma m_{0i}\mathbf{v}_i/\Sigma m_{0i}$? How will the anti- view professor explain that each m_{0i} needs to be multiplied by its γ_i for the correct result? The center of mass of the system is its center of relativistic mass. However, we can reconcile both points of view by referring to the center-of-momentum frame rather than the center-of-mass frame. In the center-of-momentum frame, the total momentum is zero and center of relativistic mass is at rest for particles that only interact through contact forces.

IV. THE PHOTON

The photon is a problem for anti-relativistic mass proponents. They reason that since the photon has no rest mass and there is no such thing as relativistic mass (at least not for an individual particle), a single photon never has mass. Although a photon is attracted to and does attract objects with mass, they say it has no gravitational mass. (Thus they must replace GMm with $GM(E/c^2)$ in gravitational force equations.) A photon can transfer mass between a source and an absorber, but they state that it carries no mass in doing so. It does take a force to start a photon, stop a photon, or change its direction, but they give it no inertial mass. The magnitude of its momentum is $p = (E/c^2)c$, but again they don't allow E/c^2 to be the mass of an individual photon. Despite holding that each photon has no mass, they do include an amount E_i/c^2 for each photon when summing to find the total mass of a system. In the anti- view, this amount of nonmass must be used to find the total mass whether the system contains one photon (as in gamma decay), two photons (as in a decay of a π^0 or in a pair annihilation), or many photons (as in a cavity or the universe). However, if the system consists only of photons and all those photons are moving in the same direction, they calculate the mass of the system to be zero. How could this anti- view of photons seem simple or consistent to an introductory student?

V. IS RELATIVISTIC MASS CONFUSING?

Objections raised to the use of relativistic mass are not convincing. Let us primarily consider those raised in Adler's paper, as Okun mostly restates Adler's arguments in a more polemic form, while adding other material.

Adler's first objection is that using relativistic mass invites confusion because another course in the department may not use it. If this is somehow a serious problem, instructors should be able to solve it by letting the students know that the pro- and anti- views are just different ways of looking at the same situation. It is simply my contention that the pro- view is a less confusing and more consistent approach for introductory students.

His second argument is that the pro- view is the confusing one because there are supposedly several relativistic masses—a value multiplied by velocity to give momentum, a transverse value, a longitudinal value, and a gravitational value. I agree that the transverse and longitudinal labels are confusing and that they should be eliminated. However, my reason for doing so is the consistency of the Eq. (1) definition (or its energy or momentum equivalents), while other definitions are based on what I believe to be erroneous concepts discussed below.

VI. RELATIVISTIC MASS AND GRAVITATION

Adler's third argument is that Eq. (1) cannot be used for calculating gravitational forces in general relativity. If true, this argument offers no problem because this is a discussion of special relativity, not general relativity. But just as we have to adjust our thinking to travel from classical physics to special relativity, so we need to make another adjustment in going from the flat space-time of special relativity to the curved space-time of general relativity. Indeed, in the fullness of general relativity, we can say there's no such thing as a real gravitational force. A gravitational force is merely a fictitious force used to apply flat space-time concepts to curved space-time. (This is similar to the use of $m\mathbf{a}$, centrifugal, or Coriolis forces to apply inertial reference frame concepts to noninertial frames.) Thus Misner, Thorne, and Wheeler entitled Chap. 7 of their tome *Gravitation*, "Incompatibility of Gravity and Special Relativity."

Nonetheless, Okun presents equations for the gravitational force, using E/c^2 for the supposedly massless photon and other particles.[6] Since his equations multiply E/c^2 by other terms, he sets up a straw man by defining the product of E/c^2 and those terms as a "gravitational mass," and then gives objections to his definition. Wouldn't it be more consistent to accept the equivalence of inertial mass and gravitational mass (both equal to E/c^2), and then to understand the other terms as relativistic corrections to Newton's classical law of gravitation?

One of his objections, for instance, leads to the famous factor of 2 in the deflection of light passing close to a massive astronomical body. Giving the photon a gravitational mass of E/c^2 and applying Newton's law of gravitation does give just half the observed deflection. But, as pointed out in essays by Clifford Will,[7] this partial deflection is relative to straight lines in the flat space-time of special relativity. Then the curvature of space-time in the vicinity of the massive body bends the locally straight lines by just enough (relative to lines far from the body) to give the other half of the observed deflection.

VII. DOES RELATIVISTIC MASS DEPEND ON ANGLE?

Adler's fourth argument is that the inertia *I must* equal $|\mathbf{F}|/|\mathbf{a}|$, and that equating the mass to this inertia leads to a mass that unsatisfactorily depends on the angle between the force and velocity. Although we do not agree that inertia in relativity must equal $|\mathbf{F}|/|\mathbf{a}|$, we do agree that relativistic mass does not have this angular dependence. This dependence often shows up as the confusing statement that there are differing transverse and longitudinal masses, a confusion that arises if one uses $|\mathbf{F}| = m|\mathbf{a}|$ to define mass when the mass is not constant.

The most general form of Newton's second law for a particle acted upon by a net force $\mathbf{F}$ is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{5}$$

Substituting $\mathbf{p} = m\mathbf{v}$ into Eq. (5) with $\mathbf{a} = d\mathbf{v}/dt$ gives

$$\mathbf{F} = m\mathbf{a} + \mathbf{v}\frac{dm}{dt}. \tag{6}$$

Therefore, we see that $\mathbf{F} = m\mathbf{a}$ is true *only* when m is constant. Equation (1) shows us that the relativistic mass m is constant *only* when the speed is constant, and we know that the speed is constant *only* when the net force and velocity are perpendicular (transverse). That's why Eq. (1) also gives the so-called transverse relativistic mass, a quantity that *is* the relativistic mass and thus needs no distinguishing name of its own. The related ideas that we must also have a so-called longitudinal relativistic mass or a supposed variation of relativistic mass with angle are erroneous because their derivations incorrectly attempt to use $m = |\mathbf{F}|/|\mathbf{a}|$ even when $\mathbf{F} \neq m\mathbf{a}$.

Another argument (not used by Adler) against relativistic mass goes as follows, "Although $\mathbf{p} = m_{\rm rel}\mathbf{v}$ gives the momentum, it is not a good idea to use the relativistic mass $m_{\rm rel}$ because the kinetic energy is not equal to $\frac{1}{2}m_{\rm rel}v^2$ and the force is not in general equal to $m_{\rm rel}\mathbf{a}$." This is a specious argument. We could just as well say, "Although $\mathbf{p} = \gamma m_{\rm inv}\mathbf{v}$ gives the momentum, it is not a good idea to use the invariant mass $m_{\rm inv}$ because the kinetic energy is not equal to $\frac{1}{2}m_{\rm inv}v^2$ and the force is not in general equal to $m_{\rm inv}\mathbf{a}$ (or $\frac{1}{2}\gamma m_{\rm inv}v^2$ and $\gamma m_{\rm inv}\mathbf{a}$)." Kinetic energy simply does not equal $\frac{1}{2}mv^2$ in relativistic physics (unless a special "kinetic relativistic mass" $m_{\rm kin} = 2(\gamma - 1)m_0c^2/v^2$ were defined—an abomination shunned by pro- and anti- proponents). And the net force does not equal $m\mathbf{a}$ unless m is constant.

VIII. SYSTEM MASS AND "REAL" MASS

His fifth argument against the use of relativistic mass is the anti- view that, although the kinetic energy of a single particle gives no mass increase, a system can have a mass increase produced by the kinetic energy of the particles that make it up. Again, wouldn't an introductory student find this concept confusing and seemingly inconsistent? In the pro- view, this is an argument for the use of relativistic mass, not against it. In this paper, I contrast the two views of several systems.

His sixth argument is essentially that the relativistic mass is not the "real" mass of the object. Although answering this argument is to some extent a question of semantics, it reminds me of a jazz song of some years back that asked, in the true spirit of relativity, "Real as compared to what?" Yes, a comoving observer detects no change in the mass of an object (and we should emphasize that point to our students). However, that observer also detects no changes in the length of the object in the direction of motion, in time intervals on the object, in frequencies emitted by the object, and so on—but that does not mean that we should do away with length contraction, time dilation, or the Doppler effect. (Well, not to me it doesn't. But, for true consistency, shouldn't it for those who wish to do away with relativistic mass?) Relative to the comoving observer, the object's velocity is zero. Does that mean we should say that the momentum and kinetic energy of the object are "really" zero?

With the great gift I have for being able to tell other parents what they should have told their children, I think the answer to Adler's title, "Does mass really depend on velocity, Dad?" is that we use the term *relativity* because many quantities have values that are not absolute, but are *relative* to the observer. An observer moving with the object still measures m_0, but an observer whose speed relative to the object is v measures $m_0/(1 - v^2/c^2)^{1/2}$. From that point of view, there's no inertial reference frame that's any more "real" than any other. Both observers are correct and both can use their measurements and Eq. (1) to tell what the other measured.

IX. AN ANONYMOUS AUTHOR ANSWERS ADLER

I'm the unnamed author whose earlier paraphrased arguments are disputed in Sec. V. of Adler's paper (the "Modern Physics Book" referred to was a manuscript version of Ref. 5). I believe that using $\mathbf{F} = m\mathbf{a} + \mathbf{v}\,dm/dt$ rather than $\mathbf{F} = m\mathbf{a}$ counters his objection to point 1. I agree that the energy argument should also be stressed. His response to point 2 is basically his anti- view that it is more "profound" to apply $E = mc^2$ to just rest mass and rest energy rather than all mass and energy. I believe that his response to point 3 strengthens my arguments, as he agrees that using $m = E/c^2$ for the mass of an individual photon is the easiest way to show introductory students the effect of gravity on photon frequency. Also, despite his contention, the pro- view of E/c^2 as the photon relativistic mass *does* use Eq. (1). For example, solving that equation for the rest mass of any photon gives zero, as it ought.

X. POTENTIAL ENERGY

The pro- and anti- views relate potential energy to mass in a similar manner. Before beginning a discussion of potential energy, a possible point of confusion should be cleared up. We need to remember what "owns" the potential energy. For example, I am guilty of inaccuracy (at least) if I tell a class, "This spring in my hand has a gravitational potential energy of *mgh*." That potential energy is *not* a property of the spring alone, but of the spring–earth system. The mass of the spring does not increase by mgh/c^2 when it is lifted through a height h at constant speed. And I doubt that Okun believes that the electron in the hydrogen atom has its rest mass decreased by 27.2 eV/c^2 although he gives the -27.2-eV potential energy of the hydrogen atom to its electron.[8]

Getting back to the spring, if we stretch it a distance x its potential energy increase $\frac{1}{2}kx^2$ *is* a property of the spring and its mass does increase by an amount $\frac{1}{2}kx^2/c^2$. Of course, for a common lab spring, the mass increase will be immeasurably small, of the order 10^{-17} kg. But the potential energy increase, of order 1 J, is easily measured. (I point these values out to those anti- proponents who ask, "If you say that mass and energy are truly equivalent, why not just do away with one of them?" Both concepts are useful.)

In the anti- view, what happens if we do 1 J of work on the spring to increase its elastic potential energy? Its mass increases. What results from an increase in kinetic energy? In the anti- view, it depends on the kind of kinetic energy. If we do the 1 J of work to increase the spring's translational kinetic energy, its mass does not increase. But if we do the 1 J of work to increase its rotational or vibrational kinetic energy about its motionless center of mass, its mass does increase. Proponents of the anti- view cannot apply $\Delta m = \Delta E/c^2$ to all types of energy. They are not inconsistent in this, but they add another point of confusion for new students.

XI. AN APPLICATION OF $E = mc^2$

The most common popular applications of $E = mc^2$ are to nuclear and particle physics. For example, adding a neutron to a proton to form a deuteron is conceptually the simplest fusion reaction. Both pro- and anti- proponents agree that 2.2 MeV is available from this reaction because the rest mass of a deuteron is 2.2 MeV/c^2 less than the sum of the rest masses of a proton and a neutron. We find the difference in the two views by examining how they might arrive at the approximately $-2\ \text{MeV}/c^2$ mass change.

Lacking an exact model of the nuclear potential, let us assume that the potential energy for a deuteron is approximately -36 MeV relative to infinite separation, and that the proton and neutron each have an average kinetic energy of about 17 MeV. The pro- view says that the decrease in mass of 2 MeV/c^2 results from a change in mass of -36 MeV/c^2 from the decrease in the potential energy plus two increases of 17 MeV/c^2 in the relativistic masses of the two nearly equal-rest-mass particles. Adding the mass changes,

$$(-36 + 17 + 17)\ \text{MeV}/c^2 = -2\ \text{MeV}/c^2.$$

The anti- view holds that there's a $-36\ \text{MeV}/c^2$ change for the potential energy, but no individual mass increases of the two particles. Nonetheless, the total mass change of a deuteron is $-2\ \text{MeV}/c^2$. That is,

$$(-36 + 0 + 0)\ \text{MeV}/c^2 \text{ yields } -2\ \text{MeV}/c^2.$$

The anti- view is consistent, but I repeat that I believe introductory students would find it confusing.

The deuteron has no real excited state. However, if we excite another type of nucleus, both views agree that its rest mass increases by $\Delta m = E_{\text{ex}}/c^2$, where E_{ex} is the excitation energy. Exciting a nucleus involves changes in both the potential and kinetic energies. Both views give the same mass change resulting from the potential energy change. Again, the pro- view says that the Δm includes, in principle, the $\Delta m_i = \Delta K_i/c^2$ occurring for the individual nucleons and the anti- view says there's no Δm_i, but part of the Δm is from $\Delta E_i/c^2$.

XII. CAN MASS BE CONVERTED TO ENERGY?

When a proton and neutron combine to form a deuteron, the rest mass decreases by 2.2 MeV/c^2 and most of the 2.2 MeV of energy released is carried off by a gamma-ray photon. Let the neutron and proton make up the system before the fusion; the slowly recoiling deuteron and the 2.2-MeV gamma ray comprise it after. Since an individual photon has energy but no mass in the anti- view and a 2.2-MeV/c^2 decrease in rest mass has occurred, the anti- description makes it easy to slip into the common statement, "Mass has been converted to energy." Even accepting the anti- view, this statement has a serious defect. The reader can easily interpret the statement to mean that this system's mass has decreased and that energy has been created in the process. This leads to the incorrect conclusion that conservation of mass and energy do not hold for *isolated* systems in nuclear reactions.

In the pro- view, the statement, "Mass has been converted to energy" is incorrect because mass and energy are equivalent. For a given system, we cannot increase one by decreasing the other. The deuteron has less mass and energy because the gamma ray took mass and energy from it. During the preparation of the final draft of this paper, an article appeared that discusses this point in greater detail.[9]

Particle–antiparticle annihilation is sometimes given as the ultimate example of this supposed conversion of mass to energy. Consider the usual fate of positronium in its frame of reference: The electron and positron annihilate one another with the production of two equal-energy, oppositely directed photons. The isolated system consists of the two particles before annihilation and the two photons after. *From both points of view, neither the system mass nor its energy change.* Their values are conserved at 1.022 MeV/c^2 and 1.022 MeV. In the pro- view after, the two photons (each with 0.511-MeV/c^2 mass) give a system mass of 1.022 MeV/c^2. In the anti- view after, the two photons (each with zero mass) give a system mass of 1.022 MeV/c^2 because their total momentum is zero.

XIII. RELATIVISTIC MASS IN TEACHING

Must we use relativistic mass in teaching our introductory students? Of course not. We can always let it appear incognito by using γm_0 (without the subscript) or E/c^2 whenever relativistic mass is needed. But why should we hide it? Is there a desire for absolutes and an accompanying uneasiness with the idea that many measured properties depend on motion relative to the observer?

Although many who use four-vectors include relativistic mass, Adler seems to believe that relativistic mass is essentially irrelevant if one uses the four-vector approach to teach introductory relativity. But what is this $\Sigma E_i/c^2$ term in calculating the invariant mass of a system? And how many sophomore and junior students will understand a four-vector approach in the limited time available? Such students often use two-vectors well, but still have trouble with three-vectors. Won't beginning their modern physics instruction with four-vectors make things unnecessarily difficult and obscure the subtle concepts of relativity by moving even further from their experience?

Relativistic mass paints a picture of nature that is beautiful in its simplicity. We should continue to use relativistic mass along with consistent interpretations of Newton's second law and $E = mc^2$ in introductory courses. Insisting on its removal as a useful tool from all textbooks, as Okun does,[10] is a form of unnecessary censorship.

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[1] C. G. Adler, "Does mass really depend on velocity, dad?" Am. J. Phys. **55**, 739–743 (1987).

[2] L. B. Okun, "The concept of mass (mass, energy, relativity)," Sov. Phys. Usp. **32**, 629–640 (1989), translated from Usp. Fiz. Nauk. **158**, 511–530 (1989).

[3] L. B. Okun, "The concept of mass," Phys. Today **42** (6), 31–36 (1989).

[4] Letters in response to Ref. 3, Phys. Today **43** (5), 13–15, 115–117 (1990).

[5] T. R. Sandin, *Essentials of Modern Physics* (Addison-Wesley, Reading, MA, 1989).

[6] English Ref. 2, p. 632; Ref. 3, p. 34, and Ref. 4, p. 117.

[7] For example, Clifford Will, "The renaissance of general relativity," in *The New Physics*, edited by Paul Davies (Cambridge U.P., Cambridge, 1989), p. 12.

[8] English Ref. 2, p. 633.

[9] Ralph Baierlein, "Teaching $E = mc^2$: An exploration of some issues," Phys. Teach. **29**, 170–175 (1991).

[10] Reference 4, p. 117.

The bound states of a segmented potential

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The method of potential segmentation previously developed for scattering situations, is extended to the computation of one-dimensional bound states. The energy eigenvalues are determined by the zeros of a specific transcendental function while the corresponding eigenfunctions are supplied with the aid of a series of 2×2 matrix equations.

In a recent communication to this Journal[1] we outlined the procedure of potential segmentation as it applies to the problem of one-dimensional quantum scattering, emphasizing the ease and elegance of this method for the computation of transmission probabilities. Here we wish to show that the same procedure is readily extended to the calculation of energy eigenvalues and eigenfunctions associated with one-dimensional bound-state problems involving potentials of arbitrary shape. Such a typical binding potential, shown in Fig. 1, is taken to be generally nonconstant in the range $L < x < R (\equiv L + a)$ and to be of fixed value V_L, V_R (nonequal in general) in the left and right bounding regions respectively. On breaking up the nonconstant part of the potential into n segments having "average" heights V_j as outlined in Ref. 1 and of equal widths $w = a/n$ (see Fig. 1) and taking over all definitions and symbols appearing in Ref. 1, we arrive immediately at the series of 2×2 matrix continuity conditions

$$M[L,\alpha_L]\begin{pmatrix}A_L\\B_L\end{pmatrix} = M[L,\alpha_1]\begin{pmatrix}A_1\\B_1\end{pmatrix}$$

$$M[L+w,\alpha_1]\begin{pmatrix}A_1\\B_1\end{pmatrix} = M[L+w,\alpha_2]\begin{pmatrix}A_2\\B_2\end{pmatrix}$$

$$\vdots$$

$$M[L+(n-1)w,\alpha_{n-1}]\begin{pmatrix}A_{n-1}\\B_{n-1}\end{pmatrix} = M[L+(n-1)w,\alpha_n]\begin{pmatrix}A_n\\B_n\end{pmatrix},$$

$$M[L+nw,\alpha_n]\begin{pmatrix}A_n\\B_n\end{pmatrix} = M[R,\alpha_R]\begin{pmatrix}A_R\\B_R\end{pmatrix}. \quad (1)$$

Here, the only difference from the corresponding scattering equations (5), (7), and (8) of Ref. 1 is that in the left and right regions of the potential we have replaced the pa-