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Field-theory aspects of condensed matter physics

Examples V

To be discussed Thursday 23rd January in the examples class

I. Diffusion constant in holography

Consider the hydrodynamic expansion for a conserved $U(1)$ current to first order,

$$J^\mu = \rho u^\mu - D P^{\mu\nu} \partial_\nu \rho + \dots \qquad P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \,, \tag{1}$$

where ρ is the corresponding $U(1)$ charge density, u^μ is the four-velocity and D is the diffusion constant. Note that using the Einstein relation

$$\frac{\kappa}{T} = D \frac{\partial \rho}{\partial \mu} \,, \tag{2}$$

the expansion may also be written as

$$J^\mu = \rho u^\mu - \kappa P^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \dots \,. \tag{3}$$

μ is the chemical potential.

a) Show that in the fluid rest frame, this gives Fick's law $\vec{J} = -D \vec{\nabla} \rho$.

b) The background gauge field that is the quantum field theory source for J^μ has the time component $A_t = \mu + a_t$. Using the hydrodynamic expansion, derive the Kubo formula for κ/T. It is convenient to go to the fluid rest frame, to Fourier transform and to consider a momentum along the x^3-direction only. Use the formula for linear response for changes of the J^3 component in terms of the retarded Green's function.

c) Use the AdS/CFT correspondence to evaluate the retarded Green's function and show that this yields the universal result

$$D = \frac{1}{2\pi T} \tag{4}$$

for the diffusion constant in the limit of infinite N and infinite coupling. Hint: The calculation is easily performed by combining various results from chapter 12 of the book Ammon+Erdmenger, Gauge/Gravity Duality.

II. Entropy of the field theory dual to a Schwarzschild black hole in AdS space

Consider the following metric of a $(d+1)$-dimensional asymptotically AdS Schwarzschild black hole

$$ds^2 = \frac{L^2}{r^2} \left(f(r) d\tau^2 + \frac{dr^2}{f(r)} + dx^i dx_i \right) \,,$$

with $f(r) = 1 - \left(\frac{r}{r_H}\right)^d$ and L the AdS radius.

For $d = 4$, evaluate the entropy using the Bekenstein-Hawking formula that relates the entropy to the area of the black hole horizon,

$$S = \frac{A}{4G}\,, \tag{5}$$

where G is the Newton constant.

Evaluate the area A from

$$A = \int d^3x \sqrt{g_{3d}|_{r=r_h}}\,\mathrm{Vol}(S^5) \tag{6}$$

for the $\mathrm{AdS}_5 \times S^5$ case, where g_{3d} is the determinant of the metric in the three directions x^1, x^2, x^3. With $\mathrm{Vol}(S^5) = \pi^3 L^5$ and $G = \pi L^3/2N^2$ in the AdS/CFT correspondence for $\mathrm{AdS}_5 \times S^5$, show that

$$S = \frac{\pi^2}{2} N^2 T^3 \mathrm{Vol}(\mathbb{R}^3)\,. \tag{7}$$