Prof. Dr. Johanna Erdmenger
Dr. Christian Northe

Field-theory aspects of condensed matter physics

Examples IV

To be discussed Tuesday 17th December in the examples class

Berry curvature in quantum field theory

Based on https://arxiv.org/pdf/1701.05587.pdf by Baggio, Niarchos and Papadodimas

For the Schrödinger equation

$$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle\,, \tag{1}$$

the Berry connection is given by

$$\mathbf{A}_i{}^n = \langle n(\lambda)\frac{\partial}{\partial\lambda^i}|n(\lambda)\rangle\,, \tag{2}$$

and the Berry curvature by

$$\mathbf{F}_{ij}^{(n)} = \partial_i\mathbf{A}_j{}^n - \partial_j\mathbf{A}_i{}^n\,. \tag{3}$$

a) Show that using an orthonormal basis $|m\rangle$, the Berry curvature may be written as

$$\mathbf{F}_{ij}^{(n)} = \sum_{n\neq m}\frac{\langle n|\partial_i H|m\rangle\langle m|\partial_j H|n\rangle \;-\; (i\leftrightarrow j)}{(E_m-E_n)^2}\,. \tag{4}$$

b) Consider the Lagrangian of vacuum electrodynamics,

$$\mathcal{L} \;=\; -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}\,. \tag{5}$$

Show that with $F^{\pm}{}_{\mu\nu} = F_{\mu\nu} \pm i\tilde{F}_{\mu\nu}$, (5) may be rewritten as

$$\mathcal{L} = \frac{i}{64\pi}\tau(F^+)^2 - \frac{i}{64\pi}\bar{\tau}(F^-)^2\,, \qquad \tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}\,. \tag{6}$$

By considering photon states $|\vec{p},\epsilon\rangle$ with momentum $\vec{p}$ and helicity $\epsilon = \pm$, use (4) to show that the Berry curvature is

$$\mathbf{F}_{\tau\bar{\tau}}{}^{(\vec{p},\vec{p}',\epsilon,\epsilon')} = \frac{\epsilon}{8}\frac{1}{(\mathrm{Im}\tau)^2}\delta_{\vec{p},\vec{p}'}\delta_{\epsilon,\epsilon'}\,. \tag{7}$$

Using this result, deduce that for a multi-photon state containing n_+ photons of positive and n_- photons of negative helicity, the Berry curvature is

$$\mathbf{F}_{\tau\bar{\tau}}{}^{(n_+,n_-)} \;=\; \frac{1}{8}(n_+ - n_-)\frac{1}{(\mathrm{Im}\tau)^2}\,. \tag{8}$$