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Field-theory aspects of condensed matter physics

Examples I

To be discussed Tuesday 5th November in the examples class

I. **Gaussian path integral**

For free scalar field theory in d dimensions, the generating functional is given by the path integral

$$Z_0[J] = N \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^d x \left[-\tfrac{1}{2}\phi(-\Box + m^2 - i\epsilon)\phi + J\phi\right]\right] . \tag{1}$$

Note that the integrals over the fields ϕ are Gaussian since ϕ is at most quadratic in (1). Perform the Gaussian integral in (1) to show that

$$Z_0[J] = \exp\left[\frac{i}{2} \int \mathrm{d}^d x \mathrm{d}^d y \, J(x)\Delta_F(x-y)J(y)\right] , \tag{2}$$

where Δ_F is the Feynman propagator for a scalar field

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \tag{3}$$

satisfying the differential equation

$$(-\Box + m^2)\Delta_F(x-y) = \delta^d(x-y) . \tag{4}$$

In other words, the Feynman propagator is a Green's function for the Klein-Gordon equation.

II. **Convergence of perturbative expansion**

In the lecture we discussed that for interacting theories, the generating functional is given by

$$Z[J] = N \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^d x \, (\mathcal{L}_0 + \mathcal{L}_{\text{int}} + J\phi)\right] . \tag{5}$$

This is no longer Gaussian and we cannot perform the integration explicitly. Whenever the coupling constants in $\mathcal{L}_{\text{int}}$, such as g in ϕ^4 theory, are small, we may use *perturbation theory*. The starting point of perturbation theory is to write

$$\begin{aligned} Z[J] &= N \exp\left[i \int \mathrm{d}^d x \, \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^d x \, (\mathcal{L}_0 + J\phi)\right] \\ &= \exp\left[i \int \mathrm{d}^d x \, \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J] . \end{aligned} \tag{6}$$

To demonstrate the equivalence of (5) and (6), consider the following simplified example of an ordinary one-dimensional integral which we can perform analytically and compare to the results which we obtain from perturbation theory using (6).

Let us consider the integral

$$f(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{1}{2}m^2x^2 - \frac{\lambda}{4!}x^4 + jx} \,. \tag{7}$$

(i) For $j = 0$ but $\lambda \in \mathbb{R},\ \lambda > 0$ evaluate $f(\lambda)$ exactly. The result is

$$f(\lambda) = \sqrt{\frac{3m^2}{\lambda}} e^{\frac{3m^4}{4\lambda}} K_{1/4}\left(\frac{3m^4}{4\lambda}\right) \tag{8}$$

with $K_\nu(x)$ being the modified Bessel function of the second kind.

(ii) Show that the integral (7) may be rewritten as

$$\begin{aligned} f(\lambda) &= \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{1}{2}m^2x^2 + jx} \sum_{k=0}^{\infty} \frac{(-\lambda x^4)^k}{k!(4!)^k} \\ &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!(4!)^k} \int_{-\infty}^{\infty} \mathrm{d}x \, x^{4k} e^{-\frac{1}{2}m^2x^2 + jx} \,, \end{aligned} \tag{9}$$

assuming that we can exchange the infinite sum and the integral in the last step.

(iii) Show that

$$\int_{-\infty}^{\infty} \mathrm{d}x \, x^{2n} e^{-\frac{1}{2}m^2x^2} = \sqrt{2\pi} \frac{(2n)!}{n! \, 2^n \, m^{2n+1}} \,, \tag{10}$$

e.g. by considering $\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-1/2m^2x^2 + jx}$ and taking derivatives with respect to j.

(iv) Argue why the steps (ii) and (iii) to evaluate the integral (7) are similar to those involved from (5) to (6).

(v) For $j = 0$, compare the partial sums

$$f_n(\lambda) = \sqrt{2\pi} \sum_{k=0}^{n} \frac{(-\lambda)^k (4k)!}{k! \, (2k)! \, (4!)^k 2^{2k} m^{4k+1}} \tag{11}$$

as a function of n to the exact result obtained from (i). What about $\lim_{n\to\infty} f_n(\lambda)$?

(Hint: Look at https://arxiv.org/pdf/1201.2714.pdf)