String Theory II - Exercise Sheet 4

Due to Thu Jan 25 2017 10-12 Uhr M1.03.020

Problem 4.1: Supercurrent and the Noether trick

Noether's theorem states that if an action is invariant under a global ($\epsilon = \text{const.}$) symmetry transformation on the fields $\delta_\epsilon \phi_i = \epsilon \delta \phi_i$, i.e. if the variation of the Lagrangian is at most a total derivative,

$$\delta_\epsilon S = \int d^d x \epsilon \partial_a \tilde{J}^a \tag{1}$$

then on-shell (i.e. on solutions of the Euler-Lagrange equations), the Noether current

$$J^a_{\text{Noether}} = \tilde{J}^a - \sum_i \delta\phi_i \frac{\delta_L \mathcal{L}}{\delta \partial_a \phi_i} \tag{2}$$

is conserved.[1] We constructed the Noether current for the worldsheet $N = (1, 1)$ supersymmetry of the string in the lecture.

Often, instead of constructing (2) directly, it is more convenient consider local transformation parameters $\epsilon(x)$. By continuity with (1), the variation of S now generally reads

$$\delta_\epsilon S = \int d^d x \left(\epsilon \partial_a \tilde{J}^a + (\partial_a \epsilon) A^a \right) , \tag{3}$$

with A^a being some vector field.

1. By comparing (3) with the general form of the variation of the action $S = \int d^d x \mathcal{L}[\phi_i, \partial_a \phi_i]$, show that

$$A^a = \sum_i \delta\phi_i \frac{\delta_L \mathcal{L}}{\delta \partial_a \phi_i} , \tag{4}$$

   i.e. the conserved current is $J^a_{\text{Noether}} = \tilde{J}^a - A^a$ can be read off from (3).

[1] Here δ_L denotes the left derivative for Grassmann valued fields. Note that with this convention, the fermionic $\delta\phi_i$ have to be multiplied from the left.

2. Use the above version of the Noether trick and the localized version of the global supersymmetry transformations of the RNS string in superconformal gauge,

$$
\begin{aligned}
S_{RNS} &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_a X^\mu \partial^a X_\mu + i\alpha'\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right) , && (5)\\
\delta_\epsilon X^\mu &= i\sqrt{\frac{\alpha'}{2}}\bar{\epsilon}(\sigma^a)\psi^\mu , && (6)\\
\delta_\epsilon \psi^\mu &= \frac{1}{\sqrt{2\alpha'}}\rho^a \epsilon(\sigma^a)\partial_a X^\mu , && (7)
\end{aligned}
$$

show that the supercurrent is given by the expression derived in the lecture,

$$J_F^a = \frac{i}{2\pi\alpha'}\sqrt{\frac{\alpha'}{2}}\rho^b\rho^a\psi^\mu\partial_a X_\mu , \tag{8}$$

I.e. explicitly calculate the variation $\delta_\epsilon S_{RNS}$ under local supersymmetry transformations, show that it is of the form (3) and read off the form of A^a. The form of $\tilde{J}^a$ was given in the lecture.

Problem 4.2: $N=(1,1)$ Supersymmetry algebra in 1+1 D

By acting on X^μ and ψ^μ, show that the commutator of two global supersymmetry transformations on-shell (i.e. upon use of the equations of motion) closes onto a translation:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -\xi^a \partial_a , \quad \xi^a = \frac{i}{2}(\bar{\epsilon}_1\rho^a\epsilon_2 - \bar{\epsilon}_2\rho^a\epsilon_1) \tag{9}$$

Hint: For the action on X^μ, the calculation is straight forward. For the action on ψ^μ, use the Fierz identity for general Majorana spinors Ψ, χ, ϕ and λ,

$$(\bar{\Psi}\lambda)(\bar{\phi}\chi) = -\frac{1}{2}\left[(\bar{\Psi}\chi)(\bar{\phi}\lambda) + (\bar{\Psi}\bar{\rho}\chi)(\bar{\phi}\bar{\rho}\lambda) + (\bar{\Psi}\rho^a\chi)(\bar{\phi}\rho_a\lambda)\right] , \tag{10}$$

to rearrange spinors and add an appropriate term proportional to the Dirac equation $\rho^a\partial_a\psi^\mu = 0$ (which vanishes on-shell) to cancel the superfluous terms from the Fierz identity.