# String Theory II - Exercise Sheet 3

Due to Thu Jan 18 2018 10-12 Uhr M1.03.020

## Problem 3.1: Spinors in Two Dimensions

The RNS formulation of the superstring requires supplementing the bosonic embedding functions $X^\mu$ with two-dimensional fermionic superpartners $\psi^\mu$. The Clifford algebra in two-dimensional flat Minkowski space-time $ds^2 = -dt^2 + dx^2 = dx^a dx^b \eta_{ab}$ is generated by matrices $\rho^a$ with the property

$$\left\{\rho^a, \rho^b\right\} = 2\eta^{ab} \,. \tag{1}$$

Here $\eta_{ab} = \mathrm{diag}(-1, 1)$, and $\eta^{ab}$ is the inverse of $\eta_{ab}$. Obviously, the minimal matrix representation of (1) must be 2x2 matrices, and hence the minimal spinor representation on which (1) acts must be a 2-component spinor. A general 2-component spinor has 2 complex or 4 real degrees of freedom.

1. Show that $\rho^0 = i\sigma^2$ and $\rho^1 = \sigma^1$, where $\sigma^1$ and $\sigma^2$ are the first and second Pauli matrix, is a representation of (1).

2. Show that $\Gamma = \rho^0\rho^1$ is the chirality matrix which squares to one, is hermitean and anticommutes with the other "gamma" matrices $\rho^a$. Construct its Weyl eigenspinors.

3. Define a charge conjugation matrix $C = \rho^0$. A Majorana spinor satisfies $\bar{\lambda} = \lambda^\dagger \rho^0 = \lambda^T C$. Show that this is equivalent to taking the spinor $\lambda = \lambda^*$ to be real.

4. Show that in two dimensions the Majorana condition is compatible with the Weyl condition, i.e. that one can construct a definite chirality eigenspinor which is real. Is this the smallest representation possible of (1)?

5. Show that in light cone coordinates such that $ds^2 = 2d\sigma^+ d\sigma^-$, the Dirac kinetic term for a massless Majorana fermion $\psi$ takes the form

$$S = \frac{1}{2\pi} \int d^2x \, i\bar{\psi}\rho^a \partial_a \psi = \frac{i}{\pi} \int d^2x \left(\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-\right) \,. \tag{2}$$

   Show furthermore that $S$ is Hermitean. Derive the equations of motion and show that $\psi_\pm$ are left- and right-moving fields, respectively. What are the conformal weights $(h, \bar{h})$ of $\psi_\pm$ after Wick rotation $t = -i\tau$ to Euclidean signature?