String Theory II - Exercise Sheet 02

Due to Thu Jan 11 2018 10-12 Uhr M1.03.020

Problem 2.1: Covariant Quantization of the Open String

In the lecture we reviewed the covariant quantization of the closed string. Consider an open string of length π on a single Dp brane. Derive the mode expansion for the DD and NN directions and use the equal time canonical commutation relation

$$[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{1}$$

to derive the commutation relations

$$\begin{aligned} \left[x^a, p^b\right] &= i\eta^{ab} && (2)\\ \left[\alpha_n^a, \alpha_m^b\right] &= n\eta^{ab}\delta_{n+m,0} && (3)\\ \left[\alpha_n^I, \alpha_m^J\right] &= n\delta^{IJ}\delta_{n+m,0}\,. && (4) \end{aligned}$$

Then derive the Virasoro constraints (at world sheet time $\tau = 0$)

$$\begin{aligned} L_n &= -\frac{1}{2\pi}\int_0^\pi d\sigma\left(e^{in\sigma}T_{++} + e^{-in\sigma}T_{--}\right) && (5)\\ &= \frac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_{n-m}^a\alpha_m^b\eta_{ab} : + \frac{1}{2} : \sum_{m\neq 0} \alpha_{n-m}^I\alpha_m^J\delta_{IJ} : \text{, where} && (6)\\ T_{\pm\pm} &= -\frac{1}{\alpha'} : \left[(\partial_\pm X^a)(\partial_\pm X^b)\eta_{ab} + (\partial_\pm X^I)(\partial_\pm X^J)\delta_{IJ}\right] : \,. && (7) \end{aligned}$$

Here $\alpha_0^a = \sqrt{2\alpha'}p^a$. Assuming a normal ordering constant $a = 1$, derive from L_0 the mass formula for the open string on a Dp brane,

$$M^2 = -p_a p^a = \frac{1}{\alpha'}\left(\sum_{m>0}\alpha_{-m}^a\alpha_m^b\eta_{ab} + \sum_{m>0}\alpha_{-m}^I\alpha_m^J\delta_{IJ} - 1\right)\,. \tag{8}$$

Construct the open string spectrum up to the massless level. Show that the states you constructed are physical, i.e. that they obey

$$L_n|\text{phys}\rangle = 0\,, \quad n > 0\,. \tag{9}$$

Give a physical explanation of the conditions obtained from (9)

Problem 2.2: Virasoro Constraints & U(1) Gauge Symmetry on a Dp Brane

In the previous problem you should have encountered the state

$$A_a \alpha_{-1}^a |0, p\rangle \,. \tag{10}$$

Consider the state

$$|\psi\rangle = L_{-1}|0, p\rangle \,. \tag{11}$$

Show that

1. it can be written in the from

$$|\psi\rangle = \sqrt{2\alpha'} p_a \alpha_{-1}^a |0; p > \tag{12}$$

2. it being a physical state $L_{n>0}|\psi >= 0$ and $L_0|\psi >= |\psi >$ implies $p_a p^a = 0$,
3. it has vanishing norm $< \psi|\psi >= 0$,
4. it is orthogonal to any physical state, $\langle \text{phys}|\psi\rangle = 0$.

Due to the above properties, any complex linear combination of (11) with (10) is going to be a physical massless state with the same norm and transition amplitude to any other physical state as (10). The physical massless mode (10) is hence only defined as an equivalence class up to linear combination with physical null states such as (11). The latter correspond to the gauge degrees of freedom in the string Hilbert space, as they imply an equivalence relation on the polarizations

$$A_a \sim A_a + i\tilde{\lambda} p_a \,. \tag{13}$$