String Theory II - Exercise Sheet 01

Due to Thu Dec 21 2017 10-12 Uhr M

Problem 1.1: Supersymmetry and Stability

The $\mathcal{N}=1$ supersymmetry algebra in 3+1 dimensions is generated by a single Weyl spinor $Q_\alpha = (Q_1, Q_2)^T$ with complex conjugate $\bar{Q}_{\dot{\alpha}} = Q^\dagger_\alpha$, translation generator P^μ and Lorentz generator $M^{\mu\nu}$. The algebra reads (nonvanishing conditions only)

$$
\begin{aligned}
\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu && (1)\\
[M^{\mu\nu}, Q_\alpha] &= \frac{1}{2}\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta && (2)\\
[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho} && (3)\\
[M^{\mu\nu}, P^\rho] &= \eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu\,. && (4)
\end{aligned}
$$

Here

$$
\begin{aligned}
\sigma^\mu_{\alpha\dot{\beta}} &= \left(-\mathbb{1}, \sigma^i\right)_{\alpha\dot{\beta}} && (5)\\
\bar{\sigma}^{\mu\dot{\alpha}\beta} &= \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\beta\delta}\sigma^\mu_{\delta\dot{\gamma}} && (6)\\
\epsilon_{21} &= \epsilon^{12} = 1\,, \quad \epsilon_{12} = \epsilon^{21} = -1, \quad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0\,, && (7)\\
\sigma^{\mu\nu} &= \frac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)\,. && (8)
\end{aligned}
$$

$\sigma^{\mu\nu}$ is the generator of the Lorentz algebra (3) in the Weyl spinor representation. Show that

1. $\bar{\sigma}^\mu = (\sigma^0, -\sigma^i)$
2. $\mathrm{Tr}\left(\sigma^\mu\bar{\sigma}^\nu\right) = -2\eta^{\mu\nu}$
3. that $P^\mu = -\frac{1}{4}\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\}\bar{\sigma}^{\mu\dot{\beta}\alpha}$ and from here that P^0 is a positive operator, $\langle\Psi|P^0|\Psi\rangle \geq 0 \quad \forall \quad |\Psi>$.

The last relation implies that the supersymmetric system has a stable ground state.

Problem 1.2: Space-Time Symmetries of the Polyakov action

Use the trick discussed in the exercise session on Dec. 15 to derive the conserved currents for Poincaré and Lorentz transformations

$$P^a_\mu = -T\eta^{ab}\partial_b X_\mu \tag{9}$$
$$M^a_{\mu\nu} = X_\mu P^a_\nu - X_\nu P^a_\mu \tag{10}$$

as well as the corresponding conserved charges for a string of length ℓ described by the Polyakov action in conformal gauge

$$S_{Pol} = -\frac{T}{2}\int d^2\sigma(\partial_a X^\mu)(\partial^a X_\mu)\,. \tag{11}$$

Use the Poisson bracket structure between canonical coordinates X^μ and momenta $\Pi_\mu = P^\tau_\mu$

$$\{A[X,\Pi],B[X,\Pi]\}_{P.B.} = \int d\sigma''\left(\frac{\delta A}{\delta X^\mu(\sigma'')}\frac{\delta B}{\delta \Pi_\mu(\sigma'')} - \frac{\delta A}{\delta \Pi^\mu(\sigma'')}\frac{\delta B}{\delta X_\mu(\sigma'')}\right) \tag{12}$$

to show that the conserved charges of Poincaré and Lorentz symmetry fulfill the classical version of the algebra (3) and (4),

$$\{.,.\} = -i\,[.,.]\ . \tag{13}$$