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Introduction to Gauge/Gravity Duality

Examples IV

To hand in Tuesday 15th May

I. **Coordinates of** AdS_{d+1}

Lorentzian AdS_{d+1} can be defined by the locus

$$-L^2 = \eta_{ab} X^a X^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d} \left(X^i\right)^2 , \tag{1}$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab} X^a X^b$ with $\eta = \mathrm{diag}(-1, 1, 1, \ldots, 1, -1)$. In the following we parametrize the locus (1) in different ways.

a) Draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$. (2 points)

b) The *global coordinates* (ρ, τ, Ω_i) are defined by

$$\begin{aligned} X^{d+1} &= L \cosh\rho \, \sin\tau , & (2)\\ X^0 &= L \cosh\rho \, \cos\tau , & (3)\\ X^i &= L \sinh\rho \, \Omega_i , & (4) \end{aligned}$$

with $i = 1, ..., d$ and $\sum_{i=1}^{d} \Omega_i^2 = 1$. Using this parametrization calculate the induced metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (where $x^\mu \in \{\rho, \tau, \Omega_i\}$) for AdS_{d+1} in global coordinates. (3 points)

c) Replace ρ by $r \equiv L \sinh\rho$ and show that the metric can be written in the form

$$ds^2 = -H(r) dt^2 + H(r)^{-1} dr^2 + r^2 d\Omega_{d-1}^2, \tag{5}$$

where $d\Omega_{d-1}^2 = \sum_{i=1}^{d} d\Omega_i d\Omega_i$ is the metric of the unit $(d-1)$–sphere, S^{d-1}. (2 points)

d) The *Poincaré patch coordinates* (x^i, u) with $i = 1, ..., d$ are defined by

$$\begin{aligned} X^{d+1} + X^d &= u , & (6)\\ -X^{d+1} + X^d &= v , & (7)\\ X^i &= \frac{u}{L} x^i . & (8) \end{aligned}$$

Use the defining equation (1) to eliminate v in terms of u and x^i and show that the induced metric for (u, x^i) with $i = 1, ..., d$ takes the form

$$ds^2 = L^2 \frac{du^2}{u^2} + \frac{u^2}{L^2} dx^i dx_i \,. \tag{9}$$

Finally introduce $z = \frac{L^2}{u}$ and show that the metric is given by

$$ds^2 = \frac{L^2}{z^2} \left(dz^2 + dx^i dx_i \right) \tag{10}$$

Which part of the AdS spacetime is not covered by these coordinates (Hint: z takes only positive values (Why?)). (3 points)

II. **Curvature of AdS and Cosmological constant**

Let us consider AdS_{d+1} in the *Poincaré patch* given by the coordinates $(z, t, \vec{x})$ and the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{z^2} \left(dz^2 - dt^2 + d\vec{x}^2 \right) \,, \tag{11}$$

where $\vec{x}$ are the $d - 1$ spatial dimensions.

a) Calculate the metric $g^{\mu\nu}$, the Christoffel symbol $\Gamma^\mu_{\nu\rho}$, the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ as well as the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R.

(3 points)

b) Show that AdS_{d+1} solves the vacuum Einstein field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, where Λ is the cosmological constant and $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}$ is the Einstein tensor. Determine the value of the cosmological constant Λ!

(2 points)