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Introduction to Gauge/Gravity Duality

Examples II

To hand in Tuesday 24th April in the examples class

I. **Gauge Theory**

Consider a $SU(N)$ gauge theory in 3+1 dimensions. The theory is assumed to contain a real scalar field $\Phi(x)$ which transforms in the adjoint representation of the gauge group, i.e.

$$\Phi \to e^{-i\Lambda}\Phi e^{i\Lambda}\,, \quad \Phi = \Phi^a T^a\,, \quad \Lambda = \Lambda^a T^a.$$

a) Show that the operator $\mathcal{O} \equiv \mathrm{Tr}(\Phi^2)$ is gauge invariant. (1 point)
b) Write down the classical action for the theory involving $\Phi(x)$. (1 point)
c) Show that this action is gauge invariant. (1 point)
d) Calculate the free-field propagator of the field Φ as the Green's function of its equation of motion to linear order in the fields. For this purpose, write out the indices of the matrices T^a, i.e. $\Phi^i{}_j = \Phi^a (T^a)^i{}_j$, and calculate $\langle \Phi^i{}_j(x)\Phi^k{}_l(0)\rangle$. (2 points)

II. **Retarded Green's function**

In the lecture we considered Δ_F, the Feynman propagator for a scalar field,

$$\Delta_F(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}\,. \tag{1}$$

A further important two-point function is the *retarded Green's function* G_R given by

$$G_R(x-y) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-y)}}{-(k^0+i\epsilon)^2+\vec{k}^2+m^2}\,. \tag{2}$$

Compare the pole structure of (2) and of (1) in the complex k^0 plane. Show that $G_R(x-y)$ satisfies $(-\Box + m^2)G_R(x-y) = \delta^d(x-y)$ and that $G_R(x-y)$ vanishes for $x^0 < y^0$. Moreover, check explicitly by using the mode expansion of the free scalar field that $G_R(x-y)$ may be written as

$$G_R(x-y) = -i\,\Theta(x^0-y^0)\,\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]|0\rangle\,, \tag{3}$$

where Θ is the step function.

(5 points).