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Introduction to Gauge/Gravity Duality

Examples I

To hand in Tuesday 17th April in the examples class

I. **Fermions in quantum field theory**

a) Write down the action for a free massive Dirac fermion field Ψ in 3+1 dimensions. (2 points)

b) From this action, show that varying with respect to $\bar{\Psi}$ gives the equation of motion

$$(-i\partial\!\!\!/ + m)\Psi(x) = 0 \qquad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \,. \tag{1}$$

Derive the corresponding equation of motion for $\bar{\Psi} = \Psi^\dagger \gamma^0$. What is the relation to (1)? (3 points)

c) By acting with $i\partial\!\!\!/ + m$ on the left of equation (1) and using the Clifford algebra for the γ matrices, show that each component of $\Psi(x)$ satisfies the Klein-Gordon equation

$$(\Box - m^2)\phi(x) = 0 \,.$$

(5 points).

II. **Quantization of free scalar field and equations of motion**

a) Show that the equations of motion $(\Box - m^2)\phi(x) = 0$ of a free scalar field are satisfied by

$$\phi(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{k}}{2\omega_k} \left[a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right] \Big|_{k^0 = \omega_k} , \tag{2}$$

where $\omega_k = (\vec{k} \cdot \vec{k} + m^2)^{1/2}$ and $kx = -k^0 x^0 + \vec{k}\vec{x}$.

b) Derive the equations of motion for a scalar field with mass m and the interaction Lagrangian $\mathcal{L}_{\text{int}} = -\frac{g}{4!}\phi(x)^4$.

c) Consider two non-interacting real scalar fields ϕ_1 and ϕ_2 with common mass m. Show that the Lagrangian can be written in terms of the *complex scalar field* $\phi = 1/\sqrt{2}\,(\phi_1 + i\phi_2)$ and its complex conjugate, $\phi^* = 1/\sqrt{2}\,(\phi_1 - i\phi_2)$ in the form

$$\mathcal{L}_{\text{free}}(\phi, \partial\phi) = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \,. \tag{3}$$

Derive the equations of motion for ϕ and ϕ^* assuming that ϕ and ϕ^* are independent fields. Are the equations of motion consistent with the ones for ϕ_1 and ϕ_2?

(5 points).