Gauge Gravity Duality

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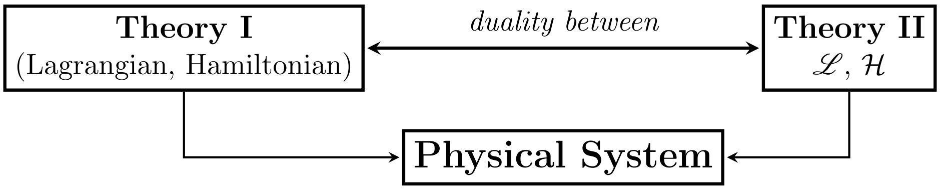
I. Introduction: Duality



Here we have:

Theory I: Gauge Theory	**Theory II**
(special example for QFT)	general relativity
without gravity	with gravity

$\mathcal{N} = 4$: $SU(N)$ Super-Yang-Mills theory which is a supersymmetric, non-abelian gauge theory.

II. Basics

1. Gauge Theories

This is an example of a field theory.

Reminder: Classical Mechanics

$$L = L(x, \dot{x})$$

For this lecture energy is conserved (so $\mathscr{L}$ does not depend on the time t). Now replace $x, \dot{x}$ by a function of x: $\phi(x)$. This is of physical significance

$$\mathscr{L} = \mathscr{L}\left(\phi(x), \dot{\phi}(x)\right) \,, \qquad x = (t, \vec{x})$$

Examples: $\vec{E}$, $\vec{B}$ (electric and magnetic fields)
Scalar field: Higgs field

$$\vec{B} = \operatorname{rot} \vec{A} \,, \quad \vec{E} = -\operatorname{grad} \varphi - \frac{\partial \vec{A}}{\partial t}$$

$A_\mu = (\varphi, \vec{A})$ is called the *gauge potential.*

Quantum Field Theory:
2^{nd} quantization: Fouriermodes of fields satisfy non-trivial commutation relations: the wave function of quantum mechanics is quantized

$$\text{Fourier:} \quad \phi(x) = \frac{1}{(2\pi^{d-1})} \int \frac{\mathrm{d}^{d-1}k}{2\omega_k} \left[a(\vec{k})e^{-\mathrm{i}kx} + a^\dagger(\vec{k})e^{\mathrm{i}kx)}\right]_{k^0=\omega_k} \tag{2.1}$$

where kx denotes the relativistic scalar product $k^\mu x_\mu$. Here we have a QFT in d dimensions (1 time dimension, $d-1$ space dimensions). The dependency of $k_0 = \omega_k$ is derived from the energy equation:

$$E^2 = \vec{p}^{\,2} - m^2 \quad \Rightarrow \quad \omega_k = \sqrt{\vec{k}^2 + m^2}$$

In equation (2.1) we have the non-trivial commutator

$$\left[a(\vec{k})\,,a^\dagger(\vec{k}')\right] = 2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\vec{k}-\vec{k}')$$

We call a and $a^\dagger$ creation and annihilation operators.

In electrodynamics we have the abelian[1] gauge group $U(1)$ $(\to\ A_\mu)$. Consider a complex scalar field $\phi(x)$ transforming under a *local* $U(1)$ transformation as

$$\phi(x)\ \to\ e^{\mathrm{i}\vartheta(x)}\phi(x)\ ,\quad \mathscr{L}(A_\mu,\phi,\partial_\mu\phi)$$

The derivative transforms as

$$\partial_\mu\phi(x)\ \to\ \partial_\mu\left(e^{\mathrm{i}\vartheta(x)}\phi(x)\right) = e^{\mathrm{i}\vartheta(x)}\left[\partial_\mu\phi(x)+\mathrm{i}\big(\partial_\mu\vartheta(x)\big)\phi(x)\right]$$

so $\partial_\mu\phi$ doesn't transform as ϕ itself!
Way out: introduce a *connection* A_μ to define a *covariant* derivative

$$\mathrm{D}_\mu\phi(x) := (\partial_\mu + \mathrm{i}A_\mu)\,\phi(x) \qquad \text{with } A_\mu \to A_\mu - \partial_\mu\phi$$

which satisfies the transformation law $\mathrm{D}_\mu\phi\ \to\ e^{\mathrm{i}\vartheta(x)}\,\mathrm{D}_\mu\phi$. The covariant derivative now may be used to construct an invariant Lagrangian.

A further useful gauge invariant quantity is called the *field strength tensor* $F_{\mu\nu}$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad\to\quad F_{\mu\nu} + \underbrace{[\partial_\mu\,,\partial_\nu]}_{=0}\vartheta$$

$$\Rightarrow\quad \mathscr{L}\ =\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + \mathrm{D}^\mu\phi\ \mathrm{D}_\mu\phi$$

Now consider the non-abelian gauge group $SU(N)$

a) Fields transforming in the *fundamental representation* of the gauge group: elements of a N-dimensional vectorspace, where $(T^a)_i{}^j$ are the $(N^2{-}1)$ *generators* (hermitian $N\times N$ matrices; $\Rightarrow$ $e^{\mathrm{i}\vartheta^a T^a}$ unitary) of $SU(N)$

$$q_i(x) \quad\to\quad \left(e^{\mathrm{i}\vartheta^a(x)T^a}\right)_i^{\ j} q_j(x)\ ,\qquad i,j = 1\ldots N$$

[1]Note that the Young-Mills-gauge group $SU(N)$ is *not* abelian.

b) Field transforming in the *adjoint representation*: elements of the (N^2-1)-dim algebra $su(N)$

$$\phi_i{}^j := \phi^a (T^a)_i{}^j \quad \rightarrow \quad \left(e^{\mathrm{i}\vartheta^b T^b}\right)_i^{\;k} \phi_k{}^l \left(e^{-\mathrm{i}\vartheta^c T^c}\right)_l^{\;j}$$

For obtaining the infinitesimal transformation, approximate the exponentials to the first order.

Infinitesimally, conjugation by a group element $e^{\mathrm{i}\vartheta^a T^a}$ involves the commutator of the $SU(N)$-generators

$$\left[T^a\,,T^b\right] = \mathrm{i} f^{abc}\, T^c$$

so we obtain the transformation law

$$\begin{aligned}\phi^a T^a \quad \rightarrow \quad & \phi^a T^a + \mathrm{i}\left(\vartheta^b T^b\,\phi^a T^a - \phi^a T^a\,\vartheta^b T^b\right) \\ & = \phi^a T^a - \mathrm{i}\,\phi^a \vartheta^b \left[T^a\,,T^b\right] = \phi^a T^a + f^{abc}\phi^a\,\vartheta^b T^c\end{aligned}$$

Again we can define a covariant derivation using a non-abelian field strength $A_\mu = A_\mu^a\, T^a$

$$(\mathrm{D}_\mu)_i{}^j = \delta_i^j\,\partial_\mu + \mathrm{i} g A_\mu^a (T^a)_i{}^j \quad \Rightarrow \quad F_{\mu\nu} = -\frac{\mathrm{i}}{g}\left[\mathrm{D}_\mu\,,\mathrm{D}_\nu\right]$$

where g is the coupling constant.

For a non-abelian gauge theory we have the action

$$S[A] = \int \mathrm{d}^{3+1}x\ \mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}]$$

and by redefining $A \rightarrow gA$ we get

$$= \frac{1}{g^2}\int \mathrm{d}^{3+1}x\ \mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}]$$

In the most important QFT's (such as $U(1)$ and $SU(N)$ gauge theories in 3+1 dimensions) the quantization procedure (i. e. introducing commutation relations for the Fourier modes) leads to infinities when calculating physical observables. The method of dealing with these infinities is called *renormalization*. This procedure introduces an *energy scale* μ at which physical observables are defined. Then the question arises: What happens if I change this scale? For example:

QED: This theory becomes non-interacting when $\mu \to 0$

QCD: ($SU(3)$-gauge theory) This theory becomes non-interacting when $\mu \to \infty$ and strongly interacting[2] when $\mu \to 0$!

Remember that if $c = \hbar = 1$ the length scale $l \propto E^{-1}$ so the limes $\mu \to 0$ is equivalent to the limes $l \to \infty$. As you see, the first item is nothing new at all (in electrodynamics the Coulomb-potential goes by $\frac{1}{r}$). But in QCD we have a force[3] that increases its strength the larger the distances become which is associated with *confinement*.[4]

Later we consider the case of a $SU(N)$-gauge theory in the limit $N \to \infty$. As shown by G. 't Hooft in 1974, in this limit the quantized gauge theory simplifies significantly.

2. A very brief review of general relativity

Background:

AdS/CFT correspondence

Anti-de Sitter space
a *gravity theory*

conformal field theory[5]
$SU(N)$-gauge theory, $N \to \infty$

General Relativity is the gauge theory of coordinate transformations $\Leftrightarrow$ physics does not depend on the coordinate system chosen! Einstein set gravity equal to geometry[6] as a result of the spacetime being a differentiable manifold $\mathcal{M}$ (with d dimensions).

At each point $p \in \mathcal{M}$ we have a *tangent space* $\mathrm{T}_p\mathcal{M}$ spanned by the tangent vectors

$$\partial_\mu := \frac{\partial}{\partial x^\mu}$$

Any vector $V \in \mathrm{T}_p\mathcal{M}$ may be written as

$$V = V^\mu \, \partial_\mu$$

[2] In this case we have $g > 1$ so the application of perturbation theory is not possible.
[3] Quantum Chromodynamics describe the strong interactions, the fundamental force between quarks and gluons.
[4] The fact that color charged particles cannot be observed singularly.
[5] A conformal symmetry is a generalized scale symmetry, i. e. for any scale μ, $\mathscr{L}_{\mathcal{N}=4}$ is invariant.
[6] Gravity is the curvature (resulting from matter) of the spacetime.

We may also define the *cotangent space* $\mathrm{T}_p^*\mathcal{M}$ consisting all linear maps from $\mathrm{T}_p\mathcal{M} \to \mathbb{R}$. ∂_μ induces a *dual basis* $\mathrm{d}x^\nu$ of the cotangent space in the way the equation

$$\mathrm{d}x^\nu(\partial_\mu) = \delta^\nu_\mu \qquad \Rightarrow \qquad W \in \mathrm{T}_p^*\mathcal{M} : \quad W = W_\nu \, \mathrm{d}x^\nu$$

holds.

$SU(N)$-gauge theory	general relativity
covariant derivative D_μ	covariant derivative ∇_μ
gauge field A_μ	Christoffel symbol $\Gamma^\mu{}_{\nu\lambda}$
field strength tensor $F_{\mu\nu}$	Riemann curvature tensor $R^\mu{}_{\nu\rho\sigma}$
Action $S = \int \mathrm{d}^4x \, F^{\mu\nu}F_{\mu\nu}$	Einstein-Hilbert action $S = \int \mathrm{d}^4x \sqrt{-g}R$ (R denotes the Ricci tensor)
Equation of motion $\mathrm{D}^\mu F_{\mu\nu} = 0$	Einstein equations

Table 2.1: Comparison of two gauge theories

We would like to formulate a gauge theory of coordinate transformations. The first step is to define the coordinate transformation as

$$x \to x' : \; \partial_\mu \to \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\,\partial_\nu \,, \quad V = V^\mu \, \partial_\mu = V'^\mu \, \partial'_\mu \quad \Rightarrow \quad V'^\mu = V^\nu \, \frac{\partial x'^\mu}{\partial x^\nu}$$

For the cotangent space we can derive

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu \,, \quad W = W_\nu \, \mathrm{d}x^\nu = W'_\nu \, \mathrm{d}x'^\nu \qquad \Rightarrow \quad W'_\mu = W_\nu \, \frac{\partial x^\nu}{\partial x'^\mu}$$

The next step is to define a *tensor of rank* (r, s):

$$T^{(r,s)} : \underbrace{\mathrm{T}_p^*\mathcal{M} \otimes \cdots \otimes \mathrm{T}_p^*\mathcal{M}}_{r \text{ times}} \otimes \underbrace{\mathrm{T}_p\mathcal{M} \otimes \cdots \otimes \mathrm{T}_p\mathcal{M}}_{s \text{ times}} \to \mathbb{R}$$

$$T^{(r,s)} = \underbrace{T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}}_{\text{functions of } p \in \mathcal{M}} \; \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_s}$$

A particularly important $(0, 2)$-tensor is the *metric*. At each $p \in \mathcal{M}$ this is a non-degenerate symmetric bilinear form g, i. e.

$$g : \begin{cases} \mathrm{T}_p\mathcal{M} \otimes \mathrm{T}_p\mathcal{M} \to \mathbb{R} \\ (u, v) \mapsto g(u, v) \end{cases}$$

$$g(u, v) = g(v, u)$$

so we can write the *line element* as

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

To obtain the covariant derivative, we consider

$$\begin{aligned} \partial'_\mu W'_\nu &= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} \left(\frac{\partial x^\sigma}{\partial x'^\nu} W_\sigma \right) \\ &= \underbrace{\frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \left(\frac{\partial}{\partial x^\rho} W_\sigma \right)}_{\text{covariant term}} + \underbrace{W_\sigma \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x^\sigma}{\partial x^\rho \partial x^\nu}}_{\text{needs to be cancelled}} \end{aligned}$$

Now we replace $\partial_\mu \to \nabla_\mu$ which should have the following properties:

- ∇_μ maps (r, s)-tensors to $(r, s+1)$-tensors
- ∇_μ is linear: $\nabla_\mu(S + T) = \nabla_\mu S + \nabla_\mu T$
- ∇_μ satisfies a Leibniz-rule: $\nabla_\mu(ST) = (\nabla_\mu S)T + S(\nabla_\mu T)$

This implies the form of the *Levi-Civita connection*, the Christoffel symbols Γ. We have

$$\begin{aligned} \nabla_\mu V^\nu &= \partial_\mu V^\nu \; + \; \Gamma^\nu{}_{\mu\lambda} V^\lambda \\ \nabla_\mu W_\nu &= \partial_\mu W_\nu \; - \; \Gamma^\lambda{}_{\mu\nu} W_\lambda \\ \Gamma^\lambda{}_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} \; + \; \partial_\nu g_{\rho\mu} \; - \; \partial_\rho\, g_{\mu\nu} \right) \end{aligned}$$

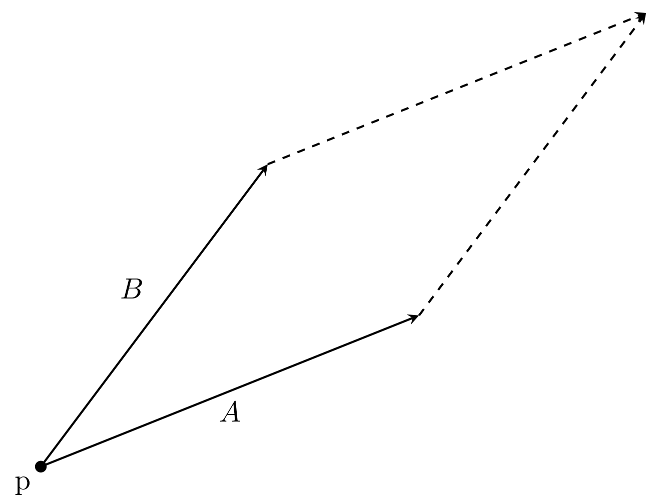
Curvature: The first step to talk about curvature is to think about parallel transport: A *parallel transport* of a vector V along a path $x^\mu(\lambda)$ (where λ is the affine parameter) is defined by the vanishing covariant derivative

$$0 = \nabla_\rho V^\mu \qquad \Rightarrow \qquad 0 = \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\, \nabla_\rho V^\mu = \frac{\mathrm{d}V}{\mathrm{d}\lambda} + \Gamma^\mu{}_{\rho\sigma}$$

A *geodesic* is a curve $x^\mu(\lambda)$ along which the tangend vector $V^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ is parallel transported. This satisfies

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0$$

what we call the *geodesic equation.* In general, parallel transport of a vector along a *closed loop* in a curved spacetime will lead to a different vector than before. Consider



If we compare the parallel transport of a vector V first along A and then along B to the same vector first parallel transported along B and then along A. The difference δV between the two vectors is given by

$$\delta V^\rho = R^\rho{}_{\mu\alpha\beta} V^\mu A^\alpha B^\beta$$

where $R^\rho{}_{\mu\alpha\beta}$ denotes the *Riemann curvature tensor*. We also have

$$\begin{aligned}[\nabla_\alpha\, , \nabla_\beta]\; V^\rho &= \partial_\alpha (\nabla_\beta V^\rho) + \Gamma^\rho \alpha_\mu \, \nabla_\beta V^\mu - \Gamma^\sigma{}_{\alpha\beta} \, \nabla_\sigma V^\mu \;-\; (\beta \leftrightarrow \alpha) \\ &=: R^\rho{}_{\mu\alpha\beta} V^\mu - \underbrace{(\Gamma^\sigma{}_{\alpha\beta} - \Gamma^\sigma{}_{\beta\alpha})}_{\text{vanishes if } \Gamma^\lambda{}_{\mu\nu} \,=\, \Gamma^\lambda{}_{\nu\mu}} \nabla_\sigma V^\rho\end{aligned}$$

For a manifold with a metric $g_{\mu\nu}$, we have $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{(\mu\nu)}$[7] so the second therm vanishes. In therms of the Christoffel symbols we have

$$R^\rho{}_{\mu\alpha\beta} = \partial_\alpha \, \Gamma^\rho{}_{\beta\mu} - \partial_\beta \, \Gamma^\rho{}_{\alpha\mu} + \Gamma^\rho{}_{\alpha\sigma} \, \Gamma^\sigma{}_{\beta\mu} - \Gamma^\rho{}_{\beta\sigma} \, \Gamma^\sigma{}_{\alpha\mu}$$

which leads us to some index symmetries of $R_{\mu\nu\alpha\beta} = g_{\mu\sigma} \, R^\sigma{}_{\nu\alpha\beta}$

$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta} = R_{\alpha\beta\mu\nu}$$

[7] The brackets denote the total symmetrization of this indices:

$$\Gamma^\lambda{}_{(\mu\nu)} = \frac{1}{2} \left(\Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{\nu\mu} \right)$$

The Riemann tensor also satisfies the *Bianchi identity*

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0$$

and

$$\nabla_{[\lambda} R_{\mu\nu]\alpha\beta} = 0 \tag{2.2}$$

Here $[\lambda\mu\nu]$ means total anti-symmetrization. Further we define the *Ricci-tensor*

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu} = R_{\nu\mu}$$

and the *Ricci-scalar*

$$R := R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

which can be connected with equation (2.2) via

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R$$

Next we look at the Einstein-field equations which relate gravity (curvature) and matter (curves the spacetime).

Parenthesis: Einstein equation
Einstein's first idea was to set

$$R_{\mu\nu} = T_{\mu\nu}$$

($T_{\mu\nu}$ denotes the *energy-momentum-tensor*[8]), but since energy and momentum are conserved quantities, we have

$$\nabla^{\mu} T_{\mu\nu} = 0$$

which implies

$$\frac{1}{2} \nabla_{\nu} R = \nabla^{\mu} R_{\mu\nu} = \nabla^{\mu} T_{\mu\nu} = 0$$

so a constantly curved spacetime would be the only possible solution! The "corrected" equations are

$$G_{\mu\nu} = T_{\mu\nu} \ , \qquad G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{(Einstein tensor)}$$

but—in addition to the Ricci-scalar—we also get another therm $\Lambda g_{\mu\nu}$ where Λ is the *cosmological constant*,[9] so the full version of the Einstein equation reads

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}}_{=G_{\mu\nu}} + \Lambda g_{\mu\nu} = \kappa^2\, T_{\mu\nu}$$

with $\kappa^2 = 8\pi\, G$ (G denotes the *Newton constant*).

The *Einstein-Hilbert-action*, whose Euler-Lagrange equations lead us to the Einstein equations, is given by

$$S_{\text{EH}} = \frac{1}{2\kappa^2} \int \mathrm{d}^d x\, \sqrt{-g}\, (R - 2\Lambda)$$

Examples for solutions to the Einstein equations for $T_{\mu\nu} = 0$ (i. e. in the vacuum) are the *maximally symmetric spacetimes*. These satisfy

$$R_{\mu\nu\sigma\rho} = \frac{R}{d(d-1)}\, (g_{\mu\sigma}\, g_{\nu\rho} - g_{\nu\sigma}\, g_{\mu\rho})$$

i. e. they meet a special condition to the curvature.

Let us first consider Riemannian manifolds. Then the maximally symmetric spacetimes are

a) Euclidean flat space

b) spherical solutions

c) hyperbolic solutions

The line element of these spaces is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}\chi}{1 - k\chi^2} + \chi^2\, \mathrm{d}\Omega^2_{d-1}\,, \qquad k \in \{0, \pm 1\}$$

[9] Originally Einstein introduced this constant to allow static solutions for the equations since in his times is was a commonly accepted “fact” that our universe is a static construct which always existed and never expands or shrinks. Only a few years later the expansion of the universe was discovered by E. Hubble, so Einstein—who could have predicted this dynamic years before—referred the introduction of Λ as his “größte Eselei.” But nowadays we again consider a cosmological constant because the expansion of our universe accelerates which contradicts our expectations—and leads us to the so called *dark energy* that should be the reason for $\Lambda > 0$.

where $\mathrm{d}\Omega_{d-1}$ denotes the line element of the unit sphere S^{d-1} that can be constructed via

$$\mathrm{d}\Omega_1 = \mathrm{d}\theta_1$$
$$\mathrm{d}\Omega_j^2 = \mathrm{d}\theta_j^2 - \sin^2\theta_j\,\mathrm{d}\Omega_{j-1}^2\,, \quad \theta_1 \in [0, 2\pi),\ \theta_j \in [0, \pi),\ j = 2 \dots d{-}1$$

Depending on k we get

k = 0: Euclidean space, χ is the radial coordinate. Example for $d = 2$:

$$\Rightarrow \chi = r,\ \theta_1 = \phi,\ \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2$$

k = 1: Make a coordinate transformation $\chi = \sin\phi$, $\phi \in [0, \phi)$, then we have

$$\mathrm{d}s^2 = \mathrm{d}\phi^2 + \sin^2\phi\ \mathrm{d}\Omega_{d-1}^2$$

which is the metric of the *unit sphere* S^{d-1} in d dimensions.

k = −1: $\chi = \sinh\psi$, $\psi \in [0, \infty)$, so we get the line element of a *hyperboloid*

$$\mathrm{d}s^2 = \mathrm{d}\psi^2 + \sinh^2\psi\ \mathrm{d}\Omega_{d-1}^2$$

(This case we call *Euclidean Anti-de Sitter space.*

In Lorentzian spacetime, we have the following solution of Einstein's equations in the vacuum for maximally symmetric spacetimes:

$\boldsymbol{\Lambda} = \mathbf{0}$: Minkowski space

$\boldsymbol{\Lambda} > \mathbf{0}$: de Sitter space

$\boldsymbol{\Lambda} < \mathbf{0}$: Anti-de Sitter space

Let us consider the case of Anti-de Sitter space: AdS_{d+1} is embedded into $(d{+}2)$-dimensional Minkowski space

$$-(X^0)^2 + \sum_{i=1}^{d}(X^i)^2 - (X^{d+1})^2 = -L^2$$

where L denotes the so called *Anti-de Sitter radius.* This space has a conformal boundary at infinity.

For the AdS-space there are two typical parametrizations:

Global AdS coordinates	Local (Poincaré) coordinates
(cover entire AdS space)	(cover only half of the space)
$X^0 = L\cosh\rho\,\cos\tau$	$X^0 = \frac{L^2}{2r}\Big(1-\frac{r^2}{L^4}\big(\vec{x}^2-t^2+L^2\big)\Big)$
$X^{d+1} = L\cosh\rho\,\sin\tau$	$X^i = \frac{r}{L}x^i$
$X^i = L\sinh\rho\,\sin\tau\,\Omega_i$	$X^d = \frac{L^2}{2r}\Big(1+\frac{r^2}{L^4}(\vec{x}^2-t^2+L^2)\Big)$
	$X^{d+1} = \frac{r}{L}t$
$\Rightarrow \mathrm{d}s^2 = L^2\big(-\cosh^2\rho\,\mathrm{d}\tau^2+\mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega^2_{d-1}\big)$	$\Rightarrow \mathrm{d}s^2 = \frac{L^2}{r^2}\mathrm{d}r^2+\frac{r^2}{L^2}\big(-\mathrm{d}t^2+\mathrm{d}\vec{x}^2\big) = \frac{L^2}{r^2}\mathrm{d}r^2+\frac{r^2}{L^2}\Big(\underbrace{\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}_{d\text{-dim Minkowski metric}}\Big)$

3. Classical scalar field theory

We consider a real scalar field $\varphi(x)$ in flat d-dimensional Minkowski spacetime $\mathbb{R}^{d-1,1}$ with $d-1$ spatial directions. The coordinates of $\mathbb{R}^{d-1,1}$ are denoted by X^μ where μ takes values from 0 to $d-1$. We set $c=1$ so the metric of Minkowski-space becomes

$$\mathrm{d}s^2 = -\big(\mathrm{d}X^0\big)^2+\sum_{i=1}^{d-1}\big(\mathrm{d}X^i\big)^2 = \eta_{\mu\nu}\,\mathrm{d}X^\mu\,\mathrm{d}X^\nu$$

The Symmetries of Minkowski-space are the *Poincaré transformations*

$$\begin{aligned} x\to\quad x' &= \Lambda x + a \\ x'^{\,\mu} &= \Lambda^\mu{}_\nu\,x^\nu + a^\mu \end{aligned}$$

where Λ is a *Lorenz transformation* and a is a translation.

A real scalar field is a map which origins a real number $\varphi(x)$ to each spacetime point x. If we transform x via

$$x \quad\to\quad x' = \Lambda\,x$$

we obtain for our field

$$\begin{aligned} \varphi \quad &\to\quad \phi' = \varphi(\Lambda^{-1}\,x) \\ \underbrace{L(x,\dot{x})}_{\text{classical mechanics}} \quad &\to\quad \underbrace{\mathscr{L}(\varphi,\partial_\mu\varphi)}_{\text{field theory, infinite number of variables!}} \end{aligned}$$

The *dynamics* of the scalar field is specified by an *action functional* $S[\varphi]$ which can be written as an integral over the *lagrangian density* $\mathscr{L}(\varphi, \partial_\mu \varphi)$

$$S[\varphi] = \int \mathrm{d}t \; \mathrm{d}^{d-1}x \; \mathscr{L}(\varphi, \partial_\mu \varphi) := \int \mathrm{d}^d x \; \mathscr{L}(\varphi, \partial_\mu \varphi)$$

For example, the action functional of a massive real scalar field with no interactions reads

$$\begin{aligned} S[\varphi] = \int \mathrm{d}^d x \; \mathscr{L}_0 &= -\frac{1}{2} \int \mathrm{d}^d x \left(\underbrace{\eta^{\mu\nu} \; \partial_\mu \varphi \; \partial_\nu \varphi}_{\text{kinetic term}} + \underbrace{m^2 \, \varphi^2}_{\text{mass term}} \right) \\ &= -\frac{1}{2} \int \mathrm{d}^d x \left(-\left(\partial_t \, \varphi(t, \vec{x})\right)^2 + \left(\nabla \varphi(t, \vec{x})\right)^2 + m^2 \, \varphi(t, \vec{x})^2 \right) \end{aligned} \tag{2.3}$$

The principle of sationary action (Hamilton) gives the Lagragian equations of motion.

Functional derivative $\dfrac{\delta S[\varphi]}{\delta \varphi(x)}$

The *functional derivative* is defined by

$$\frac{\delta \varphi(x)}{\delta \varphi(y)} = \delta^{(d)}(x - y) \qquad \text{dirac delta distribution}$$

$$\Rightarrow \quad \frac{\delta}{\delta \varphi(x)} \int \mathrm{d}^d y \, \varphi(y) = \int \mathrm{d}^d y \; \frac{\delta \varphi(y)}{\delta \varphi(x)} = \int \mathrm{d}^d y \, \delta^{(d)}(x - y) = 1$$

$$\Rightarrow \quad \frac{\delta S[\varphi]}{\delta \varphi(x)} = \frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \, \partial_\mu \varphi} \right) \stackrel{!}{=} 0 \qquad \text{Lagrange equation}$$

For our example (2.3) the Lagrange equations of motion become

$$\left(\Box - m^2\right) \varphi(x) = 0 \tag{2.4}$$

where $\Box = \partial^\mu \, \partial_\mu = -\partial_t^2 + \triangle$ is the *d'Alembert operator*. Equation (2.4) is called the *Klein-Gordon equation*. So far, we considered a field in a potential $V(\varphi) = 0$, but more interesting physics is obtained by considering *interactions*, i. e. $V(\varphi) \neq 0$. Generally, $V(\varphi)$ will be a polynomial $V(\varphi) = \varphi^\alpha$, $\alpha > 2$. Then we have $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}$, $\mathscr{L}_{\text{int}} = -\frac{g_n}{n!} \varphi(x)^n$. g_n is the associated coupling constant which messures the strength of the interaction.

In electrodynamics—as an example for a classical field theory—the fields read

$$A_\mu = (\phi, \vec{A})$$

so we can derive the equations of motion from the action

$$S[A_\mu] = \int \mathrm{d}^4x \, F^{\mu\nu} F_{\mu\nu} \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Rightarrow \quad 0 = \frac{\delta S[A]}{\delta A^\mu(x)} \qquad \Rightarrow \quad \underset{\text{covariant derivative}}{\mathrm{D}^\mu F_{\mu\nu} = 0}$$

To *quantize the scalar field theory*, we note that the Fourier decomposition of φ reads

$$\varphi(x) = \frac{1}{(2\phi)^{d-1}} \int \frac{\mathrm{d}^{d-1}k}{2\omega_k} \left[a(\vec{k})\, e^{-\mathrm{i}kx} + a^*(\vec{k})\, e^{\mathrm{i}kx} \right]_{k^0=\omega_k}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and $kx = -k^0x^0 + \vec{k} \cdot \vec{x}$ satisfies the Klein-Gordon equation.

The starting point for a general *quantization of free fields*, which satisfy $(\Box - m^2)\varphi = 0$, is to replace the Fourier modes with *operators* $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$. The field $\varphi(x)$ then also becomes an operator denoted by $\hat{\varphi}(x)$.

$$\hat{\varphi}(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^{d-1}k}{2\omega_k} \left[\hat{a}(\vec{k})\, e^{-\mathrm{i}kx} + \hat{a}^\dagger(\vec{k})\, e^{\mathrm{i}kx} \right]_{k^0=\omega_k}$$

The operators $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ satisfy the commutation relations

$$\begin{aligned} \left[\hat{a}(\vec{k}) \,, \hat{a}^\dagger(\vec{k}') \right] &= 2\omega_k (2\pi)^{d-1}\, \delta^{(d-1)}(\vec{k} - \vec{k}') \\ \left[\hat{a}(\vec{k}) \,, \hat{a}(\vec{k}') \right] &= 0 \\ \left[\hat{a}^\dagger(\vec{k}) \,, \hat{a}^\dagger(\vec{k}') \right] &= 0 \end{aligned}$$

These commutation relations coincide with those of a harmonic oscillator in quantum mechanics at frequency ω_k. Therefore we may interpret the $\hat{a}$, $\hat{a}^\dagger$ as creation and annihilation operators. The vacuum state $|0\rangle$ is annihilated by $\hat{a}$ for all $\vec{k}$ ($\hat{a}(\vec{k})\,|0\rangle = 0$). We choose $\langle 0|0\rangle = 1$. A single-particle state with momentum $\vec{k}$, $|\vec{k}\rangle$, is created by acting on the vacuum state with the *creation* operator $\hat{a}^\dagger(\vec{k})$ ($|\vec{k}\rangle = \hat{a}^\dagger(\vec{k})\,|0\rangle$).

A quantum field theory combines the ideas of classical field theory with the ideas of quantum mechanics. In particular, the propagation of a mode $|\vec{k}\rangle$ width momentum $\vec{k}$ in space may be related to the concepts of Huyghen's principle and Green's functions in classical field theory.

$$\varphi(y) = \int \mathrm{d}^d x \; G(x,y)\, \varphi(x)$$

For our example we have

$$(-\Box + m^2)\, G(x,y) = \delta^{(d)}(x-y)$$

The goals we want to reach are

- Quantize interacting QFT's
- Define generating functionals

→ Introduce path integral quantization of (free) fields

Within quantum mechanics, the path integral sums over all possible paths which start at the position q_i at time t_i and end at q_f at time t_f.

In quantum field theory this translates into summing over all field configurations φ in configuration space. The integration measure becomes

$$\mathcal{D}\varphi = (\text{factor}) \prod_{t_i \le t \le t_f} \prod_{\vec{x}\,\in\,\mathbb{R}^{d-1}} \mathrm{d}\varphi(t,\vec{x})$$

We are able to use this to give a formula for the transition from an initial state $|\varphi_i, t_i\rangle$ to a final state $|\varphi_f, t_f\rangle$, where

$$\begin{aligned}
\underbrace{\hat{\varphi}(t,\vec{x})}_{\text{operator}}\, |\varphi_i, t_i\rangle &= \varphi_i(\vec{x})\, |\varphi_i, t_i\rangle \\
\overbrace{\hat{\varphi}(t,\vec{x})}\, |\varphi_f, t_f\rangle &= \varphi_f(\vec{x})\, |\varphi_f, t_f\rangle
\end{aligned}$$

so we obtain

$$\langle \varphi_f, t_f | \varphi_i, t_i \rangle = N \int \mathcal{D}\varphi\, \exp\left[\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t \int_{\vec{x}\,\in\,\mathbb{R}^{d-1}} \mathrm{d}^{d-1}x\, \mathscr{L}_0(\varphi, \partial_\mu\varphi)\right]$$

where N is a normalization factor and $\mathscr{L}_0$ the *free* Lagrangian, i. e. without any interactions (e. g. $\mathscr{L}_0 = -\frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2$). The expression for $\langle \varphi_f, t_f | \varphi_i, t_i \rangle$ applies to the case of a free field.

We integrate over all field configurations $\varphi(t,\vec{x})$ satisfying the boundary conditions $\varphi(t_i,\vec{x}) = \varphi_i(\vec{x})$ and $\varphi(t_f,\vec{x}) = \varphi_f(\vec{x})$. It is not clear if this integral exists in a mathematical sense! We may improve convergence by replacing m^2 by $m^2 - \mathrm{i}\varepsilon$ and taking $\varepsilon \to 0$ at the end of all calculations.

From now on we restrict to *vacuum-to-vacuum* amplitudes width $t_i \to -\infty$, $t_f \to +\infty$ and consider $\varphi_i(\vec{x}) = \varphi_f(\vec{x}) = 0$. We write $\langle 0, +\infty | 0, -\infty \rangle =: \langle 0|0 \rangle$, so we have the condition

$$\langle 0|0\rangle = N \int \mathcal{D}\varphi \, \exp\left[\mathrm{i}\int \mathrm{d}^d x \, \mathscr{L}_0(\varphi, \partial_\mu \varphi)\right] \stackrel{!}{=} 1$$

which fixes our normalization constant N.

For describing physical processes we are interested in so called *correlation functions* of the form

$$\langle 0| \, \tau \hat{\varphi}(x_1)\hat{\varphi}(x_2) \ldots \, \hat{\varphi}(x_n) \, |0\rangle =: \langle \varphi(x_1) \ldots \, \varphi(x_n) \rangle =: G^{(n)}(x_1, \ldots \, x_n)$$

where τ denotes the *time-sorted product*. For two operators, τ becomes

$$\tau \hat{\varphi}(x)\hat{\varphi}(y) = \Theta(x^0 - y^0)\hat{\varphi}(x)\hat{\varphi}(y) + \Theta(y^0 - x^0)\hat{\varphi}(y)\hat{\varphi}(x)$$

i. e. the operators are sorted the way the time of their arguments decreases from left to right. Then we have

$$\langle \varphi(x_1) \ldots \, \varphi(x_n) \rangle = N \int \mathcal{D}\varphi \; \hat{\varphi}(x_1) \ldots \, \hat{\varphi}(x_n) \, \exp\left[\mathrm{i}\int \mathrm{d}^d x \, \mathscr{L}(\varphi, \partial_\mu \varphi)\right]$$

In order to calculate the correlation functions, it is convenient to introduce the *generation functional*

$$Z_0[J] := \langle \exp\left[\mathrm{i}\int \mathrm{d}^d x \; J(x)\varphi(x)\right] \rangle$$

(this may be compared to the partition function in statistical mechanics) where $J(x)$ is called a *source dual* to the field $\varphi(x)$.

$Z_0[J]$ is extremely useful! We may write

$$\langle \varphi(x_1) \ldots \, \varphi(x_n) \rangle = \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \, \delta J(x_n)} \right|_{J=0}$$

Written in full we have

$$Z_0[J] = N \int \mathcal{D}\varphi \, \exp\left[\mathrm{i}\int \mathrm{d}^d x \left[\mathscr{L}_0(\varphi, \partial_\mu \varphi) + J(x)\varphi(x)\right]\right]$$

Let us consider the example of the free massive scalar field,

$$Z_0[J] = N \int \mathcal{D}\varphi \, \exp\left[\mathrm{i}\int \mathrm{d}^d x \left[-\frac{1}{2}\varphi(-\Box + m^2 - \mathrm{i}\varepsilon)\varphi + J\varphi\right]\right]$$

Note that the integral over φ is almost quadratic in φ. This allows us to perform the integration:

$$Z_0[J] = N \exp\left[\frac{\mathrm{i}}{2}\int \mathrm{d}^d x \int \mathrm{d}^d y\, J(x)\Delta_\mathrm{F}(x-y)J(y)\right]$$

where Δ_F is the *Feynman-propagator* for a scalar field

$$\Delta_\mathrm{F}(x-y) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{e^{\mathrm{i}k(x-y)}}{k^2+m^2-\mathrm{i}\varepsilon}$$

which satisfies $(-\Box + m^2)\Delta_\mathrm{F}(x-y) = \delta^{(d)}(x-y)$, so Δ_F is the Green's function of the Klein-Gordon equation.

$$G^{(2)}(x,y) = \langle\varphi(x_1),\varphi(x_2)\rangle = -\mathrm{i}\Delta_\mathrm{F}(x_1-x_2)$$

In the massless case we have for $d=4$

$$\Delta_\mathrm{F}(x-y) = \frac{1}{4\pi^2}\frac{1}{(x-y)^2}\ , \qquad (x-y)^2 = (x^\mu - y^\mu)(x_\mu - y_\mu)$$

To treat *interacting fields* we add some terms such as

$$\mathscr{L}_\mathrm{int} = -\frac{g_n}{n!}\varphi^n$$

to the Lagrangian, which leads us to

$$Z[J] = N\int \mathcal{D}\varphi\, \exp\left[\mathrm{i}\!\int \mathrm{d}^d x\, \left(\mathscr{L}_0 + \mathscr{L}_\mathrm{int} + J\varphi\right)\right]$$

This is no longer Gaussian. However, if the coupling constants are small ($g \ll 1$), we may use *perturbation theory.* The starting point is to write

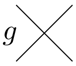
$$\begin{aligned} Z[J] &= N\exp\left[\mathrm{i}\!\int \mathrm{d}^d x\, \mathscr{L}_\mathrm{int}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right]\int \mathcal{D}\varphi\, \exp\left[\mathrm{i}\!\int \mathrm{d}^d x\, (\mathscr{L}_0 + J\varphi)\right] \\ &= N\exp\left[\mathrm{i}\!\int \mathrm{d}^d x\, \mathscr{L}_\mathrm{int}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J] \end{aligned}$$

The expansion in a power series may be visualized in a graphical way: the *Feynman diagrams.*

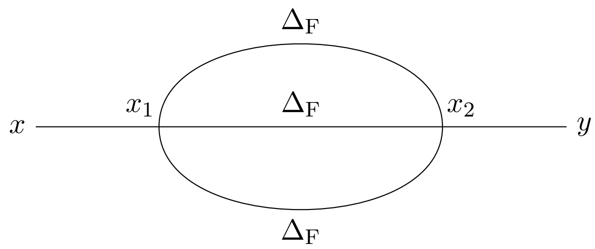
For example let's consider a φ^4 theory in $d=4$ dimensions.

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_\mathrm{int}\ , \qquad \mathscr{L}_0 = -\frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\,\varphi^2\ , \qquad \mathscr{L}_\mathrm{int} = -\frac{g}{4!}\varphi^4$$

To draw a diagram, we have to follow the three *Feynman rules*, which are

1) Propagator $\mathrm{i}\Delta_{\mathrm{F}}(x-y) \rightarrow$
2) Interaction vertices $\rightarrow$ g 
3) An integration has to be performed on all vertices.

Possible Feynman diagrams for our example theory are:



$\mathcal{O}(1)$

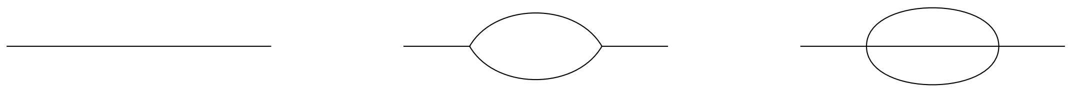
$\mathcal{O}(g^2)$

Later we will look at examples for Feynman diagrams in a $SU(N)$ theory.

Side remark: There are many interesting QFT's in physics where $g > 1$, i. e. perturbation theory is not applicable. These thermes are referred to as "strongly coupled." Generally, *collective* phenomena occur, i. e. it is hard to identify single-particle excitations (instead one should look at *collective excitations*). In the AdS/CFT correspondence, we are forced to consider a particular limit in which the CFT is indeed strongly coupled which makes calculations in the CFT harder. However, we can use the dual classical gravity theory (which is easy to solve since it is classical) to make non-trivial predictions about strongly coupled CFT's.

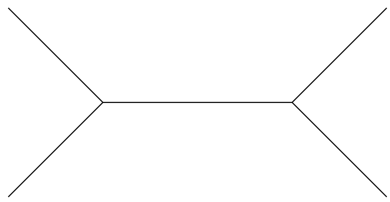
! In addition to its intrinsic interest of relating QFT and gravity, the AdS/CFT correspondence provides a useful tool for making predictions about strongly coupled QFT's!

Let's go back to perturbation theory for now. Loosely speaking, every closed loop in a Feynman diagram corresponds to an order of $\hbar$.



$\mathcal{O}(\hbar^0)$ $\mathcal{O}(\hbar)$ $\mathcal{O}(\hbar^2)$

In a classical theory, there are only *tree diagrams*



Fore the AdS/CFT correspondence we are interested in perturbation theory for the $SU(N)$ non-abelian gauge theory

$$S = \mathrm{Tr}\left[\frac{1}{g^2}\int \mathrm{d}^4x\ F^{\mu\nu}F_{\mu\nu}\right]$$

This is rather complicated, so let us begin with a simpler toy model with a scalar field ϕ in the adjoint representation of the gauge group $SU(N)$ ($\phi^i{}_j := \phi^a(T^a)^i{}_j$).

$$\mathscr{L} = \underbrace{-\frac{1}{2}\,\mathrm{Tr}\,(\partial_\mu\phi\,\partial^\mu\varphi)}_{\text{kinetic term}} + \underbrace{g\,\mathrm{Tr}\left(\phi^3\right) + g^2\,\mathrm{Tr}\left(\phi^4\right)}_{\text{interactions}}$$

Note that the theory is massless ($m = 0$). It is convenient to rescale $\phi \to \tilde{\phi} = g\phi$, so our Lagrangian becomes

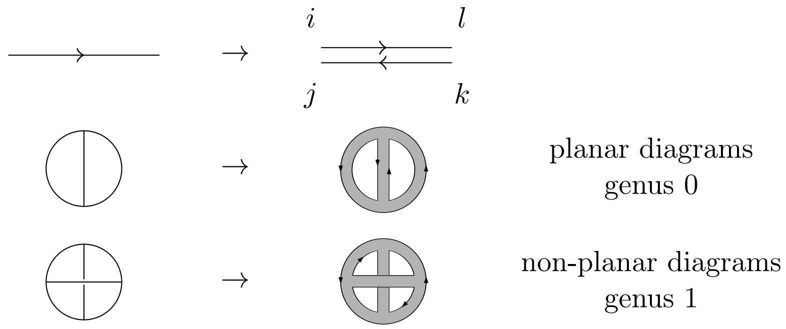
$$\mathscr{L} = \frac{1}{g^2}\left(-\frac{1}{2}\,\mathrm{Tr}\,(\partial_\mu\phi\,\partial^\mu\varphi) + \mathrm{Tr}\left(\phi^3\right) + g^2\,\mathrm{Tr}\left(\phi^4\right) + \mathrm{Tr}\left(\phi^4\right)\right)$$

An important result of G. 't Hooft (1974) was that the perturbative expansion may be organized in a topological fashion in the limit $N \to \infty$. To have this limit well-defined, we introduce the 't Hooft coupling $\lambda := g^2 N$.

The propagator takes the form

$$\langle\tilde{\phi}^i{}_j\,\tilde{\phi}^k{}_l\rangle = \delta^i_l\,\delta^j_k\,\frac{g^2}{4\pi^2\,(x-y)^2}\ , \qquad (d=4)$$

't Hooft suggested a double-line notation for these propagators:



't Hooft found out that in the limit $N \to \infty$ only the genus 0, i. e. the planar diagrams, survive!

4. Conformal Symmetry

In a CFT, the fields transform covariantly under conformal symmetry transformations. Let us first look at such symmetry transformations: Consider a line element for a metric, i. e.

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

Conformal symmetry transformations are those which leave the metric invariant up to an arbitrary positive spacetime-dependent factor[10]

$$\begin{aligned} x \quad &\to \quad \tilde{x} = f(x) \qquad \text{with} \\ g_{\mu\nu}(x) \quad &\to \quad \underbrace{\Omega^{-2}(x)}_{\text{scalar function}}\, g_{\mu\nu}(x) \end{aligned}$$

Note that this is not a Lorentz transformation since it doesn't act on the indices of $g_{\mu\nu}$. Now we want to find infinitesimal transformations of x^μ, so we write

$$\mathrm{d}s'^2 = e^{2\sigma(x)}\,\mathrm{d}s^2\,, \quad \text{i. e.} \quad \Omega(x) = e^{-\sigma(x)}$$

Consider $g_{\mu\nu} = \eta_{\mu\nu}$ (Minkowsky spacetime). Then an infinitesimal transformation is given by

$$x^\mu \quad \to \quad \tilde{x}^\mu = x^\mu + \varepsilon^\mu(x)$$

[10] An interpretation of conformal transformations is that they preserve angles locally.

Under general coodinate transformations, we have $\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$; also we have infinitesimally that $\Omega(x) = 1 - \sigma(x)$. Therefore (for conformal transformations) we have

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\sigma(x)\,\eta_{\mu\nu}$$

which is called the *conformal Killing equation*. The conformal coordinate transformations are solutions to this equation.

Next step is to take the trace of the equation which leads us to

$$\partial \cdot \varepsilon := \partial_\mu \varepsilon^\mu = d\,\sigma(x)$$

where d denotes the number of dimensions. This implies

$$(\eta_{\mu\nu} - \partial^\rho\,\partial_\rho + (d-2)\,\partial_\mu\,\partial_\nu)\,\partial \cdot \varepsilon = 0$$

The case $d = 2$ is special, so let's first solve the equation for $d > 2$. The solutions are

$$\begin{array}{ccccccc} \varepsilon^\mu(x) & = & \underbrace{x^\mu}_{\mathcal{O}(1)} & + & \underbrace{\omega^\mu{}_\nu\, x^\nu + \xi\, x^\mu}_{\mathcal{O}(x)} & + & \underbrace{\left(b^\mu\, x^2 - 2(b\cdot x)x^\mu\right)}_{\mathcal{O}(x^2)} \\ d=4 & \Rightarrow & 4 & + & 6+1 & + & 4 \end{array}$$

so we have 15 free parameters (ω denotes an antisymmetric matrix and ξ a scalar).

Now let us look at a finite transformation. There are

$$\begin{array}{lccc} \text{scale transformations:} & x^\mu & \to & \lambda\, x^\mu \\ \text{special conformal transformations:} & x^\mu & \to & \dfrac{x^\mu + b^\mu\, x^2}{1 + 2(bx) + b^2\, x^2} \end{array}$$

These transformations form a group, the *conformal group*: $\begin{cases} \text{Lorentzian:} & \text{SO(d,2)} \\ \text{Euclidean:} & \text{SO(d+1,1)} \end{cases}$

In addition, for a finite conformal transformation it is useful to introduce the *inversion* which is a reflection at the unit circle

$$x^\mu \quad \to \quad \frac{x^\mu}{x^2}$$

This is not connected to the identity, so it is an element of $O(d,2)$ but not $SO(d,2)$ (for Lorentzian signature). However, we may show that a special conformal transformation may be written as an inversion, a translation and another inversion (i. e. with an even number of inversions).

What happens in $d = 2$? We take Euclidean signature and obtain

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2 \underbrace{\sigma(x)}_{=\frac{1}{d}\,\partial\cdot\varepsilon} \eta_{\mu\nu} \,, \qquad \mu, \nu \in \{1, 2\}$$

$$\left.\begin{aligned} && \partial_1 \varepsilon_2 + \partial_2 \varepsilon_1 &= 0 \\ 2\,\partial_1 \varepsilon_1 = \partial_1 \varepsilon_1 + \partial_2 \varepsilon_2 &\quad \Rightarrow & \partial_1 \varepsilon_1 &= \partial_2 \varepsilon_2 \end{aligned}\right\} \quad \text{Cauchy-Riemann equations}$$

$\Rightarrow$ Every holomorphic function is a solution to the equations

$\Rightarrow$ Conformal group is infinite dimensional

$\Rightarrow$ There is an infinite number of conserved quantities!

Also the conformal symmetry leads to an algebraic structure. Here we have

Generators:	$a_\mu \to P_\mu$	associated with momentum
	$\omega_{\mu\nu} \to J_{\mu\nu}$	rotation or Lorentz-transformation
Generators of scale transformations:	D	
Special conformal transformation:	K_μ	

These generators form an algebra including the Poincaré-algebra which involves commutators of P_μ and $J_{\mu\nu}$. The further commutators are

$$\begin{aligned} [J_{\mu\nu}\,, K_\rho] &= \mathrm{i}(\eta_{\mu\rho}\, K_\nu - \eta_{\nu\rho}\, K_\mu) \\ [D\,, P_\mu] &= \mathrm{i} P_\mu \\ [D\,, K_\mu] &= -\mathrm{i} K_\mu \\ [D\,, J_{\mu\nu}] &= 0 \\ [K_\mu\,, K_\nu] &= 0 \\ [K_\mu\,, P_\nu] &= 2\mathrm{i}(\eta_{\mu\nu}\, D - J_{\mu\nu}) \end{aligned}$$

These form the algebra $so(d, 2)$ in d dimensions ($d > 2$). Now we consider how fields (more specifically: a scalar field $\varphi(x)$) transforms covariantly under these transformations. This yields a representation of the conformal algebra. To obtain this, we consider the transformations at $x = 0$; applying the translation operator P_μ then gives the transformation for general x. For the Lorentz-transformation we have

$$[J_{\mu\nu}\,, \varphi(0)] = J_{\mu\nu}\, \varphi(0)$$

$J_{\mu\nu}$ is a finite-dimensional representation of the Lorentz group reflecting the spin of φ (0 in this case).

$$[D\,,\varphi(0)] = -\mathrm{i}\Delta\varphi(0)$$

where Δ is the scale dimension of $\varphi(0)$, i. e. under scale transformations apply

$$\begin{aligned}\varphi(x) \to \varphi'(x') &= \lambda^{-\Delta}\varphi(x)\\ x \to x' &= \lambda x\end{aligned}$$

A $\varphi(x)$ which has a fixed scaling dimension is an eigenstate of D: In a conformal algebra it is sufficient to consider particular fields, the *conformal primary fields* which satisfy $[K_\mu\,,\varphi(0)] = 0$.

This defines an irreducible multiplet. All other fields in the same multiplet—the *descendants of* φ—are obtained by acting with P_μ on the conformal primary fields. P_μ is our 'Ladder opperatur'—we will shortly encounter a similar structure for the *supersymmetry*-algebra.

By applying translations we can deduce the transformation properties of $\varphi(x)$ from those of $\varphi(0)$. We find

$$\begin{aligned}[P_\mu\,,\varphi(x)] &= \mathrm{i}\,\partial_\mu\varphi(x)\\ [D\,,\varphi(x)] &= -\mathrm{i}\Delta\varphi(x) + x^\mu\,\partial_\mu\varphi(x) =: \delta\varphi(x)\\ [J_{\mu\nu}\,,\varphi(x)] &= J_{\mu\nu}\,\varphi(x) - \mathrm{i}(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu)\varphi(x)\\ [K_\mu\,,\varphi(x)] &= \left(\mathrm{i}(x^2\,\partial_\mu - 2x_\mu x^\rho\,\partial_\rho + 2x_\mu\Delta) - 2x^\nu J_{\mu\nu}\right)\varphi(x)\end{aligned}$$

In a CFT, for the correlation functions of $\varphi(x)$ we must have

$$\begin{aligned} &\sum_{i=1}^{n}\langle\varphi_1(x_1)\dots\,\delta\varphi_i(x_i)\dots\,\varphi_n(x_n)\rangle &&= 0 \qquad \text{(ward identity)}\\ \Rightarrow\qquad &\sum_{i=1}^{n}\left(x_i^\mu\,\frac{\partial}{\partial x_i^\mu} - \mathrm{i}\Delta\right)\langle\varphi_1(x_1)\dots\,\varphi_n(x_n)\rangle &&= 0\end{aligned}$$

For the two-point functions this implies

$$\langle\varphi_1(x_1)\varphi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\,\langle\varphi_1(\lambda x_1)\varphi_2(\lambda x_2)\rangle$$

which implies that

$$\langle\varphi_1(x_1)\varphi_2(x_2)\rangle = \begin{cases}\dfrac{C}{(x_1-x_2)^{2\Delta}} & \Delta_1 = \Delta_2 = \Delta\\[2ex] 0 & \text{otherwise}\end{cases}$$

5. Fermions (in QFT)

5.1. Dirac fermions in classical field theory

A Dirac field $\Psi(x)$ transforms under Lorentz-transformations $x \to \Lambda\, x$ as

$$\Psi(x) \quad \to \quad \Psi(x') = \exp\left(\frac{1}{8}\omega_{\mu\nu}\left[\gamma^\mu\,,\gamma^\nu\right]\right)\Psi(\Lambda^{-1}x)$$

where $\omega_{\mu\nu}$ denotes the antisymmetric parameter matrix belonging to Λ. The γ^μ are the *Dirac matrices* satisfying the *Clifford algebra*

$$\{\gamma^\mu\,,\gamma^\nu\} = -2\eta^{\mu\nu}\,\mathbb{1}\;, \qquad \{A\,,B\} = AB + BA$$

Let us construct the Lagrangian for the fermionic fields $\Psi(x)$. In the bosonic case the Lagrangian reads

$$\mathscr{L}(\varphi, \partial_\mu\varphi) = -\frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - m^2\varphi^2 \qquad \text{(free bosonic theory)}$$

For a free fermionic theory we have

$$\mathscr{L} = \mathrm{i}\overline{\Psi}\,\partial\!\!\!/\Psi - m\overline{\Psi}\Psi\;, \qquad \text{where } \partial\!\!\!/ = \gamma^\mu\,\partial_\mu\;, \quad \overline{\Psi} = \Psi^\dagger\gamma^0$$

which results in the Euler-Lagrange equations

$$0 \stackrel{!}{=} \frac{\delta S[\Psi,\overline{\Psi}]}{\delta\Psi} = (-\mathrm{i}\,\partial\!\!\!/ + m)\Psi$$

and similarly for $\overline{\Psi}$.

Some remarks about the γ-matrices in $d = 4$: A basis for these may be written in terms of Pauli matrices. We denote $\sigma^\mu = (-\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (-\mathbb{1}, -\vec{\sigma})$, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ so we are able to write

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

γ^0 is a Hermitian matrix while γ^1, γ^2 and γ^3 are anti-Hermitian. Furthermore it is very useful to define γ_5 [11] as

$$\gamma^5 := \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

[11] That it is not actually denoted as γ^4 is a result of different notations used in the past: Some physicists numbered the γ^μ with $\mu \in \{0,1,2,3\}$ but some others used $\mu \in \{1,2,3,4\}$. To not confuse some people the matrix is denoted as γ^5 since it's an unique name independent of the used convention.

5.2. Dirac fermions and Weyl fermions

Using γ^5 we may introduce left- and right-handed *Weyl-fermions*. Therefore we will need the projection operators

$$\begin{aligned} \mathcal{P} &= \frac{\mathbb{1}+\gamma_5}{2}\,, & \overline{\mathcal{P}} &= \frac{\mathbb{1}-\gamma_5}{2} \\ \Psi_\mathrm{L} &= \mathcal{P}\Psi\,, & \Psi_\mathrm{R} &= \overline{\mathcal{P}}\Psi \end{aligned}$$

This decomposes into two-component spinors while Dirac spinors have four components:

$$\Psi_\mathrm{L} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}\,, \quad \Psi_\mathrm{R} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}$$

For Ψ_L the free Lagrangian is

$$\mathscr{L} = -\mathrm{i}\psi^\dagger\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

The classical Dirac Lagrangian has a global $U(1)$ symmetry corresponding to the transformation

$$\Psi \quad\to\quad e^{\mathrm{i}\alpha}\,\Psi\,, \qquad \overline{\Psi} \quad\to\quad e^{-\mathrm{i}\alpha}\,\overline{\Psi}$$

This leaves the Dirac Lagrangian invariant so we may associate it with the conserved current $J^\mu = \overline{\Psi}\,\gamma^\mu\,\Psi$. For a massless theory there is an additional symmetry

$$\Psi \quad\to\quad e^{\mathrm{i}\alpha\gamma_5}\,\Psi\,, \qquad \overline{\Psi} \quad\to\quad \overline{\Psi}\,e^{-\mathrm{i}\alpha\gamma_5}$$

with conserved current

$$J_5^\mu = \overline{\Psi}\,\gamma^\mu\gamma_5\,\Psi$$

which is called the *axial current*.

6. Supersymmetry

If we head back to our *supersymmetry*-algebra, we can rearrange the generators in the following way

$$\begin{aligned} J_{\mu\nu} &= \ldots\,, & \mu,\nu &= 1\ldots d{-}1 \\ J_{d\,d+1} &= -D \\ J_{\mu\,d} &= \frac{1}{2}(K_\mu - P_\mu) \\ J_{\mu\,d+1} &= \frac{1}{2}(K_\mu + P_\mu) \end{aligned}$$

that we are able to see the structure of $SO(d, 2)$.

At the moment, all the generators we have (J, P, K, D) are bosonic which means they cannot change the symmetry (symmetric or antisymmetric) of the states they are acting on. For the supersymmetry, we now may introduce some *fermionic* operators[12] and a *supercharge* Q with

$$\Lambda = \exp(\mathrm{i}\overline{\Sigma} Q) \ , \quad Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix} \ , \quad \alpha = 1, 2 \ , \ \ \dot\alpha = \dot 1, \dot 2 \ , \ \ d = 4$$

where $\overline{\Sigma}$ is a fermionic generator. This restricted Lie-algebra fulfills a Jacobi-identity and is graded so the Lie-brackets become

$$[A, B\} = \begin{cases} \{A\, , B\} & A \text{ and } B \text{ fermionic} \\ [A\, , B] & \text{else} \end{cases}$$

with the (anti-)commutation relations

$$\begin{aligned} \left\{Q_\alpha\, , \bar{Q}_{\dot\alpha}\right\} &= 2(\sigma^\mu)_{\alpha\dot\alpha}\, \bar{P}_\mu \\ [Q\, , P] = \left[\bar{Q}\, , P\right] &= \{Q\, , Q\} = \left\{\bar{Q}\, , \bar{Q}\right\} = 0 \end{aligned}$$

or, in general

$$\begin{aligned} [Q_\alpha\, , J^{\mu\nu}] &= (\sigma^{\mu\nu})_\alpha{}^\beta\, Q_\beta \\ \left[\bar{Q}_{\dot\alpha}\, , J^{\mu\nu}\right] &= \varepsilon_{\dot\alpha\dot\beta}\, (\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\gamma}\, \bar{Q}^{\dot\gamma} \end{aligned}$$

Here the $\varepsilon_{\dot\alpha\dot\beta}$ is used to raise/lower spinor-indices. Also we have the *R-symmetry* which can be used to transform different supercharges into each other. In the chase of $\mathcal{N} = 1$ the symmetry group is isomorphic to $U(1)$:

$$\begin{aligned} Q_\alpha &\rightarrow e^{\mathrm{i}\omega}\, Q_\alpha \\ \bar{Q}_{\dot\alpha} &\rightarrow e^{-\mathrm{i}\omega}\, \bar{Q}_{\dot\alpha} \end{aligned}$$

[12] The idea behind this introduction the Coleman-Mandula theorem which stated that all additional symmetries commute with the Poincaré-group. This means that it's not possible to have a symmetry mixing up spins and other internal symmetries. An answer to the Colman-Mandula theorem was given by Haag, Łopuszański and Sohnius who showed that it's indeed possible to find such symmetries but they must be generated by fermionic charges—a loophole in the Coleman-Mandula theorem.

6.1. Extended SUSY

As indicated above, it is possible to introduce multiple supercharges Q^a_α, $a \in \{1, \dots \mathcal{N}\}$. The algebraic relations between the different Q^a_α are

$$\begin{aligned} \left\{Q^a_\alpha \, , Q^b_\beta\right\} &= \varepsilon_{\alpha\beta} \, Z^{ab} \\ \left\{\bar{Q}_{a\,\dot\alpha} \, , \bar{Q}_{b\,\dot\beta}\right\} &= \varepsilon_{\dot\alpha\dot\beta} \, \bar{Z}_{ab} \end{aligned}$$

where $Z^{ab} = -Z^{ba}$ is the *central charge* which commutes with all generators. In this setting, the R-symmetry becomes

$$\begin{array}{cclcl} Q^a_\alpha & \to & R^a{}_b \, Q^b_\alpha & \to & \text{fundamental representation} \\ \bar{Q}_{a\,\dot\alpha} & \to & \bar{Q}_{b\,\dot\alpha} \, (R^\dagger)^b{}_a & \to & \text{anti-fundamental representation} \end{array}$$

6.2. Massless Representation

In the massless case, we represent our states through $|p^\mu, \lambda\rangle$ where the *helicity* λ is the eigenvalue with respect to $J_{12} \propto J_3 = J_z$. Since the mass of the particles is zero, we are not able to find a system in which $p^\mu = (E, 0, 0, 0)$ but rather $p^\mu = (E, 0, 0, E)$, so we have

$$\left\{Q^a_\alpha \, , \bar{Q}_{b\,\dot\beta}\right\} = 2\delta^a_b \underbrace{(\sigma^\mu)_{\alpha\dot\beta} \, P_\mu}_{=(-\sigma^0+\sigma^3)E} = 4\delta^a_b \, E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This leads us to

$$\begin{aligned} \left\{Q^a_2 \, , \bar{Q}_{b\dot2}\right\} = 0 \quad &\Rightarrow \quad \langle p^\mu, \lambda| \overbrace{\{\dots\}}^{a=b} |p^\mu, \lambda\rangle = \left\|Q^a_2 \, |p^\mu, \lambda\rangle\right\|^2 + \left\|\bar{Q}_{a\dot2} \, |p^\mu, \lambda\rangle\right\|^2 = 0 \\ &\Rightarrow \quad Q^2_a = \bar{Q}_{a\dot2} = 0 \end{aligned}$$

as well as

$$\left\{Q^a_1 \, , \bar{Q}_{b\dot1}\right\} = 4\delta^a_b \, E \quad \Rightarrow \quad a^b = \frac{Q^b_1}{2\sqrt{E}} \, , \quad a^\dagger_b = \frac{\bar{Q}_{b\dot1}}{2\sqrt{E}} \, , \quad \left\{a^b \, , a^\dagger_c\right\} = \delta^b_c$$

and

$$\begin{aligned} \left[Q^a_1 \, , J^{12}\right] &= (\underbrace{\sigma^{12}}_{=\frac{1}{2}\sigma_3})_\alpha{}^\beta \, Q^a_\beta = \frac{1}{2} Q^a_1 \\ \left[\bar{Q}^a_{\dot\alpha} \, , J^{12}\right] &= -\frac{1}{2}(\sigma_3)^{\dot\beta}_{\dot1} \, \bar{Q}^a_{\dot\beta} = \frac{1}{2}\bar{Q}^a_{\dot1} \end{aligned}$$

λ	$\mathcal{N}=1$			$\mathcal{N}=2$	$\mathcal{N}=4$	
1	$a^\dagger\,\vert\Omega\rangle$	$a^\dagger\,\vert\Omega\rangle$		$a_1^\dagger a_2^\dagger\,\vert\Omega\rangle$	$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger\,\vert\Omega\rangle$	
$\frac{1}{2}$	$\vert\Omega\rangle$	$\vert\Omega\rangle$	$a^\dagger\,\vert\Omega'\rangle$	$a_1^\dagger\,\vert\Omega\rangle \quad a_2^\dagger\,\vert\Omega\rangle$	$a_{b_1}^\dagger a_{b_2}^\dagger a_{b_3}^\dagger\,\vert\Omega\rangle$	4 states
0			$\vert\Omega'\rangle$	$\vert\Omega\rangle \quad a_1^\dagger a_2^\dagger\,\vert\Omega'\rangle$	$a_{b_1}^\dagger a_{b_2}^\dagger\,\vert\Omega\rangle$	6 states
$-\frac{1}{2}$	$a^\dagger\,\vert\Omega'\rangle$			$a_1^\dagger\,\vert\Omega'\rangle \quad a_2^\dagger\,\vert\Omega'\rangle$	$a_b^\dagger\,\vert\Omega\rangle$	4 states
-1	$\vert\Omega'\rangle$			$\vert\Omega'\rangle$	$\vert\Omega\rangle$	

Table 2.2: Some examples of applying $\mathcal{N}$ creation operators to generate a complete set of supersymmetric states. For $\mathcal{N}=1$ there are $2+2$ states resulting in a vector multiplet (left-hand side) and a chiral multiplet $(\lambda, \phi^1, \phi^2)$ (right-hand side). For $\mathcal{N}=2$ we also obtain a vector multiplet A^μ and a chiral multiplet $(\lambda^1, \lambda^2, \phi^1, \phi^2)$. $\mathcal{N}=4$ is the *maximally supersymmetric state* with the multiplets A^μ (vector), λ^a_α (fermion) and ϕ^i (scalar).

$\Rightarrow$ Q_1^b lowers λ by $\frac{1}{2}$ $\quad\Rightarrow\quad a^b$

$\Rightarrow$ $\bar{Q}_{1b}$ raises λ by $\frac{1}{2}$ $\quad\Rightarrow\quad a_b^\dagger$

Next step is to construct the spectrum. Therefor we start with the vacuumstate $|\Omega\rangle$ with minimal λ and subsequently raise λ by applying $a_b^\dagger$ which results in $2^{\mathcal{N}}$ different states $\{\lambda_i\}$. To obtain CPT-invariance, we also have to add the states with $\{-\lambda_i\}$.

In table 2.2 there are some examples given how to construct the spectrum for $\mathcal{N}=1, 2, 4$. A closer look at the example $\mathcal{N}=4$ reveals

$$\begin{aligned}
\delta\phi &= [\phi\,, Q] \propto \lambda \\
\delta\lambda &\propto F + Q^2 \\
\delta\bar{\lambda} &\propto \mathrm{D}_\mu\phi \\
\delta A &\propto \bar{\lambda}
\end{aligned}$$

The Lagrangian reads

$$\mathscr{L} = \mathrm{Tr}\Big[\frac{1}{4g_{\mathrm{YM}}^2}F^2 - \mathrm{D}_\mu\phi^i\,\mathrm{D}^\mu\phi^i - \mathrm{i}\lambda^a\,\bar{\sigma}^\mu\,\mathrm{D}_\mu\,\lambda_a$$
$$+\frac{\vartheta}{16\pi^2}\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} + g_{\mathrm{YM}}\,C^{ab}{}_i\,\lambda_a\left[\phi^i\,,\lambda_b\right]$$
$$+\,g_{\mathrm{YM}}\,C_{iab}\,\bar{\lambda}^a\left[\phi^i\,,\bar{\lambda}_b\right] + \frac{g_{\mathrm{YM}}^2}{2}\left[\phi^i\,,\phi^j\right]\left[\phi^i\,,\phi^j\right]\Big]$$

where C^{ab} denote the *Glebsch-Gordon coefficients*—a generalization of $(\sigma^\mu)_{\alpha\dot{\beta}}$—, g_{YM} is the *Yang-Mills coupling* and A^μ, λ^a and ϕ are taken from the adjoint representation. This results in a well-defined (renormalizable) quantum field theory in $d = 3{+}1$ dimensions which is a conformal field theory—even when quantized.

Let us shortly recapitulate the Anti-de Sitter space. This hyperbolic space is a solution to the Einstein-equations with a constant negative curvature and is determined by

$$-X_0^2 + X_1^2 + \cdots X_d^2 - X_{d+1}^2 = L^2$$

The symmetry of AdS_{d+1} has a d dimensional *boundary* which is a (conformal compactification of) a d dimensional Minkowsky-space (1 time,$d{-}1$ space dimensions). It is convenient to think of the conformal field theory in d dimensions to be defined on the *boundary* of a $d+1$ dimensional Anti-de Sitter space.

The AdS/CFT correspondence[13] makes a number of extremely non-trivial statements! In particular, a quantum field theory in flat space is conjectured to be equivalent to a gravity theory in one dimension higher (note that the symmetries and the number of freedoms of both theories agree)—this is called the *holomorphic principle*. The conjecture is motivated by the considerations of *D-branes* (D stands for *Dirichlet*) in *string theory*. We aim at writing the action for D-branes. These are a generalization of the action for a point particle (relativistically invariant).

Paticle: The action of a non-relativistic free particle reads

$$S_{\text{non-rel}} = \int \frac{m}{2}\dot{x}^2\,\mathrm{d}t\,, \quad \dot{x} = v$$

Equation of motion: $\quad m\ddot{x} = 0 \quad \Rightarrow \quad \vec{v} = \text{constant}$

[13] AdS/CFT correspondence was first conjectured in 1997 by J. Maldacena (*The Large N Limit of Superconformal field theories and supergravity*).

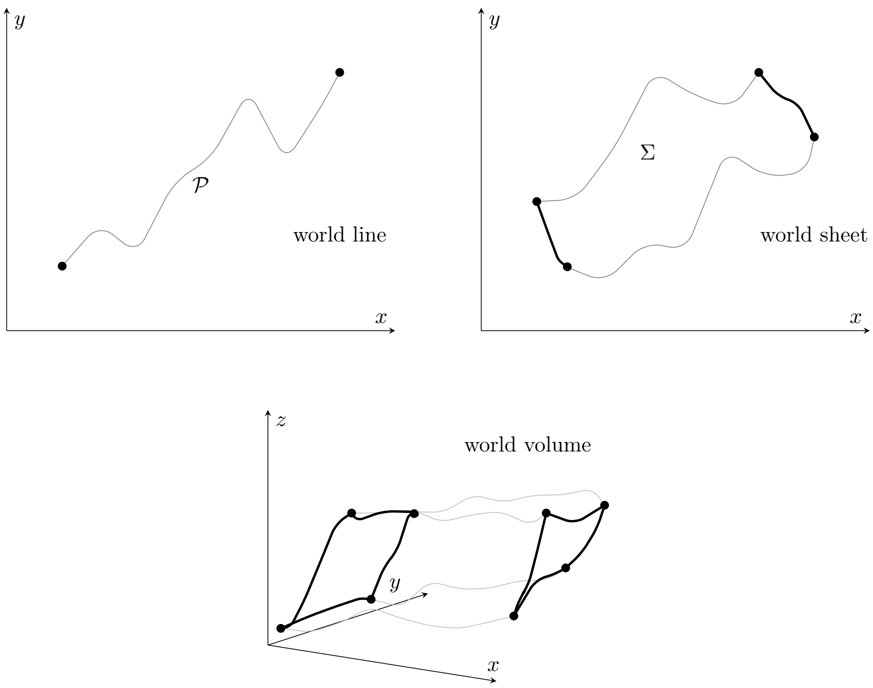


Figure 2.1: Illustration of the worldpath of a particle (upper left), a string (upper right) and a membrane (bottom).

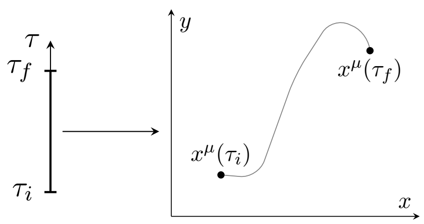
For a relativistic one we have

$$S = -mc \int_{\mathcal{P}} \mathrm{d}s = -mc^2 \int_{t_1}^{t_2} \mathrm{d}t \sqrt{1 - \frac{v^2}{c^2}} \qquad - \mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \left(\mathrm{d}x^1\right)^2 + \cdots$$

$$\mathrm{d}x = v\ \mathrm{d}t$$

$$\mathrm{d}s = c\sqrt{1 - \frac{v^2}{c^2}}$$

As parametrization we use the parameter τ



so we have

$$x^\mu = x^\mu(\tau)\ , \qquad \mathrm{d}s^2 = \eta_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,\mathrm{d}\tau^2$$

$$\Rightarrow \quad S = \int \mathrm{d}\tau \sqrt{\eta_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}$$

which results in the equation of motion

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0\ , \qquad\qquad p^\mu = m u^\mu$$

where

$$u^\mu = \gamma(c, \vec{v})\ , \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

is the four-velocity. For a charged particle, we also have to add the action of the gauge field:

$$S = -mc\int_{\mathcal{P}} \mathrm{d}s + \frac{q}{c}\int_{\mathcal{P}} A_\mu(x)\ \mathrm{d}x^\mu \qquad\qquad \text{(Abelian symmetry)}$$

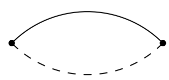
Now we want to do the same for objects extended in space, e. g. strings as the simplest example. The worldline of the relativistic particle becomes a world*sheet* (area) as seen in figure 2.1. To embed it into spacetime, we now need to parameters

$$x^\mu = x^\mu(\tau, \sigma)$$

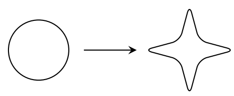
The action of the string then is the *Nambu-Goto action*:

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma\ \sqrt{-\det\big(\partial_\alpha x^\mu\,\partial_\beta x^\nu\,g_{\mu\nu}\big)}$$

There are two different kinds of strings: Open strings and closed ones . In the low energy limit, the fundamental oscillation heavily depends on which type the string actually is:



dipole moment $p^\mu \leftrightarrow A^\mu$



$Q_{\mu\nu} \leftrightarrow g_{\mu\nu}$ metric so the lowest energy mode of an open string can be approximated as a dipole whereas the lowest energy mode of a closed string is a quadrupole in first order—which we will later set in correspondence to the gauge field or the metric respectively.

Next step is to look at the action of a membrane (p-brane) which is a charged object in p space and one time direction. The parameters (τ, σ) become a p+1-dimensional parameter vector ξ^i, so the action reads

$$S_{\text{DBI}} = -\overbrace{T_p}^{\text{Tension of the brane}} \int \mathrm{d}^{p+1}\xi \,\sqrt{-\det\big(P[g] + 2\pi\alpha' F_{\alpha\beta}\big)}$$

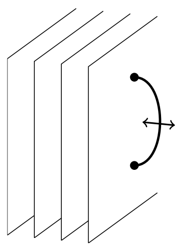
and is called the *Dirac-Born-Infeld action*, where

$$P[g]_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta}\, g_{\mu\nu}\ , \qquad \alpha, \beta \in \{0, \ldots p\}$$

is the pullback of the worldvolume metric to the brane. Now we consider a Dp-brane and more specifically $p = 3$. 'D' stands for Dirichlet boundary conditions. Considering D3-branes in the opend and closed string pictures was used by Maldacena to motivated the AdS/CFT conjecture.

Conjecture: $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills theory in 3+1 dimensions with 't Hooft coupling $\lambda = 2\pi g_{\text{YM}}^2 N$ large and fixed and $N \to \infty$ is equivalent (dual) to 10 dimensional supergravity on the space $\text{AdS}_5 \times S^5$. (Note: We have to consider 10 = 9+1 dimensions since superstring theory is consistent only in this number of dimensions).

The DBI-action describes D-branes in the open string picture, i. e. we may consider the following picture:



It turns out that in a low energy limit where only the lowest fluctuations (i. e. the massless ones) contribute, a stack of N coincident D3-branes is described by the action of $\mathcal{N} = 4$ $SU(N)$ Super-Yang-Mills theory: This may be seen by expanding the square root in the DBI-action and keeping only the lowest order terms in α' (essentially this corresponds to taking the length of the strings to zero).

The first term in the expansion for the gauge fields gives $-\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}$. The pull-back of the six perpendicular coordinates to the worldvolume of the D3-brane become the six scalar fields of $\mathcal{N}=4$ SYM. The fermionic contribution to the DBI-action (not shown) gives the action for the Weyl fermions of $\mathcal{N}=4$ SYM.

In the closed string picture, the D3-branes are heavy charged objects which curve the space around them (solitonic solutions of supergravity). They are solutions to the Einstein-equations in 9+1 dimensions. Our ansatz for a solution of 10 dimensional gravity is

$$\mathrm{d}s^2 = H(r)^{-\frac{1}{2}}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + H(r)^{\frac{1}{2}}\,\delta_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j$$

where

$$r^2 = y_1^2 + y_2^2 + \cdots\, y_6^2$$

gives the radial coordinate on the perpendicular 6 dimensional space, x^i denote the coordinates on the D3-brane in 3+1 dimensions and y^i the coordinates perpendicular to the D3-brane respectively. Insert this ansatz on the undetermined function $H(r)$, namely $\Box H(r) = 0$ for $r \neq 0$, so $H(r)$ is a harmonic function,

$$H(r) = 1 + \left(\frac{L}{r}\right)^4$$

A stringtheory argument to be explained later gives

$$\begin{aligned} L^4 &= 4\pi\, g_\mathrm{S}\, N\left(\alpha'\right)^2 \qquad \text{(prop. to 't Hooft coupling } \lambda\text{)}\\ g_\mathrm{S} &= 2g_\mathrm{YM}^2 \end{aligned}$$

Two asymptotic regions are:

$r \gg L$: $H(r) \to 1$ which gives the flat-space limit

$r \ll L$: $H(r) \to \left(\frac{L}{r}\right)^4$ which is called the *near-horizon region* (corresponds to the low energy-limit)

Inserting $H(r) = \left(\frac{L}{r}\right)^4$ into the ansatz gives, introducing a new coordinate $z = \frac{L^2}{r}$, the metric becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{r^2}{L^2}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \frac{L^2}{r^2}\,\delta_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j\\ &= \frac{L^2}{z^1}\underbrace{\left(\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \mathrm{d}z^2\right)}_{\mathrm{AdS}_5} + L^2\,\mathrm{d}\Omega_5^2 \end{aligned}$$

where the last term comes from the S_5 space fulfilling a $SO(6)$ symmetry, which is isomorphic to $SU(4)$—the symmetry group of SUSY!