

①

Classical scalar field theory

We consider a real scalar field $\varphi(x)$ in flat d -dimensional Minkowski-spacetime $\mathbb{R}^{d-1,1}$ with $d-1$ spatial directions.

$$\begin{pmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

The coordinates on $\mathbb{R}^{d-1,1}$ are denoted by x^μ , $\mu \in \{0, \dots, d-1\}$, we set $c=1$.

Metric of Minkowski space:

$$ds^2 = -(dx^0)^2 + \sum_{i=0}^{d-1} (dx^i)^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu$$

The symmetries of Minkowski space are the Poincaré transformations:

$$x \rightarrow x' = \Lambda x + q$$

↑
 Lorentz transformations Translation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + q^\mu$$

A real scalar field is a map which assigns a real number $\varphi(x)$ to each spacetime point x .

$$\text{When } x^\mu \rightarrow x' = \Lambda x$$

$$\varphi \rightarrow \varphi' = \varphi(\Lambda^{-1}x)$$

$$\begin{array}{ccc} L(x, \dot{x}) & \rightarrow & \mathcal{L}(\varphi, \partial_\mu \varphi) \\ \text{classical mechanics} & & \text{field theory} \\ & & \text{infinite number of variables} \end{array}$$

The dynamics of the scalar field is specified by an action functional $S[\varphi]$ which can be written as an integral over the Lagrangian density $\mathcal{L}(\varphi, \partial_\mu \varphi)$

$$S[\varphi] = \int dt d^{d-1}x \mathcal{L}(\varphi, \partial_\mu \varphi) \equiv \int d^d x \mathcal{L}(\varphi, \partial_\mu \varphi)$$

Example:

massive real scalar field; free theory (no interactions)

$$S[\varphi] = \int d^d x \mathcal{L}_0 = -\frac{1}{2} \int d^d x \underbrace{(\eta_{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi)}_{\text{kinetic term}} + \underbrace{m^2 \varphi^2}_{\text{mass term}}$$

$$= -\frac{1}{2} \int d^d x [-(\partial_t \varphi(t, \vec{x}))^2 + (\vec{\nabla} \varphi(t, \vec{x}))^2] + m^2 \varphi(t, \vec{x})^2$$

The principle of stationary action (Hamilton) gives the Lagrangian equation of motion.

2) Functional derivative:

Defined by	$\frac{\delta S[\Psi, \partial_\mu \Psi]}{\delta \Psi(x)} = 0 \quad (*)$
$\frac{\delta \Psi(y)}{\delta \Psi(x)} = \delta^{(4)}(y-x)$ delta distribution	$\rightarrow \frac{\delta}{\delta \Psi(x)} \int d^4y \Psi(y) = \int d^4y \frac{\delta \Psi(y)}{\delta \Psi(x)} = \int d^4y \delta(y-x) = 1$
analogue to $\frac{dy}{dx} = \delta_{xy}$	

$$(*) \frac{\delta S[\Psi]}{\delta \Psi(x)} = \frac{\partial \mathcal{L}}{\partial \Psi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \Psi)} \right) = 0$$

(Lagrangian equation)

For our example:

$$\mathcal{L}_0(\Psi, \partial_\mu \Psi) = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \Psi(x) \partial_\nu \Psi(x) - \frac{1}{2} m^2 \Psi(x)^2$$

The Lagrangian equation of motion becomes:

$$(\square - m^2) \Psi(x) = 0 \quad (**)$$

where $\square \equiv \partial^\mu \partial_\mu = -\partial_t^2 + \nabla^2$ is the of'Allembert Operator. The equation $(**)$ is referred to as the Klein-Gordon equation.

So far, we considered a field in a potential $V(\Psi) = 0$. More interesting physics is obtained by considering interactions, i.e. $V(\Psi) \neq 0$.

Generally $V(\Psi)$ will be a polynomial $V(\Psi) = \Psi^\alpha$; $\alpha > 2$.

Then we have $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$; $\mathcal{L}_{\text{int}} = -\frac{g_n}{n!} \Psi(x)^n$

g_n is the associated coupling constant which measures the strength of the interaction

(Example for classical field theory is Electrodynamics, Fields will be $A_\mu = (\Phi, \vec{A})$)

Equation of motion:

$$\begin{aligned} \text{Action } S[A_\mu] &= \int d^4x F^{\mu\nu} F_{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ 0 &\equiv \frac{\delta S[A_\mu]}{\delta A^\mu(x)} \Rightarrow D^\mu F_{\mu\nu} = 0 \\ &\quad (\text{covariant derivative}) \end{aligned}$$

(3) Quantizing the scalar field theory:

We note that the Fourier decomposition

$$\hat{\psi}(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}k}{2\omega_k} [\hat{a}(k)e^{-ikx} + \hat{a}^*(k)e^{ikx}] \Big|_{k^0=\omega_k}$$

$$\text{where } \omega_k = (\vec{k}^2 + m^2)^{\frac{1}{2}} \text{ and } k \cdot x = -k^0 x^0 + \vec{k} \cdot \vec{x}$$

satisfies the Klein-Gordon equation.

Exercise: Derive the equation of motion for an interacting scalar field theory, of $\mathcal{L}_{\text{int}} = -\frac{g^4}{4!} \phi^4$

Symmetries and conserved currents in classical field theory, Noether Theorem

$\rightarrow T_{\mu\nu}$: energy-matter-tensor

Quantisation of free fields, with equation of motion

$$(\square - m^2)\phi = 0$$

The starting point of quantizing the real scalar field is to promote the Fourier modes to operators $\hat{a}(k)$ and $\hat{a}^*(k)$.

The field $\phi(x)$ then also becomes an operator, denoted by $\hat{\phi}(x)$.

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{d^{d-1}k}{2\omega_k} [\hat{a}(k)e^{-ikx} + \hat{a}^*(k)e^{ikx}] \Big|_{k^0=\omega_k}$$

The operators $\hat{a}(k)$ and $\hat{a}^*(k)$ satisfy the commutation relations

$$[\hat{a}(k), \hat{a}^*(k')] = 2\omega_k (2\pi)^{d-1} \delta^{d-1}(\vec{k} - \vec{k}')$$

$$[\hat{a}(k), \hat{a}(k')] = 0 = [\hat{a}^*(k), \hat{a}^*(k')]$$

These commutation relations coincide with those of a harmonic oscillator in quantum mechanics at frequency ω_k . Therefore we may interpret the \hat{a}, \hat{a}^* as creation and annihilation operators.

The vacuum state $|0\rangle$ is annihilated by \hat{a} for all k ,

$$\hat{a}(k)|0\rangle = 0.$$

$$\text{We chose } \langle 0|0 \rangle = 1$$

A single particle state with momentum $\vec{k}, |\vec{k}\rangle$ is created by acting on the vacuum state with the creation operator $\hat{a}^*(k)$, $|\vec{k}\rangle = \hat{a}^*(k)|0\rangle$.

4)



propagator

A particle is created at x_1 with momentum \vec{k} , propagates to x_2 where it is annihilated again.

A QFT combines the ideas of classical field theory with the ideas of quantum mechanics. In particular, the propagation of a mode $|k\rangle$ with momentum \vec{k} in space may be related to the concepts of Huygen's principle and Green's function in classical field theory.

$$G(y) = \int d^d x \ G(x, y) \ \Psi(x)$$

$$\text{For our example } [-\square + m^2] G(x, y) = \delta^{(d)}(x - y)$$

- Two goals:
- Quantize interacting QFT's
 - Define generating functional

→ Introduce path integral quantization of (free) fields. Within quantum mechanics, the path integral sums over all possible paths which start at some point position q_i at time t_i and end at q_f at time t_f .

In quantum field theory, this translates into summing over all field configurations φ in configuration space. The integration measure becomes:

$$D\varphi = (\text{factor}) \prod_{t_i \leq t \leq t_f} \prod_{x \in \mathbb{R}^{d-1}} d\varphi(t, x)$$