Introduction to Gauge/Gravity Duality

Examples VI

To hand in Thursday 13th July in the examples class

I. Relation between propagators in AdS

Let us consider Euclidean AdS in the Poincaré patch.

a) Derive the equation of motion for a scalar field with mass $m$ in Euclidean $AdS_{d+1}$.

(2 points)

b) Use the ansatz $\phi(z) = z^\Delta$ for an asymptotic solution of this equation near the boundary $z \to 0$. Determine the two possible values of $\Delta_\pm$, where $\Delta_+ > \Delta_-$.  

(4 points)

c) The bulk-to-boundary propagator $K(x, z; x')$ is the solution of the equations of motion which is regular in the interior and diverges as

$$\lim_{z \to \epsilon} K(z, x; x') = \epsilon^\Delta \delta(x - x')$$

near the boundary, i.e. for $\epsilon \ll 1$. The bulk-to-bulk propagator is given by the solution of the equation of motion with a pointlike source term,

$$\left(\Box_{x,z} - m^2\right) G(z, x; z', x') = \frac{1}{\sqrt{g}} \delta(x - x')\delta(z - z'),$$

where $\Box_{x,z}$ is the scalar Laplacian in Euclidean $AdS_{d+1}$ acting only on $x$ and $z$. Moreover the bulk-to-bulk propagator is regular in the interior.

Show that the bulk-to-boundary propagator $K(x, z; x')$ can be calculated from the bulk-to-bulk propagator $G(z, x; z', x')$ by

$$K(z, x; x') = \lim_{z' \to \epsilon} \frac{\Delta_+ - \Delta_-}{\epsilon^\Delta} G(z, x; z', x').$$

Hint: Do not use the explicit solution for $G(z, x; z', x')$, but Green's second identity!

(4 points)

(please turn over)
II. Saturation of unitarity bound

Consider a free scalar field with mass $m$ in the Poincare patch of Euclidean $AdS_{d+1}$ with radius $R$.

a) Show that the usual bulk action

$$S_1 = -\frac{1}{2} \int d^d x \, dz \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right),$$

evaluated for solutions of the form $\phi(x, z) \sim z^\Delta e^{ikx}$ near $z = 0$, is finite if $\Delta > d/2$. This means that the solution with $\Delta = \Delta_+$ ($\Delta_+$ being the larger root of $\Delta(\Delta - d) = m^2 R^2$) are normalizable with respect to the action $S_1$.

(4 points)

b) Consider the bulk action

$$S_2 = -\frac{1}{2} \int d^d x \, dz \sqrt{g} \phi \left( -\Box_g + m^2 \right) \phi.$$

Show that $S_2$ can be obtained by adding a boundary term to the action $S_1$ (hint: integration by parts) and that the equations of motion are the same as for the action $S_1$.

(2 points)

c) Show, that the action $S_2$ evaluated for solutions of the form $\phi(x, z) \sim z^\Delta e^{ikx}$ near $z = 0$ is finite if $\Delta > (d - 2)/2$. Conclude that for

$$-\frac{d^2}{4} < m^2 R^2 < -\frac{d^2}{4} + 1$$

both solutions, i.e. $\Delta = \Delta_+$ and $\Delta = \Delta_-$ are normalizable with respect to the action $S_2$.

(4 points)

For more additional information see arXiv: hep-th/9905104.