II. Near-Horizon limit of M2–branes

Let us consider the near horizon limit of M2–branes in 11-dimensional supergravity. The supergravity solution of M2–branes reads

$$ds^2 = H(r)^{-2/3} \left(-dt^2 + dx^2 + dy^2\right) + H(r)^{1/3} \left(dr^2 + r^2 d\Omega^7\right),$$

$$F_{(4)} = dt \wedge dx \wedge dy \wedge dH^{-1},$$

where $H(r)$ is given by

$$H(r) = 1 + \frac{L^6}{r^6}, \quad \text{where} \quad L^6 = 32\pi^2 N_f^6$$

and $F_{(4)}$ is a four-form.

a) Take the near-horizon limit $r \rightarrow 0$ and calculate the metric and the four-form $F_{(4)}$ in this limit. (6 points)

b) Use the coordinate transformation $z = \frac{r^3}{2 L^3}$ and compute the metric as well as the four-form $F_{(4)}$ in the coordinates $(z, t, x, y, \Omega^7)$. Which manifold is described by this metric? (4 points)

II. Penrose-Brown-Henneaux Transformation

The AdS metric in Poincaré coordinates may be written as

$$ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu} dx^\mu dx^\nu,$$

where the $x^\mu$, $\mu \in \{0, 1, 2, 3\}$, are the coordinates parallel to the boundary of AdS and $\rho$ is the radial direction. We consider coordinate transformations of this metric under the Penrose-Brown-Henneaux diffeomorphism, which is given by

$$\rho = \rho'(1 - 2\sigma(x')) \quad x'^\mu = (x')^\mu + a^\mu(x', \rho'),$$

demanding that

$$g'_{55} = g_{55} \quad \text{and} \quad g'_{\mu5} = g_{\mu5}. \quad (3)$$
Note that the index 5 stands for the $\rho$ direction. Show that the conditions (3) imply

$$\partial_5 a^\mu = \frac{L^2}{2} g^{\mu\nu} \partial_\nu \sigma$$  \hspace{1cm} (4)

and

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + 2\sigma \left(1 - \rho \frac{\partial}{\partial \rho}\right) g_{\mu\nu} + \nabla_\mu a_\nu + \nabla_\mu a_\nu.$$  \hspace{1cm} (5)

(5 points)

Explain why this transformation induces a conformal transformation $g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu}(x)$ at the boundary.

(5 points)