Prof. Dr. Johanna Erdmenger
Nina Miekley

Introduction to Gauge/Gravity Duality

Examples III

To hand in Thursday 8th June in the examples class

I. **Fermions in quantum field theory**

a) Write down the action for a free massive Dirac fermion field Ψ in 3+1 dimensions. (2 points)

b) From this action, show that varying with respect to $\bar{\Psi}$ gives the equation of motion

$$(-i\partial\!\!\!/ + m)\Psi(x) = 0 \qquad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu\,. \tag{1}$$

Derive the corresponding equation of motion for $\bar{\Psi} = \Psi^\dagger \gamma^0$. What is the relation to (1)? (3 points)

c) By acting with $i\partial\!\!\!/ + m$ on the left of equation (1) and using the Clifford algebra for the γ matrices, show that each component of $\Psi(x)$ satisfies the Klein-Gordon equation

$$(\Box - m^2)\phi(x) = 0\,.$$

(5 points).

II. **Irreducible representations of massless SUSY multiplets**

The algebra for $\mathcal{N}$ supercharge operators is given by

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \qquad , \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab} = -2\epsilon_{\alpha\beta} Z^{ba},$$

with $\alpha, \beta \in \{1,2\}$, $a, b \in \{1, 2, ...\mathcal{N}\}$ and σ^μ are components of the four vector $(-\mathbf{1}, \sigma^i)$ of 2×2 matrices with the standard Pauli matrices σ^i.

For studying massless representations, we choose the Lorentz frame with $P^\mu = (E, 0, 0, E)$ and $E > 0$. We use the mostly minus metric.

a) Show that the SUSY algebra relation reduces to

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = -2(\sigma^\mu P_\mu)_{\alpha\dot{\beta}} \delta^a_b = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^a_b.$$

(3 points)

b) Using the result in (a) and the second SUSY algebra relation given above, show that $Q_2^a = 0$ and $Z^{ab} = 0$. (2 points)

c) The remaining operators Q_1^a and $\bar{Q}_{\dot{1}a}$ with $a = 1, 2, ..., \mathcal{N}$ are used to represent the field content of SUSY theories with $\mathcal{N}$ supercharges as follows:

- Q_1^a lowers helicity λ by $\frac{1}{2}$, i.e. $Q_1^a|\lambda\rangle = |\lambda - \frac{1}{2}\rangle$;
- $\bar{Q}_{\dot{1}a}$ raises helicity λ by $\frac{1}{2}$, i.e. $\bar{Q}_{\dot{1}a}|\lambda\rangle = |\lambda + \frac{1}{2}\rangle$.

Determine all the states of a gauge multiplet for $\mathcal{N} = 1, 2, 3, 4$ by starting from the highest helicity states $|\lambda\rangle = |1\rangle$ and applying products of Q_1^a operators on $|\lambda\rangle$ for all possible values of a.

(Hint: For $\mathcal{N} = D$, there are 2^D states in total.) (5 points)