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Introduction to Gauge/Gravity Duality

Examples I

To hand in Tuesday 9th May

I. **Gauge Theory**

Consider a $SU(N)$ gauge theory in 3+1 dimensions. The theory is assumed to contain a real scalar field $\Phi(x)$ which transforms in the adjoint representation of the gauge group, i.e.

$$\Phi \to e^{-i\Lambda}\Phi e^{i\Lambda}\,, \quad \Phi = \Phi^a T^a\,.$$

a) Show that the operator $\mathcal{O} \equiv \mathrm{Tr}(\Phi^2)$ is gauge invariant. (1 point)
b) Write down the classical action for this theory. (1 point)
c) Show that this action is gauge invariant. (1 point)
d) Calculate the free-field propagator of the field Φ as the Green's function of its equation of motion to linear order in the fields. For this purpose, write out the indices of the matrices T^a, i.e. $\Phi^i{}_j = \Phi^a (T^a)^i{}_j$, and calculate $\langle \Phi^i{}_j(x)\Phi^k{}_l(0)\rangle$. (2 points)

II. **Coordinates of** AdS_{d+1}

Lorentzian AdS_{d+1} can be defined by the locus

$$-L^2 = \eta_{ab}X^aX^b = -\left(X^{d+1}\right)^2 - \left(X^0\right)^2 + \sum_{i=1}^{d}\left(X^i\right)^2\,,$$

where $X \in \mathbb{R}^{2,d}$ and $ds^2 = \eta_{ab}X^aX^b$ with $\eta = \mathrm{diag}(-1,1,1,\ldots,1,-1)$. In the following we parametrize the locus (2) in different ways.

a) Draw a picture of AdS_2 embedded in $\mathbb{R}^{2,1}$. (2 points)

b) The *global coordinates* (ρ, τ, Ω_i) are defined by

$$\begin{aligned} X^{d+1} &= L\cosh\rho\,\sin\tau\,, \\ X^0 &= L\cosh\rho\,\cos\tau\,, \\ X^i &= L\sinh\rho\,\Omega_i\,, \end{aligned}$$

with $i = 1, ..., d$ and $\sum_{i=1}^{d}\Omega_i^2 = 1$. Using this parametrization calculate the induced metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (where $x^\mu \in \{\rho, \tau, \Omega_i\}$) for AdS_{d+1} in global coordinates. (3 points)

c) Replace ρ by $r \equiv L \sinh \rho$ and show that the metric can be written in the form

$$ds^2 = -H(r)dt^2 + H(r)^{-1}dr^2 + r^2 d\Omega_{d-1}^2,$$

where $d\Omega_{d-1}^2 = \sum_{i=1}^{d} d\Omega_i d\Omega_i$ is the metric of the unit $(d-1)$–sphere, S^{d-1}. (2 points)

d) The *Poincaré patch coordinates* (x^i, u) with $i = 1, ..., d$ are defined by

$$\begin{aligned} X^{d+1} + X^d &= u\,, \\ -X^{d+1} + X^d &= v\,, \\ X^i &= \frac{u}{L}x^i\,. \end{aligned}$$

Use the defining equation (2) to eliminate v in terms of u and x^i and show that the induced metric for (u, x^i) with $i = 1, ..., d$ takes the form

$$ds^2 = L^2\frac{du^2}{u^2} + \frac{u^2}{L^2}dx^i dx_i\,.$$

Finally introduce $z = \frac{L^2}{u}$ and show that the metric is given by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + dx^i dx_i\right)$$

Which part of the AdS spacetime is not covered by these coordinates (Hint: z takes only positive values (Why?)). (3 points)

III. **Curvature of AdS and Cosmological constant**

Let us consider AdS_{d+1} in the *Poincaré patch* given by the coordinates $(z, t, \vec{x})$ and the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right)\,,$$

where $\vec{x}$ are the $d-1$ spatial dimensions.

a) Calculate the metric $g^{\mu\nu}$, the Christoffel symbol $\Gamma^\mu_{\nu\rho}$, the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ as well as the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R.

(3 points)

b) Show that AdS_{d+1} solves the vacuum Einstein field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, where Λ is the cosmological constant and $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu}$ is the Einstein tensor. Determine the value of the cosmological constant Λ!

(2 points)