Wilson lines and anomalous dimensions in AdS/CFT

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Based on work with: Besken, D'Hoker, Hegde, Hijano, Sivaramakrishnan, Snively

- I will be discussing computations in the AdS_3/CFT_2 correspondence
- Emphasis on universal features of gravitational interactions, and relation to CFT via Virasoro algebra
- Much activity in using conformal bootstrap ideas to develop AdS/CFT dictionary and understand emergence of holography, and this talk fits into that general program

Self-energy

- Given a particle, how is its mass renormalized by gravitational interactions? Typically, cutoff dependent and hence non-universal. Situation more favorable in AdS_3

Particle of mass m in flat space is conical deficit: $\Delta\phi = 8\pi G m$

Think of placing this particle in AdS. Hold $\Delta\phi$ fixed

Asymp AdS metric of total energy M:

$$ds^2 = -(r^2 - 8GMl^2)dt^2 + \frac{l^2 dr^2}{r^2 - 8GMl^2} + r^2 d\phi^2$$

$$\Delta\phi = 2\pi(1 - \sqrt{-8GM})$$

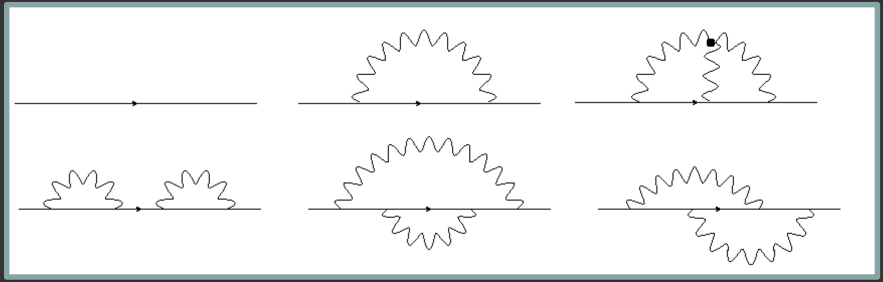
Equating: $M = -\frac{1}{8G} + m - 2Gm^2$

rewrite in terms of $c = \frac{3l}{2G},\ Ml = -\frac{c}{12} + 2h,\ ml = 2h_0$

$$\Rightarrow h = h_0 - \frac{6h_0^2}{c}$$

valid for $h, c \to \infty$, $\frac{h}{c}$ fixed corrections?

- Consider computing dimension from boundary 2-point function: $\langle O(x)O(x')\rangle \sim |x-x'|^{-4h}$
- Diagrammatic expansion



- There is a natural expectation for the exact answer. Turning on gravity enhances symmetry from SL(2) to Virasoro
- Need to recall how reps of SL(2) turn into those of Virasoro. Various ways of thinking about this (e.g. Bershadsky/Ooguri)

SL(2) current algebra: $J^a(z)J^b(0) \sim \frac{(k/2)\eta^{ab}}{z^2} + \frac{i\epsilon^{ab}{}_c J^c(0)}{z}$

$$T_{\mathrm{SL}(2)} = \frac{1}{k-2}\eta_{ab}J^aJ^b \ , \quad c_{\mathrm{SL}(2)} = \frac{3k}{k-2}$$

Spin-j primary: $h_{\mathrm{SL}(2)}[\Phi_j] = -\frac{j(j+1)}{k-2}$

constraints: $J^-(z) = k \ , \quad J^0(z) = 0$

Modified stress tensor: $T = T_{\mathrm{SL}(2)} + \partial J^3 + T_{\mathrm{gh}} \ , \quad c = \frac{3k}{k-2} + 6k - 2$

New scaling dimension: $h(j,c) = -j + \frac{m+1}{m} j(j+1)$

$$c = 1 - \frac{6}{m(m+1)} \ , \quad k = \frac{m+2}{m+1}$$

Large c expansion (m-> -1): $h(j,c) = -j - \frac{6j(j+1)}{c} - \frac{78j(j+1)}{c^2} + \ldots$

previous result: $h = h_0 - \frac{6h_0^2}{c}$

- Correspondence under $h_0 = -j$

  $j^2 \to j(j+1)$ due to corrections to point particle limit

- Computation of correlator by Feynman diagrams in covariant gauge is very challenging

- Leading order: $\langle O(x)O(x')\rangle \sim e^{-2h_0 L}$ L=geodesic length in bulk

- Essentially want to "quantize L". Tractable in d=3 due to lack of local degrees of freedom. Full gravity phase space is known:

$$ds^2 = d\rho^2 + e^{2\rho}dzd\overline{z} - 4GT(z)dz^2 - 4G\overline{T}(\overline{z})d\overline{z}^2 + 16G^2T(z)\overline{T}(\overline{z})e^{-2\rho}dzd\overline{z}$$

"Banados metric"

T(z) identified with CFT stress tensor *operator* (cf Fitzpatrick, Kaplan et al; Hartman et al)

- Promotes L to an operator, and now we can try to make sense of $\langle O(x)O(x')\rangle \sim \langle e^{-2h_0 L}\rangle \sim |x-x'|^{-4h}$

- Advantageous to holomorphically factorize:

$$e^{-2h_0 L} = \left| \langle h_0^{out} | P e^{\int A} | h_0^{in} \rangle \right|^2$$

(cf Ammon, Castro, Iqbal)

SL(2) connection: $A = (L_1 + \frac{6}{c} T(z) L_{-1}) dz$

$$L_{\pm 1, 0} = \mathrm{SL(2)\ generators,\ independent\ of\ T(z)}$$

$$L_0 | h_0^{in} \rangle = h_0 | h_0^{in} \rangle \ , \quad L_{-1} | h_0^{in} \rangle = 0$$

$$L_0 | h_0^{out} \rangle = -h_0 | h_0^{out} \rangle \ , \quad L_1 | h_0^{out} \rangle = 0$$

- Justified by checking that for arbitrary fixed T(z), formula gives correct geodesic length

- More generally, take expectation value in CFT vacuum state and use standard CFT formulas for CFT stress tensor correlators. Still highly nontrivial due to path ordering and integration

- Note that Wilson line $\langle h_0^{out}|Pe^{\int A}|h_0^{in}\rangle$ is built out of CFT stress tensors, and is expected to be a basic CFT object: the “Virasoro vacuum OPE block”

$$O(z)O(0) \sim (\text{stress tensors}) \ + \ (\text{other operators})$$

Virasoro vacuum OPE block

- Wilson line packages the Virasoro vacuum block in a form convenient for expansion in 1/c

- We want to take expectation value in CFT vacuum and establish

$$G(z) = \langle h_0^{out}|Pe^{\int A}|h_0^{in}\rangle \sim z^{-2h}$$

$$\sim z^{-2h_0(j)}\left(1 - \frac{2h_1(j)}{c}\ln z - \frac{2h_2(j)}{c^2}\ln z + \frac{2h_1(j)^2}{c^2}(\ln z)^2 + \ldots\right)$$

$$h = \sum_{n=0}^{\infty} \frac{h_n(j)}{c^n}$$

$$h_0(j) = -j \ , \quad h_1(j) = -6j(j+1) \ , \quad h_2(j) = -78j(j+1) \ , \ldots$$

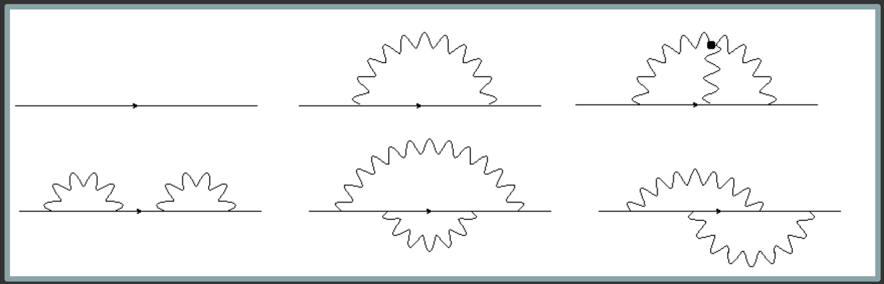
- expansion gives series of nested integrals

$$G_j(z_1,z_2)=\langle Pe^{\int_{z_1}^{z_2}a(y)dy}\rangle=\sum_{n=0}^{\infty}\int_{z_1}^{z_2}dy_n\int_{z_1}^{y_n}dy_{n-1}\dots\int_{z_1}^{y_2}dy_1\langle a(y_n)\dots a(y_1)\rangle$$

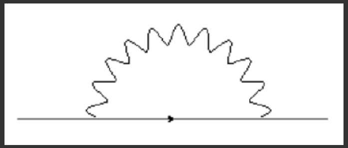
$$a=(L_1+\tfrac{6}{c}T(y)L_{-1})dy$$

$$T(y_1)T(y_2)\dots T(y_n)\;\rightarrow\;\langle T(y_1)T(y_2)\dots T(y_n)\rangle$$

- evaluate order by order in 1/c



1/c contribution



$$G^{(1)}(z) = G^{(0)}(z)\frac{36j}{c}\int_0^z dy_1 \int_0^{y_1} dy_2 \frac{y_2(z-y_1)\big(2jy_1(z-y_2)-y_2(z-y_1)\big)}{z^2(y_1-y_2)^4}$$

divergence coming from short distance singularity of

$$\langle T(y_1)T(y_2)\rangle = \frac{c/2}{(y_1-y_2)^4}$$

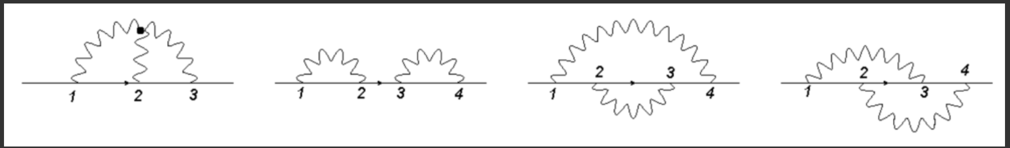
regulate via replacement: $\langle T(y_1)T(y_2)\rangle = \frac{c/2}{\big((y_1-y_2)^2+\epsilon^2\big)^2}$

integrals can now be evaluated:

$$G^{(1)}(z) = G^{(0)}(z)\frac{36j}{c}\left[\frac{(2j-1)\pi z^3}{120\epsilon^3} + \frac{z^2}{12\epsilon^2} - \frac{(j+1)\pi z}{12\epsilon} + \frac{j+1}{3}\ln\frac{z}{\epsilon} + \frac{2j-1}{18} + O(\epsilon)\right]$$

get expected result: $G^{(1)}(z) = G^{(0)}(z)\frac{36j}{c}\left[\frac{j+1}{3}\ln z\right]$

- 1/c^2 contribution



Previous regulator gives ambiguous results.
Better to use a form of dim reg Hikida/Uetoko

- Let stress tensor have dimension $2-\epsilon$

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^{4-2\epsilon}}$$ etc

Also introduce a vertex renormalization

$$Pe^{\int A} \to Pe^{\alpha\int A}$$

divergent part fixed by demanding finiteness, and finite part fixed by Ward identity

$$\langle T(z')Pe^{\alpha\int A}\rangle \sim \left[\frac{h}{(z-z')^2} + \frac{1}{z'-z}\partial_z\right]\langle Pe^{\alpha\int A}\rangle$$

- Detailed computation with many nontrivial cancellations and "conspiracies" indeed yields expected result

$$h = -j - \frac{6j(j+1)}{c} - \frac{78j(j+1)}{c^2} + \dots$$ Hikida/Uetoko

- We carried this out to one further order (3 loops)

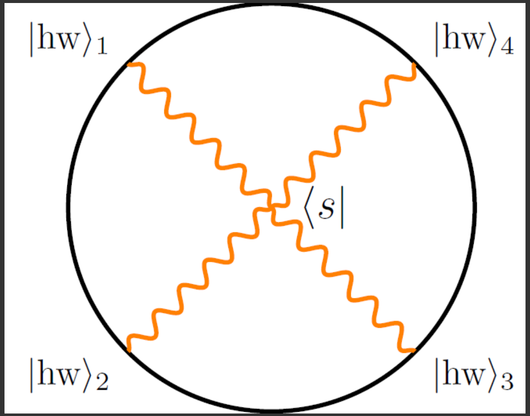
$$h = -j - \frac{6j(j+1)}{c} - \frac{78j(j+1)}{c^2} - \frac{1230j(j+1)}{c^3} + \dots$$ ✓

Besken,D'Hoker, Hegde,PK

- Story seems to hold together, but question remains as how to establish result at all orders due to lack of regulator that preserves all symmetries, most notable Virasoro

- Story generalizes to higher point functions. E.g. 4-point function:

Besken, Hegde, Hijano,PK;
Bhatta, Raman, Suryanarayana



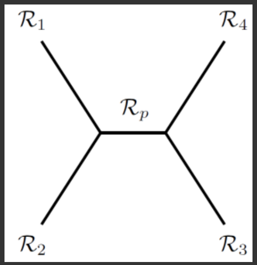
- primaries labeled by highest weight states in SL(2)

  Propagate these into bulk: $Pe^{\int_{x_i}^{x_b} A}|\mathrm{hw}\rangle_i$

- interaction vertex given by singlet state in tensor product

$$\langle O_1(z_1)O_2(z_2)P_pO_3(z_3)O_4(z_4)\rangle = \langle s|\left[Pe^{\int_{x_1}^{x_b} A}\right]_1 \ldots \left[Pe^{\int_{x_4}^{x_b} A}\right]_4 |\mathrm{hw}\rangle_1 \ldots |\mathrm{hw}\rangle_4$$

singlets constructed by combining pairs of external reps into an ``exchanged rep”



Space of singlets identified with space of conformal blocks

- Gauge invariance implies:
  - object transforms correctly under conformal group
  - diagram independent of location of vertex
  - no separate exchange diagrams
- space of singlets to include constrained by crossing
- obvious higher spin extension: SL(2) → SL(N)

- As before, replacing strings of stress tensors by their quantum correlation functions is expected to convert these from global blocks to Virasoro blocks, organized in a 1/c expansion

  Some explicit checks   Fitzpatrick, Kaplan, Li, Wang

Late time block

PK, Sivaramakrishnan, Snively

- Another way to describe a 4-point block is to evaluate the Wilson line in a nontrivial CFT primary state

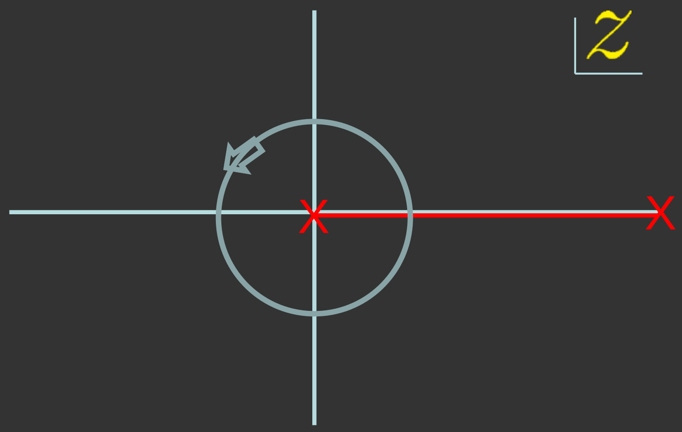
$$G(z) = \langle h_2 | \langle h_1 | P e^{\int A} | h_1 \rangle | h_2 \rangle$$

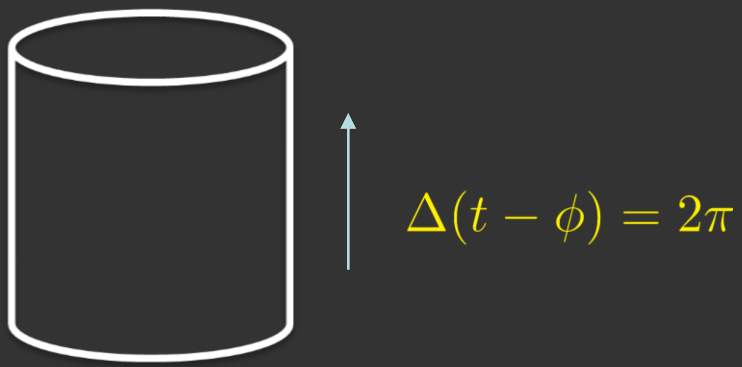
$$|h_1\rangle = SL(2) \text{ state}$$

$$|h_2\rangle = \text{ CFT state}$$

  - heavy-light block: $h_2 \sim c$
  - light-light block: $h_2 \sim c^0$
- If $h_2 > \frac{c}{24}$ then we get a contribution to the correlator of a probe in a black hole background

- As originally pointed out by Maldacena, the late time behavior of such correlators is relevant for the BH info paradox
- Blocks considered here have a 1/c expansion, but this breaks down when t =O(c). Need to resum





Taking z once around the branch cut, at fixed $\bar{z}$ advances time on the Lorentzian cylinder

- A first step is to determine the late time behavior of Virasoro blocks. For heavy-light blocks this is analytically intractable, but numerics indicate exponential falloff followed by power law decay Chen, Hussong, Kaplan,Li. Full correlator should not decay to zero
- Analytical progress can be made for (less interesting) case of light-light blocks

$$c, t \to \infty \ , \quad h_{1,2}, \ \frac{t}{c} \ \ \text{fixed}$$

will be related to exponentiation of 1-graviton exchange diagrams

- For $t = 2\pi n$ we consider Wilson line that wraps n times around the branch point

$$G(z) = \langle P e^{\int_0^z (L_1 + \frac{6}{c} T(y) L_{-1}) dy} \rangle$$

- Think of expanding exponential and doing integrals. Path ordering only matters when more than one T(y) is in same sheet
- Key point: in late time limit, have a diffuse gas of T(y) insertions, so path ordering can be omitted

- Integrals can then be done and result is

$$G(t) = e^{-i(h+h')t}\langle h, h'| e^{e^{-it}L_1} \underbrace{e^{-\frac{12it}{c}L\cdot L'}}_{\text{new factor}} e^{e^{it}L_{-1}} |h, h'\rangle$$

SL(2) invariant dot product: $L \cdot L' = \eta^{mn} L_m L'_n$

rewrite in terms of quadratic Casimirs

$$2L \cdot L' = (L + L')^2 - L^2 - L'^2$$

Now have expansion $G(t) = \sum_n a_n e^{-iE_n t}$

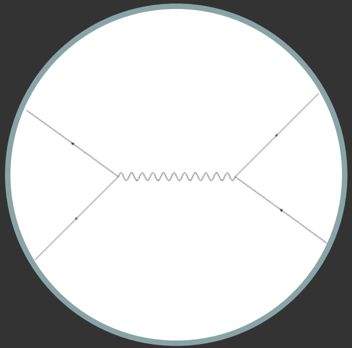
$$E_n = h + h' + n + \tfrac{1}{2}\gamma_n$$

corresponds to double trace operators/2-particle states with anomalous dim:

$$\gamma_n = \frac{12}{c}\left(C_2(h + h' + n) - C_2(h) - C_2(h')\right)$$

$$C_2(h) = h(h-1)$$

- Compare this to anomalous dimension computed from graviton exchange diagram



  such computations are greatly simplified by using inversion formula of Caron-Huot

- Input is the double commutator computed from analytic continuation around branch cuts

$$G_4(x_i) = \langle [O_1(x_1), O_2(x_2)], [O_3(x_3), O_4(x_4)] \rangle$$

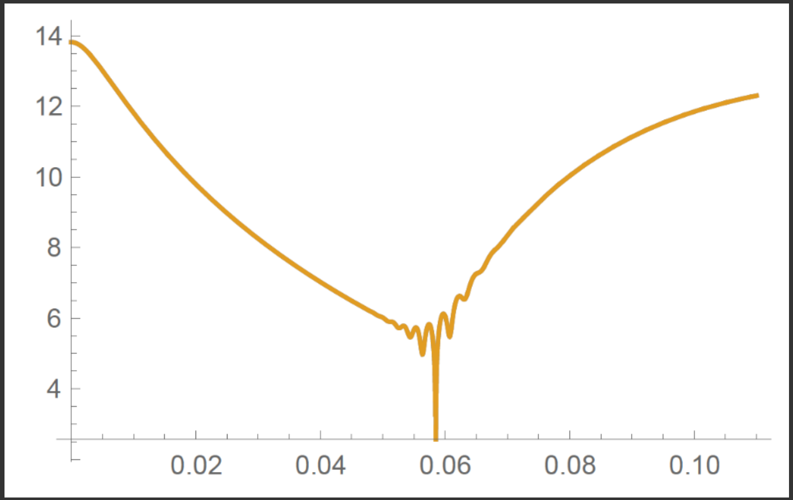
- This is then integrated against certain conformal blocks, and OPE data is extracted from singularities

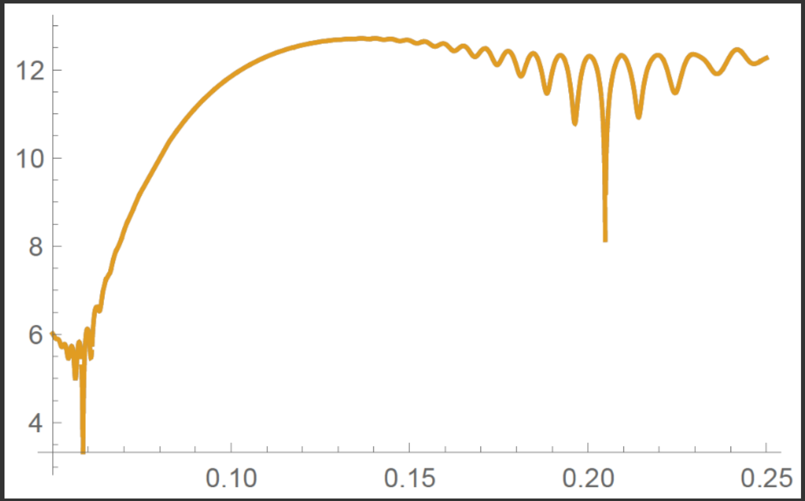
$$D(h,\overline{h}) \sim \int_0^1 \int_0^1 \frac{dz d\overline{z}}{z^2 \overline{z}^2} g_h(z) g_{1-\overline{h}}(\overline{z}) G_4(z,\overline{z})$$

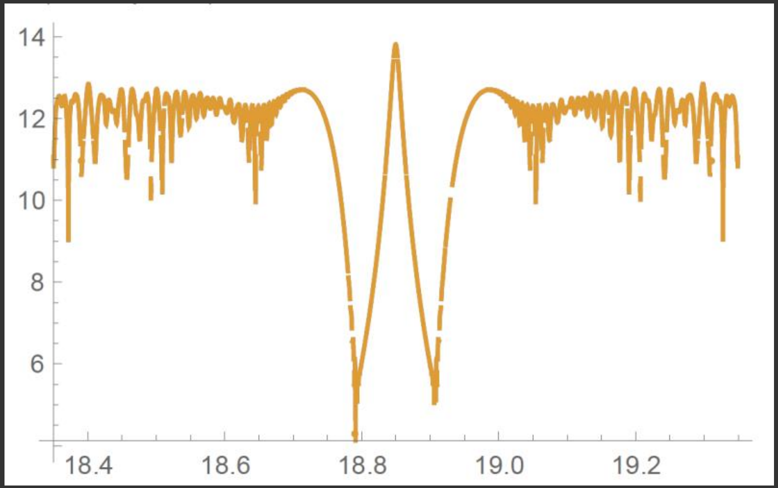
$$D(h,\overline{h}) \sim \frac{C_{h_i h_j h}}{h - h_{phys}}$$

- Major simplification is that one doesn’t need to compute full Witten diagram: double traces cancel out of double commutator so only stress tensor block survives
- reproduces $\gamma_n = \frac{12}{c}\left(C_2(h+h'+n) - C_2(h) - C_2(h')\right)$

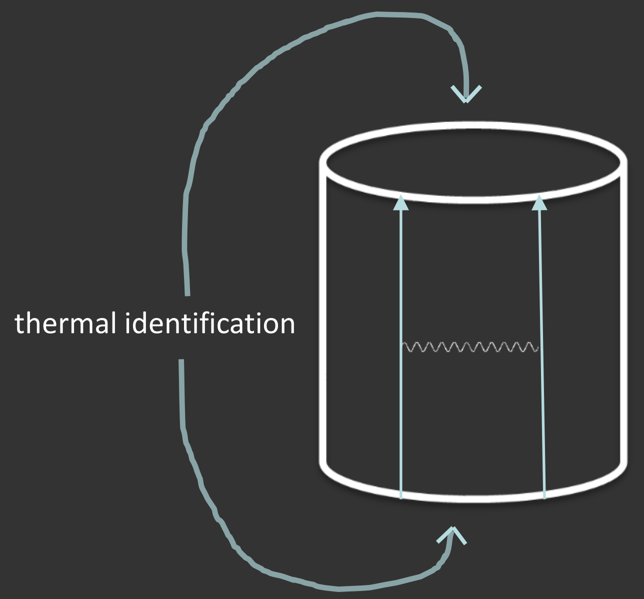
- plots of log(G) versus 6t/c







- Anomalous dimensions can also be extracted from corrections to the thermal partition function



$$Z(\beta) = \sum_n d_n e^{-\beta(E_n^0 + \gamma_n)}$$

$$= Z_0(\beta) - \beta \sum_n d_n \gamma_n + \dots$$

Conclusion

- Wilson lines can be used to compute anomalous dimension and late time blocks

- Results suggest simple structure, not fully understood