Characters of the conformal algebra in higher-spin holography

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From joint works with Thomas Basile & Euihun Joung

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Higher-spin holography and Representation theory

Higher-spin holography

Basic idea behind the conjectured duality:
[Konstein-Vasiliev-Zaikin, Sezgin-Sundell, Sundborg, Witten, Mikhailov]

Free (or integrable) CFTs have an infinite number of higher-order conformal symmetries.

Higher-spin holography

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$\overset{\text{Noether theorem}}{\Longrightarrow}$

Their spectrum contains an infinite tower of traceless conserved currents with unbounded spin (including spin two).

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$\overset{\text{AdS/CFT dictionary}}{\Longrightarrow}$

Free (or integrable) CFTs should be dual to "higher-spin gravity" theories whose spectrum contains an infinite tower of gauge fields with unbounded spin (including spin two).

Higher-spin holography

Remarks:

- Although the boundary dual is free, the n-point correlators are non-vanishing
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- Provides the simplest examples of AdS/CFT correspondence
  $\Rightarrow$ These holographic dualities
  - might be explicitly provable (even with mathematical level of rigor)
  - are Gaussian starting points for describing gravity duals to interacting CFTs with softly broken higher-spin symmetries

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  - might be explicitly provable (even with mathematical level of rigor)
  - are Gaussian starting points for describing gravity duals to interacting CFTs with softly broken higher-spin symmetries
- Free CFTs have no $1/N$ corrections
  $\Rightarrow$ Bulk higher-spin theories should be quantum gravity theories which are UV-finite in the strongest possible form: without quantum correction

Higher-spin holography

Simplest examples: Free vector models

[Klebanov-Polyakov, Sezgin-Sundell, ...]

Boundary CFT$_d$:

Composite operators in the singlet sector, made of free fields in the vector representation of some internal compact symmetry group.

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Vector models

$\Rightarrow$ all singlet primary operators are bilinear in the fundamental fields, among which an infinite tower of conserved currents.

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Bulk higher-spin theory around AdS$_{d+1}$:

Spectrum contains an infinite tower of massless higher-spin fields.

Higher-spin holography

Higher-spin symmetry is so huge that the dynamics is essentially fixed by the symmetries.

$\Rightarrow$ The representation theory of the conformal algebra already knows a lot about higher-spin gravity (linearized spectrum, symmetry algebra, vacuum one-loop quantities, etc)

Outline & Summary

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Representation theory of algebra $\mathfrak{so}(d,2)$:

- CFT$_d$ elementary fields
  - *Singleton* = ultrashort irrep [Dirac (1963)]
  - *Higher-spin algebra* = enveloping algebra of singleton symmetries [Eastwood (2002) Vasiliev (2003)]

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Representation theory of higher-spin algebra:

- *Fundamental* = Singleton (Free CFT)
- *Adjoint* = Higher-spin algebra (Symmetries)
- *Twisted adjoint* (Spectrum)
  = Singleton $\otimes$ Singleton (Boundary, Bilinears)
  = $\oplus$ Massless fields with Dirichlet bdy cond (Bulk)
  [Flato, Fronsdal (1978)]

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- Classification of irreducible (generalized) Verma $\mathfrak{so}(d,2)$-modules [Gavrilik, Klimyk, 1975; Shaynkman, Tipunin, Vasiliev, 2004]
- Character formulae of Verma $\mathfrak{so}(d,2)$-modules [F. Dolan, 2005]

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$\Longrightarrow$ Characters & partition functions of spin-s **CFT$_d$ fields:**

- *Current* (irreducible module)
- *Shadow* (ir/reducible for odd/even d)
- *Fradkin-Tseytlin* (irreducible but only defined for even d)

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classical (tree) relation [XB, Grigoriev, 2012]

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1-loop relation [Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013]

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    - **Free theory** [Fradkin, Tseytlin, 1985]
        - Higher-spin (s>2) generalization of ($d = 4$) Maxwell photon (s=1) and Weyl graviton (s=2)
        - Kinetic operator with $2s + d - 4$ derivatives (local for even d)
        - *Non-unitary irreducible representation* of conformal algebra (for $s > 1$ or $d > 4$)

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     - *Non-unitary irreducible representation* of conformal algebra (for $s > 1$ or $d > 4$)
   - **Interacting theory** [Tseytlin, 2002; Segal, 2003; XB, Joung, Mourad, 2011; Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013]
     - *classical theory* as induced gravity: logarithmically divergent piece of the 1-loop effective action of a conformal scalar field minimally coupled to a background of higher-spin shadow fields
     - *quantum theory*: vanishing Weyl anomaly? (and higher-spin versions?)
     - *holographic duality*: logarithmically divergent piece of the on-shell action of the bulk higher-spin gravity

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   - **Regularization** à la Zeta [Giombi, Klebanov, Safdi, 2014]

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   - **Regularization** à la Zeta [Giombi, Klebanov, Safdi, 2014]
   - **Cancellations for unbroken higher-spin gravity** (Dirichlet boundary condition) at 1-loop of
     - Vacuum bubble diagrams (for boundary S^d)
       [Giombi, Klebanov, 2013; Giombi, Klebanov, Safdi, 2014; ...]
     - Casimir energy (for boundary $\mathbb{R} \times S^{d-1}$)
       [Giombi, Klebanov, Tseytlin, 2014; ...]

   are checks (at 1-loop) that unbroken higher-spin gravity might be quantum exact.

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       [Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013; Tseytlin, 2013]
     - Casimir energy
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   suggest that (bosonic) conformal higher-spin gravity might remain consistent at quantum level (like $\mathcal{N} = 4$ conformal supergravity).

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   suggest that (bosonic) conformal higher-spin gravity might remain consistent at quantum level (like $\mathcal{N} = 4$ conformal supergravity).

   Caveat: The type-B Weyl anomaly (Weyl-squared c-coefficient) does not seem to share this remarkable cancellation property but some assumptions in the computation remain questionable [Tseytlin, 2013] e.g. the factorization of the kinetic operator in any Einstein background [Metsaev, 2014; Nutma, Taronna, 2014]

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   - **Regularization of higher-spin algebra character**
     [Basile, XB, Joung, 2018]

     $$\boxed{\text{Adjoint} \quad \sim \quad \text{Singleton} \otimes \text{Singleton}^*}$$

     requires symmetrization over all Cartan generators.

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     suggests that the theory might be somewhat "topological" (asymptotic charges as only dynamical degrees of freedom).

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     suggests that the theory might be somewhat "topological"
     (asymptotic charges as only dynamical degrees of freedom).

     Caveat: relies on some regularization prescription (consistent with vanishing of type-A Weyl anomaly and Casimir energy)

Conformal higher-spin gravity

Free theory: off-shell

The free conformal spin-s gauge theory is described *off-shell* by the

- **Shadow field**, i.e. conformal primary field h_s that is a symmetric tensor field of scale dimension $\Delta(h_s) = 2 - s$ and rank s, quotiented by the gauge transformation:

$$\delta h_{\mu_1\mu_2\ldots\mu_s} = \partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} - \eta_{(\mu_1\mu_2}\alpha_{\mu_3\ldots\mu_s)}$$

- **Higher-spin Weyl tensor**, i.e. conformal primary field C_s that is the gauge-invariant irreducible tensor field of scale dimension $\Delta(C_s) = 2$ labelled by a rectangular Young diagram with rows of length s and built out of s derivatives of the shadow field:

$$C_{\mu_1\ldots\mu_s\,,\,\nu_1\ldots\nu_s} = \partial_{\nu_1}\cdots\partial_{\nu_s}h_{\mu_1\ldots\mu_s} + \cdots$$

- **Fradkin-Tseytlin-Segal action** [$d=4$: Fradkin & Tseytlin, 1985; $d>4$: Segal, 2002]

$$S[h_s] = \int h_s\,\Box^{\frac{2s+d-4}{2}}h_s + \cdots = (-1)^s\int C_s\,\Box^{\frac{d-4}{2}}C_s + \cdots$$

Free theory: on-shell

The free conformal spin-s gauge theory is described *on-shell* by the

- **Higher-spin Bach tensor**, i.e. conformal primary field B_s that is the gauge-invariant symmetric tensor field of scale dimension $\Delta(B_s) = s + d - 2$ built out of $s + d - 4$ derivatives of the Weyl-like tensor:

$$B_{\mu_1 \dots \mu_s} = \partial^{\nu_1} \cdots \partial^{\nu_s} \Box^{\frac{d-4}{2}} C_{\mu_1 \dots \mu_s \, , \, \nu_1 \dots \nu_s} + \cdots$$

- **Fradkin-Tseytlin field**, i.e. shadow field subject to the higher-spin Bach equation $B_s = 0$.

Interacting theory

Conformal higher-spin gravity as induced higher-spin gravity

[Tseytlin, 2002; Segal, 2003; XB, Joung, Mourad, 2011;
Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013]

Interacting theory

The interacting conformal higher-spin theory is described *off-shell* by the

- **Free scalar singleton**, i.e. free conformal scalar field φ of scale dimension $\Delta(\varphi) = d/2 - 1$ minimally coupled via

$$S_M[\varphi\,;h] = \int d^dx \left(-\frac{1}{2}(\partial\varphi)^2 + \sum_s h_s j_s \right) = \frac{1}{2}\int d^dx\, \varphi(\partial^2 + \hat{h})\varphi$$

  to a background of higher-spin shadow fields through the

- **Conformal currents**, i.e. conformal primary field j_s that is a traceless conserved symmetric current of scale dimension $\Delta(j_s) = s + d - 2$ and rank s, bilinear in the conformal scalar field

$$j_{\mu_1\dots\mu_s} = \varphi\,\partial_{\mu_1}\cdots\partial_{\mu_s}\varphi + \cdots$$

- **Induced action:** logarithmically divergent piece of the effective action (in even d)

$$W_\Lambda[h] = -\frac{1}{2}\,\mathrm{Tr}_\Lambda \log(\partial^2 + \hat{h}) = \log(\Lambda/m^2)\, S_W[h] + \text{Laurent series in } \Lambda$$

Classification of irreducible representations

Classification

- Primary field on $\mathbb{R} \times S^{d-1}$ with all its descendants corresponds to a (generalized) Verma $\mathfrak{so}(d,2)$-module $\mathcal{V}(\Delta, \vec{s})$ labelled by the scaling dimension Δ and the $\mathfrak{so}(d)$-spin $\vec{s}$.
- Exhaustive classification of (ir)reducible Verma $\mathfrak{so}(d,2)$-modules and submodules [Gavrilik, Klimyk, 1975; Shaynkman, Tipunin, Vasiliev, 2004]

$\Longrightarrow$ Recursive computation of all relevant $\mathfrak{so}(d,2)$-modules

Odd dimension d

$d = 3$ realizations of irreducible $\mathfrak{so}(3,2)$-modules

CFT	Irreducible module	Verma module description
Conservation law	$\mathcal{D}(s+2, s-1)$	$\mathcal{V}(s+2, s-1)$
Conformal current	$\mathcal{D}(s+1, s)$	$\mathcal{V}(s+1, s)/\mathcal{D}(s+2, s-1)$

Conservation law $\equiv \partial_{\mu_1} j^{\mu_1 \mu_2 \ldots \mu_s} = 0$

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Pure gauge shadow	$\mathcal{D}(2-s, s)$	$\mathcal{V}(2-s, s)/\mathcal{D}(s+1, s)$

Higher-spin Cotton tensor $\partial^{2s-1} h_s = 0 \Leftrightarrow h_s = \partial \varepsilon_{s-1} + \eta_2 \alpha_{s-2}$

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Pure gauge shadow	$\mathcal{D}(2-s, s)$	$\mathcal{V}(2-s, s)/\mathcal{D}(s+1, s)$
Conformal Killing	$\mathcal{D}(1-s, s-1)$	$\mathcal{V}(1-s, s-1)/\mathcal{D}(2-s, s)$

Conformal Killing tensor $\equiv \partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} = \eta_{(\mu_1\mu_2}\alpha_{\mu_3\ldots\mu_s)}$

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Pure gauge shadow	$\mathcal{D}(2-s,s)$	$\mathcal{V}(2-s,s)/\mathcal{D}(s+1,s)$
Conformal Killing	$\mathcal{D}(1-s,s-1)$	$\mathcal{V}(1-s,s-1)/\mathcal{D}(2-s,s)$
CFT	Reducible module	Contragredient module
Shadow field	$\mathcal{S}(2-s,s)$	$\sim\mathcal{V}(2-s,s)/\mathcal{D}(2-s,s)$

Even dimension d

$d = 4$ realizations of irreducible $\mathfrak{so}(4,2)$-modules

CFT	Irreducible module	Verma module description
Conservation law	$\mathcal{D}(s+3, s-1)$	$\mathcal{V}(s+3, s-1)$
Conformal current	$\mathcal{D}(s+2, s)$	$\mathcal{V}(s+2, s)/\mathcal{D}(s+3, s-1)$
Chiral FT field	$\mathcal{D}(2; s, \pm s)$	$\mathcal{V}(2; s, \pm s)/\mathcal{D}(s+2, s)$
FT field	$\mathcal{D}(2, s, s)$ $\oplus\mathcal{D}(2, s, -s)$	$\mathcal{U}(2, s, s)/\mathcal{V}^*(s+2, s)$
Pure gauge fixed shadow	$\mathcal{D}(2-s, s)$	$\mathcal{V}(2-s, s)/\mathcal{U}(2, s, s) \cong$ $\mathcal{U}(2-s, s)/\mathcal{V}^*(s+3, s-1)$
Conformal Killing	$\mathcal{D}(1-s, s-1)$	$\mathcal{V}(1-s, s-1)/\mathcal{U}(2-s, s)$
CFT	Reducible module	Contragredient module
Shadow field	$\mathcal{S}(2-s, s)$	$\sim \mathcal{V}(2-s, s)/\mathcal{U}(2-s, s)$

Computation of partition function

Partition function

Canonical partition function

- **Generating function**

$$\mathcal{Z}(q) \equiv \sum_n d_n q^{\Delta+n} = Tr\big(\exp(-\beta\hat{H})\big)$$

of the auxilliary variable $q = \exp(-\beta)$ defined in terms of the inverse temperature β, i.e. the inverse of the perimeter of the circle S^1 in radial quantization. It encodes the number d_n of descendants of level n associated with a primary field of scale dimension Δ.

- **Character of** $\mathfrak{so}(d,2)$

$$\chi(q, x_1, \cdots, x_r) = Tr\Big(\exp\big(-\beta(\hat{H} - \sum_{i=1}^{r}\Omega_i\hat{J}_i)\big)\Big)\,, \quad r = \text{rank}\,\mathfrak{so}(d)\,,$$

with $x_i = \exp(\beta\,\Omega_i)$ as canonical partition function on thermal AdS$_{d+1}$ / $S^1 \times S^{d-1}$ where we turn on the chemical potentials Ω_i corresponding to the angular momenta in the Cartan subalgebra of $\mathfrak{so}(d)$. Therefore, $\mathcal{Z}(q) = \chi(q, 1, \cdots, 1)$.

Partition function

Characters of lowest-weight Verma modules

- Explicit formulae [F. Dolan, 2005]

$$\chi^{\mathfrak{so}(d,2)}_{\mathcal{V}(\Delta,\vec{s})}(q,\vec{x}) = \frac{1}{(1-q)^{d-2r}} \prod_{i=1}^{r} \frac{1}{(1-q\,x_i)(1-q\,x_i^{-1})}\, q^{\Delta}\, \chi^{\mathfrak{so}(d)}_{\vec{s}}(\vec{x})$$

- Useful property (lowest versus highest weight)

$$\chi^{\mathfrak{so}(d,2)}_{\mathcal{V}(\Delta,\vec{s})}(q^{-1},\vec{x}) = (-)^d \chi^{\mathfrak{so}(d,2)}_{\mathcal{V}(d-\Delta,\vec{s})}(q,\vec{x})$$

Partition function

Character of free conformal spin-s gauge field

[Beccaria, XB, Tseytlin, 2014]

- Off-shell (any d)

$$\chi_{\mathcal{S}(2-s,s)}(q,\vec{x}) = \chi_{\mathcal{D}(1-s,s-1)}(q,\vec{x}) + (-)^d \chi_{\mathcal{D}(s+d-2,s)}(q^{-1},\vec{x})$$

- On-shell (even d)

$$\chi_{\mathcal{D}(2;s,s)}(q,\vec{x}) = \chi_{\mathcal{D}(1-s,s-1)}(q,\vec{x}) + \chi_{\mathcal{D}(s+d-2,s)}(q^{-1},\vec{x}) - \chi_{\mathcal{D}(s+d-2,s)}(q,\vec{x})$$

Partition function

Remark: The formulae also applies in the degenerate case $s = 0$ in the sense that

- Off-shell

$$\chi_{\mathcal{V}(2,0)}(q, \vec{x}) = (-)^d \chi_{\mathcal{D}(d-2,0)}(q^{-1}, \vec{x})$$

- On-shell

$$\chi_{\mathcal{D}(2,0)}(q, \vec{x}) = (-)^d \chi_{\mathcal{D}(d-2,0)}(q^{-1}, \vec{x}) - \chi_{\mathcal{D}(d-2,0)}(q, \vec{x})$$

Regularization & Cancellation

Sum over all integer spins

Sum of partition functions over the infinite tower of
Conformal currents $\Leftrightarrow$ Massless fields (Dirichlet)

- $=$ Partition function of free higher-spin gravity
- is finite in agreement with the Flato-Fronsdal theorem

$$\sum_{s=0}^{\infty} \chi_{\mathcal{D}(s+d-2,s)}(q,\vec{x}) = \Big(\chi_{\mathcal{D}(\frac{d-2}{2},0)}(q,\vec{x})\Big)^2$$

where $\mathcal{D}(\frac{d-2}{2},0)$ is the scalar singleton.

Sum over all integer spins

Sum of partition functions over the infinite tower of **Fradkin-Tseytlin fields**

- = Partition function of free conformal higher-spin gravity
- is infinite and must be regularized.
- Regularization via characters [Basile, XB, Joung, 2018]
  1. Consider that the second term

$$\sum_{s=0}^{\infty}\chi_{\mathcal{D}(s+d-2,s)}(q^{-1},\vec{x})=\Big(\chi_{\mathcal{D}(\frac{d-2}{2},0)}(q^{-1},\vec{x})\Big)^2$$

     is finite as well, then the second and third terms cancel.
  2. This cancellation reduces the regularization of the partition function of free conformal higher-spin gravity to the one of the higher-spin algebra.

$$\sum_{s=0}^{\infty}\chi_{\mathcal{D}(2;s,s)}(q,\vec{x})=\sum_{s=1}^{\infty}\chi_{\mathcal{D}(1-s,s-1)}(q,\vec{x})$$

Sum over all integer spins

Conclusion: At the level of characters of conformal higher-spin gravity,

Linearized spectrum	=	Symmetry algebra

Byproducts:

- **Regularizations consistent with higher-spin symmetries** by construction.
- **Vanishing of type-A Weyl anomaly and Casimir energy** ensured by parity property $Z(-\beta) = Z(\beta)$ valid for right-hand-side.

Conclusion and perspectives

Main results

Some suitable regularization of the divergent sum of characters over all spins suggest that the following modules might be isomorphic:

- the linearized spectrum of conformal higher-spin gravity theories (sum of all Fradkin-Tseytlin fields),
- the corresponding higher-spin algebra (sum of all conformal Killing tensors).

General conclusion

Confirmation of previous indications about conformal higher-spin gravity that, after regularization of the infinite tower of fields,

- no scale is generated at 1-loop (vanishing type-A Weyl anomaly, Casimir energy).
- the theory appears "topological" (trivial scattering, asymptotic charges as only dynamical degrees of freedom).

Perspectives

In order to confirm the relevance of our regularization prescription via characters:

- Obtain a field-theoretical version of our result about free conformal higher-spin gravity (linearized spectrum = symmetry algebra).
- Investigate its nonlinear analogue (if it survives interactions) in order to understand its deeper origin.