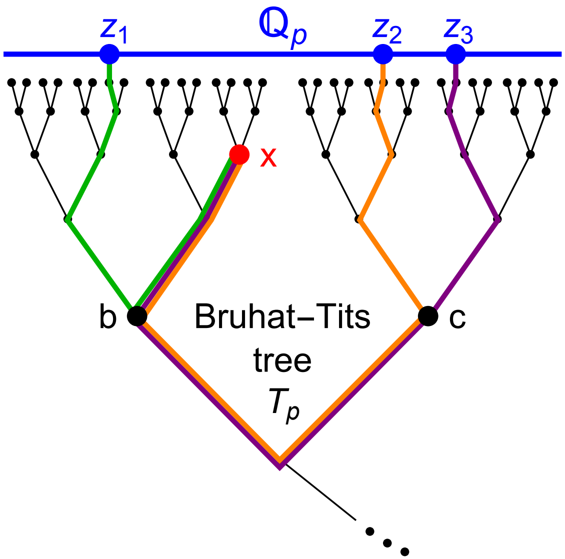
From p-adic AdS/CFT to prospects in cold atoms

Steve Gubser
Princeton

1805.07637 with C. Jepsen, Z. Ji, and B. Trundy

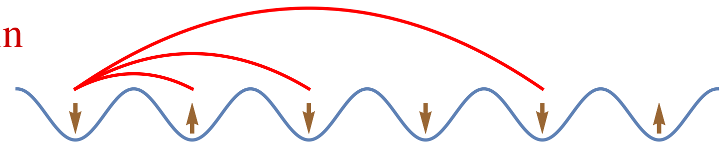
July 2018, Würzburg

1. Overview



- p-adic AdS/CFT [Gubser-Knaute-Parikh-Samberg-Witaszczyk '16, Heydeman-Marcolli-Saberia-Stoica '16] relates field theory dynamics over $\mathbb{Q}_p$ to bulk dynamics on a regular tree T_p.
- $$T_p = \frac{p\text{-adic conformal group}}{\text{max compact subgroup}}$$
  is naturally discrete. Easier starting point for quantum gravity?
- Mostly today I will focus on field theory side.

- Long-range couplings among atoms in an optical lattice can be individually tuned.
- Depending on how we dial the couplings, atoms can approximate a real continuum or a p-adic continuum.



Discussions with G. Bentsen, and E. Davis and M. Schleier-Smith, cf. [Hung-Gonzalez-Tudela-Cirac-Kimble '16].

2. What are p-adic numbers?

Write $h \in \mathbb{Z}$ as

$$h = \pm 2^{v_2} 3^{v_3} 5^{v_5} \ldots$$

Then for any choice of prime p,

$$|h|_p = p^{-v_p} .$$

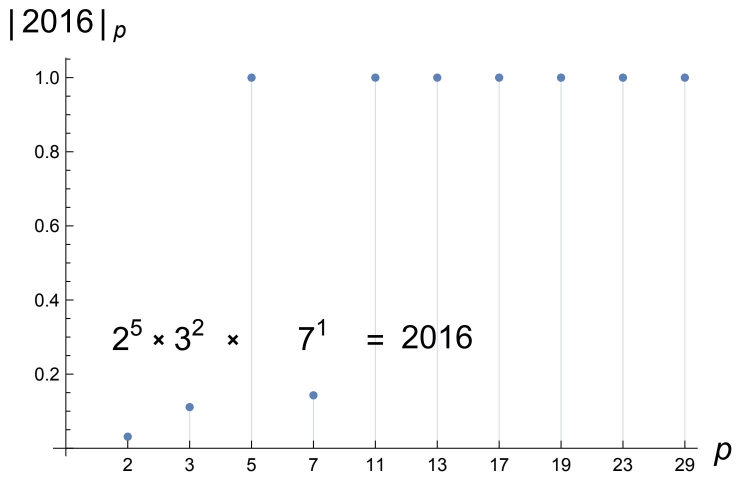
$|\cdot|_p$ is the p-adic norm.

For $x = a/b \in \mathbb{Q}$, define

$$|x|_p = \frac{|a|_p}{|b|_p} .$$

Also define $|0|_p = 0$.

Example:



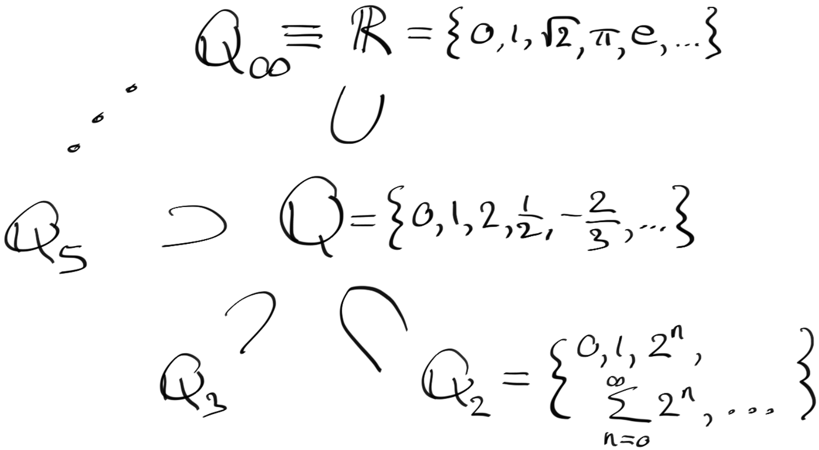
The p-adic numbers $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ wrt the norm $|\cdot|_p$.

The p-adic integers $\mathbb{Z}_p$ are the unit ball in $\mathbb{Q}_p$, i.e. $\{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}$.

Any $x \in \mathbb{Q}_p \backslash \{0\}$ has a unique base p expansion:

$$x = \underbrace{\ldots a_3 a_2 a_1 a_0}_{\text{this part is in } \mathbb{Z}_p} \,.\, \underbrace{a_{-1} a_{-2} \ldots a_{v_p}}_{\text{fractional part } \{x\}} \equiv \sum_{n=v_p}^{\infty} a_n p^n$$

where $a_n \in \{0, 1, 2, \ldots, p-1\}$ and $a_{v_p} \neq 0$.



$\mathbb{R}$ is called Archimedean because if $|a| > |b| > 0$, $\exists n \in \mathbb{Z}$ so that $|nb| > |a|$.

The $\mathbb{Q}_p$ have instead the ultrametric property: $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

$\mathbb{R}$ and the $\mathbb{Q}_p$ are the only “good” completions of $\mathbb{Q}$ (Ostrowski’s theorem).

Example for $p = 2$:

$$x = \ldots 11111 = \sum_{n=0}^{\infty} 2^n = \frac{1}{1-2} = \frac{1}{-1} = -1\,.$$

Arithmetic operations for the 2-adics works just like you expect in base 2:

$-1 + 1 = 0$ in $\mathbb{Q}_2$:

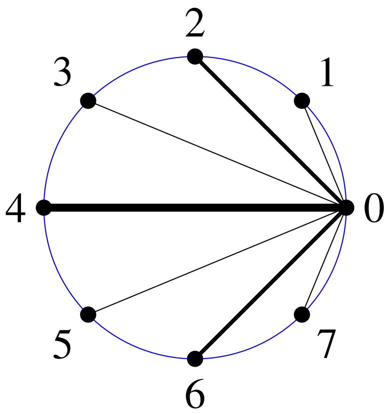
$$\begin{array}{rcl} \ldots \overset{}{1}\,\overset{}{1}\,\overset{1}{1}\,\overset{1}{1}\,\overset{1}{1}\,1 & = & -1 \\ + \qquad\qquad\qquad\qquad 1 & = & 1 \\ \hline \ldots 0\,0\,0\,0 & = & 0 \end{array}$$

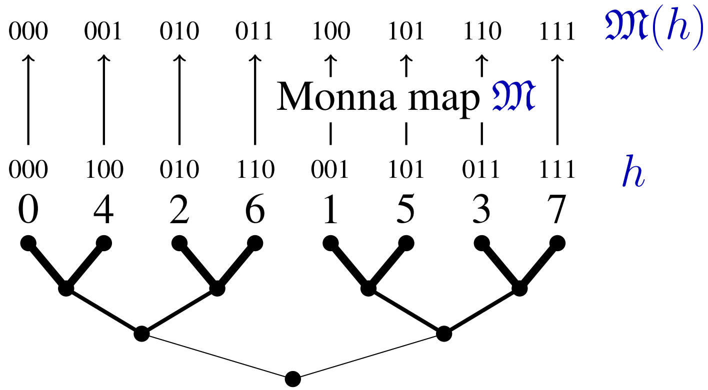
Crucial: We carry to the left. So $\mathbb{Q}_2$ is *not the same* as just writing real numbers backward in base 2.

3. Hierarchical models

Consider the *furthest neighbor* 1-d Ising model, aka the Dyson hierarchical model:

$$H = -J_* \sum_{m,n} \frac{\sigma_m \sigma_n}{|m-n|_2^{s+1}} \qquad \text{where each } \sigma_m = \pm 1.$$





After using the digit-reversing Monna map, far apart spins are close and vice versa.

The 2-adic norm formalizes this alternative notion of closeness: $|i-j|_2 = 2^{-d(i,j)/2}$ where $d(i,j)$ is distance *on the tree*.

The tree appears to badly break translation invariance, but this is an illusion: $h \to h+1$ *preserves* the tree structure.

Dyson '69: Furthest neighbor Ising model has a finite temperature phase transition.

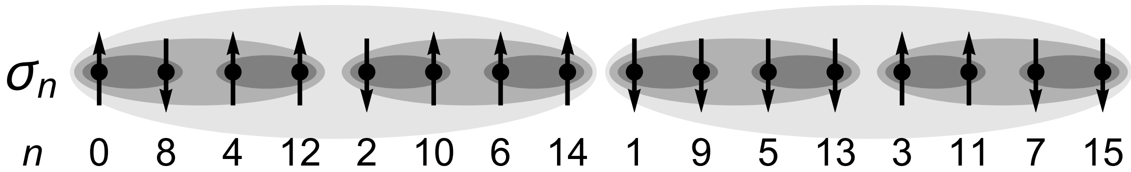
Missarov-Lerner '89 (also Bleher-Sinai '75 and others) showed that the critical theory at the transition is characterized by ϕ^4 theory over $\mathbb{Q}_2$:

$$S = -\int_{\mathbb{Q}_2\times\mathbb{Q}_2} dx\,dy\, \frac{1}{2}\frac{\phi(x)\phi(y)}{|x-y|_2^{s+1}} + \int_{\mathbb{Q}_2} dx\,\left[\frac{r}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right] \quad \text{where} \quad \phi : \mathbb{Q}_2 \to \mathbb{R}$$

and s is a parameter.

Ordinary conformal invariance is *not* realized, but $PGL(2,\mathbb{Q}_2)$ is.

- $z \to \frac{az+b}{cz+d}$ with $a, b, c, d, z \in \mathbb{Q}_2$.
- These LFTs map spin clusters to spin clusters, but sometimes changing the size of the clusters by a power of 2.



- $\langle \mathcal{O}(z)\mathcal{O}(0)\rangle \propto \frac{1}{|z|_2^{2\Delta}}$, similar to usual CFTs.

4. Ingredients for field theory over $\mathbb{Q}_p$

- We have length but not direction: $\mathbb{Q}_p$ isn't naturally ordered.
- We have measure: $\mathrm{Vol}(p^v \mathbb{Z}_p) \equiv p^{-v}$, and Vol is invariant under translations.
- We have integration à la Lebesgue following from Vol.
- We have translation invariance and plane waves: $\chi(kx) = e^{2\pi i \{kx\}}$.
- We have Fourier transforms and a non-local version of derivatives:

$$\phi(x) = \int_{\mathbb{Q}_p} dk\, \chi(kx) \tilde{\phi}(k) \qquad \text{(Note } \phi : \mathbb{Q}_p \to \mathbb{R}\text{)}$$

$$D^s \phi(x) = \int_{\mathbb{Q}_p} dk\, \chi(kx) |k|^s \tilde{\phi}(k) = \frac{1}{\Gamma_p(-s)} \int_{\mathbb{Q}_p} dy\, \frac{\phi(y) - \phi(x)}{|y-x|_p^{s+1}}\,.$$

- The sum of any number of soft momenta is still soft: the "ultra-metric" property.

$$\text{If all } |k_i|_p \leq \Lambda, \text{ then} \qquad \left| \sum_i k_i \right|_p \leq \Lambda\,.$$

- We can have scale-invariance, but it's discrete in steps of $k \to pk$.

5. Approximating p-adic interactions in a physical system

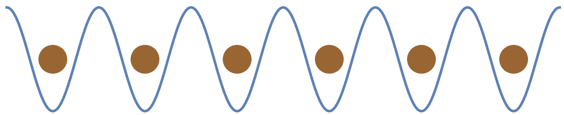
Cold atom systems allow unprecedented control over the Hamiltonian. E.g. non-local interactions for spin chains with spin $1/2$ on each site:

$$\hat{H} = -\sum_{m,n} J_{mn} \vec{\sigma}_m \cdot \vec{\sigma}_n \qquad \text{and variants, e.g. TIM and XY}$$

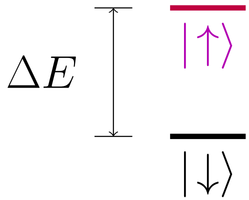
Recent proposals, incl. [Hung et al 2016] and ongoing work in M. Schleier-Smith's lab, naturally allows for translationally invariant $J_{mn} = J_{m-n}$ (up to endpoint effects).

Details of AMO at the level I can explain:

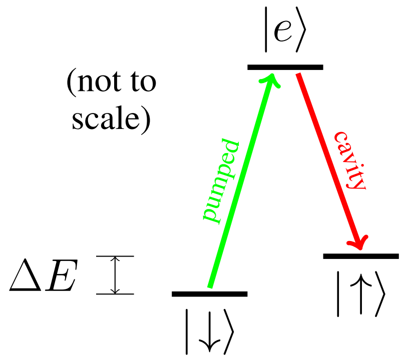
- Each site in a 1d optical lattice contains (ideally) one atom.



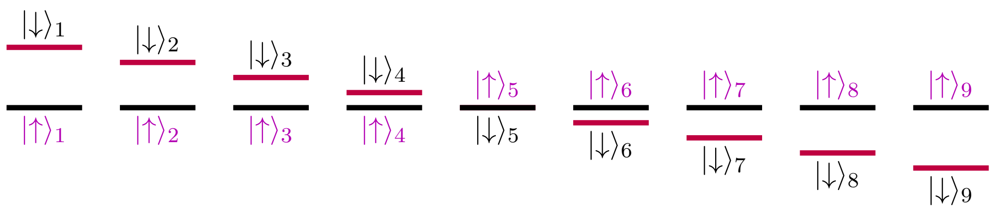
- Two low-lying states (e.g. with hyperfine splitting) provide single-atom pseudo-spin Hilbert space.



- $|\downarrow\rangle \leftrightarrow |e\rangle$ is coupled to photons we pump into the cavity.
- $|\uparrow\rangle \leftrightarrow |e\rangle$ is coupled to cavity photons that propagate across the whole sample.



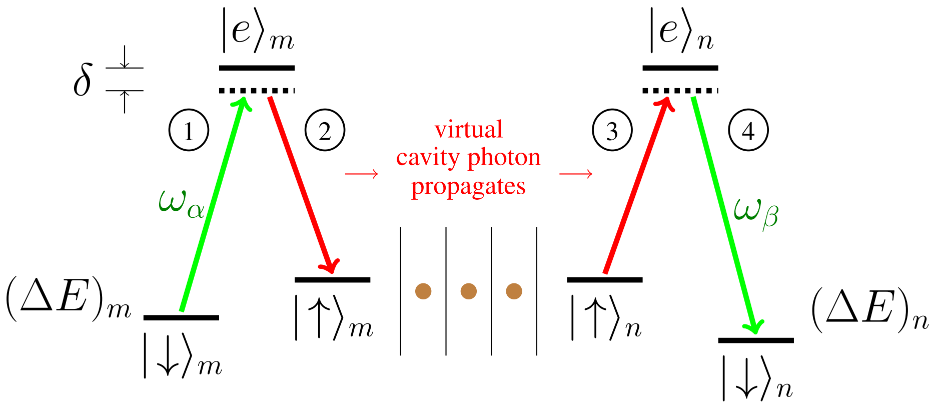
- Applied magnetic field shifts the hyperfine states unequally, so a linear gradient in B provides a site-dependent shift in energy, say in $|\downarrow\rangle$ but not $|\uparrow\rangle$.



Energy difference between low-lying states at site m is

$$(\Delta E)_m \equiv (E_\uparrow)_m - (E_\downarrow)_m = (m - m_0)\epsilon \qquad \text{with } \epsilon \text{ fixed.}$$

- $\sigma_m^+ \sigma_n^-$ interaction mediates $|{\downarrow\uparrow}\rangle_{mn} \to |{\uparrow\downarrow}\rangle_{mn}$ and proceeds in four steps:



For $|{\downarrow\uparrow}\rangle_{mn} \to |{\uparrow\downarrow}\rangle_{mn}$ to proceed, we need on-resonance condition

$$\underbrace{(m-n)\epsilon}_{\text{translationally invariant}} = (\Delta E)_m - (\Delta E)_n = \underbrace{\omega_\beta - \omega_\alpha}_{\text{experimentally controlled}}$$

We must pump in *both* ω_α photons and ω_β photons to get $|{\downarrow\uparrow}\rangle_{mn} \leftrightarrow |{\uparrow\downarrow}\rangle_{mn}$.

- Current prospect is to realize pure hopping Hamiltonian: $\sigma_m^+ \sigma_n^- + \sigma_m^- \sigma_n^+$.

- Dissipative effects associated with each new pumped mode encourage selecting frequencies parsimoniously.

Schleier-Smith's sparse coupling proposal: **Say $J_h = 0$ unless $h = 2^n$**

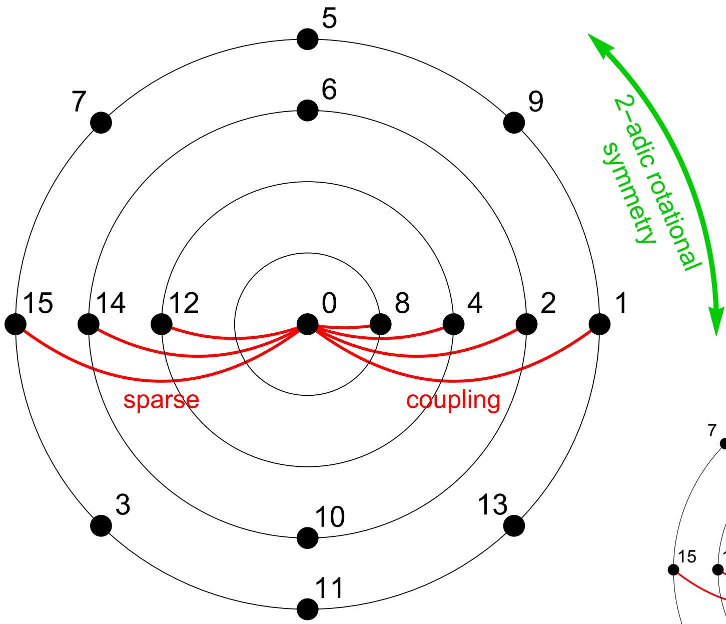
For example, with $N = 2^\ell$ spins,

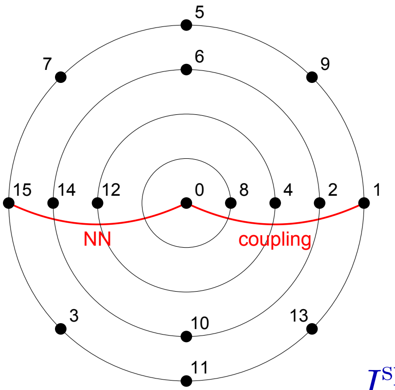
$$J_h^{\text{sparse}} \equiv J_* \sum_{n=0}^{\ell-1} 2^{ns} (\delta_{h-2^n} + \delta_{h+2^n} - 2\delta_h) \,.$$

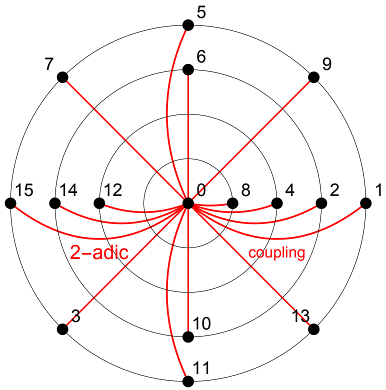
This is an approximation to

$$J_h^{2-\text{adic}} \equiv J_* |h|_2^{-s-1} \qquad \text{if } h \neq 0.$$

At first blush these couplings do not seem very similar! A primary aim of the remainder of the talk is to see that actually they are when $s > 0$.







$$J_h^{\text{sparse}} \equiv J_* \sum_{n=0}^{\ell-1} 2^{ns}(\delta_{h-2^n} + \delta_{h+2^n} - 2\delta_h)$$

As we dial s from $-\infty$ to $+\infty$ we interpolate between nearest neighbor couplings and 2-adic couplings.

$$J_h^{\text{NN}} \equiv J_*(\delta_{h+1} + \delta_{h-1} - 2\delta_h)$$

$$J_h^{2\text{-adic}} \equiv J_* |h|_2^{-s-1}$$

Problem: It's hard to get anywhere with quantum spin-$1/2$ Hamiltonians without numerics, absent some special trick like Jordan-Wigner.

Today's solution: Simplify the model to a free boson on a lattice, still with non-local interactions, treated in classical stat mech:

$$H \equiv -\frac{1}{2}\sum_{m,n} J_{m-n}\phi_m\phi_n - \sum_m b_m\phi_m \qquad Z[b] \equiv \left(\prod_{m=0}^{2^\ell-1}\int_{-\infty}^{\infty} d\phi_m\right)\delta(\tilde{\phi}_0)e^{-\beta H}\,.$$

Require $\tilde{J}_0 = 0$, while $\tilde{J}_k < 0$ for $k \neq 0$. In words:

- Interaction is ferromagnetic.
- Uniformly shifting all the ϕ_n is a massless mode.
- We explicitly fix that Goldstone-like mode with $\delta(\tilde{\phi}_0)$ inside the path integral.

The model is completely determined once we know the two-point function

$$G_{mn} = \langle\phi_m\phi_n\rangle = \frac{1}{\beta^2 Z[0]}\left.\frac{\partial^2 Z[b]}{\partial b_m \partial b_n}\right|_{b=0}$$

Just to get the idea, consider the most trivial nearest neighbor model:

$$J_h^{\rm NN} = J_*(\delta_{h+1} + \delta_{h-1} - 2\delta_h)$$

$$J_k^{\rm NN} \equiv \frac{1}{\sqrt{N}} \sum_{h=0}^{N-1} e^{-2\pi i k h/N} J_h^{\rm NN} = -\frac{4J_*}{\sqrt{N}} \sin^2\left(\frac{\pi k}{N}\right) \qquad \text{where} \qquad N = 2^\ell$$

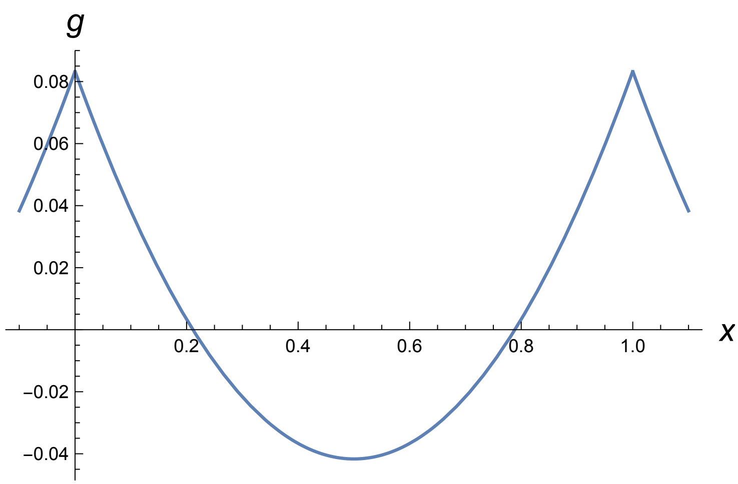
$$\tilde{G}_k^{\rm NN} = -\frac{1-\delta_k}{N\beta \tilde{J}_k} \qquad \Longrightarrow \qquad G_h^{\rm NN} \approx \frac{N}{\beta J_*} G(h/L) \qquad \text{where}$$

$$g(x) = \frac{1}{2}\left(x - \frac{1}{2}\right)^2 - \frac{1}{24}$$

for $\quad x \in [0,1)$:

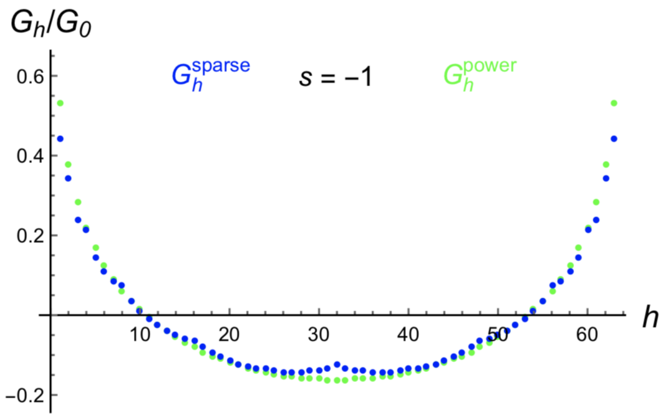
$$g''(x) = -\delta(x) + 1$$

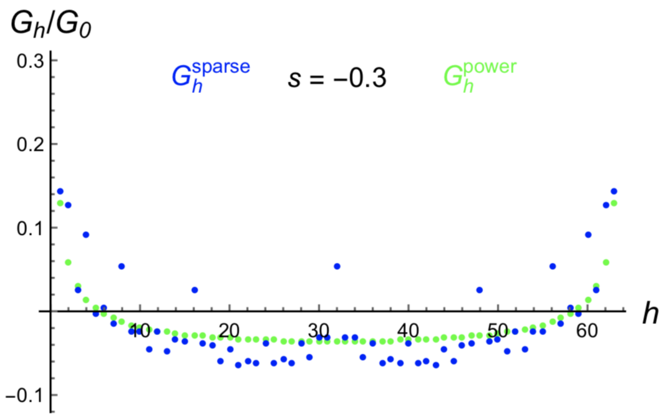
and $\quad \displaystyle\int_0^1 dx\, g(x) = 0\,.$



As we start adding in sparse long-range couplings by dialing up s, we'll see this smooth Green's function become less and less smooth.

Results for 64 spins. Archimedean side, $s < 0$:





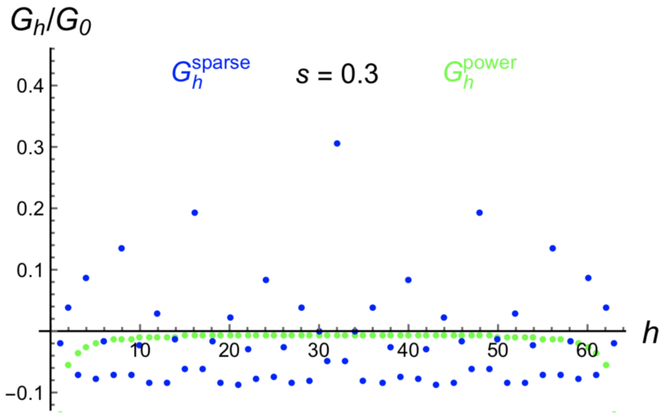
To make the best comparison between the sparse coupling results and a smooth Green's function, we introduce power-law couplings:

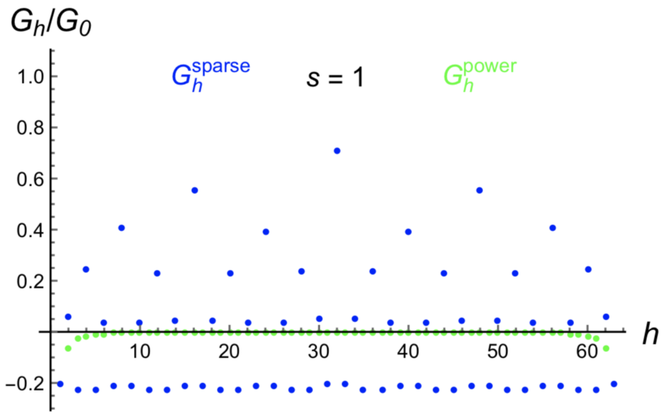
$$\tilde{J}_k^{\text{power}} \equiv -\frac{J_*}{2^s\sqrt{N}}\left[\sin\left(\frac{\pi k}{N}\right)\right]^{-s}$$

$\tilde{J}_k^{\text{power}}$ is a lattice version of the familiar power-law couplings: $J_h^{\text{power}} \sim |h|_\infty^{s-1}$ for $|h|_\infty/N \ll 1$.

$G_h < 0$ for some h seems wrong given ferromagnetic couplings. In fact, $\delta(\tilde{\phi}_0)$ enforces $\tilde{G}_0 = \frac{1}{\sqrt{N}}\sum_h G_h = 0$.

Results for 64 spins. Ultrametric side, $s > 0$:



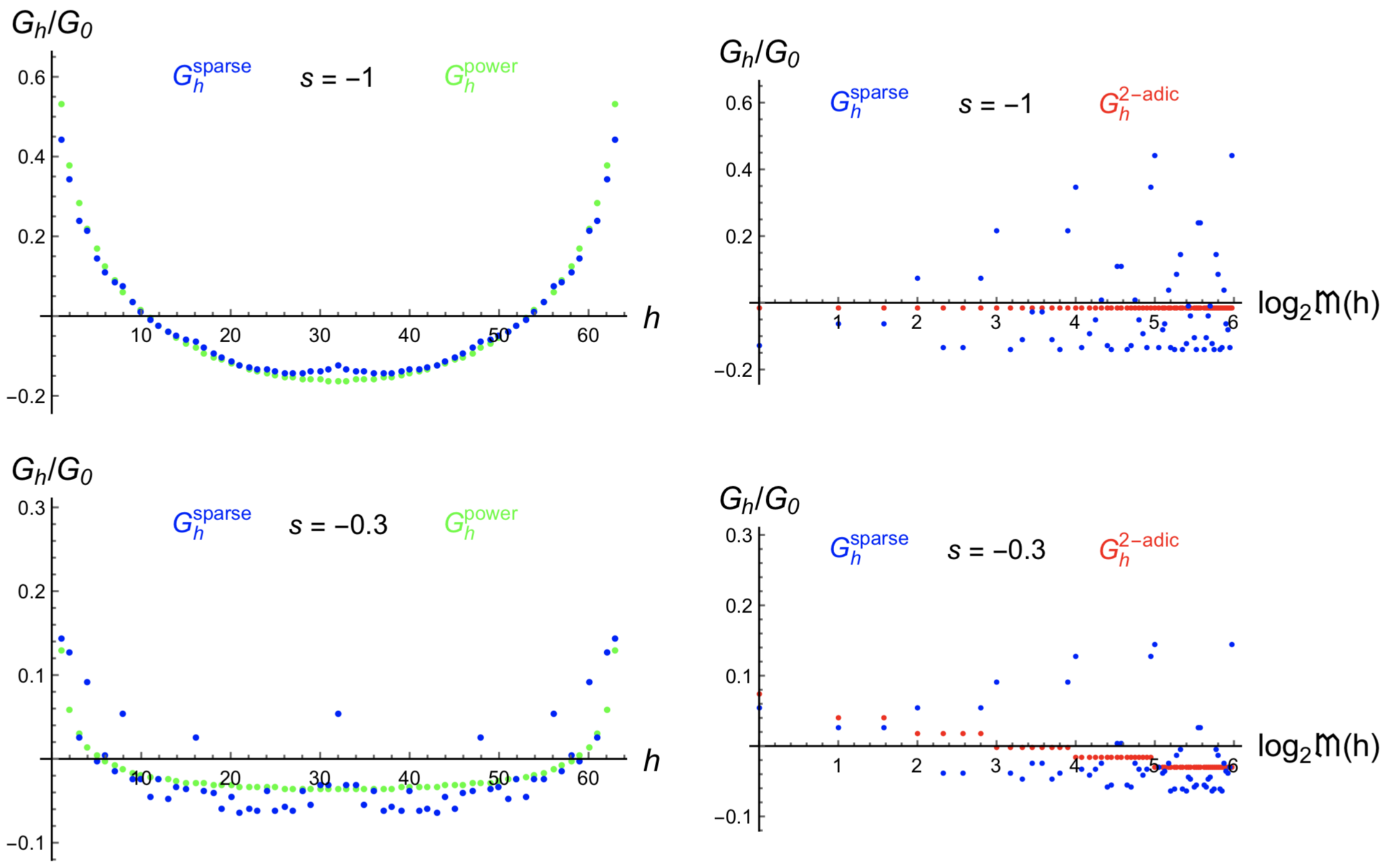


At this point it's clear that there just isn't a real continuum limit of the Green's function.

The strong response of spin number 32 happens for the very good reason that the coupling between 0 and 32 is strong.

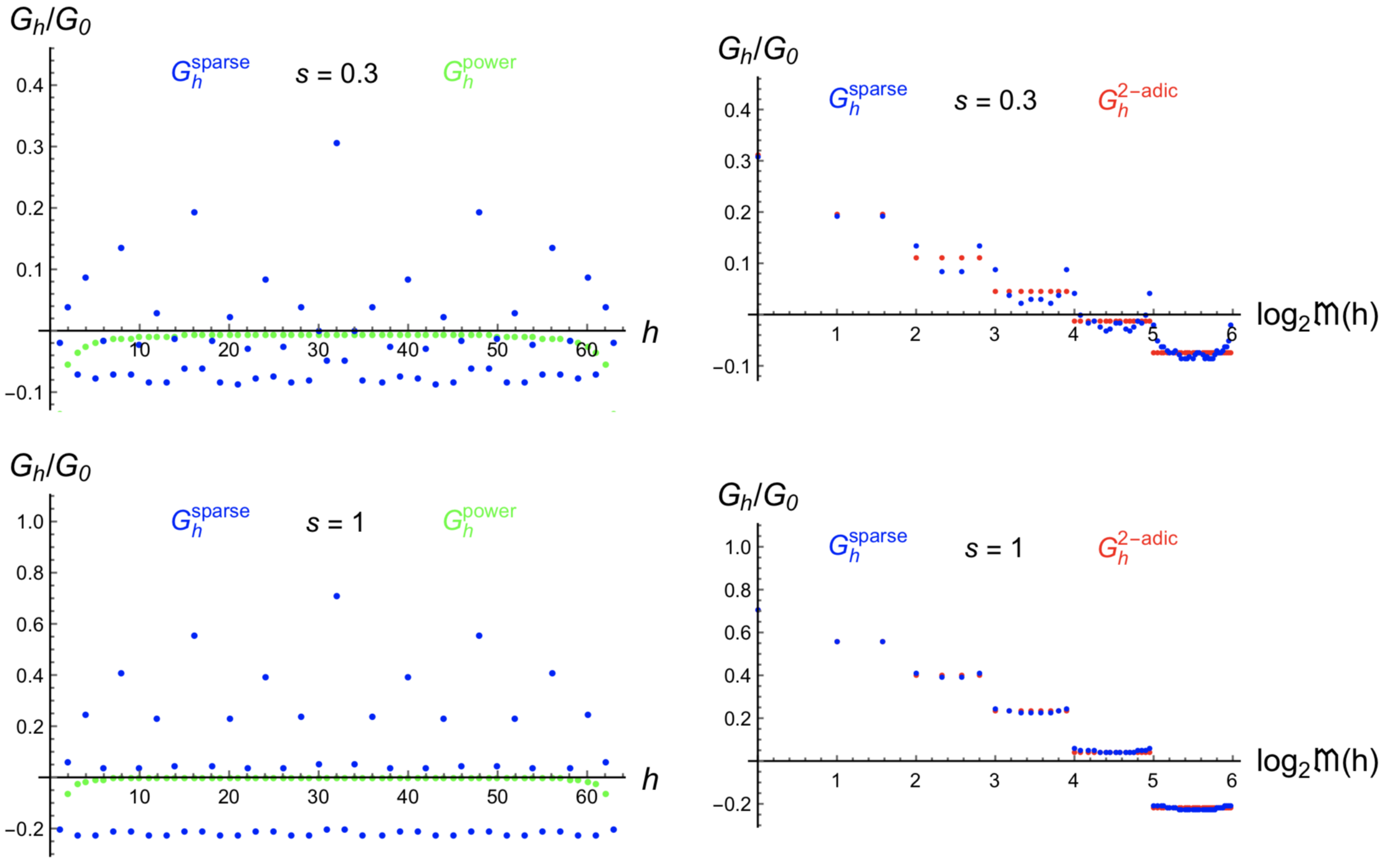
We see from the the $s = 1$ plots almost degenerate values. Let's see what happens when we pass h through the Monna map!

Results for 64 spins. Archimedean side, $s < 0$:



Organizing G_h according to the 2-adics is as useless for $s < 0$ as using the reals is for $s > 0$.

Results for 64 spins. Ultrametric side, $s > 0$:



But for $s > 0$, the 2-adic couplings neatly capture most of what's going on in the sparse coupling model.

6. Back to p-adic field theory

- By increasing density of points (i.e. $\ell \to \infty$), we can pass to a field theory over $\mathbb{Z}_p$.
- If typical correlation lengths are much less than the system size, then we can ignore the finite size effects that distinguish between $\mathbb{Z}_p$ and $\mathbb{Q}_p$.
- In short, remove UV and IR cutoffs to get free but non-local scalar field theory over $\mathbb{Q}_p$. Focus on $p = 2$ for simplicity.

$$S = -\int_{\mathbb{Q}_2} dx dy \, \frac{1}{2} \phi(x) J(x-y) \phi(y)$$

where

$$J(x) = J_* \sum_{n \in \mathbb{Z}} 2^{ns} \left[\delta(x - 2^n) + \delta(x + 2^n) - 2\delta(x) \right] \,,$$

Setting $J_* = 1/4$ for convenience,

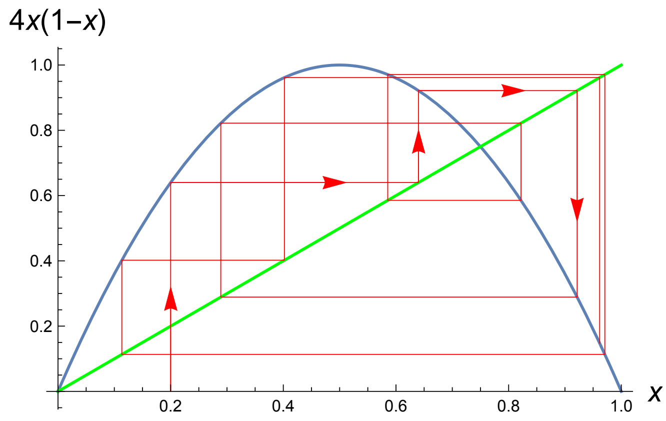
$$\tilde{J}(k) = -\frac{1}{\tilde{G}(k)} = \sum_{n \in \mathbb{Z}} 2^{ns} \frac{\chi(2^n k) + \chi(-2^n k) - 2}{4} = -\sum_{n < -v_2(k)} 2^{ns} \sin^2(\pi \{2^n k\}) \,.$$

Our aim is to inquire how smooth or ragged $\tilde{G}(k)$ and $G(x)$ are.

But first...

It's worth noting that $x_n \equiv \sin^2(\pi\{2^n k\})$ for p-adic k is quite a special class of sequences.

- $x_n = 0$ for $n \geq -v_2(k)$ because then $2^n k \in \mathbb{Z}_2$, so $\{2^n k\} = 0$.
- x_n solves the integrable limiting case of the logistical map, $x \to 4x(1-x)$.
- Often one thinks of $x_n^{\mathbb{R}} \equiv \sin^2(\pi 2^n k)$ with $k \in \mathbb{R}$ as the general solution, but actually it captures only solutions that go to 0 as $n \to -\infty$.
- $x_n = \sin^2(\pi\{2^n k\})$ is a whole other class of solutions: those which lead to total extinction.
- The 2-adic norm $|k|_2 = 2^{-v_2}$ predicts the moment of extinction.

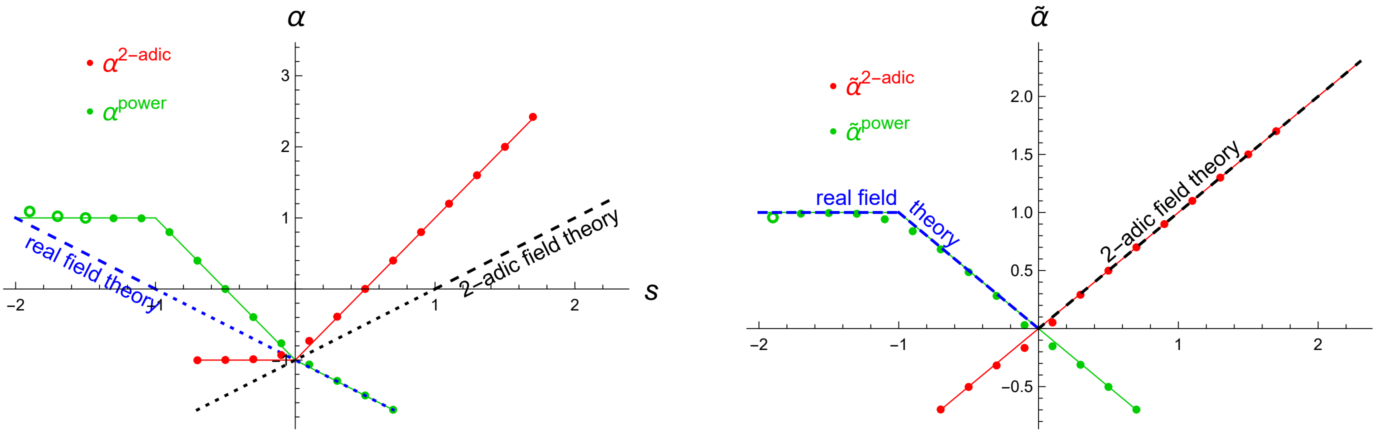


A function f is α-Hölder continuous on some domain O iff $\exists K$ such that

$$|f(x_1) - f(x_2)| < K|x_1 - x_2|^\alpha \,.$$

- For $f : \mathbb{R} \to \mathbb{R}$, the smoothest non-constant functions have $\alpha = 1$.
- For $f : \mathbb{Q}_p \to \mathbb{R}$, piecewise constant functions have $\alpha = \infty$ (!) provided the level sets are both open and closed.

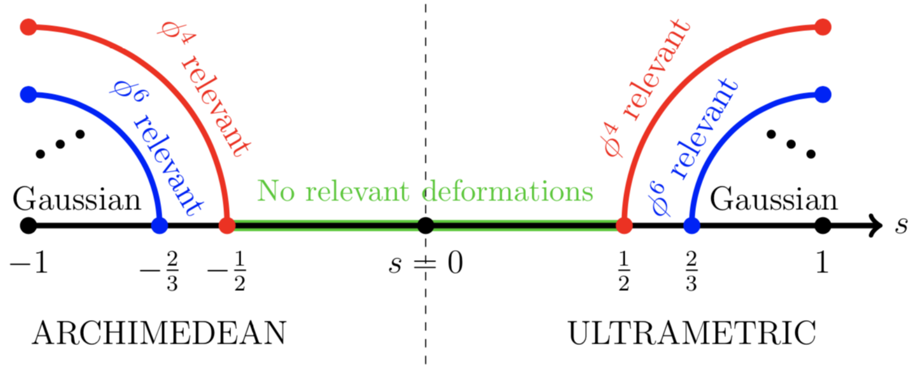
Starting from Fourier series for $\tilde{J}(k)$, we can establish lower bounds on α which match numerics for $\tilde{G}(k)$ and are clearly sub-optimal for $G(x)$.



The transition between Archimedean and 2-adic continuity is clearly at $s = 0$.

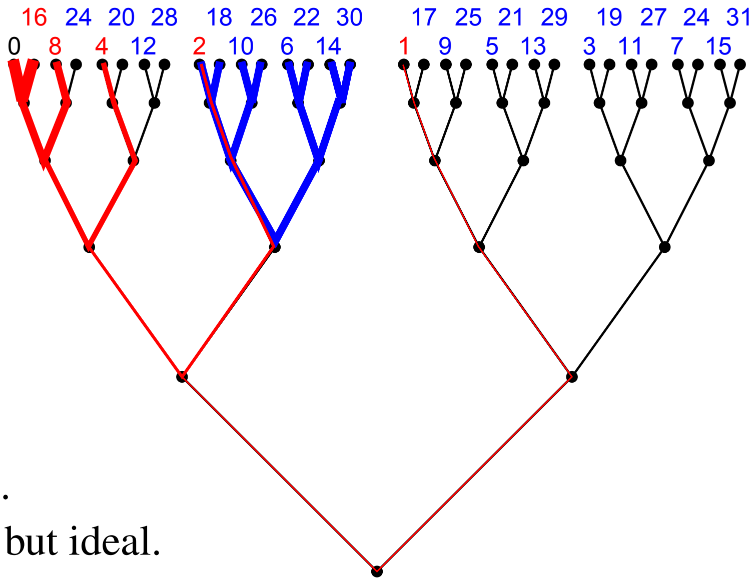
Free sparsely coupled bosonic field theory should be just the beginning!

Power counting suggests a picture as follows:



- Explicit calculations, e.g. perturbative expansion in $\epsilon = s - 1/2$, might give evidence for the beginning of a Wilson-Fisher branch.
- Sparsely coupled Ising Monte Carlo simulations could show anomalous scaling for theories further out on the WF branch.

7. Conclusions



- J_h^{sparse} and $J_h^{2-\text{adic}}$ lead to nearly the same dynamics at large s because couplings are strongly hierarchical.
- Coupling to just one spin in a tightly bound cluster is nearly the same as coupling to them all.
- Geometry emerges from interactions.
  I. Kant: Space and Time are not real but ideal.
- Is there some sort of quantum criticality at $s = 0$?
- Conjecture (from listening to M. Schleier-Smith & G. Bentsen): $s = 0$ sparse coupling provides the most efficient possible quantum scrambler.
- Entanglement and dynamical correlations are probably clearer from holographic perspective, where locality may be more manifest.
- Cold atoms promise to probe an ever-widening range of physical regimes.
  Add p-adic CFT to the list!