From $p$-adic AdS/CFT to prospects in cold atoms

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1. Overview

- \( p \)-adic AdS/CFT

[Gubser-Knaute-Parikh-Samberg-Witaszczyk '16, Heydeman-Marcolli-Saberia-Stoica '16] relates field theory dynamics over \( \mathbb{Q}_p \) to bulk dynamics on a regular tree \( T_p \).

- \( T_p = \frac{p\text{-adic conformal group}}{\text{max compact subgroup}} \) is naturally discrete. Easier starting point for quantum gravity?

- Mostly today I will focus on field theory side.

- Long-range couplings among atoms in an optical lattice can be individually tuned.

- Depending on how we dial the couplings, atoms can approximate a real continuum or a \( p \)-adic continuum.

Discussions with G. Bentsen, and E. Davis and M. Schleier-Smith, cf. [Hung-Gonzalez-Tudela-Cirac-Kimble '16].
2. What are $p$-adic numbers?

Write $h \in \mathbb{Z}$ as

$$h = \pm 2^{v_2} 3^{v_3} 5^{v_5} \ldots$$

Then for any choice of prime $p$,

$$|h|_p = p^{-v_p}.$$ 

$|\cdot|_p$ is the $p$-adic norm.

For $x = a/b \in \mathbb{Q}$, define

$$|x|_p = \frac{|a|_p}{|b|_p}.$$ 

Also define $|0|_p = 0$.

The $p$-adic numbers $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ wrt the norm $|\cdot|_p$.

The $p$-adic integers $\mathbb{Z}_p$ are the unit ball in $\mathbb{Q}_p$, i.e. $\{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. 

Example:

$$2^5 \times 3^2 \times 7^1 = 2016$$
Any $x \in \mathbb{Q}_p \setminus \{0\}$ has a unique base $p$ expansion:

$$x = a_3 a_2 a_1 a_0 \cdots a_{-1} a_{-2} \cdots a_{v_p} \equiv \sum_{n=v_p}^{\infty} a_n p^n$$

where $a_n \in \{0, 1, 2, \ldots, p-1\}$ and $a_{v_p} \neq 0$.

$\mathbb{R}$ is called Archimedean because if $|a| > |b| > 0$, $\exists n \in \mathbb{Z}$ so that $|nb| > |a|$.

The $\mathbb{Q}_p$ have instead the ultrametric property: $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

$\mathbb{R}$ and the $\mathbb{Q}_p$ are the only “good” completions of $\mathbb{Q}$ (Ostrowski’s theorem).
Example for $p = 2$:

$$x = \ldots 11111 = \sum_{n=0}^{\infty} 2^n = \frac{1}{1 - 2} = \frac{1}{-1} = -1.$$ 

Arithmetic operations for the $2$-adics works just like you expect in base 2:

$$\begin{align*}
-1 + 1 &= 0 \quad \text{in } \mathbb{Q}_2 : \\
\ldots 111111111111111111111 + 1 &= 1 \\
\ldots 00000 &= 0
\end{align*}$$

Crucial: We carry to the left. So $\mathbb{Q}_2$ is not the same as just writing real numbers backward in base 2.
3. Hierarchical models

Consider the furthest neighbor 1-d Ising model, aka the Dyson hierarchical model:

\[ H = -J_* \sum_{m,n} \frac{\sigma_m \sigma_n}{|m - n|^{s+1}_2} \]

where each \( \sigma_m = \pm 1 \).

![Diagram of hierarchical model](image)

After using the digit-reversing Monna map, far apart spins are close and vice versa.

The \( 2 \)-adic norm formalizes this alternative notion of closeness: \( |i - j|_2 = 2^{-d(i,j)/2} \) where \( d(i, j) \) is distance on the tree.

The tree appears to badly break translation invariance, but this is an illusion: \( h \rightarrow h + 1 \) preserves the tree structure.
Dyson ’69: Furthest neighbor Ising model has a finite temperature phase transition. Missarov-Lerner ’89 (also Bleher-Sinai ’75 and others) showed that the critical theory at the transition is characterized by $\phi^4$ theory over $\mathbb{Q}_2$:

$$S = -\int_{\mathbb{Q}_2 \times \mathbb{Q}_2} dx \, dy \frac{1}{2} \frac{\phi(x)\phi(y)}{|x-y|^{s+1}_2} + \int_{\mathbb{Q}_2} dx \left[ \frac{r}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

where $\phi : \mathbb{Q}_2 \to \mathbb{R}$ and $s$ is a parameter.

Ordinary conformal invariance is *not* realized, but $PGL(2, \mathbb{Q}_2)$ is.

- $z \to \frac{az+b}{cz+d}$ with $a, b, c, d, z \in \mathbb{Q}_2$.
- These LFTs map spin clusters to spin clusters, but sometimes changing the size of the clusters by a power of 2.

$$\langle \mathcal{O}(z)\mathcal{O}(0) \rangle \propto \frac{1}{|z|^{2\Delta}}$$

similar to usual CFTs.
4. Ingredients for field theory over $\mathbb{Q}_p$

- We have length but not direction: $\mathbb{Q}_p$ isn’t naturally ordered.

- We have measure: $\text{Vol}(p^v\mathbb{Z}_p) \equiv p^{-v}$, and $\text{Vol}$ is invariant under translations.

- We have integration à la Lebesgue following from $\text{Vol}$.

- We have translation invariance and plane waves: $\chi(kx) = e^{2\pi i \{kx\}}$.

- We have Fourier transforms and a non-local version of derivatives:

$$\phi(x) = \int_{\mathbb{Q}_p} dk \, \chi(kx) \tilde{\phi}(k) \quad \text{(Note } \phi : \mathbb{Q}_p \to \mathbb{R})$$

$$D^s \phi(x) = \int_{\mathbb{Q}_p} dk \, \chi(kx) |k|^s \tilde{\phi}(k) = \frac{1}{\Gamma_p(-s)} \int_{\mathbb{Q}_p} dy \, \frac{\phi(y) - \phi(x)}{|y-x|_p^{s+1}}.$$

- The sum of any number of soft momenta is still soft: the “ultra-metric” property.

$$\text{If all } |k_i|_p \leq \Lambda, \quad \text{then} \quad \left| \sum_i k_i \right|_p \leq \Lambda.$$

- We can have scale-invariance, but it’s discrete in steps of $k \to pk$. 

5. Approximating \( p \)-adic interactions in a physical system

Cold atom systems allow unprecedented control over the Hamiltonian. E.g. non-local interactions for spin chains with spin \( 1/2 \) on each site:

\[
\hat{H} = - \sum_{m,n} J_{mn} \vec{\sigma}_m \cdot \vec{\sigma}_n
\]

and variants, e.g. TIM and XY

Recent proposals, incl. [Hung et al 2016] and ongoing work in M. Schleier-Smith’s lab, naturally allows for translationally invariant \( J_{mn} = J_{m-n} \) (up to endpoint effects).

Details of AMO at the level I can explain:

- Each site in a 1d optical lattice contains (ideally) one atom.

- Two low-lying states (e.g. with hyperfine splitting) provide single-atom pseudo-spin Hilbert space.
• $|\downarrow\rangle \leftrightarrow |e\rangle$ is coupled to photons we pump into the cavity.

• $|\uparrow\rangle \leftrightarrow |e\rangle$ is coupled to cavity photons that propagate across the whole sample.

Applied magnetic field shifts the hyperfine states unequally, so a linear gradient in $B$ provides a site-dependent shift in energy, say in $|\downarrow\rangle$ but not $|\uparrow\rangle$.

Energy difference between low-lying states at site $m$ is

$$ (\Delta E)_m \equiv (E_{\uparrow})_m - (E_{\downarrow})_m = (m - m_0)\epsilon $$

with $\epsilon$ fixed.
The $\sigma^+_m \sigma^-_n$ interaction mediates $|\downarrow\uparrow\rangle_{mn} \rightarrow |\uparrow\downarrow\rangle_{mn}$ and proceeds in four steps:

1. $|\downarrow\rangle_m |e\rangle_m \rightarrow |\uparrow\rangle_m |e\rangle_m$ with transition frequency $\omega_\alpha$.
2. Virtual cavity photon propagates from $|\uparrow\rangle_m$ to $|\uparrow\rangle_n$.
3. $|\uparrow\rangle_n |e\rangle_n \rightarrow |\downarrow\rangle_n |e\rangle_n$ with transition frequency $\omega_\beta$.
4. Translationally invariant condition: $(m - n)\epsilon = (\Delta E)_m - (\Delta E)_n = \omega_\beta - \omega_\alpha$.

For $|\downarrow\uparrow\rangle_{mn} \rightarrow |\uparrow\downarrow\rangle_{mn}$ to proceed, we need the on-resonance condition

$$\left( m - n \right) \epsilon = \left( \Delta E \right)_m - \left( \Delta E \right)_n = \omega_\beta - \omega_\alpha$$

We must pump in both $\omega_\alpha$ photons and $\omega_\beta$ photons to get $|\downarrow\uparrow\rangle_{mn} \leftrightarrow |\uparrow\downarrow\rangle_{mn}$.

Current prospect is to realize pure hopping Hamiltonian: $\sigma^+_m \sigma^-_n + \sigma^-_m \sigma^+_n$. 
• Dissipative effects associated with each new pumped mode encourage selecting frequencies parsimoniously.

Schleier-Smith’s sparse coupling proposal: Say \( J_h = 0 \) unless \( h = 2^n \)

For example, with \( N = 2^\ell \) spins,

\[
J_h^{\text{sparse}} \equiv J_* \sum_{n=0}^{\ell-1} 2^{ns} (\delta_{h-2^n} + \delta_{h+2^n} - 2\delta_h).
\]

This is an approximation to

\[
J_h^{2\text{-adic}} \equiv J_* |h|_{2}^{-s-1} \quad \text{if} \ h \neq 0.
\]

At first blush these couplings do not seem very similar! A primary aim of the remainder of the talk is to see that actually they are when \( s > 0 \).
As we dial $s$ from $-\infty$ to $+\infty$ we interpolate between nearest neighbor couplings and 2-adic couplings.

$$J_{h}^{\text{sparse}} \equiv J_{\ast} \sum_{n=0}^{\ell-1} 2^{ns} \left( \delta_{h-2n} + \delta_{h+2n} - 2\delta_{h} \right)$$

$$J_{h}^{\text{NN}} \equiv J_{\ast} (\delta_{h+1} + \delta_{h-1} - 2\delta_{h})$$

$$J_{h}^{2\text{-adic}} \equiv J_{\ast} |h|_{2}^{-s-1}$$
Problem: It’s hard to get anywhere with quantum spin-$1/2$ Hamiltonians without numerics, absent some special trick like Jordan-Wigner.

Today’s solution: Simplify the model to a free boson on a lattice, still with non-local interactions, treated in classical stat mech:

\[
H \equiv -\frac{1}{2} \sum_{m,n} J_{m-n} \phi_m \phi_n - \sum_m b_m \phi_m
\]

\[
Z[b] \equiv \left( \prod_{m=0}^{2^{\ell}-1} \int_{-\infty}^{\infty} d\phi_m \right) \delta(\tilde{\phi}_0) e^{-\beta H}.
\]

Require $\tilde{J}_0 = 0$, while $\tilde{J}_k < 0$ for $k \neq 0$. In words:

- Interaction is ferromagnetic.
- Uniformly shifting all the $\phi_n$ is a massless mode.
- We explicitly fix that Goldstone-like mode with $\delta(\tilde{\phi}_0)$ inside the path integral.

The model is completely determined once we know the two-point function

\[
G_{mn} = \langle \phi_m \phi_n \rangle = \frac{1}{\beta^2 Z[0]} \left. \frac{\partial^2 Z[b]}{\partial b_m \partial b_n} \right|_{b=0}
\]
Just to get the idea, consider the most trivial nearest neighbor model:

\[
J_h^{\text{NN}} = J_*(\delta_{h+1} + \delta_{h-1} - 2\delta_h)
\]

\[
J_k^{\text{NN}} \equiv \frac{1}{\sqrt{N}} \sum_{h=0}^{N-1} e^{-2\pi i k h/N} J_h^{\text{NN}} = -\frac{4J_*}{\sqrt{N}} \sin^2\left(\frac{\pi k}{N}\right) \quad \text{where} \quad N = 2^\ell
\]

\[
\tilde{G}_k^{\text{NN}} = -\frac{1 - \delta_k}{N\beta \tilde{J}_k} \implies G_k^{\text{NN}} \approx \frac{N}{\beta J_*} G(h/L) \quad \text{where}
\]

\[
g(x) = \frac{1}{2} \left(x - \frac{1}{2}\right)^2 - \frac{1}{24}
\]

for \( x \in [0, 1) : \\
g''(x) = -\delta(x) + 1
\]

and \( \int_0^1 dx\ g(x) = 0 \).

As we start adding in sparse long-range couplings by dialing up \( s \), we’ll see this smooth Green’s function become less and less smooth.
Results for 64 spins. Archimedean side, $s < 0$:

To make the best comparison between the sparse coupling results and a smooth Green’s function, we introduce power-law couplings:

$$\tilde{J}_k^{\text{power}} \equiv - \frac{J_*}{2^s \sqrt{N}} \left[ \sin \left( \frac{\pi k}{N} \right) \right]^{-s}$$

$\tilde{J}_k^{\text{power}}$ is a lattice version of the familiar power-law couplings: $J_h^{\text{power}} \sim |h|_{\infty}^{s-1}$ for $|h|_{\infty}/N \ll 1$.

$G_h < 0$ for some $h$ seems wrong given ferromagnetic couplings. In fact, $\delta(\tilde{\phi}_0)$ enforces $\tilde{G}_0 = \frac{1}{\sqrt{N}} \sum_h G_h = 0$. 
Results for 64 spins. Ultrametric side, $s > 0$:

At this point it’s clear that there just isn’t a real continuum limit of the Green’s function.

The strong response of spin number 32 happens for the very good reason that the coupling between 0 and 32 is strong.

We see from the $s = 1$ plots almost degenerate values. Let’s see what happens when we pass $h$ through the Monna map!
Results for 64 spins. Archimedean side, $s < 0$:

Organizing $G_h$ according to the 2-adics is as useless for $s < 0$ as using the reals is for $s > 0$. 
Results for 64 spins. Ultrametric side, $s > 0$:

But for $s > 0$, the $2$-adic couplings neatly capture most of what’s going on in the sparse coupling model.
6. Back to $p$-adic field theory

- By increasing density of points (i.e. $\ell \to \infty$), we can pass to a field theory over $\mathbb{Z}_p$.

- If typical correlation lengths are much less than the system size, then we can ignore the finite size effects that distinguish between $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

- In short, remove UV and IR cutoffs to get free but non-local scalar field theory over $\mathbb{Q}_p$. Focus on $p = 2$ for simplicity.

$$S = - \int_{\mathbb{Q}_2} dx\, dy \, \frac{1}{2} \phi(x) J(x - y) \phi(y)$$

where

$$J(x) = J_\ast \sum_{n \in \mathbb{Z}} 2^{ns} \left[ \delta(x - 2^n) + \delta(x + 2^n) - 2 \delta(x) \right],$$

Setting $J_\ast = 1/4$ for convenience,

$$\tilde{J}(k) = -\frac{1}{\tilde{G}(k)} = \sum_{n \in \mathbb{Z}} 2^{ns} \chi(2^n k) + \chi(-2^n k) - 2$$

$$= -\sum_{n < -v_2(k)} 2^{ns} \sin^2(\pi \{2^n k\}).$$

Our aim is to inquire how smooth or ragged $\tilde{G}(k)$ and $G(x)$ are.
But first...

It’s worth noting that $x_n \equiv \sin^2(\pi \{2^n k\})$ for $p$-adic $k$ is quite a special class of sequences.

- $x_n = 0$ for $n \geq -v_2(k)$ because then $2^n k \in \mathbb{Z}_2$, so $\{2^n k\} = 0$.
- $x_n$ solves the integrable limiting case of the logistical map, $x \to 4x(1-x)$.
- Often one thinks of $x_n^\mathbb{R} \equiv \sin^2(\pi 2^n k)$ with $k \in \mathbb{R}$ as the general solution, but actually it captures only solutions that go to 0 as $n \to -\infty$.
- $x_n = \sin^2(\pi \{2^n k\})$ is a whole other class of solutions: those which lead to total extinction.
- The $2$-adic norm $|k|_2 = 2^{-v_2}$ predicts the moment of extinction.
A function $f$ is $\alpha$-Hölder continuous on some domain $O$ iff $\exists K$ such that

$$|f(x_1) - f(x_2)| < K|x_1 - x_2|^{\alpha}.$$ 

- For $f : \mathbb{R} \rightarrow \mathbb{R}$, the smoothest non-constant functions have $\alpha = 1$.
- For $f : \mathbb{Q}_p \rightarrow \mathbb{R}$, piecewise constant functions have $\alpha = \infty$ (!) provided the level sets are both open and closed.

Starting from Fourier series for $\tilde{J}(k)$, we can establish lower bounds on $\alpha$ which match numerics for $\tilde{G}(k)$ and are clearly sub-optimal for $G(x)$.

The transition between Archimedean and 2-adic continuity is clearly at $s = 0$. 
Free sparsely coupled bosonic field theory should be just the beginning!

Power counting suggests a picture as follows:

- Explicit calculations, e.g. perturbative expansion in $\epsilon = s - 1/2$, might give evidence for the beginning of a Wilson-Fisher branch.
- Sparsely coupled Ising Monte Carlo simulations could show anomalous scaling for theories further out on the WF branch.
7. Conclusions

- $J^{\text{sparse}}_h$ and $J^{2-\text{adic}}_h$ lead to nearly the same dynamics at large $s$ because couplings are strongly hierarchical.

- Coupling to just one spin in a tightly bound cluster is nearly the same as coupling to them all.

- Geometry emerges from interactions.
  I. Kant: Space and Time are not real but ideal.

- Is there some sort of quantum criticality at $s = 0$?

- Conjecture (from listening to M. Schleier-Smith & G. Bentsen): $s = 0$ sparse coupling provides the most efficient possible quantum scrambler.

- Entanglement and dynamical correlations are probably clearer from holographic perspective, where locality may be more manifest.

- Cold atoms promise to probe an ever-widening range of physical regimes. Add $p$-adic CFT to the list!