Holographic Complexity 101

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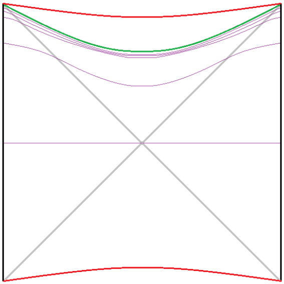
“Entanglement is not enough” (1411.0690)

Consider the thermofield double state as a realization of ER=EPR:

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_i e^{-\beta E_i/2} |i\rangle_L |\tilde{i}\rangle_R$$

- Black hole reaches thermal equilibrium quickly, $\sim t_{\mathrm{therm}}$
- Distance along maximal slices increases linearly with time



- $|\mathrm{TFD}\rangle$ continues to evolve for $\sim t_{\mathrm{comp}}$

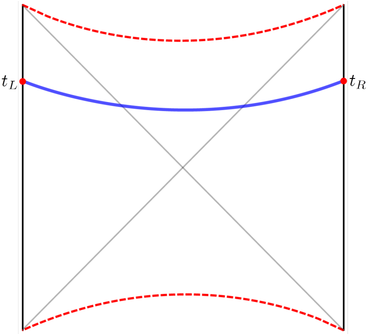
Holographic complexity

Susskind proposed "holographic complexity" as the CFT quantity that encodes the continued evolution of the ERB. [Susskind et al. '14, '16]

Two proposals for the bulk dual of complexity:

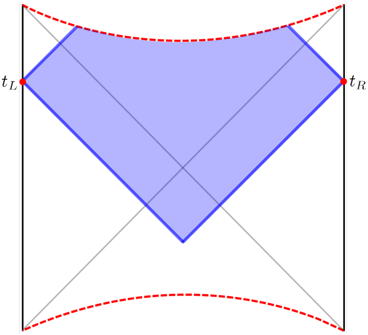
"complexity = volume" (CV)

$$\mathcal{C}_V\left(t_L, t_R\right) = \frac{V(t_L, t_R)}{Gl}$$



"complexity = action" (CA)

$$\mathcal{C}_A\left(t_L, t_R\right) = \frac{\mathcal{A}}{\pi\hbar}$$



Holographic complexity: bulk studies

- Structure of divergences in CV vs CA [Carmi, Myers, Rath '16; Reynolds, Ross '16; Bolognesi, Rabinovici, Roy '18]
- Time dependence [Carmi, Chapman, Marrochio, Myers, Sugishita '17]
- Shockwaves/quenches [Chapman, Marrochio, Myers '18 $\times 2$; Moosa '17; Ageev, Aref'eva, Bagrov, Katsnelson '18]
- Lloyd's bound [Cottrell, Montero '17]
- Complexity of formation [Chapman, Marrochio, Myers '16]
- Subregion/topological complexity [Abt, Erdmenger, Gerbershagen, Hinrichsen, Melby-Thompson, Meyer, Northe, Reyes '17,'18; Agón, Headrick, Swingle '18]
- Solitons, de Sitter [Reynolds, Ross '17 $\times 2$]

...and many more.

Computational (circuit) complexity

[Jefferson, Myers '17; Chapman, Heller, Marrochio, Pastawski '17]

- Goal: construct the optimum circuit for a given task
- Given a reference state $|\psi_0\rangle$, what is the least complex quantum circuit U that produces a given target state $|\psi_1\rangle$?

$$|\psi_1\rangle = U\,|\psi_0\rangle$$

- U consists of a sequence of gates Q_i: $U = Q_1 Q_2 \dots$
- *Circuit complexity* = length of circuit $\mathcal{D}(U)$
- *State complexity* $\mathcal{C}(\psi)$ = complexity of least complex circuit U that generates the state $|\psi\rangle$
- Defined relative to a reference state, $\mathcal{C}(\psi_0) \equiv 0$
- Depends on the set of gates, $\{Q_i\}$

A free field theory model

Consider a free scalar field as an infinite set of harmonic oscillators:

$$H = \frac{1}{2}\int \mathrm{d}^{d-1}x \left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$$
$$\to \frac{1}{2}\sum_{\vec{n}} \left\{ \frac{p(\vec{n})^2}{\delta^{d-1}} + \delta^{d-1}\left[\frac{1}{\delta^2}\sum_i \left(\phi(\vec{n}) - \phi(\vec{n}-\hat{x}_i)\right)^2 + m^2\phi(\vec{n})^2\right]\right\}$$

Simpler starting point: two oscillators at positions x_1, x_2,

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right]$$
$$= \frac{1}{2}\left(\tilde{p}_+^2 + \tilde{p}_-^2 + \tilde{\omega}_+^2\tilde{x}_+^2 + \tilde{\omega}_-^2\tilde{x}_-^2\right)$$

where $\omega = m$, $\Omega = 1/\delta$, $\tilde{x}_\pm = \frac{1}{\sqrt{2}}\left(x_1 \pm x_2\right)$, $\tilde{\omega}_+^2 = \omega^2$, $\tilde{\omega}_-^2 = \omega^2 + 2\Omega^2$.

Choosing our states

- Target state: ground state oscillators in normal-mode basis $x_\pm$

$$\psi_1(\tilde{x}_+, \tilde{x}_-) = \psi_1(\tilde{x}_+)\psi_1(\tilde{x}_-)$$
$$= \frac{(\tilde{\omega}_+\tilde{\omega}_-)^{1/4}}{\sqrt{\pi}} \exp\left[-\frac{1}{2}\left(\tilde{\omega}_+\tilde{x}_+^2 + \tilde{\omega}_-\tilde{x}_-^2\right)\right]$$

Equivalently, in physical coordinates x_1, x_2

$$\psi_1(x_1, x_2) = \frac{\left(\omega_1\omega_2 - \beta^2\right)^{1/4}}{\sqrt{\pi}} \exp\left[-\frac{\omega_1}{2}x_1^2 - \frac{\omega_2}{2}x_2^2 - \beta x_1 x_2\right]$$

where $\omega_1 = \omega_2 = \frac{1}{2}\left(\tilde{\omega}_+ + \tilde{\omega}_-\right)$, $\quad \beta \equiv \frac{1}{2}\left(\tilde{\omega}_+ - \tilde{\omega}_-\right) < 0$

- Natural reference state: factorized Gaussian

$$\psi_0(x_1, x_2) = \sqrt{\frac{\omega_0}{\pi}} \exp\left[-\frac{\omega_0}{2}\left(x_1^2 + x_2^2\right)\right]$$

Choosing our gates

Sufficient set of gates to produce ψ_1 from ψ_0:

$$Q_{ab} = e^{i\epsilon x_a p_b} \;, \qquad Q_{aa} = e^{\frac{i\epsilon}{2}(x_a p_a + p_a x_a)} = e^{\epsilon/2} e^{i\epsilon x_a p_a} \;.$$

These act on an arbitrary state $\psi\left(x_1, x_2\right)$ as follows:

$$Q_{21}\,\psi(x_1,x_2) = \psi(x_1 + \epsilon x_2, x_2) \qquad \text{shift } x_1 \text{ by } \epsilon x_2 \text{ (entangling)}$$
$$Q_{11}\,\psi(x_1,x_2) = e^{\epsilon/2}\psi\left(e^{\epsilon} x_1, x_2\right) \qquad \text{rescale } x_1 \text{ to } e^{\epsilon} x_1 \text{ (scaling)}$$

Gaussian states $=$ space of 2×2 matrices:

$$\left.\begin{array}{l} \psi_0(x_1,x_2) \simeq \exp\left[-\omega_0(x_1^2 + x_2^2)\right] \\ \psi_1(x_1,x_2) \simeq \exp\left[-\omega_1 x_1^2 - \omega_2 x_2^2 - 2\beta x_1 x_2\right] \end{array}\right\} \psi \simeq \exp[-x_i\, A_{ij}\, x_j]$$

Gates $Q_I = \exp[\epsilon\, M_I]$ act as $A' = Q_I\, A\, Q_I^T$, where $M_I \in \mathfrak{gl}(2,\mathbb{R})$

Geometrizing the problem [Nielsen '05; Nielsen et al. '06, '07]

Circuit U as path in $\mathrm{GL}(2,\mathbb{R})$

$$A_1 = U(1)\, A_0\, U^T(1)\ , \qquad U(s) = \overleftarrow{\mathcal{P}}\exp\left[\int_0^s \mathrm{d}s' Y^I(s') M_I\right]$$

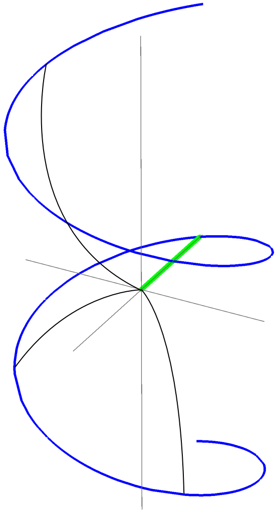
Parametrize $U \in \mathrm{GL}\,(2,\mathbb{R})$, components $\mathrm{d}Y^I = \mathrm{tr}(\mathrm{d}U U^{-1} M_I^T)$
$\to$ construct Euclidean geometry $\mathrm{d}s^2 = G_{IJ}\mathrm{d}Y^I\mathrm{d}Y^J$:

$$\begin{aligned}\mathrm{d}s^2 = {} & 2\mathrm{d}y^2 + 2\mathrm{d}\rho^2 + 2\cosh(2\rho)\cosh^2\rho\,\mathrm{d}\tau^2 \\ & + 2\cosh(2\rho)\sinh^2\rho\,\mathrm{d}\theta^2 - 2\sinh^2(2\rho)\,\mathrm{d}\tau\mathrm{d}\theta\end{aligned}$$

Circuit depth $\mathcal{D}(U)$ then becomes geometric length

$\Longrightarrow$ *optimum circuit (complexity) given by minimum geodesic*

Geodesics on circuit space



For our problem, minimal geodesic has

$$\tau(s) = 0\ ,\quad \Delta\theta = 0 \implies \theta(s) = \theta_1 = \pi$$

Minimum path given by

$$U(s) = \exp\left[\begin{pmatrix} y_1 & -\rho_1 \\ -\rho_1 & y_1 \end{pmatrix} s\right]$$

Complexity given by length of U:

$$\mathcal{C} = \mathcal{D}(U) = \int_0^1 \mathrm{d}s \sqrt{G_{IJ} Y^I(s) Y^J(s)}$$

$$= \sqrt{\sum\nolimits_I |Y^I(1)|^2} = \sqrt{2\left(\rho_1^2 + y_1^2\right)}$$

$$= \frac{1}{2}\sqrt{\ln^2\left(\frac{\tilde{\omega}_+}{\omega_0}\right) + \ln^2\left(\frac{\tilde{\omega}_-}{\omega_0}\right)}$$

Generalization to N oscillators

Reference and target states described by $N \times N$ matrices $\tilde{A}_0$, $\tilde{A}_1$

$$\psi_0\left(\tilde{x}_k\right) = \left(\frac{\omega_0}{\pi}\right)^{N/4} \exp\left[-\frac{1}{2}\tilde{x}^\dagger \tilde{A}_0 \tilde{x}\right], \qquad \tilde{A}_0 = \omega_0 \mathbb{1}$$

$$\psi_1\left(\tilde{x}_k\right) = \prod_{k=0}^{N-1} \left(\frac{\tilde{\omega}_k}{\pi}\right)^{1/2} \exp\left[-\frac{1}{2}\tilde{x}^\dagger \tilde{A}_1 \tilde{x}\right], \quad \tilde{A}_1 = \mathrm{diag}\left(\tilde{\omega}_0, \ldots, \tilde{\omega}_{N-1}\right)$$

Optimum circuit scales-up diagonal entries

$$\tilde{U}(s) = \exp\left[\tilde{Y}^{\tilde{I}} \tilde{M}_{\tilde{I}}\right], \quad \tilde{Y}^{\tilde{I}} \tilde{M}_{\tilde{I}} = \mathrm{diag}\left(\frac{1}{2}\ln\frac{\tilde{\omega}_0}{\omega_0}, \ldots, \frac{1}{2}\ln\frac{\tilde{\omega}_{N-1}}{\omega_0}\right)$$

Complexity for one-dimensional lattice of N oscillators:

$$\mathcal{C} = \sqrt{\sum\nolimits_{\tilde{I}} \left|\tilde{Y}^{\tilde{I}}(1)\right|^2} = \frac{1}{2}\sqrt{\sum_{k=0}^{N-1} \ln^2 \frac{\tilde{\omega}_k}{\omega_0}}$$

Return to field theory (continuum limit)

Leading order dominated by UV modes, $\tilde{\omega}_{\vec{k}} \sim 1/\delta \implies$

$$\mathcal{C} \approx \frac{N^{\frac{d-1}{2}}}{2} \ln\left(\frac{1}{\delta\omega_0}\right) \sim \left(\frac{V}{\delta^{d-1}}\right)^{1/2} \quad , \qquad V = N^{d-1}\delta^{d-1}$$

Compare with CV or CA proposals: $\mathcal{C}_{\text{holo}} \sim \frac{V}{\delta^{d-1}} \implies$ to connect with holography, Reimannian distance function a bad choice!

$$\tilde{\mathcal{D}}_\kappa = \int_0^1 \mathrm{d}s \sum_{\tilde{I}} \left|Y^{\tilde{I}}(s)\right|^\kappa \quad , \quad \kappa \in \mathbb{Z}_+$$

In particular, $\kappa = 1$:

$$\mathcal{C} \approx \frac{V}{\delta^{d-1}} \left|\ln \frac{1}{\delta\omega_0}\right|$$

$$\omega_0 = \begin{cases} \text{UV scale } e^{-\sigma}/\delta & \implies \quad \mathcal{C} \approx \sigma \frac{V}{\delta^{d-1}} \qquad \text{(CV)} \\ \text{IR scale } \frac{\alpha}{\ell_{\text{AdS}}} \ll \frac{1}{\delta} & \implies \quad \mathcal{C} \approx \frac{V}{\delta^{d-1}} \left|\ln \frac{\ell_{\text{AdS}}}{\alpha\delta}\right| \quad \text{(CA)} \end{cases}$$

Summary & outlook

TL;DR:

- Preliminary steps towards *defining* holographic complexity in field theory (goal: new entry in holographic dictionary)
- Circuit complexity: complexity of target state given by length of optimum circuit (constructed from fundamental gates)
- Geometrical approach: optimum circuit is minimum geodesic in circuit space

What's next?

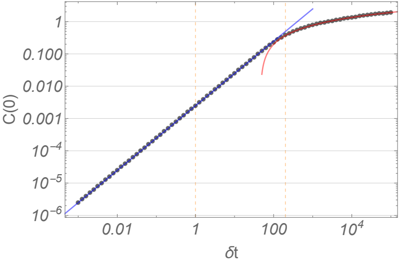
- Extension to free fermions [Hackl, Myers '18; Reynolds, Ross '17; Khan, Krishnan, Sharma '18]
- Extension to coherent states [Guo, Hernandez, Myers, Ruan '18]
- Preliminary Virasoro algebra [Caputa, Magan '18]
- Alternative approaches [Caputa et al. '17, '18; Czech '17; Hashimoto et al. '17; Yang et al. '18]

Applications/advertisement

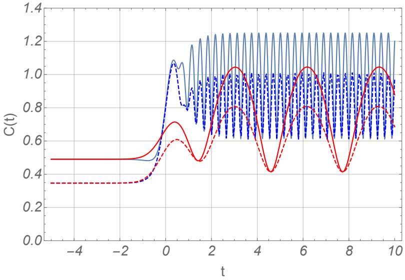
Question: Complexity, huh? What is it good for?
Answer: ~~Absolutely nothing~~ Probing dynamical quantum systems!

Applied to quantum quenches in 1807.07075
[Camargo, Caputa, Das, Heller, Jefferson '18; Alves, Camilo '18]

Universal scalings (complementary probe to entanglement)



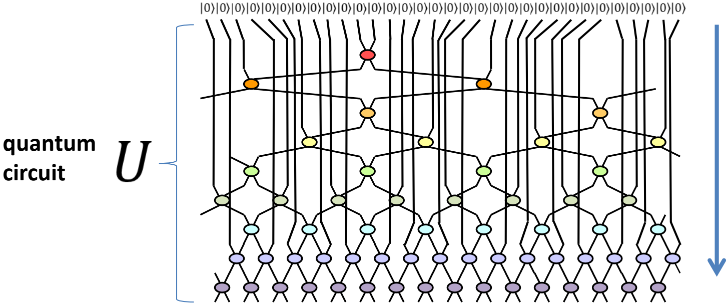
Complexity of subregions is *superadditive*: $\mathcal{C}_A + \mathcal{C}_{\bar{A}} \geq \mathcal{C}$



Applied to TFD in [Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers '18 (?)]

MERA: a quantum circuit

Connections to MERA (Mutli-scale Entanglement Renormalization Ansatz – efficiently generates ground-state wavefunction in $d = 2$ critical systems) [Vidal '15]



cMERA describes the ground state of a free scalar field