Holographic Complexity 101

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Consider the thermofield double state as a realization of ER=EPR:

\[
|\text{TFD}\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_i e^{-\beta E_i/2} |i\rangle_L |\tilde{i}\rangle_R
\]

- Black hole reaches thermal equilibrium quickly, \( \sim t_{\text{therm}} \)
- Distance along maximal slices increases linearly with time

Figure 9: Two-sided ADS black hole foliated by maximal slices.

At late time almost all of the ERB volume is very close to the final slice. Only the portion near the ends deviates from \( r_f \). Over most of the length the cross-sectional area of the

\[ |\text{TFD}\rangle \text{ continues to evolve for } \sim t_{\text{comp}} \]
Holographic complexity

Susskind proposed “holographic complexity” as the CFT quantity that encodes the continued evolution of the ERB. [Susskind et al. ’14, ’16]

Two proposals for the bulk dual of complexity:

“complexity = volume” (CV)

\[ C_V(t_L, t_R) = \frac{V(t_L, t_R)}{G_l} \]

“complexity = action” (CA)

\[ C_A(t_L, t_R) = \frac{A}{\pi \hbar} \]
Holographic complexity: bulk studies

- **Structure of divergences in CV vs CA** [Carmi, Myers, Rath '16; Reynolds, Ross '16; Bolognesi, Rabinovici, Roy '18]
- **Time dependence** [Carmi, Chapman, Marrochio, Myers, Sugishita '17]
- **Shockwaves/quenches** [Chapman, Marrochio, Myers '18 × 2; Moosa '17; Ageev, Aref’eva, Bagrov, Katsnelson '18]
- **Lloyd’s bound** [Cottrell, Montero '17]
- **Complexity of formation** [Chapman, Marrochio, Myers '16]
- **Subregion/topological complexity** [Abt, Erdmenger, Gerbershagen, Hinrichsen, Melby-Thompson, Meyer, Northe, Reyes '17,'18; Agón, Headrick, Swingle '18]
- **Solitons, de Sitter** [Reynolds, Ross '17 × 2]

...and many more.
Computational (circuit) complexity

[Jefferson, Myers '17; Chapman, Heller, Marrochio, Pastawski '17]

- **Goal**: construct the optimum circuit for a given task
- Given a reference state $|\psi_0\rangle$, what is the least complex quantum circuit $U$ that produces a given target state $|\psi_1\rangle$?

  $$|\psi_1\rangle = U |\psi_0\rangle$$

- $U$ consists of a sequence of gates $Q_i$: $U = Q_1 Q_2 \ldots$
- **Circuit complexity** = length of circuit $D(U)$
- **State complexity** $C(\psi) = $ complexity of least complex circuit $U$ that generates the state $|\psi\rangle$
- Defined relative to a reference state, $C(\psi_0) \equiv 0$
- Depends on the set of gates, $\{Q_i\}$
A free field theory model

Consider a free scalar field as an infinite set of harmonic oscillators:

\[
H = \frac{1}{2} \int d^{d-1}x \left[ \pi(x)^2 + \vec{\nabla} \phi(x)^2 + m^2 \phi(x)^2 \right]
\]

\[
\rightarrow \frac{1}{2} \sum_{\vec{n}} \left\{ \frac{p(\vec{n})^2}{\delta^{d-1}} + \delta^{d-1} \left[ \frac{1}{\delta^2} \sum_i (\phi(\vec{n}) - \phi(\vec{n} - \hat{x}_i))^2 + m^2 \phi(\vec{n})^2 \right] \right\}
\]

Simpler starting point: two oscillators at positions \(x_1, x_2\),

\[
H = \frac{1}{2} \left[ p_1^2 + p_2^2 + \omega^2 (x_1^2 + x_2^2) + \Omega^2 (x_1 - x_2)^2 \right]
\]

\[
= \frac{1}{2} \left( \tilde{p}_+^2 + \tilde{p}_-^2 + \tilde{\omega}_+^2 \tilde{x}_+^2 + \tilde{\omega}_-^2 \tilde{x}_-^2 \right)
\]

where \(\omega = m\), \(\Omega = 1/\delta\), \(\tilde{x}_\pm = \frac{1}{\sqrt{2}} (x_1 \pm x_2)\), \(\tilde{\omega}_+^2 = \omega^2\), \(\tilde{\omega}_-^2 = \omega^2 + 2\Omega^2\).
Choosing our states

- Target state: ground state oscillators in normal-mode basis \( x_\pm \)
  \[
  \psi_1(\tilde{x}_+, \tilde{x}_-) = \psi_1(\tilde{x}_+) \psi_1(\tilde{x}_-) = \frac{({\tilde{\omega}_+} + {\tilde{\omega}_-})^{1/4}}{\sqrt{\pi}} \exp \left[ -\frac{1}{2} \left( {\tilde{\omega}_+}{\tilde{x}_+}^2 + {\tilde{\omega}_-}{\tilde{x}_-}^2 \right) \right]
  \]
  Equivalently, in physical coordinates \( x_1, x_2 \)
  \[
  \psi_1(x_1, x_2) = \frac{(\omega_1\omega_2 - \beta^2)^{1/4}}{\sqrt{\pi}} \exp \left[ -\frac{\omega_1}{2} x_1^2 - \frac{\omega_2}{2} x_2^2 - \beta x_1 x_2 \right]
  \]
  where \( \omega_1 = \omega_2 = \frac{1}{2} (\tilde{\omega}_+ + \tilde{\omega}_-) \), \( \beta \equiv \frac{1}{2} (\tilde{\omega}_+ - \tilde{\omega}_-) < 0 \)

- Natural reference state: factorized Gaussian
  \[
  \psi_0(x_1, x_2) = \sqrt{\frac{\omega_0}{\pi}} \exp \left[ -\frac{\omega_0}{2} \left( x_1^2 + x_2^2 \right) \right]
  \]
Choosing our gates

Sufficient set of gates to produce $\psi_1$ from $\psi_0$:

$$Q_{ab} = e^{i\epsilon x_ap_b}, \quad Q_{aa} = e^{i\epsilon (x_a p_a + p_a x_a)} = e^{\epsilon/2} e^{i\epsilon x_a p_a}.$$ 

These act on an arbitrary state $\psi(x_1, x_2)$ as follows:

$$Q_{21} \psi(x_1, x_2) = \psi(x_1 + \epsilon x_2, x_2) \quad \text{shift } x_1 \text{ by } \epsilon x_2 \text{ (entangling)}$$
$$Q_{11} \psi(x_1, x_2) = e^{\epsilon/2} \psi(e^\epsilon x_1, x_2) \quad \text{rescale } x_1 \text{ to } e^\epsilon x_1 \text{ (scaling)}$$

Gaussian states = space of $2 \times 2$ matrices:

$$\psi_0(x_1, x_2) \simeq \exp[-\omega_0(x_1^2 + x_2^2)]$$
$$\psi_1(x_1, x_2) \simeq \exp[-\omega_1 x_1^2 - \omega_2 x_2^2 - 2\beta x_1 x_2]$$

$$\left\{ \psi \simeq \exp[-x_i A_{ij} x_j] \right\}$$

Gates $Q_I = \exp[\epsilon M_I]$ act as $A' = Q_I A Q_I^T$, where $M_I \in \mathfrak{gl}(2, \mathbb{R})$. 
Circuit $U$ as path in $\text{GL}(2, \mathbb{R})$

$$A_1 = U(1) A_0 U^T(1), \quad U(s) = \mathcal{P} \exp \left[ \int_0^s ds' Y^I(s') M_I \right]$$

Parametrize $U \in \text{GL}(2, \mathbb{R})$, components $dY^I = \text{tr}(dUU^{-1} M^T_I)$

$\rightarrow$ construct Euclidean geometry $ds^2 = G_{IJ} dY^I dY^J$:

$$ds^2 = 2dy^2 + 2d\rho^2 + 2 \cosh(2\rho) \cosh^2 \rho \, d\tau^2 + 2 \cosh(2\rho) \sinh^2 \rho \, d\theta^2 - 2 \sinh^2(2\rho) \, d\tau d\theta$$

Circuit depth $D(U)$ then becomes geometric length

$\Rightarrow$ optimum circuit (complexity) given by minimum geodesic
Geodesics on circuit space

For our problem, minimal geodesic has

$$\tau(s) = 0, \quad \Delta \theta = 0 \implies \theta(s) = \theta_1 = \pi$$

Minimum path given by

$$U(s) = \exp \left[ \begin{pmatrix} y_1 & -\rho_1 \\ -\rho_1 & y_1 \end{pmatrix} s \right]$$

Complexity given by length of $U$:

$$C = D(U) = \int_0^1 ds \sqrt{G_{IJ}Y^I(s)Y^J(s)}$$

$$= \sqrt{\sum_I |Y^I(1)|^2} = \sqrt{2 \left( \rho_1^2 + y_1^2 \right)}$$

$$= \frac{1}{2} \sqrt{\ln^2 \left( \frac{\tilde{\omega}_+}{\omega_0} \right) + \ln^2 \left( \frac{\tilde{\omega}_-}{\omega_0} \right)}$$
Generalization to $N$ oscillators

Reference and target states described by $N \times N$ matrices $\tilde{A}_0$, $\tilde{A}_1$

$$\psi_0 (\tilde{x}_k) = \left( \frac{\omega_0}{\pi} \right)^{N/4} \exp \left[ -\frac{1}{2} \tilde{x}^\dagger \tilde{A}_0 \tilde{x} \right], \quad \tilde{A}_0 = \omega_0 1$$

$$\psi_1 (\tilde{x}_k) = \prod_{k=0}^{N-1} \left( \frac{\tilde{\omega}_k}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2} \tilde{x}^\dagger \tilde{A}_1 \tilde{x} \right], \quad \tilde{A}_1 = \text{diag} (\tilde{\omega}_0, \ldots, \tilde{\omega}_{N-1})$$

Optimum circuit scales-up diagonal entries

$$\tilde{U}(s) = \exp \left[ \tilde{Y} \tilde{I} \tilde{M} \right], \quad \tilde{Y} \tilde{I} \tilde{M} = \text{diag} \left( \frac{1}{2} \ln \frac{\tilde{\omega}_0}{\omega_0}, \ldots, \frac{1}{2} \ln \frac{\tilde{\omega}_{N-1}}{\omega_0} \right)$$

Complexity for one-dimensional lattice of $N$ oscillators:

$$C = \sqrt{\sum_{\tilde{l}} \left| \tilde{Y} \tilde{l} (1) \right|^2} = \frac{1}{2} \sqrt{\sum_{k=0}^{N-1} \ln^2 \frac{\tilde{\omega}_k}{\omega_0}}$$
Leading order dominated by UV modes, $\tilde{\omega}_k \sim 1/\delta \implies$

$$C \approx \frac{N^{d-1}}{2} \ln \left( \frac{1}{\delta \omega_0} \right) \sim \left( \frac{V}{\delta^{d-1}} \right)^{1/2}, \quad V = N^{d-1} \delta^{d-1}$$

Compare with CV or CA proposals: $C_{\text{holographic}} \sim \frac{V}{\delta^{d-1}} \implies$ to connect with holography, Reimannian distance function a bad choice!

$$\tilde{D}_\kappa = \int_0^1 ds \sum_{\tilde{I}} |Y_{\tilde{I}}(s)|^\kappa, \quad \kappa \in \mathbb{Z}_+$$

In particular, $\kappa = 1$:

$$C \approx \frac{V}{\delta^{d-1}} \left| \ln \frac{1}{\delta \omega_0} \right|$$

$$\omega_0 = \begin{cases} 
\text{UV scale } e^{-\sigma}/\delta & \implies C \approx \sigma \frac{V}{\delta^{d-1}} \quad \text{(CV)} \\
\text{IR scale } \frac{\alpha}{\ell_{\text{AdS}}} \ll \frac{1}{\delta} & \implies C \approx \frac{V}{\delta^{d-1}} \left| \ln \frac{\ell_{\text{AdS}}}{\alpha \delta} \right| \quad \text{(CA)}
\end{cases}$$
Summary & outlook

TL;DR:

- Preliminary steps towards *defining* holographic complexity in field theory (goal: new entry in holographic dictionary)
- Circuit complexity: complexity of target state given by length of optimum circuit (constructed from fundamental gates)
- Geometrical approach: optimum circuit is minimum geodesic in circuit space

What’s next?

- Extension to free fermions [Hackl, Myers '18; Reynolds, Ross '17; Khan, Krishnan, Sharma '18]
- Extension to coherent states [Guo, Hernandez, Myers, Ruan '18]
- Preliminary Virasoro algebra [Caputa, Magan '18]
- Alternative approaches [Caputa et al. '17, '18; Czech '17; Hashimoto et al. '17; Yang et al. '18]
Question: Complexity, huh? What is it good for?
Answer: Absolutely nothing! Probing dynamical quantum systems!

Applied to quantum quenches in 1807.07075
[Camargo, Caputa, Das, Heller, Jefferson '18; Alves, Camilo '18]

Universal scalings (complementary probe to entanglement)

Complexity of subregions is superadditive: $C_A + C_{\overline{A}} \geq C$

Applied to TFD in [Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers '18 (?)]
Connections to MERA (Mutli-scale Entanglement Renormalization Ansatz – efficiently generates ground-state wavefunction in $d = 2$ critical systems) [Vidal '15]

cMERA describes the ground state of a free scalar field