Holographic entropy relations

Massimiliano Rota
University of California, Santa Barbara

July 31st

Gauge/Gravity Duality 2018

Julius-Maximilians-University Würzburg

1808.xxxxx with V. Hubeny and M. Rangamani

Entanglement plays a very important role in the current understanding of holography, and more generally in the quest for quantum gravity.

Entanglement plays a very important role in the current understanding of holography, and more generally in the quest for quantum gravity.

Focus on **geometric states**: states of holographic CFTs dual to a classical geometry.

Entanglement plays a very important role in the current understanding of holography, and more generally in the quest for quantum gravity.

Focus on **geometric states**: states of holographic CFTs dual to a classical geometry.

For geometric states the von Neumann entropy of an arbitrary subsystem is computed by the **RT/HRT formula**.

Entanglement plays a very important role in the current understanding of holography, and more generally in the quest for quantum gravity.

Focus on **geometric states**: states of holographic CFTs dual to a classical geometry.

For geometric states the von Neumann entropy of an arbitrary subsystem is computed by the **RT/HRT formula**.

- What is the **entanglement structure** of geometric states?
- How can we characterize it?
- Does it somehow capture the fact that they are geometric?

There are universal constraints:

- subadditivity (SA)

$$S_{\mathcal{A}} + S_{\mathcal{B}} \geq S_{\mathcal{AB}}$$

- strong subadditivity (SSA)

$$S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{C}} + S_{\mathcal{ABC}}$$

There are universal constraints:

- subadditivity (SA)

$$S_{\mathcal{A}} + S_{\mathcal{B}} \geq S_{\mathcal{AB}}$$

- strong subadditivity (SSA)

$$S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{C}} + S_{\mathcal{ABC}}$$

But geometric states also satisfy **non-universal constraints**.
Example, **monogamy of mutual information** (MMI)

Hayden et al. 11'

$$S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{A}} + S_{\mathcal{B}} + S_{\mathcal{C}} + S_{\mathcal{ABC}}$$

There are universal constraints:

- subadditivity (SA)

$$S_{\mathcal{A}} + S_{\mathcal{B}} \geq S_{\mathcal{AB}} \qquad S_{\mathcal{A}} + S_{\mathcal{B}} - S_{\mathcal{AB}} \geq 0 \qquad I_2(\mathcal{A}:\mathcal{B}) \geq 0$$

- strong subadditivity (SSA)

$$S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{C}} + S_{\mathcal{ABC}} \qquad S_{\mathcal{AC}} + S_{\mathcal{BC}} - S_{\mathcal{C}} - S_{\mathcal{ABC}} \geq 0 \qquad I_2(\mathcal{A}:\mathcal{B}|\mathcal{C}) \geq 0$$

But geometric states also satisfy **non-universal constraints**.
Example, **monogamy of mutual information** (MMI)

Hayden et al. 11'

$$S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{A}} + S_{\mathcal{B}} + S_{\mathcal{C}} + S_{\mathcal{ABC}}$$

$$S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{BC}} - S_{\mathcal{A}} - S_{\mathcal{B}} - S_{\mathcal{C}} - S_{\mathcal{ABC}} \geq 0 \qquad -I_3(\mathcal{A}:\mathcal{B}:\mathcal{C}) \geq 0$$

There are universal constraints:

- subadditivity (SA)

$$S_{\mathcal{A}} + S_{\mathcal{B}} \geq S_{\mathcal{AB}} \qquad S_{\mathcal{A}} + S_{\mathcal{B}} - S_{\mathcal{AB}} \geq 0 \qquad I_2(\mathcal{A}:\mathcal{B}) \geq 0$$

- strong subadditivity (SSA)

$$S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{C}} + S_{\mathcal{ABC}} \qquad S_{\mathcal{AC}} + S_{\mathcal{BC}} - S_{\mathcal{C}} - S_{\mathcal{ABC}} \geq 0 \qquad I_2(\mathcal{A}:\mathcal{B}|\mathcal{C}) \geq 0$$

But geometric states also satisfy **non-universal constraints**.
Example, **monogamy of mutual information** (MMI)

Hayden et al. 11’

$$S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{BC}} \geq S_{\mathcal{A}} + S_{\mathcal{B}} + S_{\mathcal{C}} + S_{\mathcal{ABC}}$$

$$S_{\mathcal{AB}} + S_{\mathcal{AC}} + S_{\mathcal{BC}} - S_{\mathcal{A}} - S_{\mathcal{B}} - S_{\mathcal{C}} - S_{\mathcal{ABC}} \geq 0 \qquad -I_3(\mathcal{A}:\mathcal{B}:\mathcal{C}) \geq 0$$

Note that **SSA is implied by SA and MMI**

$$I_2(\mathcal{A}:\mathcal{B}) - I_3(\mathcal{A}:\mathcal{B}:\mathcal{C}) = I_2(\mathcal{A}:\mathcal{B}|\mathcal{C})$$

First systematic search of new constraints of this kind: **holographic entropy cone**

Bao et al. 15'

Main results:

- proof that MMI is the only constraint for three parties
- proof that there are no new constraints for four parties
- found four new constraints for five parties
- found an infinite family of constraints, one for any odd number of parties

First systematic search of new constraints of this kind: **holographic entropy cone**

Bao et al. 15'

Main results:

- proof that MMI is the only constraint for three parties
- proof that there are no new constraints for four parties
- found four new constraints for five parties
- found an infinite family of constraints, one for any odd number of parties

However:

- static set-up (RT) only
- the constraints were found via a computer search that does not provide any guidance for finding new ones
- it does not provide an interpretation

We would like to develop:

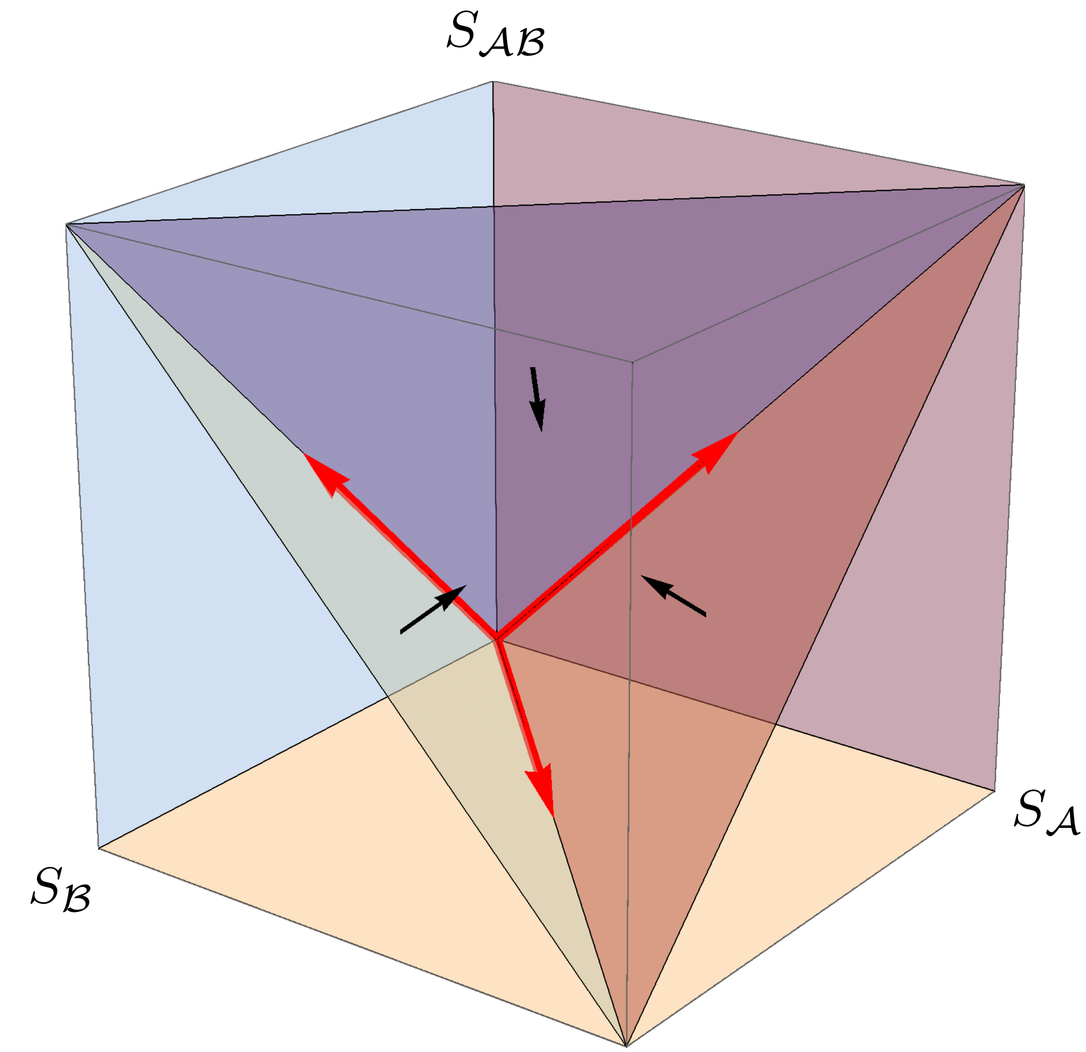
- a formulation also valid for dynamical cases (HRT)
- technique which generates candidates for new inequalities, for an arbitrary number of parties
- technique to prove inequalities from these candidates

Final goal: find an interpretation for these constraints and understand the implications for the entanglement structure of geometric states.

$\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$

Entropy vector $\vec{S}(\rho_{\mathcal{AB}}) = (S_{\mathcal{A}}, S_{\mathcal{B}}, S_{\mathcal{AB}})$

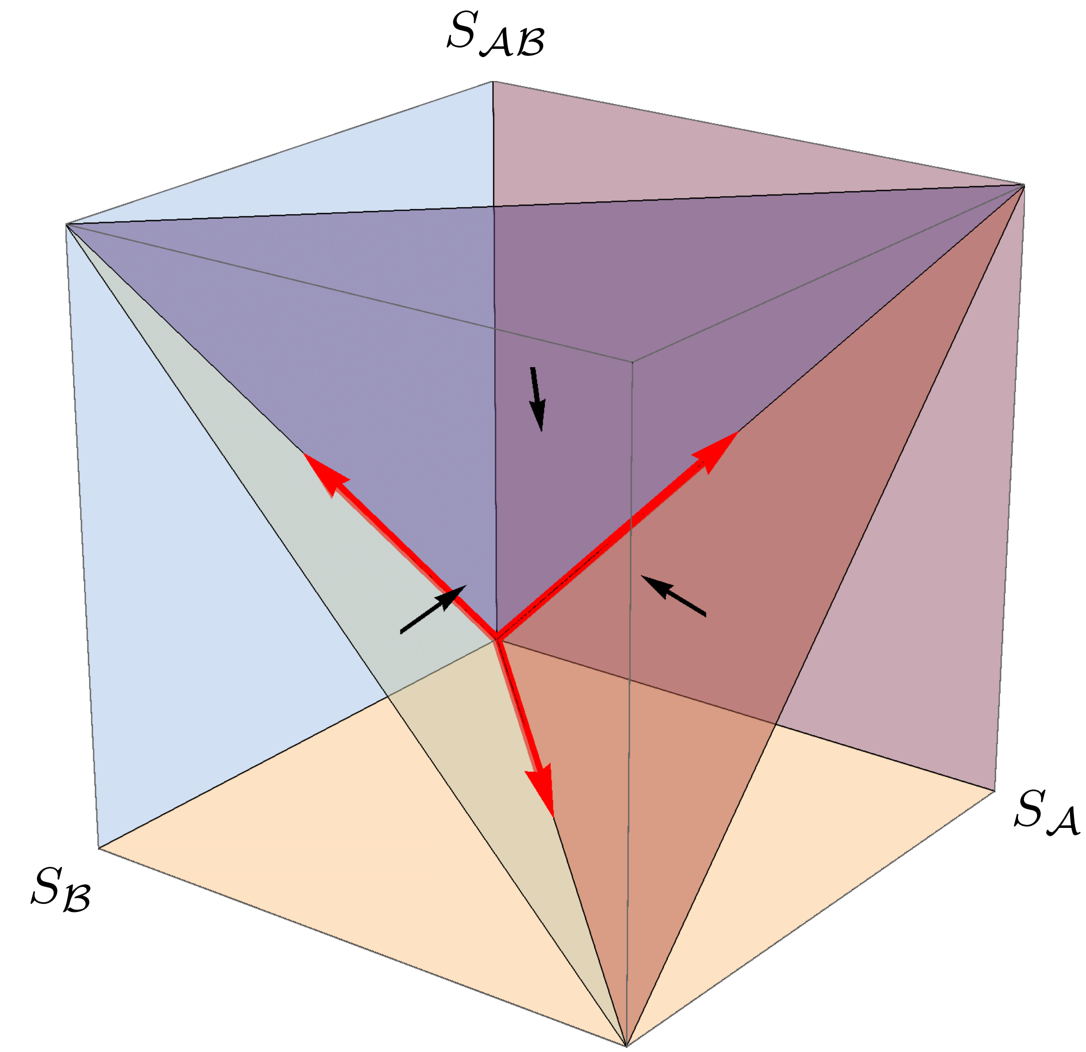
Entropy space $\mathbb{R}^3_+$



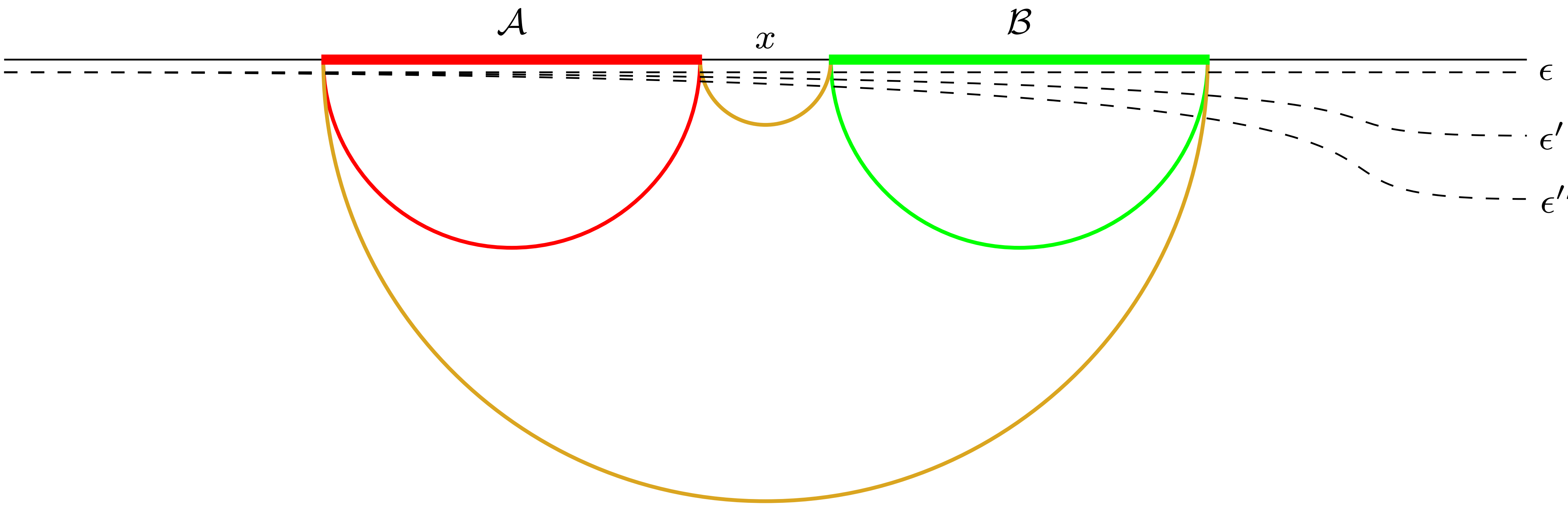
$\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$

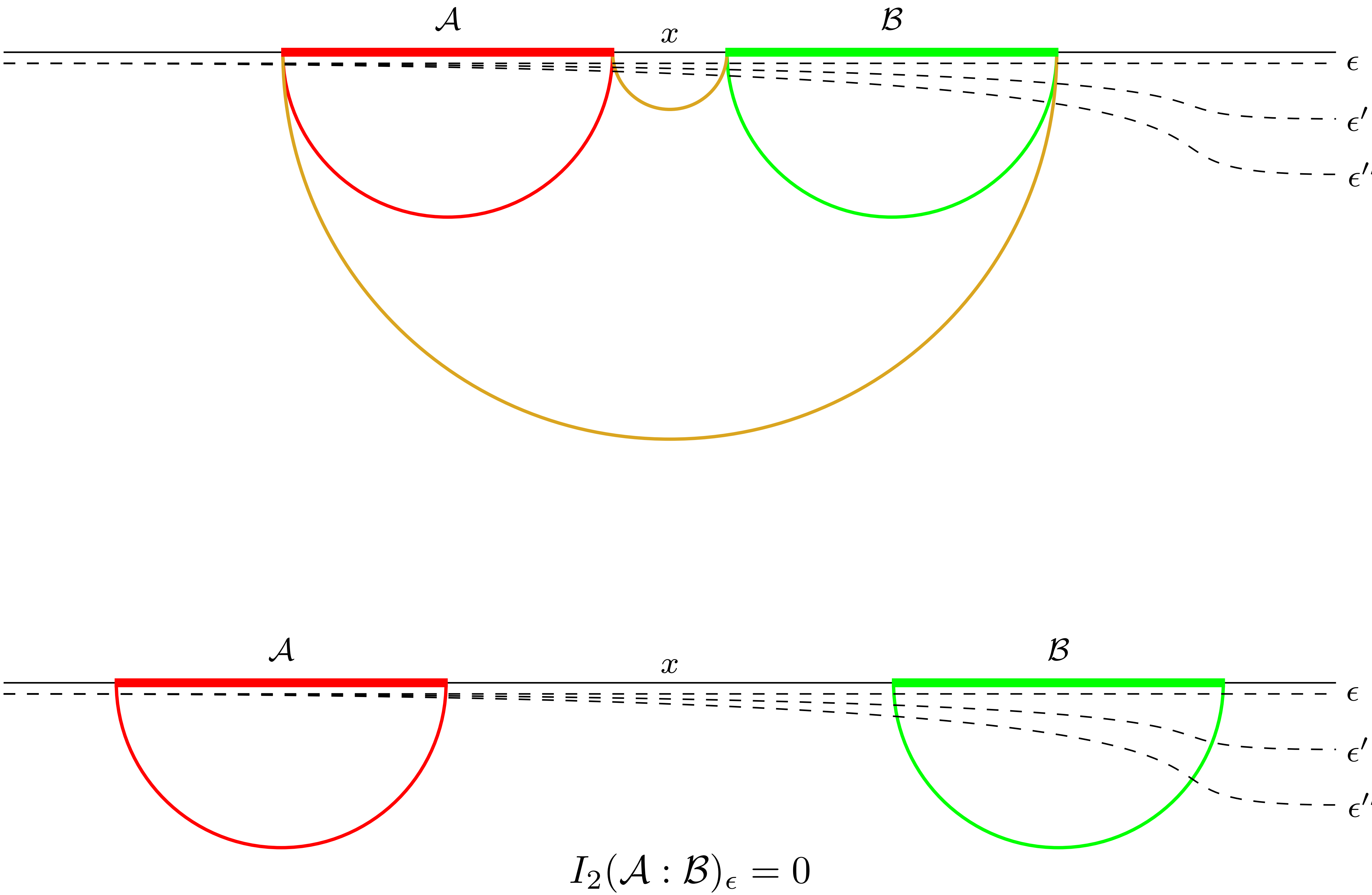
Entropy vector $\vec{S}(\rho_{\mathcal{AB}})_\epsilon = (S_\mathcal{A}, S_\mathcal{B}, S_{\mathcal{AB}})_\epsilon$

Entropy space $\mathbb{R}^3_+$



In quantum field theory all entropies are generically infinite and fixing a regulator is unphysical. Therefore entropy vectors are meaningless and we focus instead on entropy relations.


$\mathcal{A}$
x
$\mathcal{B}$
ϵ
ϵ'
ϵ''


$\mathcal{A}$
x
$\mathcal{B}$
ϵ
ϵ'
ϵ''
$\mathcal{A}$
x
$\mathcal{B}$
ϵ
ϵ'
ϵ''
$I_2(\mathcal{A}:\mathcal{B})_\epsilon = 0$

Generalization of entropy vectors and space to an arbitrary number of parties

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_{\mathsf{N}}$$

Entropy vector $\vec{S}(\rho_{\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{\mathsf{N}}}) = (S_{\mathcal{A}_1}, S_{\mathcal{A}_2}, \ldots, S_{\mathcal{A}_1\mathcal{A}_2}, S_{\mathcal{A}_1\mathcal{A}_3}, \ldots, S_{\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{\mathsf{N}}})$

Entropy space $\mathbb{R}_+^{2^{\mathsf{N}}-1}$

Generalization of entropy vectors and space to an arbitrary number of parties

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_\mathsf{N}$$

Entropy vector $\vec{S}(\rho_{\mathcal{A}_1\mathcal{A}_2...\mathcal{A}_\mathsf{N}}) = (S_{\mathcal{A}_1}, S_{\mathcal{A}_2}, ..., S_{\mathcal{A}_1\mathcal{A}_2}, S_{\mathcal{A}_1\mathcal{A}_3}, ..., S_{\mathcal{A}_1\mathcal{A}_2...\mathcal{A}_\mathsf{N}})$

Entropy space $\mathbb{R}_+^{2^\mathsf{N}-1}$

Generic linear information quantity $Q(\vec{S}) = \sum_\mathsf{J} \mathsf{Q}_\mathsf{J} S_\mathsf{J}$

Generalization of entropy vectors and space to an arbitrary number of parties

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_{\mathsf{N}}$$

Entropy vector $\vec{S}(\rho_{\mathcal{A}_1\mathcal{A}_2...\mathcal{A}_{\mathsf{N}}}) = (S_{\mathcal{A}_1}, S_{\mathcal{A}_2}, ..., S_{\mathcal{A}_1\mathcal{A}_2}, S_{\mathcal{A}_1\mathcal{A}_3}, ..., S_{\mathcal{A}_1\mathcal{A}_2...\mathcal{A}_{\mathsf{N}}})$

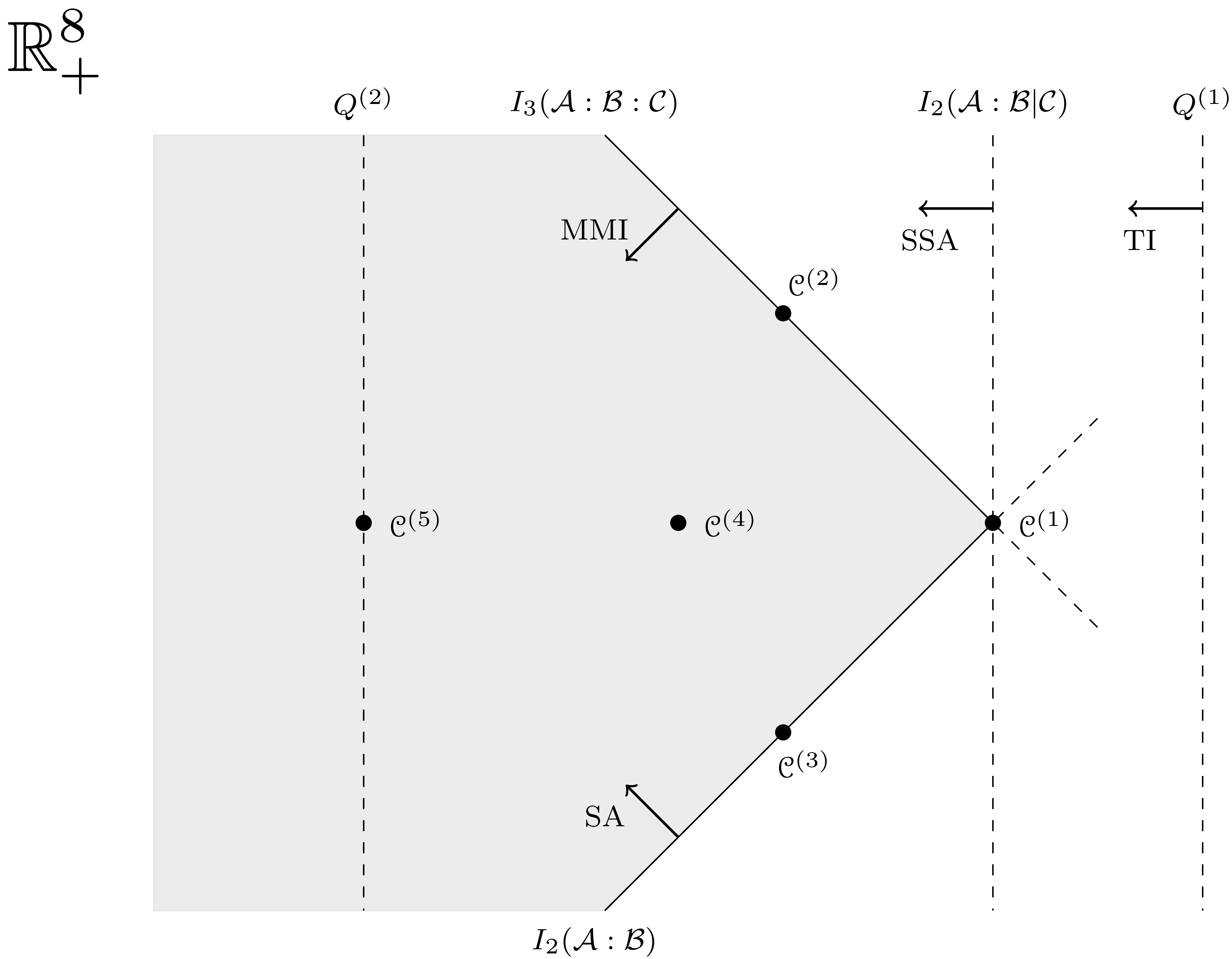
Entropy space $\mathbb{R}_+^{2^{\mathsf{N}}-1}$

Generic linear information quantity $Q(\vec{S}) = \sum_{\mathsf{J}} \mathsf{Q}_{\mathsf{J}} S_{\mathsf{J}}$

Definition 1 (faithful information quantity):
an information quantity is faithful if there exists a geometric state and a choice of subsystems such that

$$Q(\vec{S}(\rho_{\mathsf{N}})_\epsilon) = 0$$

independently from the cut-off.



$\mathbb{R}^8_+$
$Q^{(2)}$
$I_3(\mathcal{A}:\mathcal{B}:\mathcal{C})$
$I_2(\mathcal{A}:\mathcal{B}|\mathcal{C})$
$Q^{(1)}$
MMI
SSA
TI
$\mathfrak{C}^{(2)}$
$\mathfrak{C}^{(5)}$
$\mathfrak{C}^{(4)}$
$\mathfrak{C}^{(1)}$
$\mathfrak{C}^{(3)}$
SA
$I_2(\mathcal{A}:\mathcal{B})$

Definition 2 (primitive information quantity - PIQ):
an information quantity is primitive if

- it is faithful, and therefore there exists a choice of geometric state and field theory subsystems such that

$$Q(\vec{S}(\rho_{\mathsf{N}})_\epsilon) = 0$$

- but for any other faithful information quantity (and same state and subsystems)

$$Q'(\vec{S}(\rho_{\mathsf{N}})_\epsilon) \neq 0$$

for "sufficiently generic" cut-off.

Definition 2 (primitive information quantity - PIQ):
an information quantity is primitive if

- it is faithful, and therefore there exists a choice of geometric state and field theory subsystems such that

$$Q(\vec{S}(\rho_{\mathsf{N}})_\epsilon) = 0$$

- but for any other faithful information quantity (and same state and subsystems)

$$Q'(\vec{S}(\rho_{\mathsf{N}})_\epsilon) \neq 0$$

  for "sufficiently generic" cut-off.

First goal: find all PIQ for any number of parties.

The requirement of cut-off independence can be implemented efficiently by abstracting from “areas” and instead computing entropies more formally, as formal **linear combinations of connected surfaces**.

The requirement of cut-off independence can be implemented efficiently by abstracting from "areas" and instead computing entropies more formally, as formal **linear combinations of connected surfaces**.

Fix a choice of ρ_{N} and consider a generic linear information quantity $Q(\vec{S}) = \sum_{\mathsf{J}} \mathsf{Q}_{\mathsf{J}} S_{\mathsf{J}}$ where the Q_{J} are not treated as variables.

The requirement of cut-off independence can be implemented efficiently by abstracting from "areas" and instead computing entropies more formally, as formal linear combinations of connected surfaces.

Fix a choice of ρ_{N} and consider a generic linear information quantity $Q(\vec{S}) = \sum_{\mathsf{J}} \mathsf{Q}_{\mathsf{J}} S_{\mathsf{J}}$ where the Q_{J} are not treated as variables.

The set of faithful information quantities associated to ρ_{N} is the solution to a system of linear equations

$$Q(\vec{S}(\rho_{\mathsf{N}})) = \sum_{\mathsf{J}} \mathsf{Q}_{\mathsf{J}} \sum_{\mathsf{K}} c_{\mathsf{JK}} \omega_{\mathsf{K}} = \sum_{\mathsf{K}} \omega_{\mathsf{K}} \left(\sum_{\mathsf{J}} c_{\mathsf{JK}} \mathsf{Q}_{\mathsf{J}} \right) = 0$$

The requirement of cut-off independence can be implemented efficiently by abstracting from "areas" and instead computing entropies more formally, as formal **linear combinations of connected surfaces**.

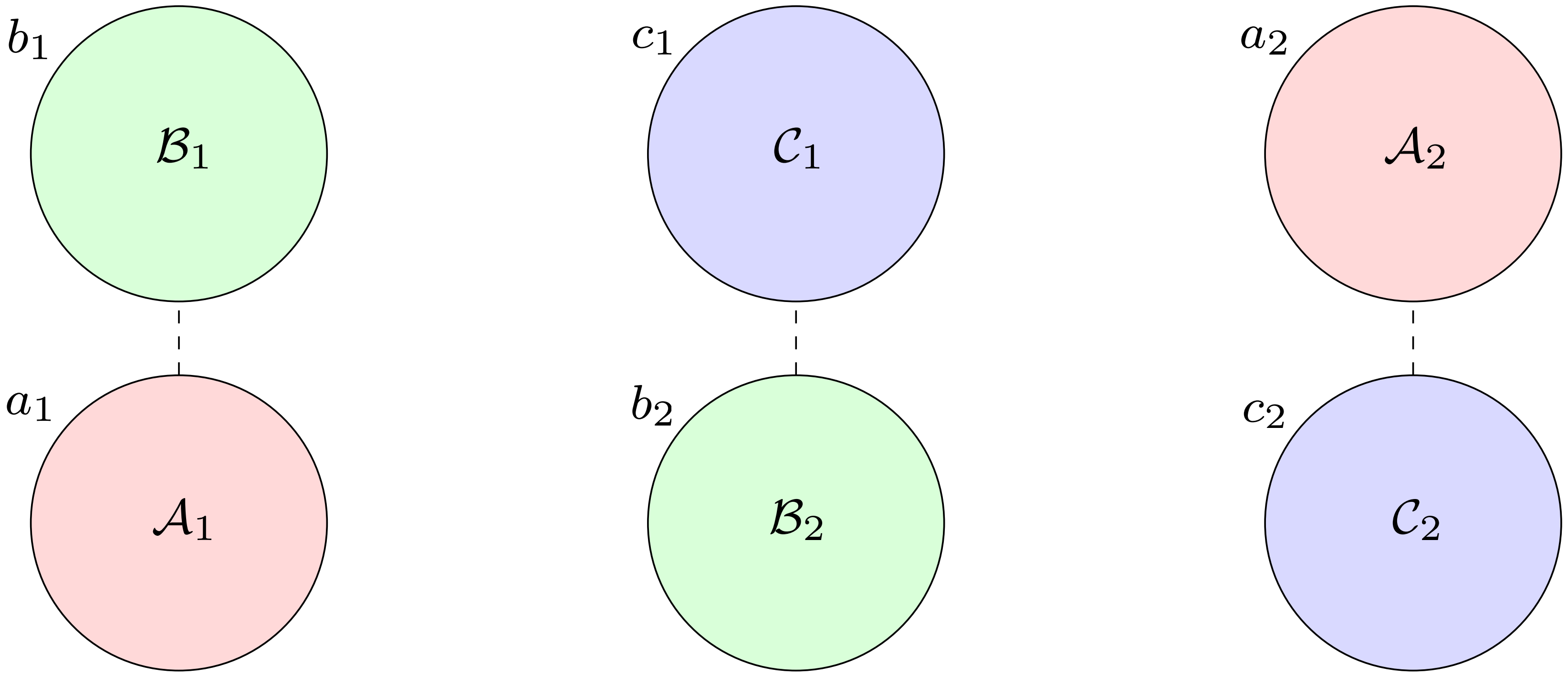
Fix a choice of ρ_{N} and consider a generic linear information quantity $Q(\vec{S}) = \sum_{\mathsf{J}} \mathsf{Q}_{\mathsf{J}} S_{\mathsf{J}}$ where the Q_{J} are not treated as variables.

The set of faithful information quantities associated to ρ_{N} is the solution to a system of linear equations

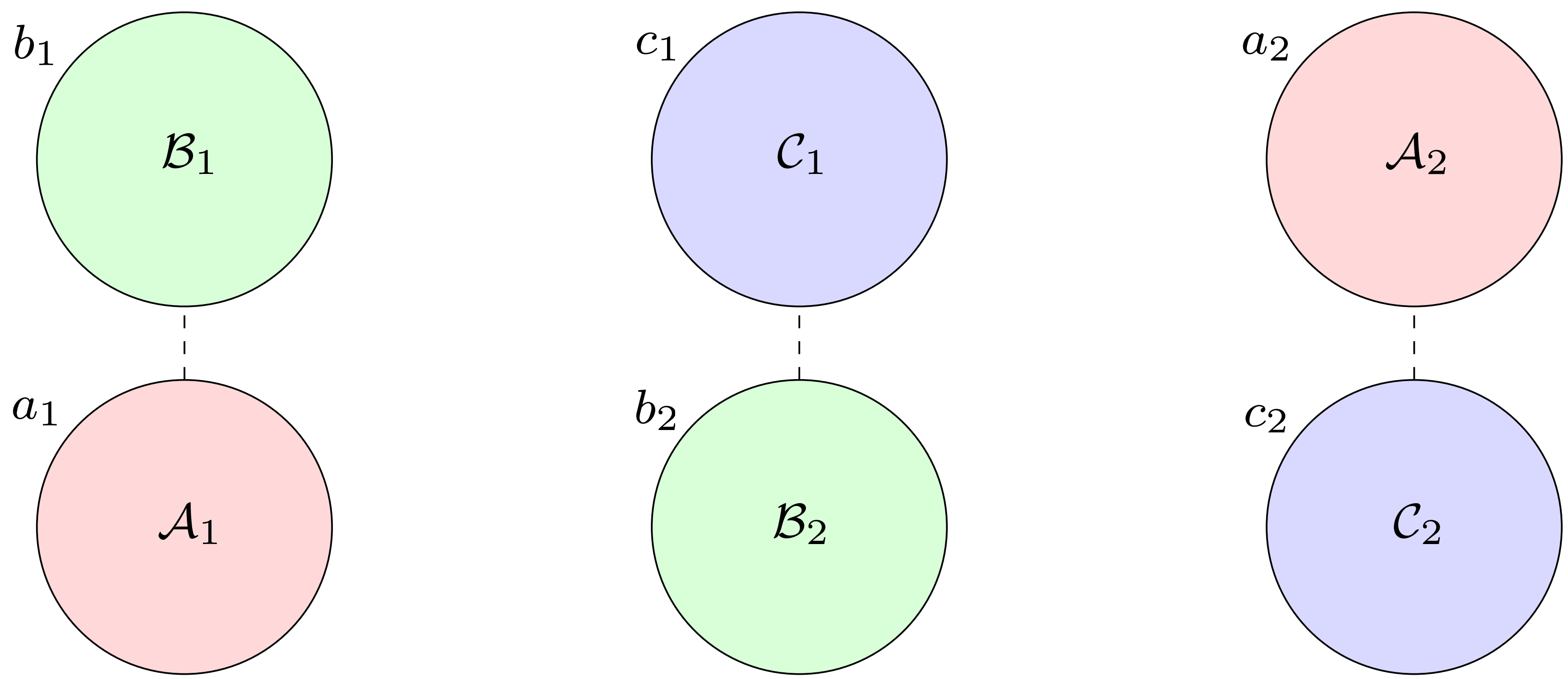
$$Q(\vec{S}(\rho_{\mathsf{N}})) = \sum_{\mathsf{J}} \mathsf{Q}_{\mathsf{J}} \sum_{\mathsf{K}} c_{\mathsf{JK}} \omega_{\mathsf{K}} = \sum_{\mathsf{K}} \omega_{\mathsf{K}} \left(\sum_{\mathsf{J}} c_{\mathsf{JK}} \mathsf{Q}_{\mathsf{J}} \right) = 0$$

The configuration ρ_{N} "generates" a PIQ if the solution to these equations is a one-dimensional space.

Example: derivation of the tripartite information



Example: derivation of the tripartite information



Surfaces	$S_{\mathcal{A}}$	$S_{\mathcal{B}}$	$S_{\mathcal{C}}$	$S_{\mathcal{AB}}$	$S_{\mathcal{AC}}$	$S_{\mathcal{BC}}$	$S_{\mathcal{ABC}}$	Relations
a_1	✓				✓			$\alpha\bar{\beta}$
b_1		✓				✓		$\beta\bar{\alpha}$
$a_1 b_1$				✓			✓	$\alpha\beta$
b_2		✓		✓				$\beta\bar{\gamma}$
c_1			✓		✓			$\gamma\bar{\beta}$
$b_2 c_1$						✓	✓	$\beta\gamma$
c_2			✓			✓		$\gamma\bar{\alpha}$
a_2	✓			✓				$\alpha\bar{\gamma}$
$a_2 c_2$					✓		✓	$\alpha\gamma$

To find all PIQ we need to **scan over all possible states and choices of subsystems**.

What we have so far:

- a framework that does not make any distinction between static and dynamical spacetimes (RT vs HRT)
- derivation of all three parties PIQ
- derivation of a **particular class of PIQ for an arbitrary number of parties** which generalizes the tripartite information
- we know that at least some of these PIQ do not correspond to new constraints

To find all PIQ we need to **scan over all possible states and choices of subsystems**.

What we have so far:

- a framework that does not make any distinction between static and dynamical spacetimes (RT vs HRT)
- derivation of all three parties PIQ
- derivation of a **particular class of PIQ for an arbitrary number of parties** which generalizes the tripartite information
- we know that at least some of these PIQ do not correspond to new constraints

What next

- complete the derivation of all PIQ for an arbitrary number of parties (in progress)
- develop a technique to extract new constraints from the PIQ (in progress)
- what is the meaning of the PIQ which do not correspond to new constraints?
- ultimately the hope is that this framework will shed light on the implications of the constraints on the entanglement structure of geometric states