Holographic entropy relations

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For geometric states the von Neumann entropy of an arbitrary subsystem is computed by the RT/HRT formula.

- What is the entanglement structure of geometric states?
- How can we characterize it?
- Does it somehow capture the fact that they are geometric?
There are universal constraints:

- subadditivity (SA)
  \[ S_A + S_B \geq S_{AB} \]

- strong subadditivity (SSA)
  \[ S_{AC} + S_{BC} \geq S_C + S_{ABC} \]
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But geometric states also satisfy non-universal constraints. Example, monogamy of mutual information (MMI) (Hayden et al. 11')

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There are universal constraints:

- **subadditivity (SA)**
  \[ S_A + S_B \geq S_{AB} \quad S_A + S_B - S_{AB} \geq 0 \quad I_2(A : B) \geq 0 \]

- **strong subadditivity (SSA)**
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- **Hayden et al. 11’**

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Note that SSA is implied by SA and MMI

\[ I_2(A : B) - I_3(A : B : C) = I_2(A : B|C) \]
First systematic search of new constraints of this kind: holographic entropy cone
Bao et al. 15’

Main results:
• proof that MMI is the only constraint for three parties
• proof that there are no new constraints for four parties
• found four new constraints for five parties
• found an infinite family of constraints, one for any odd number of parties
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However:
- static set-up (RT) only
- the constraints were found via a computer search that does not provide any guidance for finding new ones
- it does not provide an interpretation
We would like to develop:

• a formulation also valid for dynamical cases (HRT)
• technique which generates candidates for new inequalities, for an arbitrary number of parties
• technique to prove inequalities from these candidates

**Final goal**: find an interpretation for these constraints and understand the implications for the entanglement structure of geometric states.
\[ \mathcal{H}_A \otimes \mathcal{H}_B \]

Entropy vector \[ \vec{S}(\rho_{AB}) = (S_A, S_B, S_{AB}) \]

Entropy space \[ \mathbb{R}_+^3 \]
In quantum field theory all entropies are generically infinite and fixing a regulator is unphysical. Therefore entropy vectors are meaningless and we focus instead on entropy relations.
Generalization of entropy vectors and space to an arbitrary number of parties

\[ \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N \]

**Entropy vector**

\[ \vec{S}(\rho_{A_1A_2\ldots A_N}) = (S_{A_1}, S_{A_2}, \ldots, S_{A_1A_2}, S_{A_1A_3}, \ldots, S_{A_1A_2\ldots A_N}) \]

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\[ \mathbb{R}^{2^N-1}_+ \]
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Entropy space \( \mathbb{R}_+^{2N-1} \)

Generic linear information quantity \( Q(\vec{S}) = \sum_j Q_j S_j \)
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**Definition 1 (faithful information quantity):**

an information quantity is faithful if there exists a geometric state and a choice of subsystems such that

\[ Q(\vec{S}(\rho_N)_\varepsilon) = 0 \]

independently from the cut-off.
Definition 2 (primitive information quantity - PIQ): an information quantity is primitive if

- it is faithful, and therefore there exists a choice of geometric state and field theory subsystems such that

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- but for any other faithful information quantity (and same state and subsystems)

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for “sufficiently generic” cut-off.

First goal: find all PIQ for any number of parties.
The requirement of cut-off independence can be implemented efficiently by abstracting from “areas” and instead computing entropies more formally, as formal linear combinations of connected surfaces.
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Fix a choice of $\rho_N$ and consider a generic linear information quantity $Q(\tilde{S}) = \sum J Q_J S_J$ where the $Q_J$ are not treated as variables.
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The set of faithful information quantities associated to $\rho_{\mathcal{N}}$ is the solution to a system of linear equations

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The configuration $\rho_N$ “generates” a PIQ if the solution to these equations is a one-dimensional space.
Example: derivation of the tripartite information
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![Diagram of Surfaces and Relations]

<table>
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<tr>
<th>Surfaces</th>
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<th>$S_B$</th>
<th>$S_C$</th>
<th>$S_{AB}$</th>
<th>$S_{AC}$</th>
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What we have so far:
• a framework that does not make any distinction between static and dynamical spacetimes (RT vs HRT)
• derivation of all three parties PIQ
• derivation of a particular class of PIQ for an arbitrary number of parties which generalizes the tripartite information
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What next
- complete the derivation of all PIQ for an arbitrary number of parties (in progress)
- develop a technique to extract new constraints from the PIQ (in progress)
- what is the meaning of the PIQ which do not correspond to new constraints?
- ultimately the hope is that this framework will shed light on the implications of the constraints on the entanglement structure of geometric states