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*Gauge/Gravity Duality*


# Axiomatic complexity in quantum field theory

2018.07.31

Keun-Young Kim

Gwangju Institute of Science and Technology

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*Gauge/Gravity Duality* 2018


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# Axiomatic complexity in quantum field theory

Q1. What is complexity?
Q2. Why quantum field theory?
Q3. Why axiomatic?

arXiv.org > hep-th > arXiv:1803.01797

High Energy Physics – Theory

**Axiomatic complexity in quantum field theory and its applications**

Run-Qiu Yang, Yu-Sen An, Chao Niu, Cheng-Yong Zhang, Keun-Young Kim

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# What is Complexity?

## (Computational) complexity

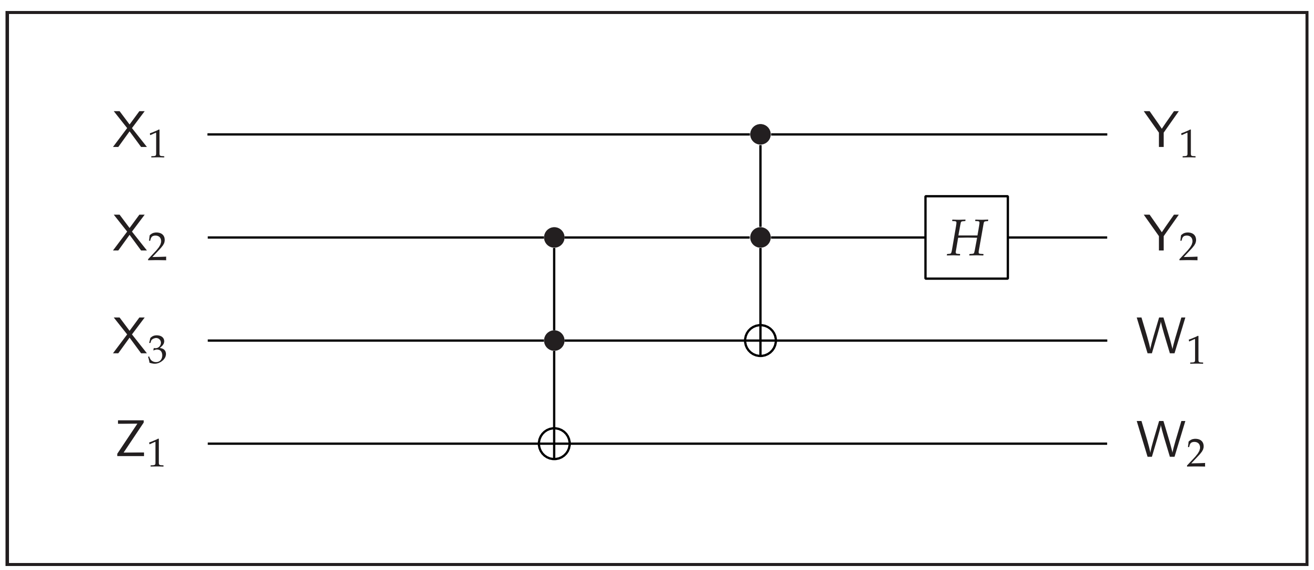
(from computer science) quantifying the difficulty of carrying out a task.

## (Circuit) complexity

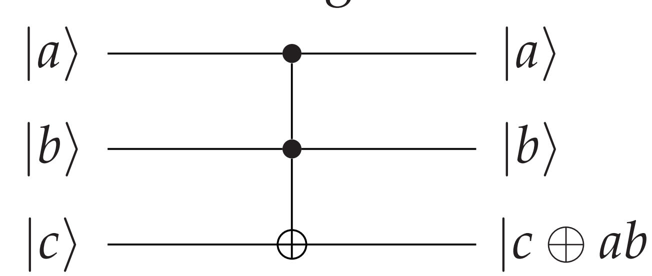
Minimal number of gates for the transformation from the reference to target state

$$|\psi_T\rangle = U|\psi_R\rangle = g_n g_{n-1} \cdots g_2 g_1 |\psi_R\rangle$$

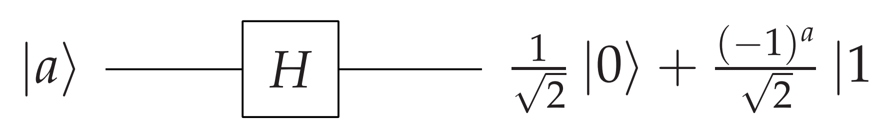
ex)




Toffoli gate




Hadamard gate



# What is Complexity?

## Complexity of quantum states

For given states $|\psi_T\rangle = U|\psi_R\rangle$ ~minimal number of gates the reference to target state

## Complexity is kind of distance

Fubini-Study distance: $d_{AB} = \arccos|\langle B|A\rangle|$ (close) $0 \sim \pi/2$ (far)

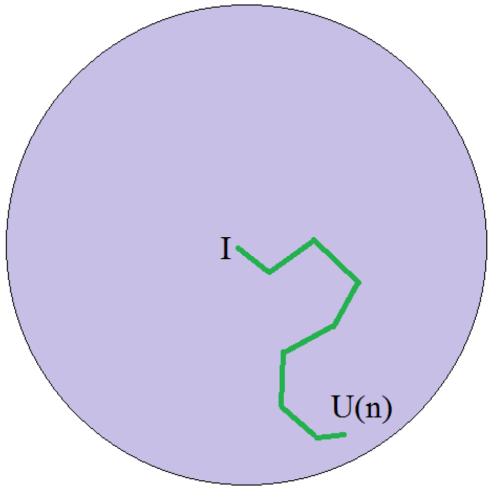
$$|0000000000\rangle \longrightarrow |0000000001\rangle$$

"Complexity distance?"

## Complexity of operator (unitary transformation)

For a given operator $U = g_n g_{n-1} \cdots g_2 g_1$ ~ minimum number of gates

$$\mathbb{I} \longrightarrow U$$



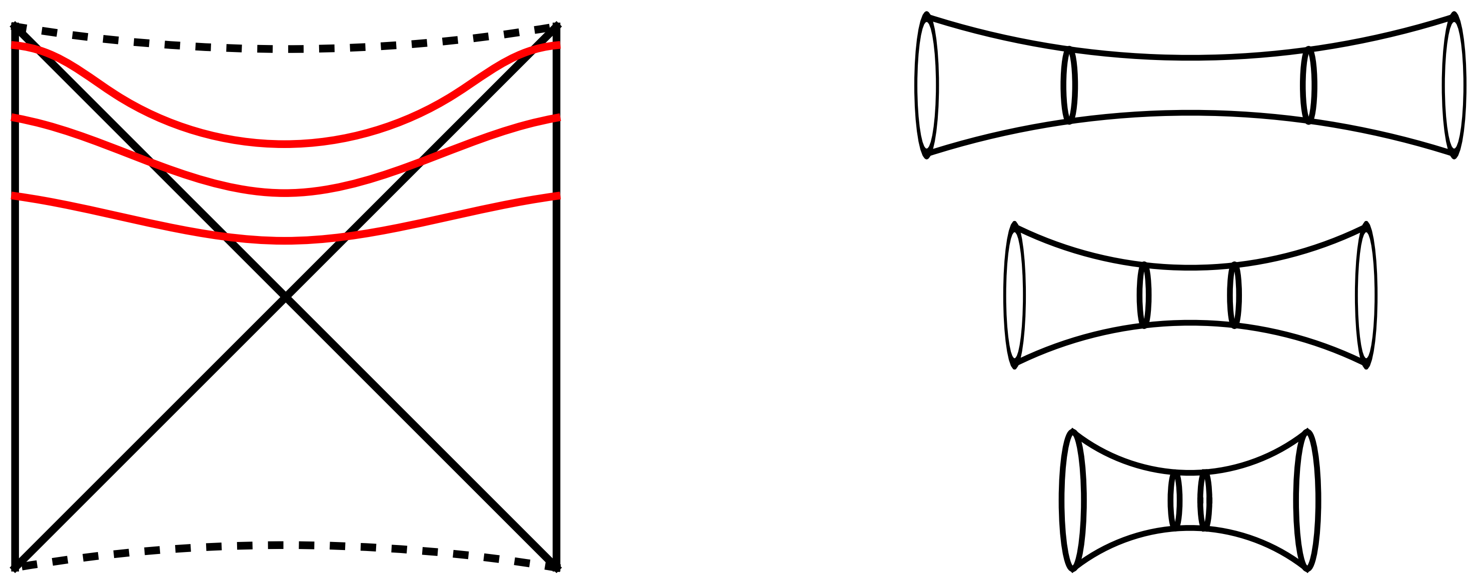

This talk focuses on the complexity of operator

# Axiomatic complexity in quantum field theory

Q1. What is complexity?
Q2. Why quantum field theory?
Q3. Why axiomatic?

# Why quantum field theory?

[Susskind: 1402.5674
Stanford and Susskind: 1406.2678]



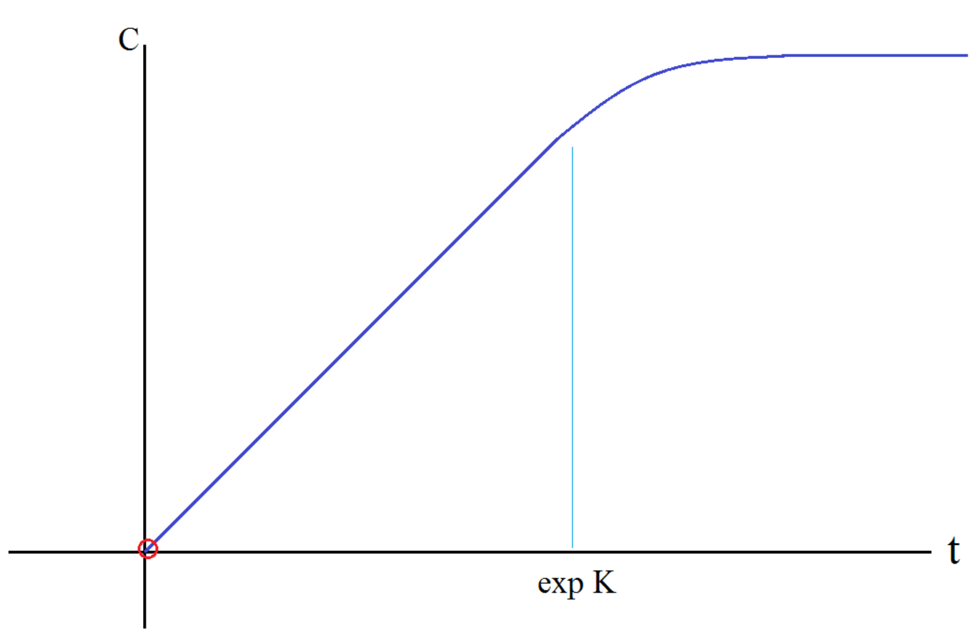
$$\mathcal{C}_V = \max_{\partial\Sigma = t_L \cup t_R} \left[ \frac{V(\Sigma)}{G_N \ell} \right]$$

Fig. from [Koji, Norihiro, Sotaro: 1707.03840]

1. Einstein-Rosen bridge increases even after thermalization
2. The field theory meaning of this? complexity?
3. Physics inside black hole?

Conjecture: complexity

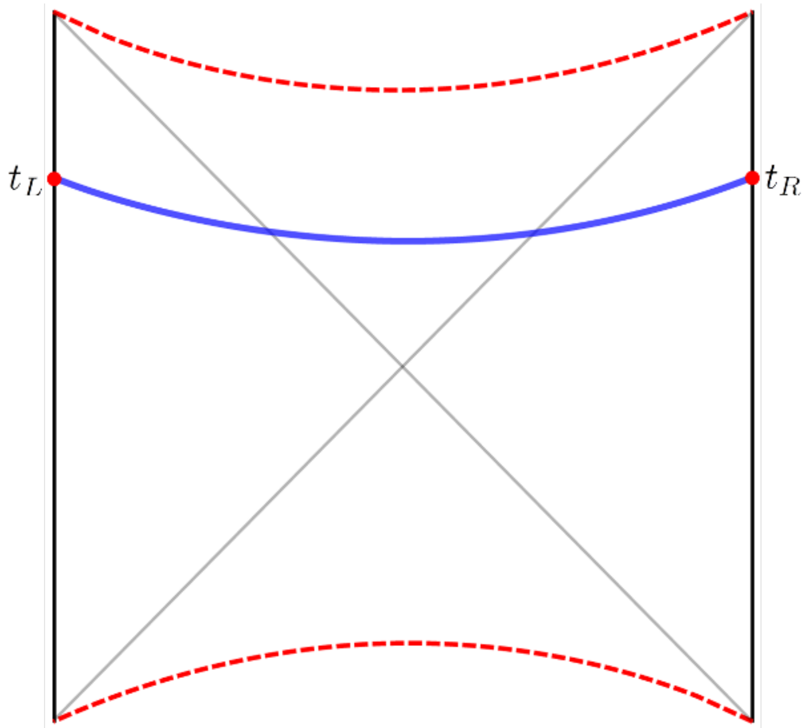
Entanglement is not enough!



# Holographic conjecture

CV (complexity-volume)

[Susskind: 1402.5674
Stanford and Susskind: 1406.2678]

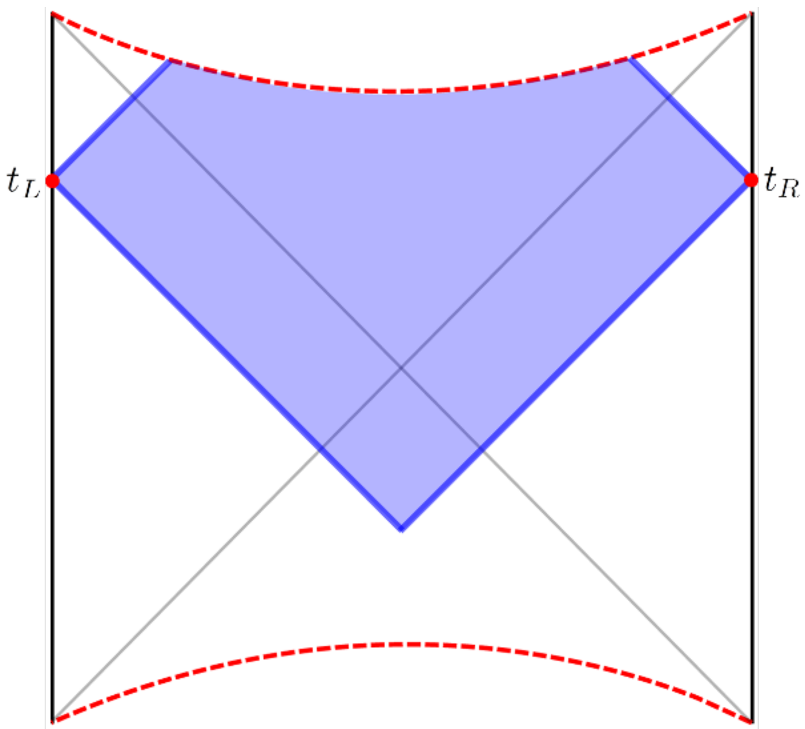



$$\mathcal{C}_V = \max_{\partial\Sigma = t_L \cup t_R} \left[ \frac{V(\Sigma)}{G_N \ell} \right]$$

- Equation of motion
- Free scale: ambiguity

CA (complexity-action)

[Brown, Roberts, Susskind Swingle and Zhao:
1509.07876, 1512.04993]




$$\mathcal{C}_A = \frac{I_{\text{WDW}}}{\pi \hbar}$$

- Boundary terms
- Singularity

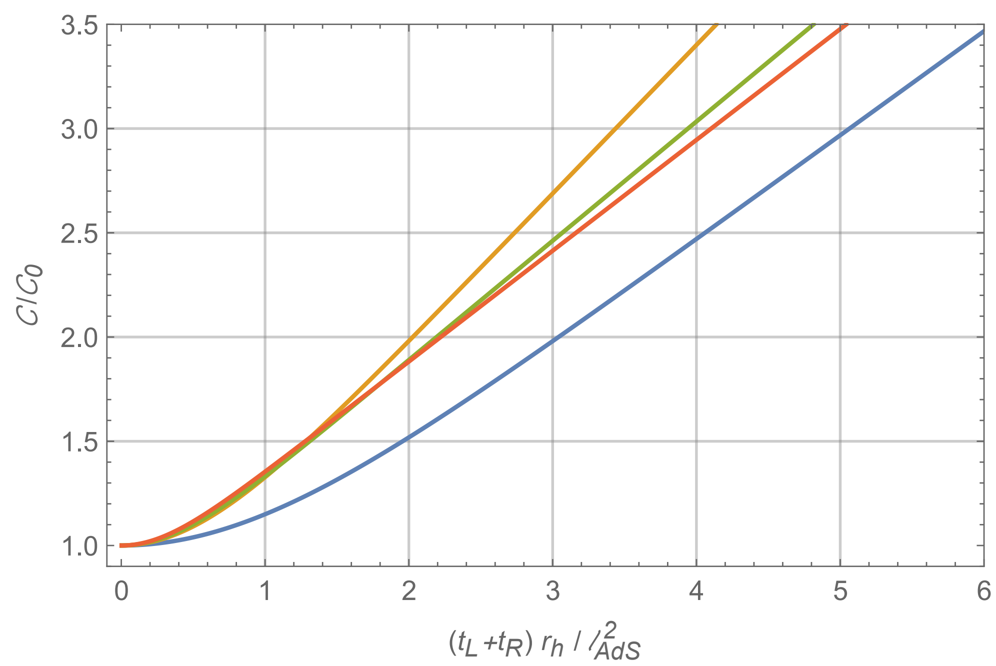
Fig. from [Jefferson, Myers: 1707.08570]

# Comparisons for duality

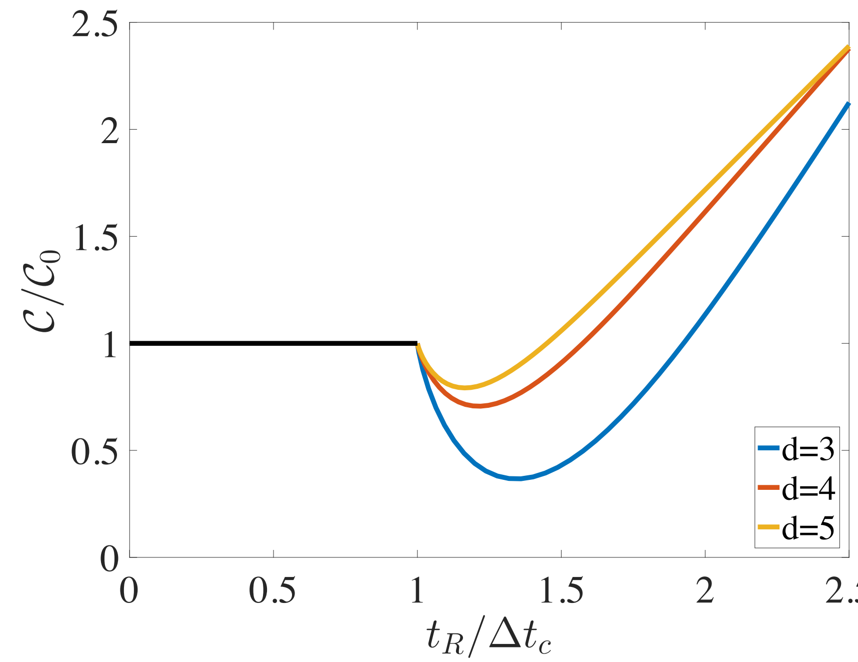
[Yang, Niu, Zhang, KK: 1710.00600]

See also [Carmi, Chapman Marrochio, Myers, and Sugishita: 1709.10184]

## CV (TFD)

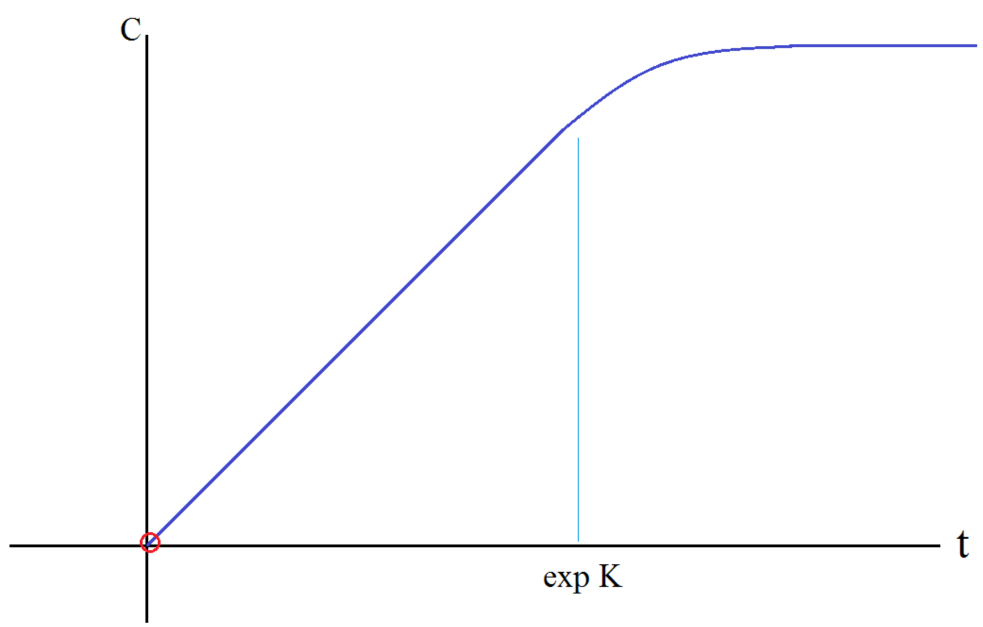



## CA (TFD)




For holographic conjectures:
The reference state is not clear

## Circuit complexity (discrete quantum circuit)




## Complexity in QFT

**?**

# Why Axiomatic?

**Axiomatic complexity in quantum field theory**

Instead of quantum circuits

gauge/gravity duality

?

# Continuous case: Nielsen's idea

## A geometric approach to quantum circuit lower bounds


Michael A. Nielsen[1,*]

[1] School of Physical Sciences, The University of Queensland, Brisbane, Queensland 4072, Australia


(Dated: February 1, 2008)

## Quantum Computation as Geometry


Michael A. Nielsen,* Mark R. Dowling, Mile Gu, Andrew C. Doherty


## The geometry of quantum computation


Mark R. Dowling[1] and Michael A. Nielsen[1,*]

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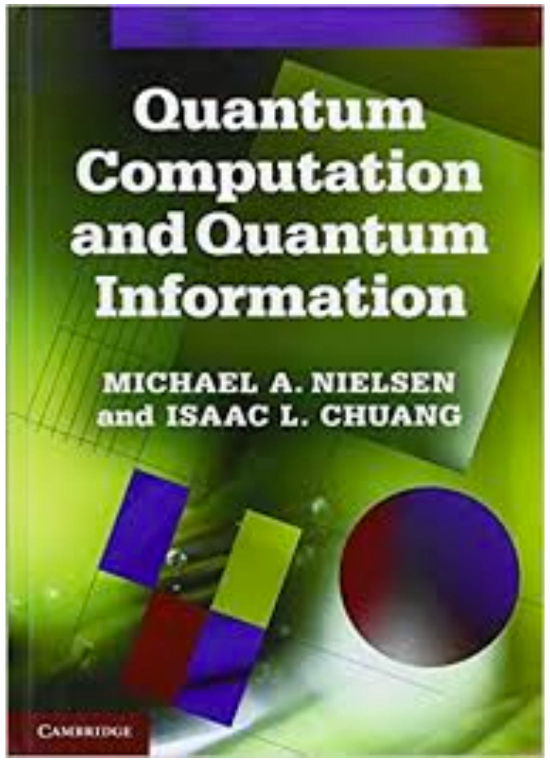


# Continuous case: Nielsen's idea

A geometric approach to quantum circuit lower bounds

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[1] *School of Physical Sciences, The University of Queensland, Brisbane, Queensland 4072, Australia*
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**Quantum Computation as Geometry**

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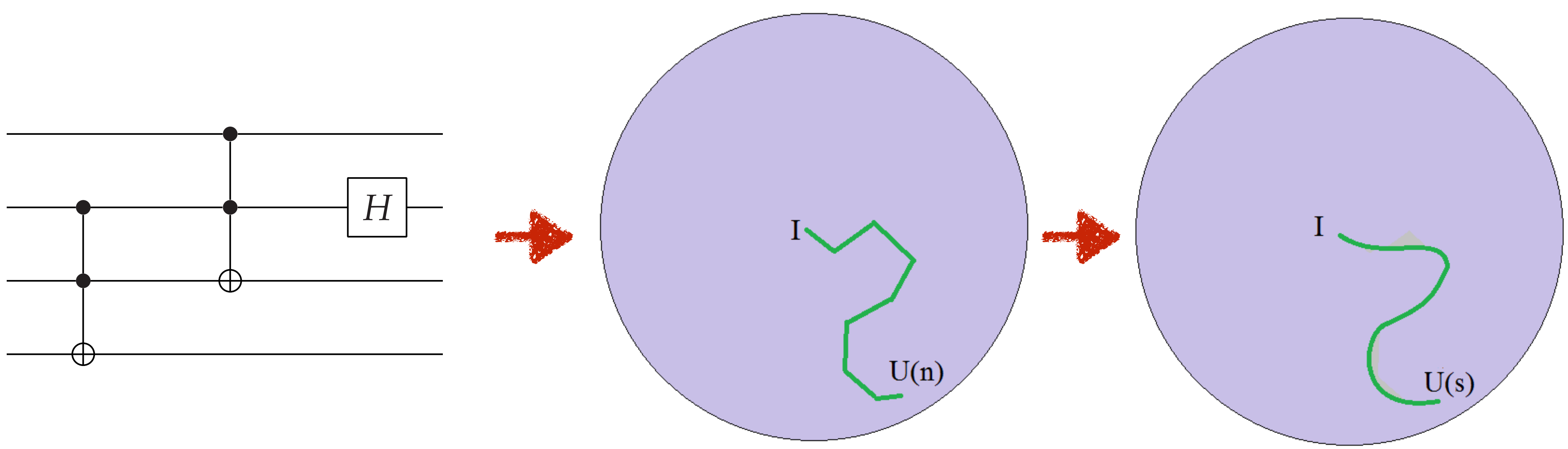
The geometry of quantum computation

Mark R. Dowling[1] and Michael A. Nielsen[1,*]
[1] *School of Physical Sciences, The University of Queensland, Brisbane, Queensland 4072, Australia*
(Dated: February 1, 2008)

Susskind and collaborators

- introduced Nielsen's idea introduced to hep-th in 2014
- have been developing the theory of complexity in QFT based on intuitions from circuit complexity




[Susskind and Zhao: 1408.2823]

# Why Axiomatic?

## Axiomatic complexity in quantum field theory

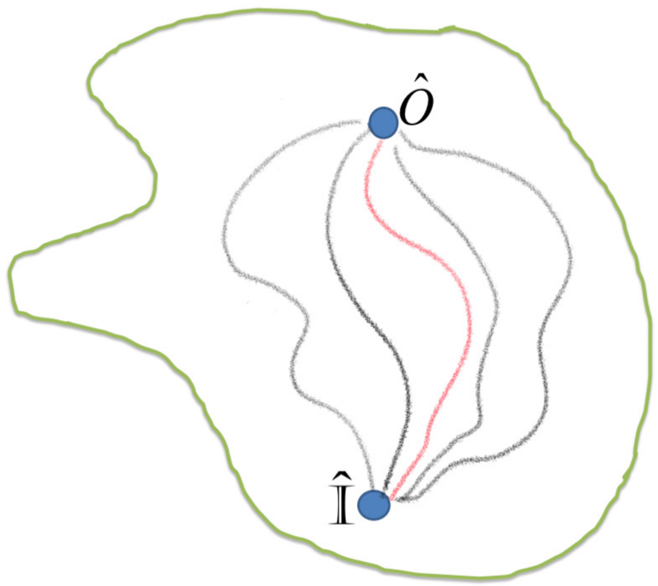
Instead of quantum circuits

gauge/gravity duality

- extract some general axioms for complexity
  from properties of computational (circuit) complexity
- take into account properties of QFT,
  some of which may not be compatible with circuit complexity

# Final result

## Complexity of SU(n) operator




4 general axioms + 2 symmetries of QFT

$$\mathcal{C}(\hat{O}) = \min\{\mathrm{Tr}\sqrt{\bar{H}\bar{H}^\dagger} \mid \forall\, \bar{H} = \ln \hat{O}\}$$

# Outline

- General axioms
- Finsler geometry from general axioms
- Constrain the geometry by QFT symmetry
- Complexity for SU(n) operator

# Three axioms for the complexity

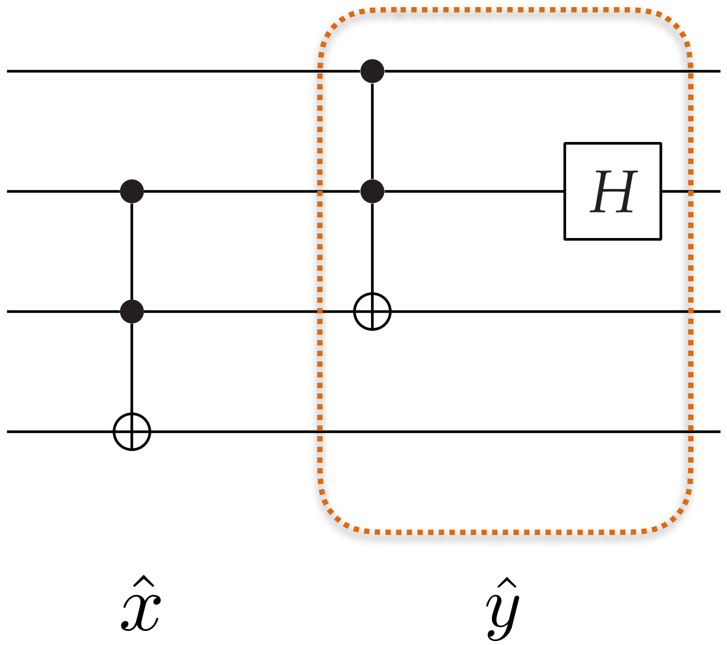
**G1** [*Nonnegativity*] $\forall \hat{x} \in \mathcal{O},\ \mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

**G2** [*Series decomposition rule (triangle inequality)*] $\forall \hat{x}, \hat{y} \in \mathcal{O},\ \mathcal{C}(\hat{x}\hat{y}) \leq \mathcal{C}(\hat{x}) + \mathcal{C}(\hat{y})$.

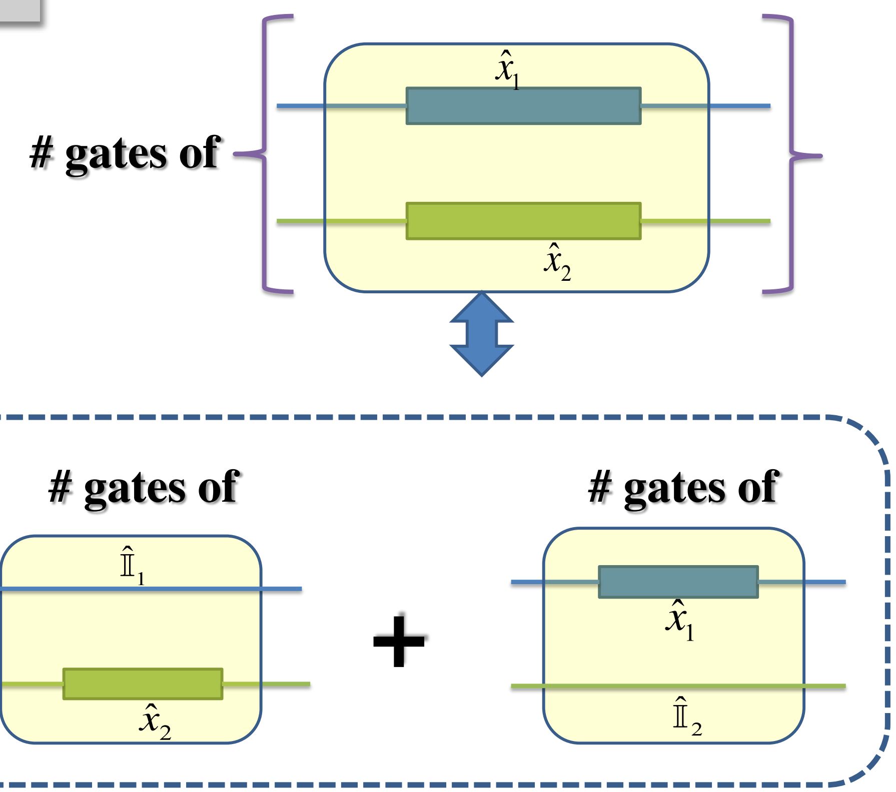
**G3** [*Parallel decomposition rule*] $\forall (\hat{x}_1, \hat{x}_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{O}$,

$$\mathcal{C}\big((\hat{x}_1, \hat{x}_2)\big) = \mathcal{C}\big((\hat{x}_1, \hat{\mathbb{I}}_2)\big) + \mathcal{C}\big((\hat{\mathbb{I}}_1, \hat{x}_2)\big).$$

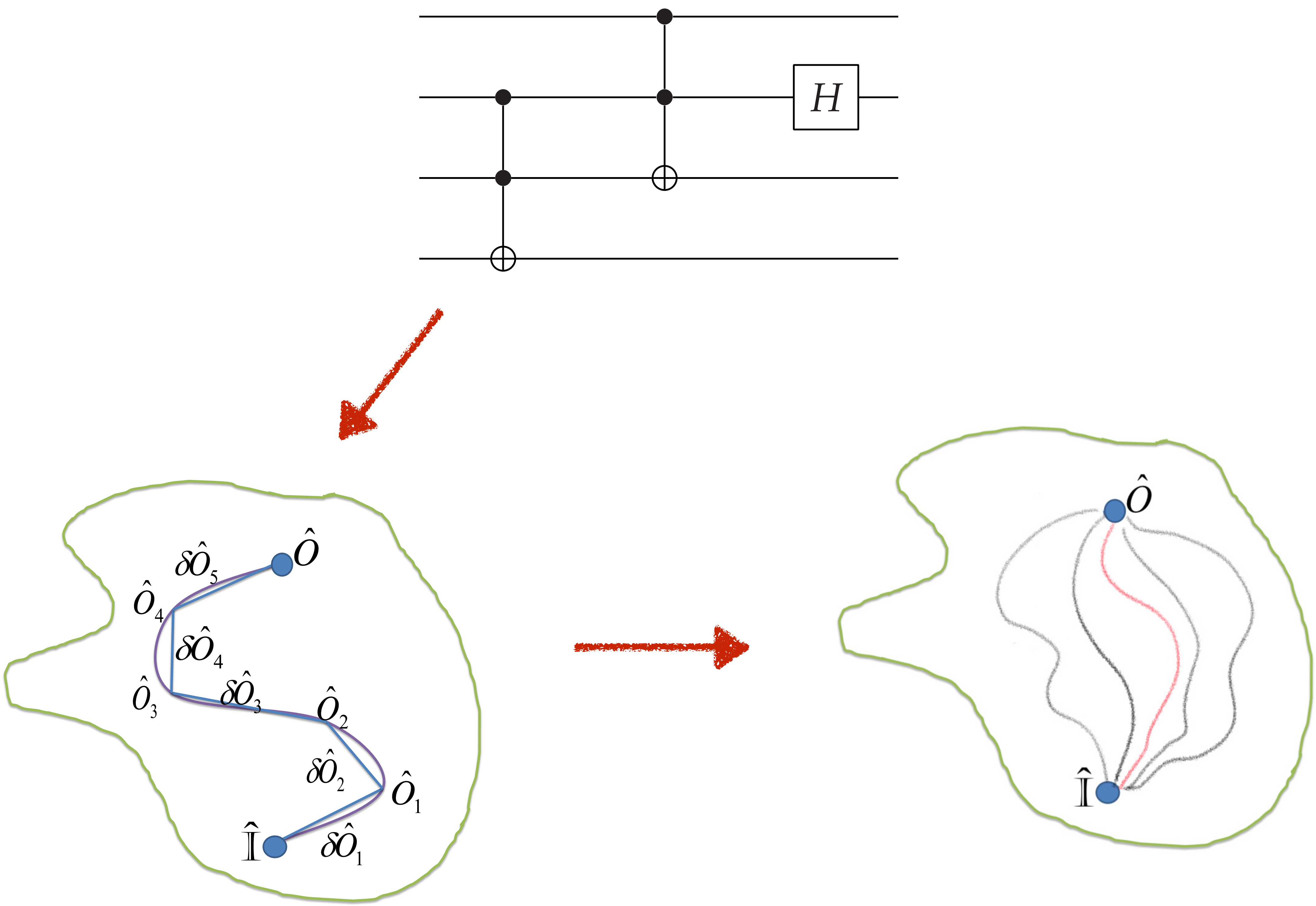
G2



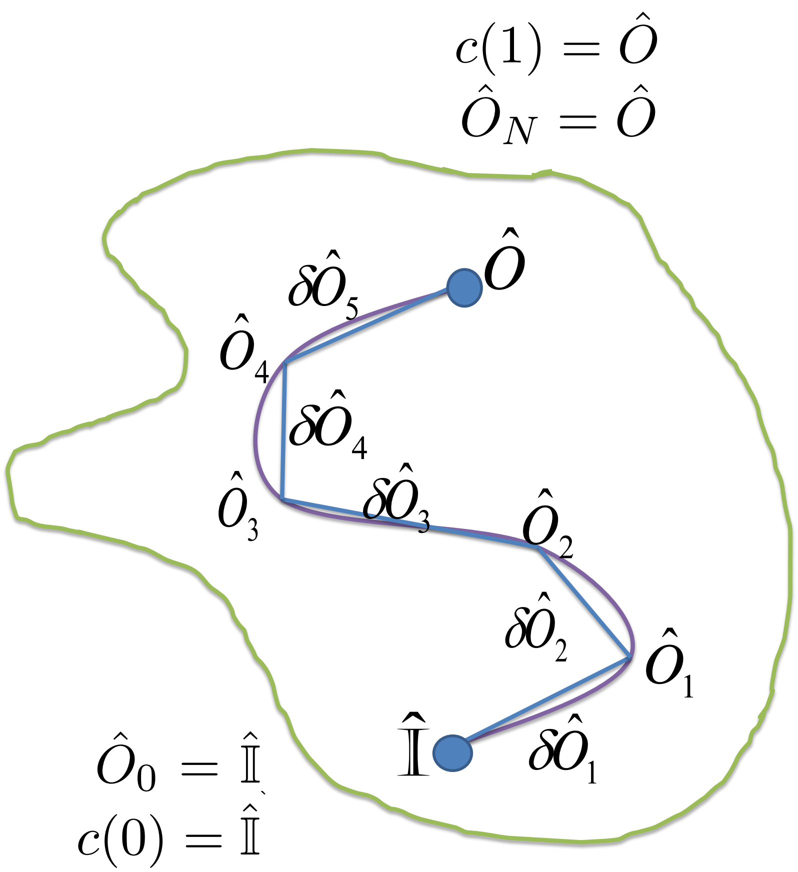

G3



# Fourth axiom for the complexity: smoothness



# Fourth axiom for the complexity: smoothness



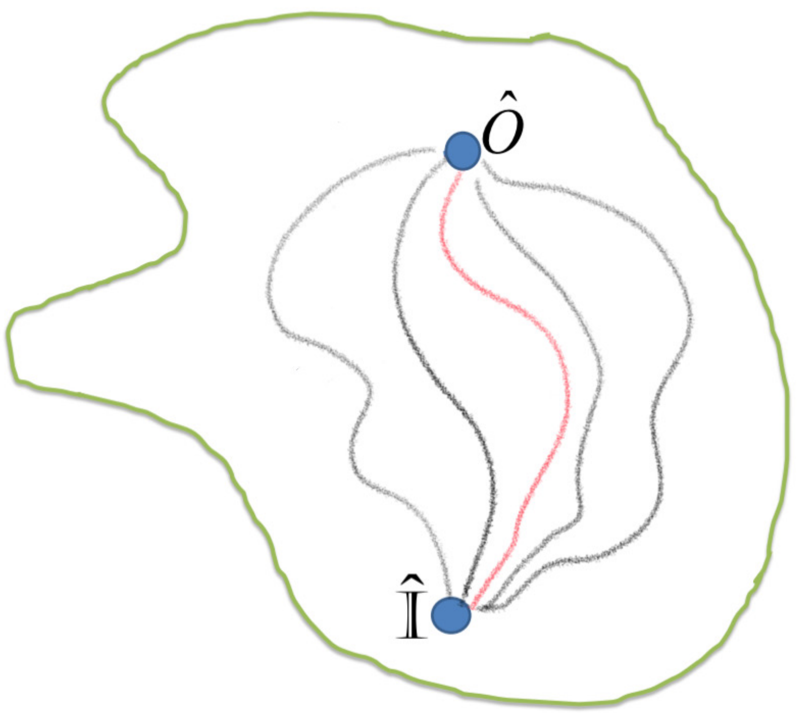

$$
\begin{aligned}
\hat{O}_n &= \delta\hat{O}_n^{(r)}\delta\hat{O}_{n-1}^{(r)}\cdots\delta\hat{O}_2^{(r)}\delta\hat{O}_1^{(r)} \\
&= \delta\hat{O}_n^{(r)}\hat{O}_{n-1}
\end{aligned}
$$

$$
\boxed{n = 1, 2, 3, \cdots, N \qquad \delta\hat{O}_n^{(\alpha)} = \exp[H_\alpha(s_n)\delta s] \qquad s_n = n/N}
$$

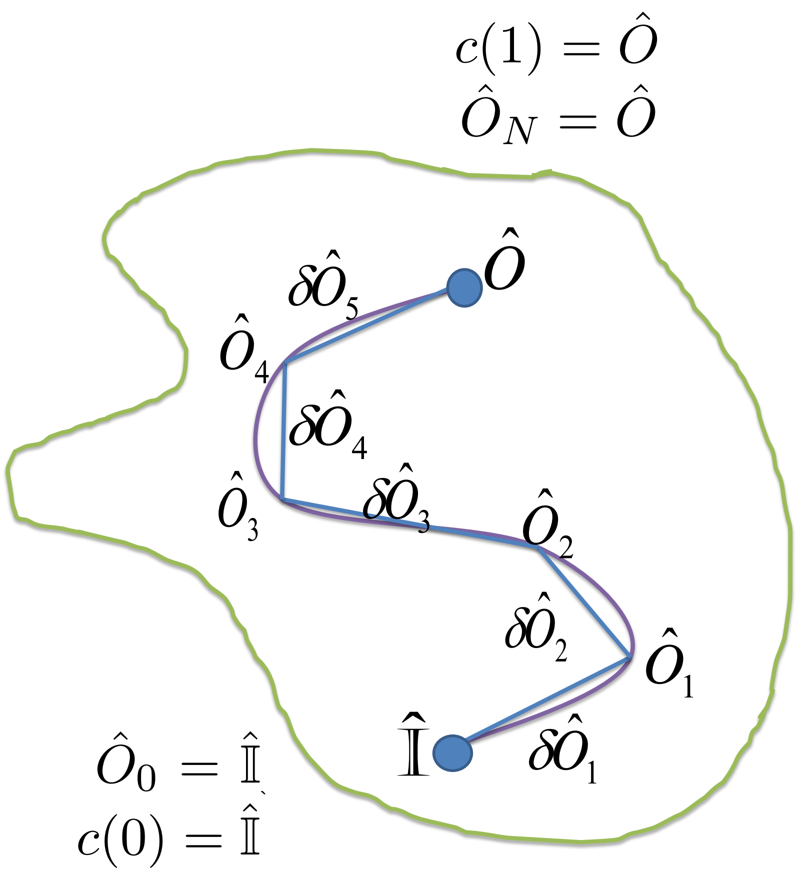
$$N \to \infty$$

$$\dot{c}(s) = H_r(s)c(s)$$

$$c(s) = \overleftarrow{\mathcal{P}}\, e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$



# Fourth axiom for the complexity: smoothness




**Right-way**

$$\hat{O}_n = \delta\hat{O}_n^{(r)}\delta\hat{O}_{n-1}^{(r)}\cdots\delta\hat{O}_2^{(r)}\delta\hat{O}_1^{(r)} = \delta\hat{O}_n^{(r)}\hat{O}_{n-1}$$

**Left-way**

$$\hat{O}_n = \delta\hat{O}_1^{(l)}\delta\hat{O}_2^{(l)}\cdots\delta\hat{O}_{n-1}^{(l)}\delta\hat{O}_n^{(l)} = \hat{O}_{n-1}\delta\hat{O}_n^{(l)}$$

$$n = 1, 2, 3, \cdots, N \qquad \delta\hat{O}_n^{(\alpha)} = \exp[H_\alpha(s_n)\delta s] \qquad s_n = n/N$$

$N \to \infty$

Right-way:

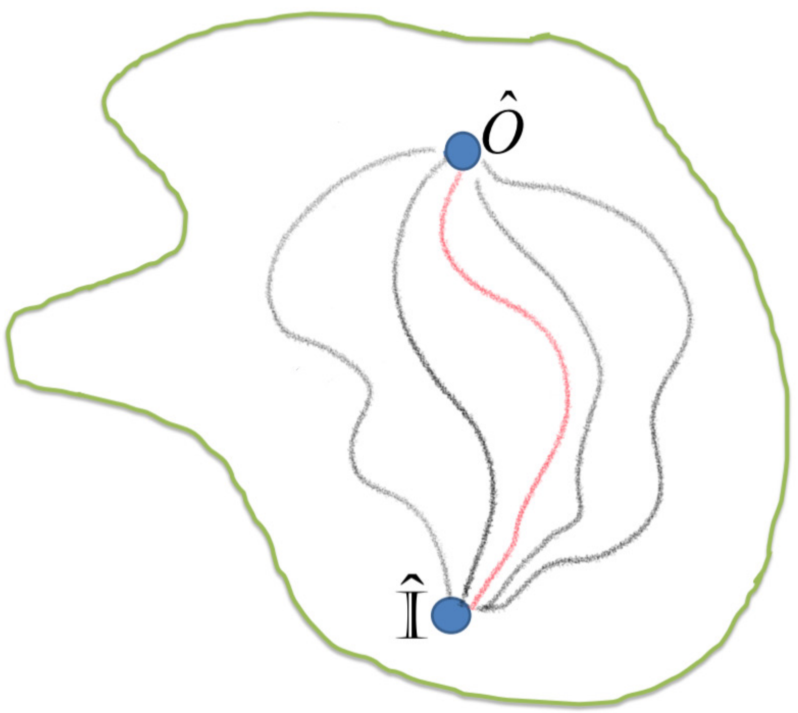
$$\dot{c}(s) = H_r(s)c(s)$$

$$c(s) = \overleftarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$

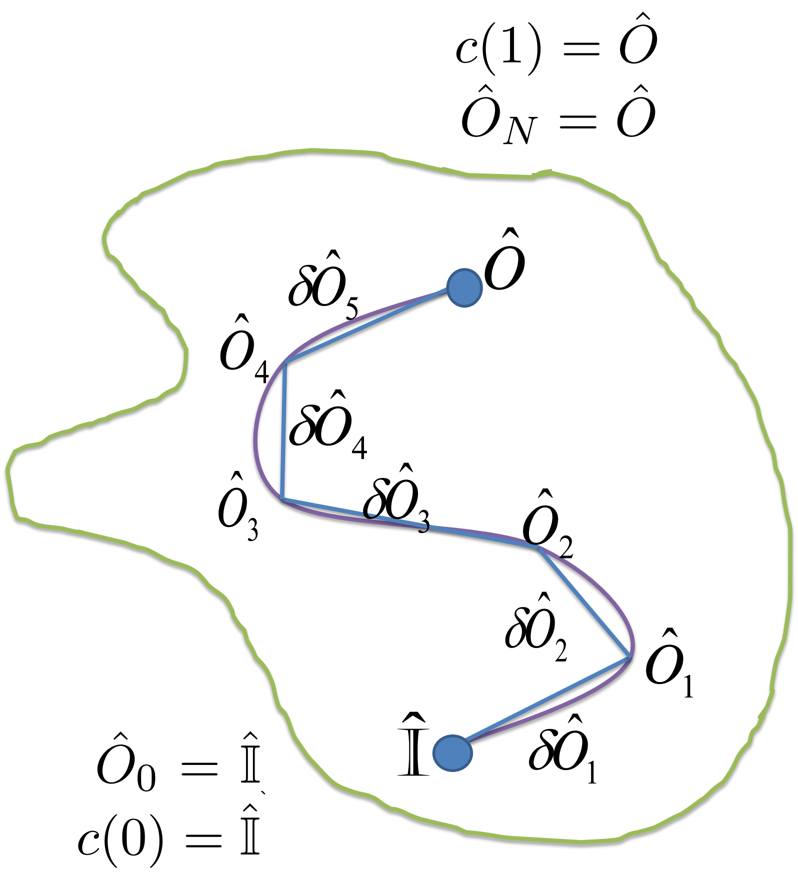
Left-way:

$$\dot{c}(s) = c(s)H_l(s)$$

$$c(s) = \overrightarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_l(\tilde{s})}$$



# Fourth axiom for the complexity: smoothness




**Right-way**

$$\begin{aligned}\hat{O}_n &= \delta\hat{O}_n^{(r)}\delta\hat{O}_{n-1}^{(r)}\cdots\delta\hat{O}_2^{(r)}\delta\hat{O}_1^{(r)} \\ &= \delta\hat{O}_n^{(r)}\hat{O}_{n-1}\end{aligned}$$

**Left-way**

$$\begin{aligned}\hat{O}_n &= \delta\hat{O}_1^{(l)}\delta\hat{O}_2^{(l)}\cdots\delta\hat{O}_{n-1}^{(l)}\delta\hat{O}_n^{(l)} \\ &= \hat{O}_{n-1}\delta\hat{O}_n^{(l)}\end{aligned}$$

$$n = 1, 2, 3, \cdots, N \qquad \delta\hat{O}_n^{(\alpha)} = \exp[H_\alpha(s_n)\delta s] \qquad s_n = n/N$$

$$N \to \infty$$

Right-way:

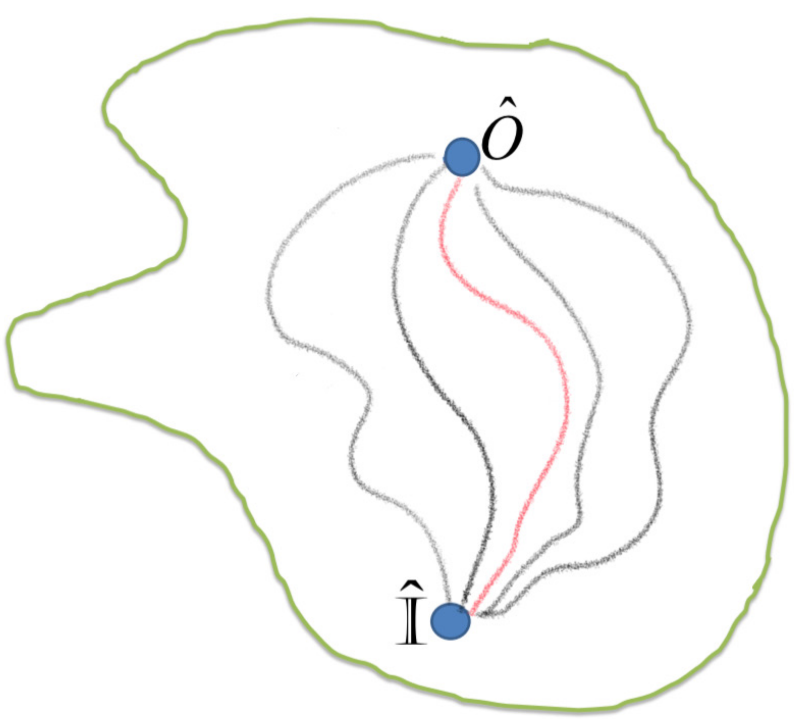
$$\begin{aligned}\dot{c}(s) &= H_r(s)c(s) \\ c(s) &= \overleftarrow{\mathcal{P}}e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}\end{aligned}$$

Left-way:

$$\begin{aligned}\dot{c}(s) &= c(s)H_l(s) \\ c(s) &= \overrightarrow{\mathcal{P}}e^{\int_0^s \mathrm{d}\tilde{s} H_l(\tilde{s})}\end{aligned}$$

**Cost (Length)**

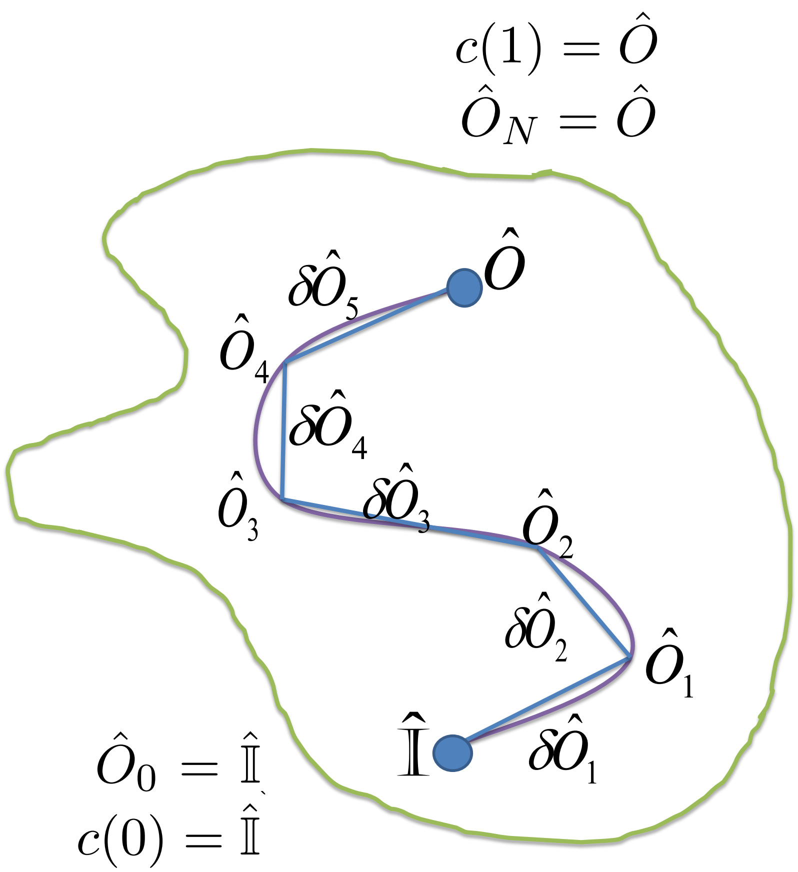
$$L_\alpha[c] := \sum_{i=1}^{N}\mathcal{C}(\delta\hat{O}_i^{(\alpha)}) \xrightarrow{N\to\infty} \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s$$




$$\textbf{G4} \quad \mathcal{C}(\delta\hat{O}^{(\alpha)}) = \mathcal{C}(\hat{\mathbb{I}}) + \tilde{F}(H_\alpha)\delta s + \mathcal{O}(\delta s^2)$$

zero by G1

# Fourth axiom for the complexity: smoothness




**Right-way**

$$\begin{aligned}\hat{O}_n &= \delta\hat{O}_n^{(r)}\delta\hat{O}_{n-1}^{(r)}\cdots\delta\hat{O}_2^{(r)}\delta\hat{O}_1^{(r)} \\ &= \delta\hat{O}_n^{(r)}\hat{O}_{n-1}\end{aligned}$$

**Left-way**

$$\begin{aligned}\hat{O}_n &= \delta\hat{O}_1^{(l)}\delta\hat{O}_2^{(l)}\cdots\delta\hat{O}_{n-1}^{(l)}\delta\hat{O}_n^{(l)} \\ &= \hat{O}_{n-1}\delta\hat{O}_n^{(l)}\end{aligned}$$

$$n = 1, 2, 3, \cdots, N \qquad \delta\hat{O}_n^{(\alpha)} = \exp[H_\alpha(s_n)\delta s] \qquad s_n = n/N$$

$$N \to \infty$$

Right-way:

$$\dot{c}(s) = H_r(s)c(s)$$
$$c(s) = \overleftarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$

Left-way:

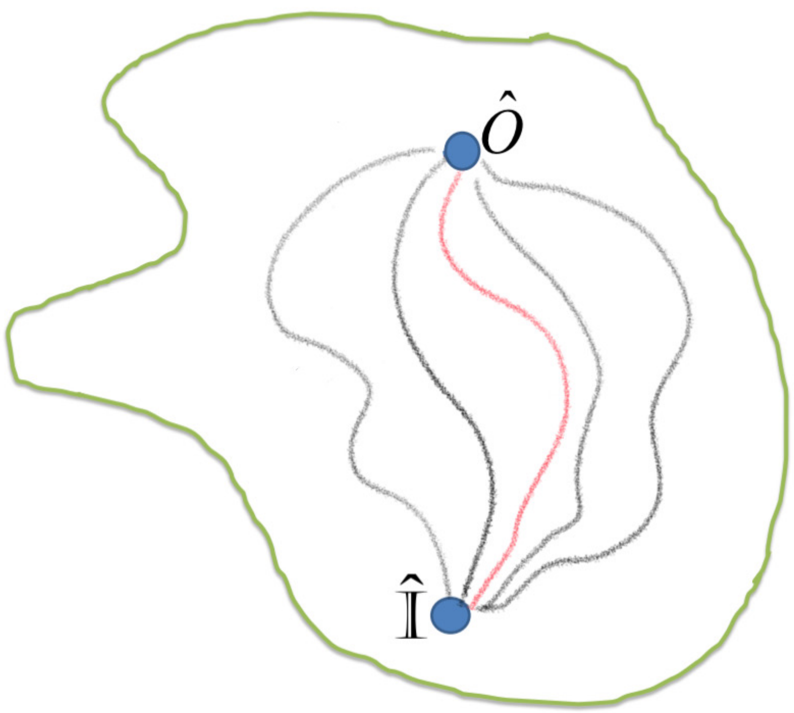
$$\dot{c}(s) = c(s)H_l(s)$$
$$c(s) = \overrightarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_l(\tilde{s})}$$

**Cost (Length)**

$$L_\alpha[c] := \sum_{i=1}^{N} \mathcal{C}(\delta\hat{O}_i^{(\alpha)}) \xrightarrow{N\to\infty} \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s$$

$$\mathbf{G4} \quad \mathcal{C}(\delta\hat{O}^{(\alpha)}) = \mathcal{C}(\hat{\mathbb{I}}) + \tilde{F}(H_\alpha)\delta s + \mathcal{O}(\delta s^2)$$

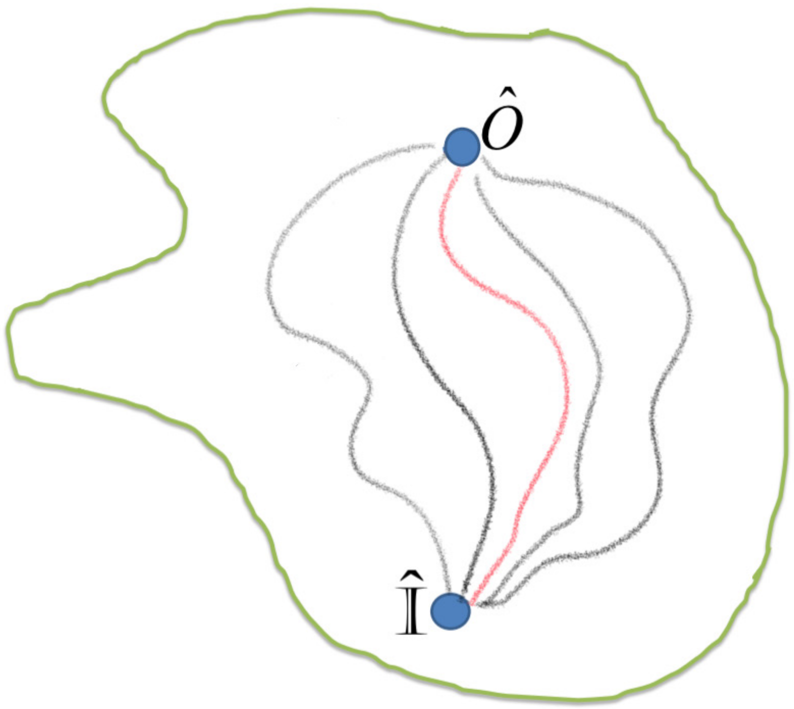
zero by $\mathbf{G1}$




**Ambiguity**

- same curve but different length $H_r$ or $H_l$
- The ambiguity will disappear if we consider symmetry of QFT

# What geometry?



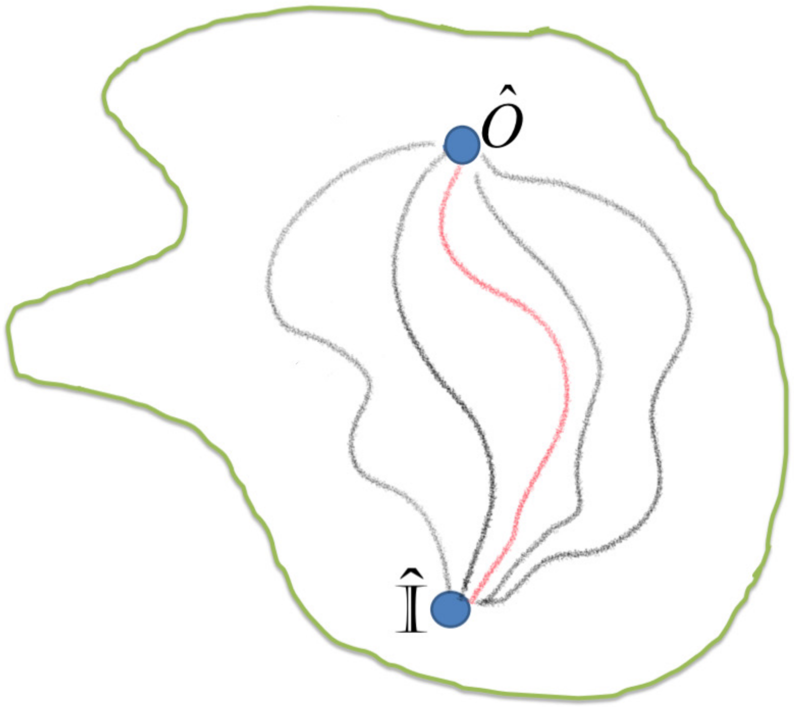

**Cost (Length)** $$L_\alpha[c] := \sum_{i=1}^{N} \mathcal{C}(\delta \hat{O}_i^{(\alpha)}) \xrightarrow{N\to\infty} \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s$$

**Complexity** $$\mathcal{C}_\alpha(\hat{O}) := \min\{L_\alpha[c] |\ \forall c(s),\ c(0) = \hat{\mathbb{I}},\ c(1) = \hat{O}\}$$

Geodesic in some geometry?

What geometry?

# What geometry?




**Cost (Length)** $$L_\alpha[c] := \sum_{i=1}^{N} \mathcal{C}(\delta\hat{O}_i^{(\alpha)}) \xrightarrow{N\to\infty} \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s$$

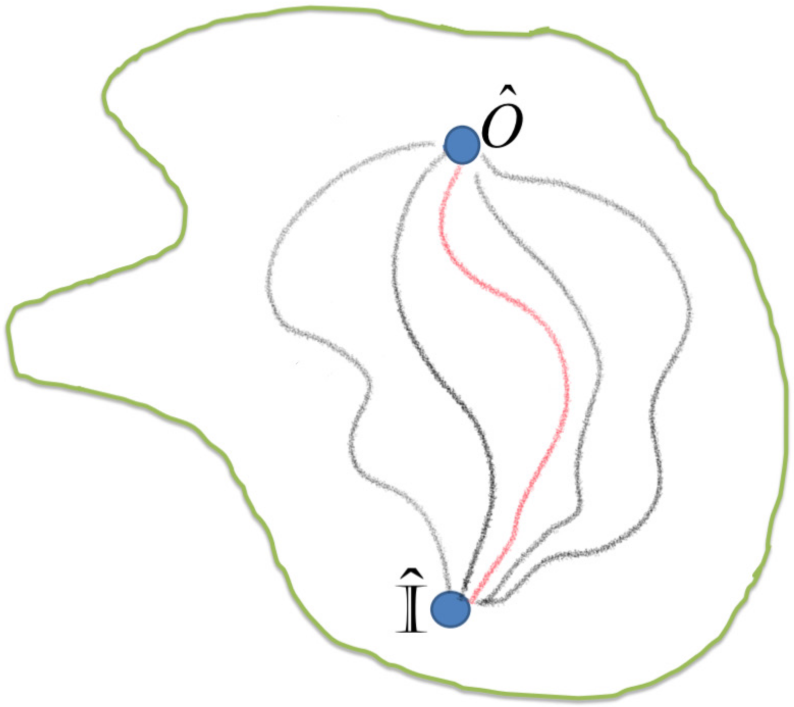
**Complexity** $$\mathcal{C}_\alpha(\hat{O}) := \min\{L_\alpha[c] |\ \forall c(s),\ c(0)=\hat{\mathbb{I}},\ c(1)=\hat{O}\}$$

Geodesic in some geometry?

What geometry?

$$\tilde{F}(H_\alpha)?$$

# What geometry?




**Cost (Length)** $L_\alpha[c] := \sum_{i=1}^{N} \mathcal{C}(\delta \hat{O}_i^{(\alpha)}) \xrightarrow{N\to\infty} \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s$

**Complexity** $\mathcal{C}_\alpha(\hat{O}) := \min\{L_\alpha[c] |\ \forall c(s),\ c(0) = \hat{\mathbb{I}},\ c(1) = \hat{O}\}$

Geodesic in some geometry?

What geometry?

$$\tilde{F}(H_\alpha)?$$

So far, it is any function.
Let us find constrains of $\tilde{F}(H_\alpha)$

# Outline

- General axioms
- Finsler geometry from general axioms
- Constrain the geometry by QFT symmetry
- Complexity for SU(n) operator

# Properties of geometric structure

Axioms of Complexity

**G1** $\forall \hat{x} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

**G2** $\forall \hat{x}, \hat{y} \in \mathcal{O}, \ \mathcal{C}(\hat{x}) + \mathcal{C}(\hat{y}) \geq \mathcal{C}(\hat{x}\hat{y})$.

**G4** $\mathcal{C}(\delta\hat{O}^{(\alpha)}) = \tilde{F}(H_\alpha)\delta s + \mathcal{O}(\delta s^2)$

$\delta\hat{O}^{(\alpha)} = \exp(H_\alpha \delta s)$

**F1** (Nonnegativity) $\tilde{F}(H_\alpha) \geq 0$ and $\tilde{F}(H_\alpha) = 0$ iff $H_\alpha = 0$

**F2** (Positive homogeneity) $\forall \lambda \in \mathbb{R}^+$, $\tilde{F}(\lambda H_\alpha) = \lambda \tilde{F}(H_\alpha)$

**F3** (Triangle inequality) $\tilde{F}(H_{\alpha,1}) + \tilde{F}(H_{\alpha,2}) \geq \tilde{F}(H_{\alpha,1} + H_{\alpha,2})$

# Properties of geometric structure

Axioms of Complexity

**G1** $\forall \hat{x} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

**G2** $\forall \hat{x}, \hat{y} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) + \mathcal{C}(\hat{y}) \geq \mathcal{C}(\hat{x}\hat{y})$.

**G4** $\mathcal{C}(\delta\hat{O}^{(\alpha)}) = \tilde{F}(H_\alpha)\delta s + \mathcal{O}(\delta s^2)$

$$\delta\hat{O}^{(\alpha)} = \exp(H_\alpha \delta s)$$

**F1** (Nonnegativity) $\tilde{F}(H_\alpha) \geq 0$ and $\tilde{F}(H_\alpha) = 0$ iff $H_\alpha = 0$

**F2** (Positive homogeneity) $\forall \lambda \in \mathbb{R}^+$, $\tilde{F}(\lambda H_\alpha) = \lambda \tilde{F}(H_\alpha)$

**F3** (Triangle inequality) $\tilde{F}(H_{\alpha,1}) + \tilde{F}(H_{\alpha,2}) \geq \tilde{F}(H_{\alpha,1} + H_{\alpha,2})$

$$F_\alpha(c, \dot{c}) := \tilde{F}(H_\alpha)$$

$$H_r = \dot{c}c^{-1} \qquad \dot{c}(s) = H_r(s)c(s)$$

$$H_l = c^{-1}\dot{c} \qquad \dot{c}(s) = c(s)H_l(s)$$

**F1′** (Nonnegativity) $F_\alpha(c, \dot{c}) \geq 0$ and $F_\alpha(c, \dot{c}) = 0$ iff $\dot{c} = 0$

**F2′** (Positive homogeneity) $\forall \lambda \in \mathbb{R}^+$, we have $F_\alpha(c, \lambda\dot{c}) = \lambda F_\alpha(c, \dot{c})$

**F3′** (Triangle inequality) $F_\alpha(c, \dot{c}_1) + F_\alpha(c, \dot{c}_2) \geq F_\alpha(c, \dot{c}_1 + \dot{c}_2)$

# Properties of geometric structure

Axioms of Complexity

**G1** $\forall \hat{x} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

**G2** $\forall \hat{x}, \hat{y} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) + \mathcal{C}(\hat{y}) \geq \mathcal{C}(\hat{x}\hat{y})$.

**G4** $\mathcal{C}(\delta\hat{O}^{(\alpha)}) = \tilde{F}(H_\alpha)\delta s + \mathcal{O}(\delta s^2)$

$$\delta\hat{O}^{(\alpha)} = \exp(H_\alpha \delta s)$$

**F1** (Nonnegativity) $\tilde{F}(H_\alpha) \geq 0$ and $\tilde{F}(H_\alpha) = 0$ iff $H_\alpha = 0$

**F2** (Positive homogeneity) $\forall \lambda \in \mathbb{R}^+$, $\tilde{F}(\lambda H_\alpha) = \lambda \tilde{F}(H_\alpha)$

**F3** (Triangle inequality) $\tilde{F}(H_{\alpha,1}) + \tilde{F}(H_{\alpha,2}) \geq \tilde{F}(H_{\alpha,1} + H_{\alpha,2})$

$$F_\alpha(c, \dot{c}) := \tilde{F}(H_\alpha)$$

$$H_r = \dot{c}c^{-1} \qquad \dot{c}(s) = H_r(s)c(s)$$

$$H_l = c^{-1}\dot{c} \qquad \dot{c}(s) = c(s)H_l(s)$$

Properties of Finsler geometry

**F1′** (Nonnegativity) $F_\alpha(c, \dot{c}) \geq 0$ and $F_\alpha(c, \dot{c}) = 0$ iff $\dot{c} = 0$

**F2′** (Positive homogeneity) $\forall \lambda \in \mathbb{R}^+$, we have $F_\alpha(c, \lambda\dot{c}) = \lambda F_\alpha(c, \dot{c})$

**F3′** (Triangle inequality) $F_\alpha(c, \dot{c}_1) + F_\alpha(c, \dot{c}_2) \geq F_\alpha(c, \dot{c}_1 + \dot{c}_2)$

# Properties of geometric structure

Axioms of Complexity

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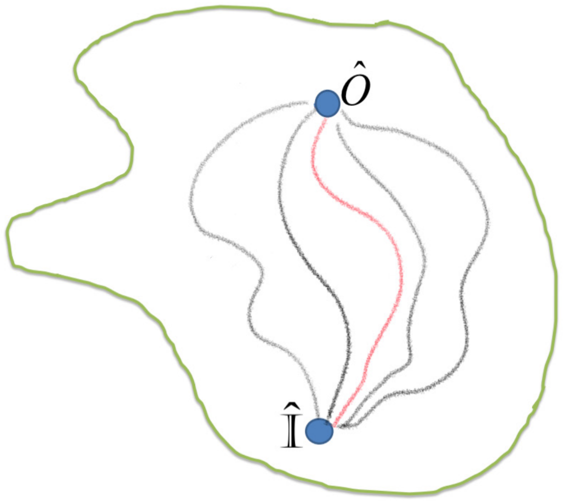
# Riemannian and Finsler Geometry

Finsler metric

**Cost (Length)**

$$L_\alpha[c] = \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s = \int_0^1 F_\alpha(c,\dot{c})\mathrm{d}s =: \int_0^1 \mathrm{d}l_\alpha$$




**Riemannian geometry**

$$\mathrm{d}l = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j}\mathrm{d}s = \sqrt{g_{ij}(x)v^iv^j}\mathrm{d}s = F(x,v)\mathrm{d}s$$

**Finsler geometry**

$$\mathrm{d}l = \sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j}\mathrm{d}s = \sqrt{g_{ij}(x,v)v^iv^j}\mathrm{d}s = F(x,v)\mathrm{d}s$$

**Metric tensor**

$$g_{ij} = \frac{\partial^2}{\partial v^i \partial v^j}\frac{F^2}{2}$$

**Cartan tensor**

$$A_{ijk}(x,v) = \frac{1}{2}\frac{\partial}{\partial v^k}g_{ij}(x,v) = \frac{1}{4}\frac{\partial^3 F^2(x,v)}{\partial v^i \partial v^j \partial v^k}$$

$A_{ijk} = 0$
for Reimannian geometry

property: $A_{ijk}(x,v)v^j = 0$

**Example**

$$F(x,v) = \sqrt{(v^1)^2 + (v^2)^2 + \sqrt{(v^1)^4 + (v^2)^4}}$$

Finsler geometry is just Riemannian geometry without the quadratic restriction

# Finsler Geometry: historical remarks

The concept was first appeared In 1854

$$L = \int dl \qquad dl = \mathcal{F}(x^1, \cdots, x^n; dx^1, \cdots, dx^n)$$

( *F* is positively homogeneous of degree 1 in $dx^n$ )

**F1′** (Nonnegativity) $F_\alpha(c, \dot{c}) \geq 0$ and $F_\alpha(c, \dot{c}) = 0$ iff $\dot{c} = 0$

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F2 is necessary for reparametrization invariance

# Finsler Geometry: historical remarks

$$L = \int dl \qquad dl = \mathcal{F}(x^1, \cdots, x^n; dx^1, \cdots, dx^n)$$

( $F$ is positively homogeneous of degree 1 in $dx^n$ )

1854: Riemann, "Habilitation" address

"Uber die Hypotheser welche der geometric zugrund liegen".

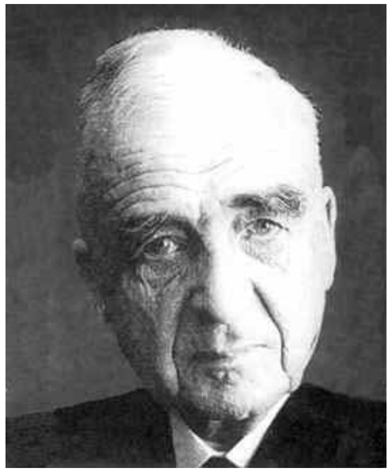
On the hypotheses, which lie at the Foundations of Geometry

"The study of metric which is the fourth root of a quartic differential form is quite time-consuming and does not throw new light to the problem."

$$F(x,v) = \sqrt{(v^1)^2 + (v^2)^2 + \sqrt{(v^1)^4 + (v^2)^4}}$$

$$\mathcal{F}^2(x, dx) = g_{ij}(x) dx^i dx^j$$

1918 : Palu Finsler's thesis



$$\mathcal{F}^2(x, dx) = g_{ij}(x, \dot{x}) dx^i dx^j$$

# Proof of F1

Axioms of Complexity

**G1** $\forall \hat{x} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

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Properties of Geometric Structure

**F1** (Nonnegativity) $\tilde{F}(H) \geq 0$ and $\tilde{F}(H) = 0$ iff $H = 0$

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**F3** (Triangle inequality) $\tilde{F}(H_1) + \tilde{F}(H_2) \geq \tilde{F}(H_1 + H_2)$

## Proof of F1

If $H = 0$ $\qquad \tilde{F}(H) \underset{\text{G4}}{=} \lim_{s \to 0^+} \mathcal{C}(\exp(Hs))/s \underset{\text{G1}}{=} 0$

If $H \neq 0$

$$0 \underset{\text{G1}}{<} \mathcal{C}(\exp(H\lambda)) = \mathcal{C}((\exp(H\lambda/N))^N) \underset{\text{G2}}{\leq} N\mathcal{C}(\exp(H\lambda/N)) \underset{\varepsilon := \lambda/N}{=} \lambda \frac{\mathcal{C}[\exp(H\varepsilon)]}{\varepsilon} \underset{\text{G4} \quad N \to \infty}{=} \lambda \tilde{F}(H)$$

# Proof of F2

**Axioms of Complexity**

**G1** $\forall \hat{x} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

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---

**Proof of F1**

$$\mathcal{C}(\exp(\lambda H \cdot \varepsilon)) = \mathcal{C}(\exp(H \cdot \lambda\varepsilon))$$

$$\Big\downarrow \text{ G4}$$

$$\tilde{F}(\lambda H)\varepsilon = \lambda \tilde{F}(H)\varepsilon$$

# Proof of F3

**Axioms of Complexity**

**G1** $\forall \hat{x} \in \mathcal{O}$, $\mathcal{C}(\hat{x}) \geq 0$ and the equality holds iff $\hat{x}$ is the identity.

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**Properties of Geometric Structure**

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---

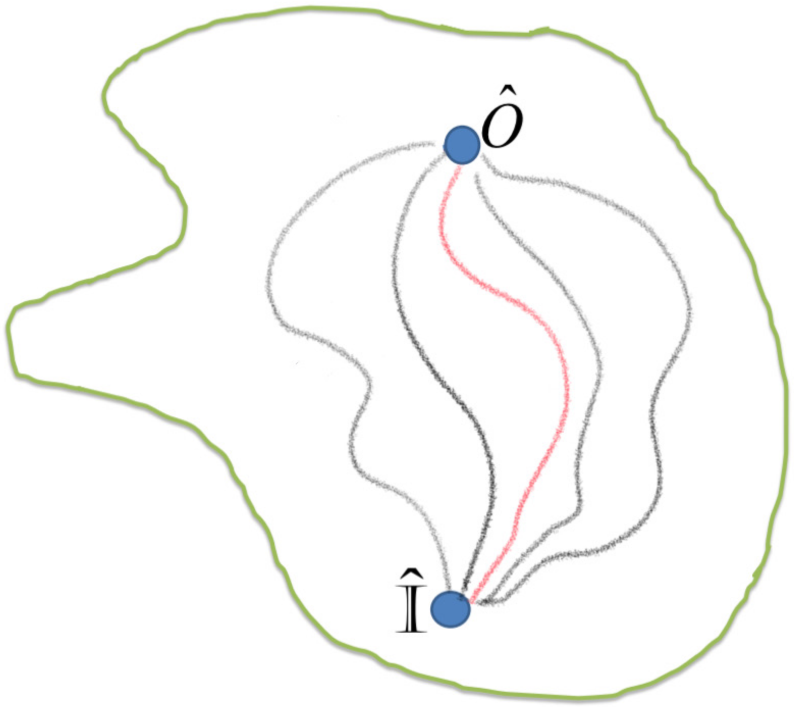
**Proof of F1**

$$\mathcal{C}(\exp(H_1\varepsilon)) + \mathcal{C}(\exp(H_2\varepsilon)) \underbrace{\geq}_{\text{G2}} \mathcal{C}(\exp(H_1\varepsilon)\exp(H_2\varepsilon)) \approx \mathcal{C}(\exp((H_1 + H_2)\varepsilon))$$

$$\Big\downarrow \text{G4}$$

$$\tilde{F}(H_1) + \tilde{F}(H_2) \geq \tilde{F}(H_1 + H_2)$$

# What geometry?




**Cost (Length)** $L_\alpha[c] := \sum_{i=1}^{N} \mathcal{C}(\delta \hat{O}_i^{(\alpha)}) \xrightarrow{N\to\infty} \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s$

**Complexity** $\mathcal{C}_\alpha(\hat{O}) := \min\{L_\alpha[c] |\ \forall c(s),\ c(0) = \hat{\mathbb{I}},\ c(1) = \hat{O}\}$

Geodesic in some geometry?

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$$\tilde{F}(H_\alpha)?$$

So far, it is any function.
Let us find constrains of $\tilde{F}(H_\alpha)$

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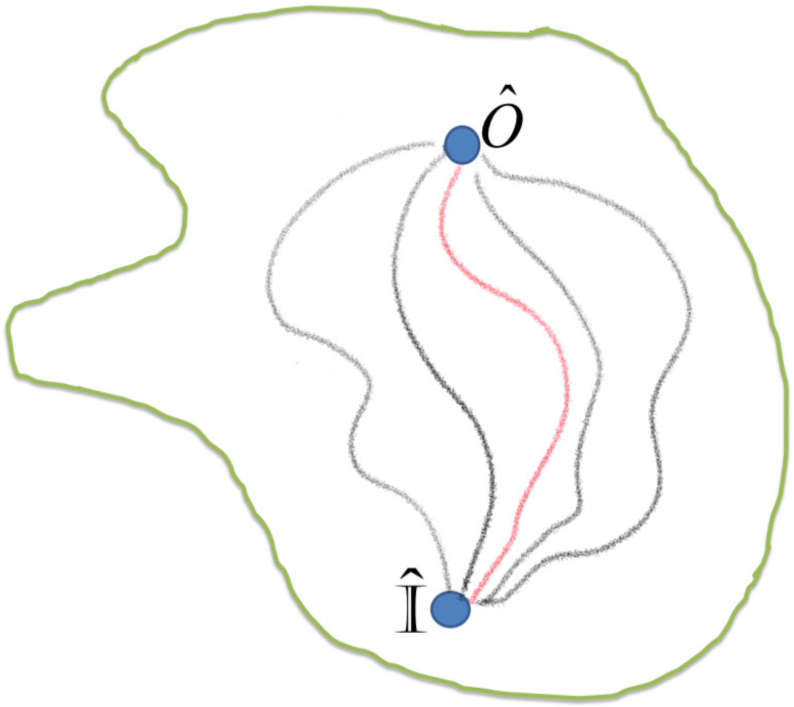


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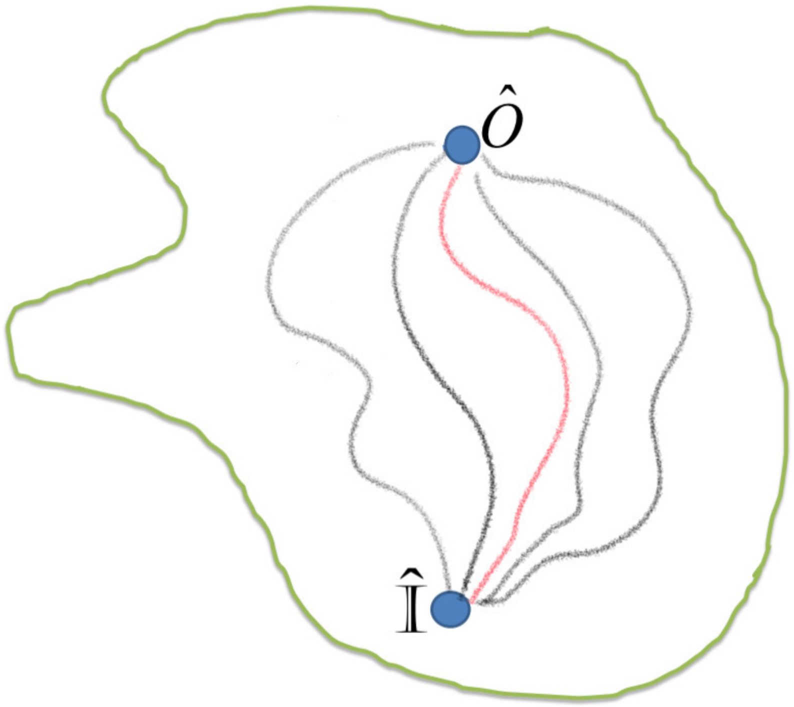


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More

# Outline

- General axioms
- Finsler geometry from general axioms
- Constrain the geometry by QFT symmetry
- Complexity for SU(n) operator

# Constraints on Finsler metric

Adjoint symmetry

$$\tilde{F}(H_\alpha) = \tilde{F}(\hat{U} H_\alpha \hat{U}^\dagger)$$

Left/right symmetry

$$\tilde{F}(H_l) = \tilde{F}(H_r)$$

Uniqueness of cost(length) :
Left/right ambiguity disappear!

Reversibility

$$\tilde{F}(H) = \tilde{F}(-H)$$

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Unitary symmetry

$$\forall \hat{U} \in \mathrm{SU}(n)$$

$$\boxed{L_\alpha[c] = L_\alpha[\hat{U} c(t) \hat{U}^\dagger]}$$

Uniqueness of cost(length) :
Left/right ambiguity disappear!

$$|\psi_T\rangle = c(t)|\psi_R\rangle$$

$$\hat{U}|\psi_T\rangle = (\hat{U} c(t) \hat{U}^\dagger)\hat{U}|\psi_R\rangle$$

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$$\tilde{F}(H) = \tilde{F}(-H)$$

Time-reversal symmetry

$$\boxed{L_\alpha[c(t)] = L_\alpha[c(-t)]}$$

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Reversibility

$$\tilde{F}(H) = \tilde{F}(-H)$$

Time-reversal symmetry

$$L_\alpha[c(t)] = L_\alpha[c(-t)]$$

**Proof**

$$L_\alpha[c] = L_\alpha[\hat{U} c(t) \hat{U}^\dagger]$$

$$c(t) := \exp(H_\alpha t)$$

$$\int_0^1 \tilde{F}(H_\alpha)\mathrm{d}t = \int_0^1 \tilde{F}(\hat{U} H_\alpha \hat{U}^\dagger)\mathrm{d}t$$

$$\tilde{F}(H_\alpha) = \tilde{F}(\hat{U} H_\alpha \hat{U}^\dagger)$$

$$\hat{U} = c \text{ for } H_l \quad \hat{U} = c^{-1} \text{ for } H_r$$

$$\tilde{F}(H_r) = \tilde{F}(H_l)$$

$$L_\alpha[c(t)] = L_\alpha[c(-t)]$$

$$c(t) := \exp(H_\alpha t)$$

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$$\tilde{F}(H) = \tilde{F}(-H)$$

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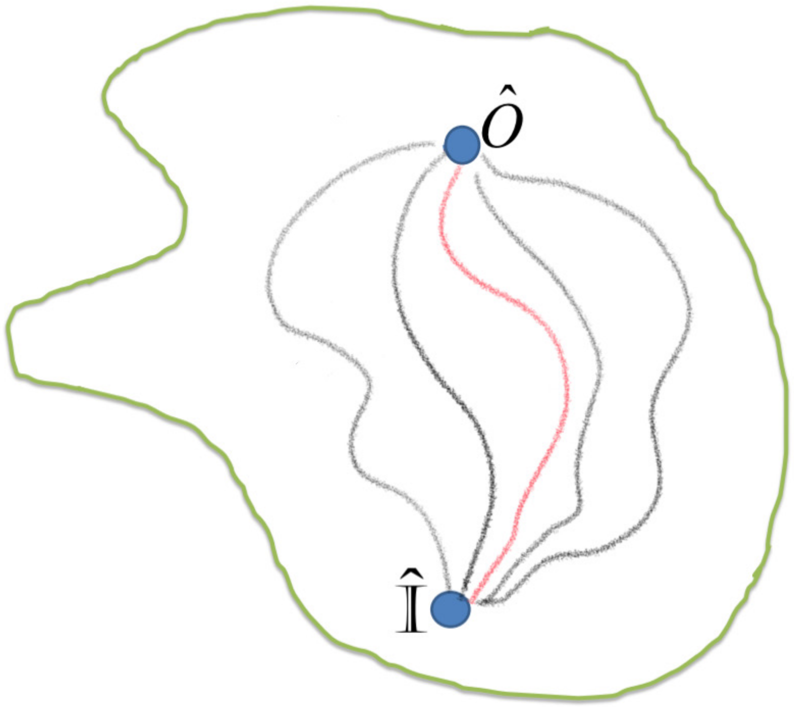


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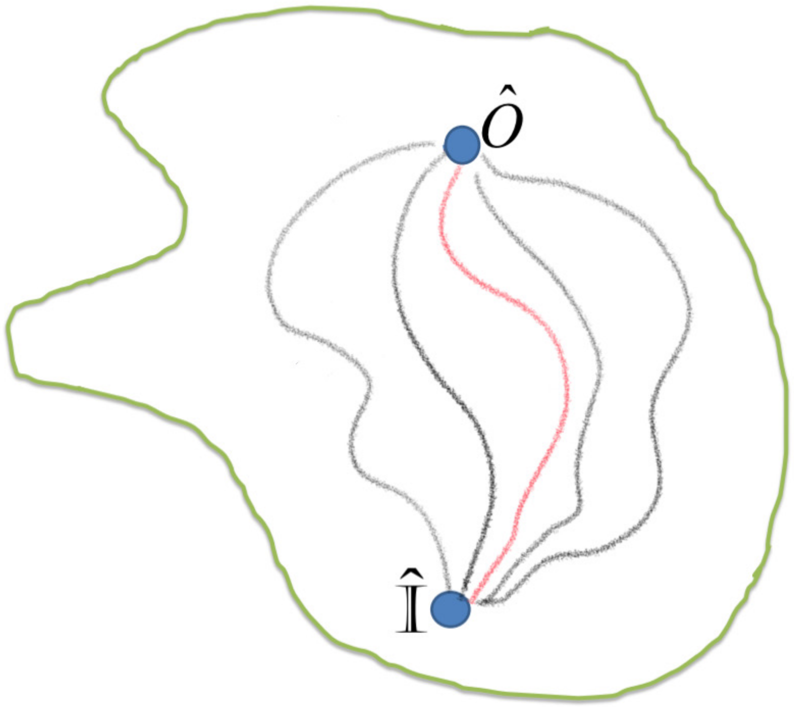
More

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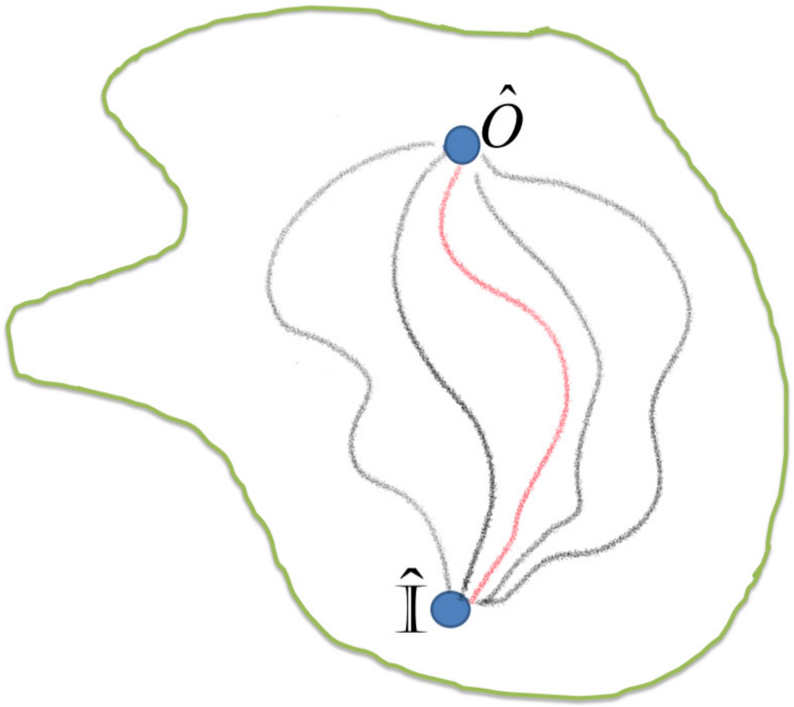
More ⟶

$$\begin{aligned}\tilde{F}(H_\alpha) &= \tilde{F}(\hat{U} H_\alpha \hat{U}^\dagger)\\ \tilde{F}(H_l) &= \tilde{F}(H_r)\\ \tilde{F}(H) &= \tilde{F}(-H)\end{aligned}$$

# What geometry?

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More ⟶

$$\tilde{F}(H_\alpha) = \tilde{F}(\hat{U} H_\alpha \hat{U}^\dagger)$$
$$\tilde{F}(H_l) = \tilde{F}(H_r)$$
$$\tilde{F}(H) = \tilde{F}(-H)$$

How much can we fix $\tilde{F}(H)$?

# Finsler metric of SU(n) operator

$$\boxed{\tilde{F}(H) = \lambda \mathrm{Tr}\left(\sqrt{HH^\dagger}\right)}$$

# Finsler metric of SU(n) operator

$$\tilde{F}(H) = \lambda \mathrm{Tr}\left(\sqrt{HH^\dagger}\right)$$

**Proof**

$$\begin{aligned}
\tilde{F}(H) &= \tilde{F}(\hat{U}H\hat{U}^{-1}) && \text{Adjoint symmetry} \\
&= \tilde{F}(\mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n)) && \text{Diagonal, order of the eigenvalues does not matter} \\
&= \sum_{j=1}^{n} f(\gamma_j) && \textbf{G3}\ [\textit{Parallel decomposition rule}]\ \forall(\hat{x}_1, \hat{x}_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{O}, \\
& && \mathcal{C}\big((\hat{x}_1, \hat{x}_2)\big) = \mathcal{C}\big((\hat{x}_1, \hat{\mathbb{I}}_2)\big) + \mathcal{C}\big((\hat{\mathbb{I}}_1, \hat{x}_2)\big) \\
&= \sum_{j=1}^{n} f(i\mathrm{Im}\gamma_j) && \text{Eigenvalues are all imaginary} \\
&= \sum_{j=1}^{n} f(i|\gamma_j|) && \tilde{F}(H) = \tilde{F}(-H) \quad \text{Reversibility} \\
&= \lambda \sum_{j=1}^{n} |\gamma_j| && \textbf{F2}\ \text{(Positive homogeneity)}\ \forall\lambda \in \mathbb{R}^+,\ \tilde{F}(\lambda H_\alpha) = \lambda\tilde{F}(H_\alpha) \\
&= \lambda \mathrm{Tr}\left(\sqrt{HH^\dagger}\right)
\end{aligned}$$

# Finsler metric of SU(n) operator: Riemannian or Finsler?

$$F(c(s),\dot{c}(s)) = \tilde{F}(H(s)) = \lambda \mathrm{Tr}\sqrt{H(s)H(s)^\dagger}$$

$$H(s) = iH^a(s)T_a, \quad H^a(s) \in \mathbb{R} \qquad T_aT_b = \frac{1}{2n}\delta_{ab}\hat{\mathbb{I}} + \frac{1}{2}\sum_{c=1}^{n^2-1}(if_{ab}{}^c + d_{ab}{}^c)T_c$$

$$\boxed{F(c,\dot{c}) = \frac{1}{\sqrt{2n}}\mathrm{Tr}\sqrt{H^a(s)H^b(s)[\delta_{ab}\hat{\mathbb{I}} + nd_{ab}{}^cT_c]}}$$

n=2: $F(c,\dot{c}) = \frac{1}{2}\mathrm{Tr}\sqrt{H^a(s)H^b(s)\delta_{ab}\hat{\mathbb{I}}} = \sqrt{H^a(s)H^b(s)\delta_{ab}}$ Riemannian

n>2 Finsler(Non-Riemannian)

Finsler geometry is just Riemannian geometry without the quadratic restriction

# Outline

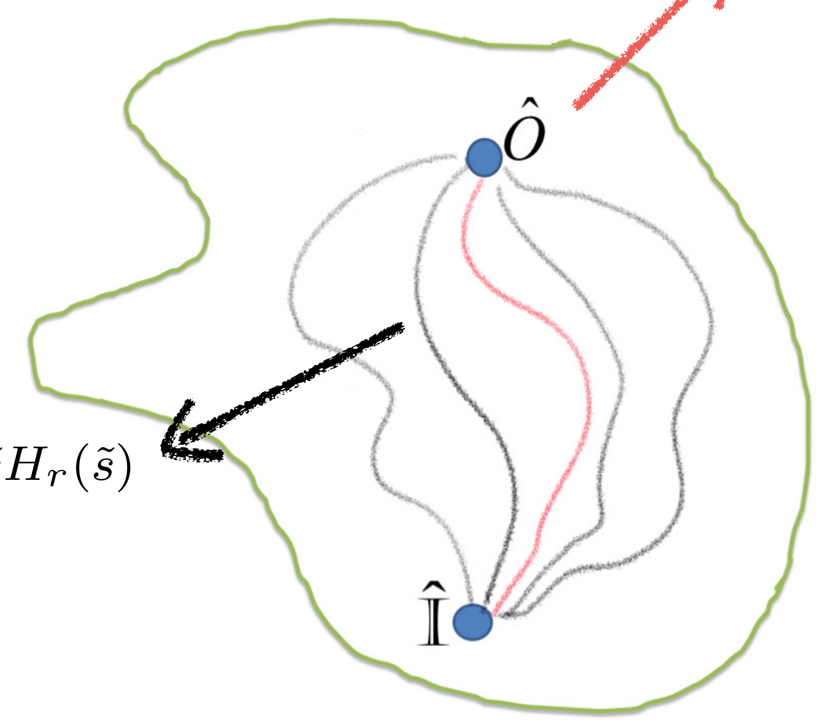
- General axioms
- Finsler geometry from general axioms
- Constrain the geometry by QFT symmetry
- Complexity for SU(n) operator

# Complexity of SU(n) operator

$$\mathcal{C}(\hat{O}) := \min\{L[c] |\ \forall c(s),\ c(0) = \hat{\mathbb{I}},\ c(1) = \hat{O}\}$$

$$L[c] = \int_0^1 \tilde{F}(H(s))\mathrm{d}s = \int_0^1 \mathrm{Tr}\sqrt{H(s)H^\dagger(s)}\mathrm{d}s$$

$$c(1) = \hat{O} = e^{\bar{H}}$$




$$c(s) = \overleftarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$

$$c(0) = \mathbb{I}$$

# Complexity of SU(n) operator

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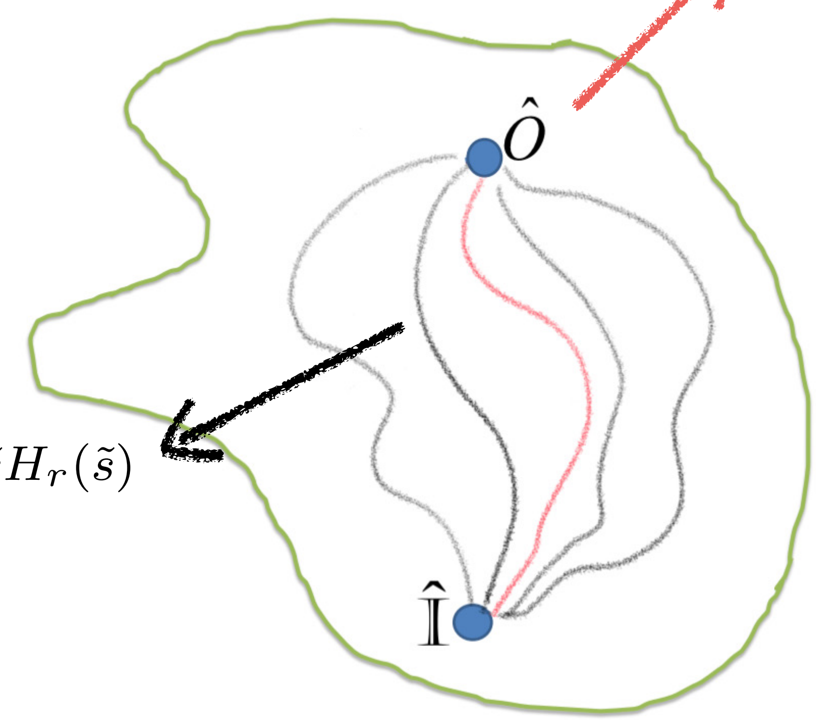



$$c(s) = \overleftarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$

$$c(0) = \mathbb{I}$$

For a bi-invariant Finsler geometry
c(s) is a geodesic iff there is a constant generator H

[Latifi and Razavi 2011,
Latifi and Toomanian 2013]

$$\boxed{\mathcal{C}(\hat{O}) = \min\{\mathrm{Tr}\sqrt{\bar{H}\bar{H}^\dagger} \mid \forall\, \bar{H} = \ln \hat{O}\}}$$

This 'min' means minimal 'geodesics'

Integral is trivial since H is constant

# Bi-invariance

## Left/right symmetry

$$\tilde{F}(H_r = \dot{c}c^{-1}) = \tilde{F}(H_l = c^{-1}\dot{c})$$

right-invariant: invariant under $c \to c\hat{U}$

left-invariant : invariant under $c \to \hat{U}c$

bi-invariance: left and right invariance

## Why bi-invariant is possible?

For this construction $U = g_n g_{n-1} \cdots g_2 g_1 W$

Right-invariance is natural because $UV = g_n g_{n-1} \cdots g_2 g_1 WV$

Left-invariance looks not possible because $VU = (V g_n g_{n-1} \cdots g_2 g_1 V^\dagger) VW$

However, left-invariance is also possible because $VU = (\tilde{g}_n \tilde{g}_{n-1} \cdots \tilde{g}_2 \tilde{g}_1) VW$

$$\tilde{g}_i = V g_i V^\dagger$$

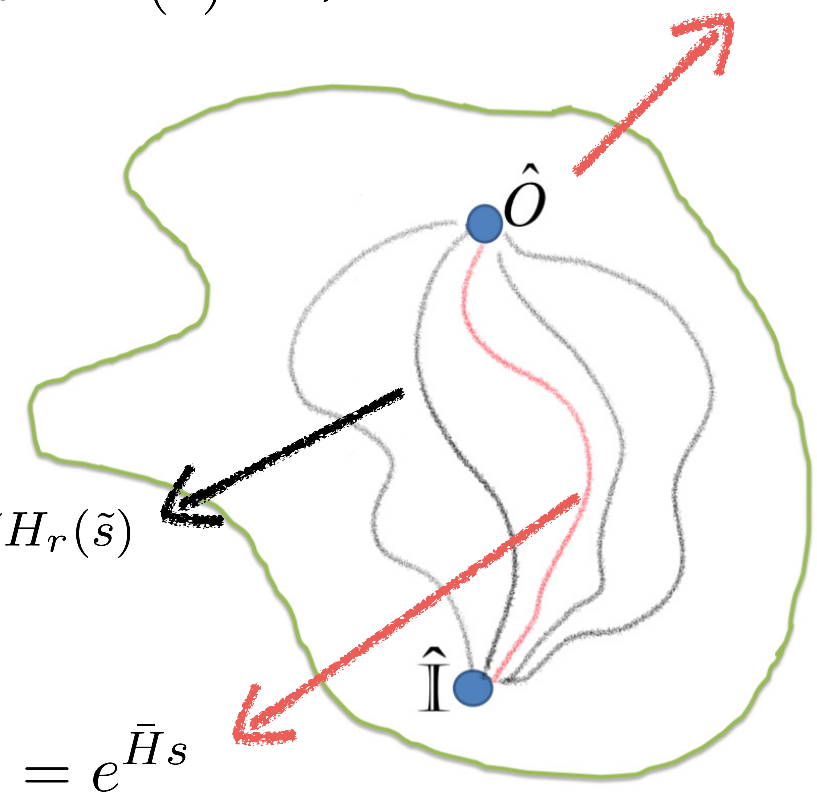
# Complexity of SU(n) operator

$$\mathcal{C}(\hat{O}) := \min\{L[c] |\ \forall c(s),\ c(0) = \hat{\mathbb{I}},\ c(1) = \hat{O}\}$$

$$\hat{O} = c(1) = \overleftarrow{\mathcal{P}} e^{\int_0^1 \mathrm{d}\tilde{s} H_r(\tilde{s})} = e^{\bar{H}}$$

$$L[c] = \int_0^1 \tilde{F}(H(s))\mathrm{d}s = \int_0^1 \mathrm{Tr}\sqrt{H(s)H^\dagger(s)}\mathrm{d}s$$

$$\boxed{\mathcal{C}(\hat{O}) = \min\{\mathrm{Tr}\sqrt{\bar{H}\bar{H}^\dagger} \mid \forall\, \bar{H} = \ln \hat{O}\}}$$




$$c(s) = \overleftarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$

$$c(s) = e^{\bar{H}s}$$

**Example: SU(2)**

$$\boxed{\mathcal{C}(\hat{O}) = 2\arccos[\mathrm{Tr}(\hat{O})/2]}$$

**Proof**

$$\hat{O} = \exp(i\theta\vec{n}\cdot\vec{\sigma}) = \hat{\mathbb{I}}\cos\theta + i(\vec{n}\cdot\vec{\sigma})\sin\theta$$

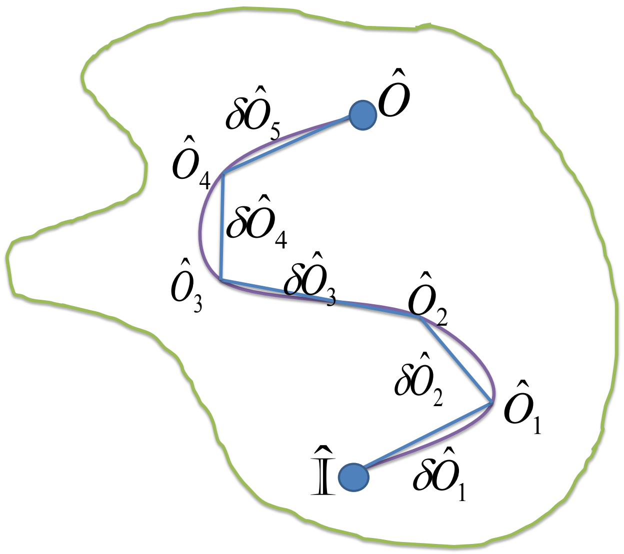
$$\bar{H} = \ln\hat{O} = i\theta_m\vec{n}\cdot\vec{\sigma} \qquad \theta_m = \arccos[\mathrm{Tr}(\hat{O})/2] + 2m\pi$$

$$\bar{H}\bar{H}^\dagger = \theta_m^2\hat{\mathbb{I}}$$

$$\mathcal{C}(\hat{O}) = \min\{\mathrm{Tr}\sqrt{\bar{H}\bar{H}^\dagger} \mid \forall\, \bar{H} = \ln\hat{O}\} = 2\theta_0 = 2\arccos[\mathrm{Tr}(\hat{O})/2]$$

# Summary and outlook

## Complexity of operator

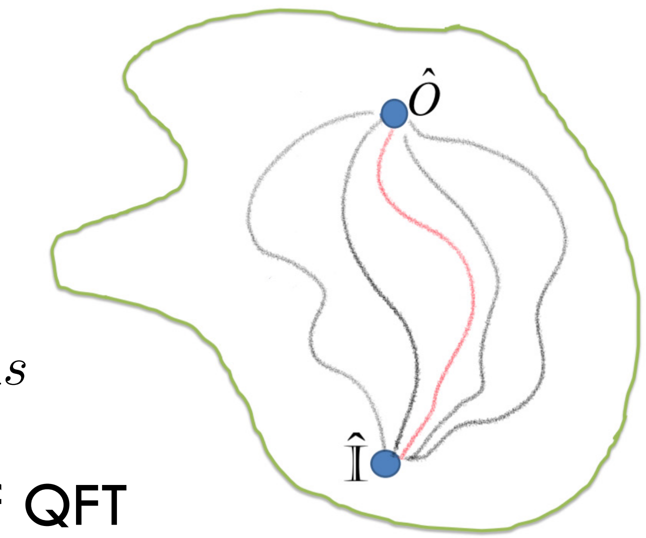



Smoothness axiom

$$c(s) = \overleftarrow{\mathcal{P}} e^{\int_0^s \mathrm{d}\tilde{s} H_r(\tilde{s})}$$

$$L_\alpha[c] = \int_0^1 \tilde{F}(H_\alpha(s))\mathrm{d}s = \int_0^1 F_\alpha(c,\dot{c})\mathrm{d}s$$

3 general axioms + 2 symmetries of QFT




$$\boxed{\mathcal{C}(\hat{O}) = \min\{\mathrm{Tr}\sqrt{\bar{H}\bar{H}^\dagger} \mid \forall\, \bar{H} = \ln \hat{O}\}}$$

## Future work

- Complexity of states, for example,

$$\mathcal{C}(\rho_1,\rho_2) := \{\mathcal{C}(\hat{O}) \,|\, \rho_2 = \hat{O}\rho_1\hat{O}^\dagger,\ \forall\hat{O} \in \mathrm{SU}(n)\}$$

- Comparison with holographic results and other field theory methods

Thank you