# Holographic Entanglement Entropy in Electric Field

**Conference Gauge/Gravity Duality 2018**

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Based on 1804.00412

# Contents

- Motivation
- Bulk dual to a system in an external field
- Response of Entanglement Entropy to DC Electric Field
  - Strip case
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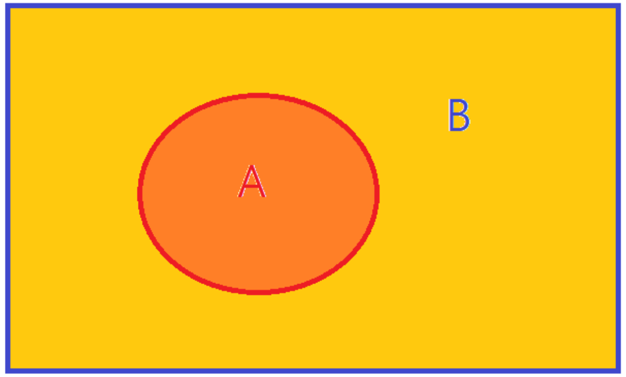
- Discussion and Future Direction

# (Holographic) entanglement entropy



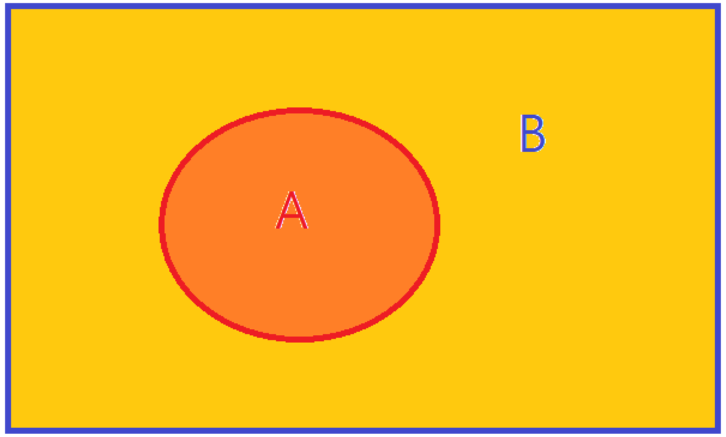

$\rho_A = \mathrm{Tr}_B \; |\Psi\rangle\langle\Psi| \qquad S_{EE} = \mathrm{Tr}\,(-\rho_A \log \rho_A)$

- Entanglement Entropy of a state
- This quantity is non-local, because of nontrivial trace on the subspace.
- In most cases, it is not easy to calculate this quantity.
  - Yesterday’s talks

# (Holographic) entanglement entropy

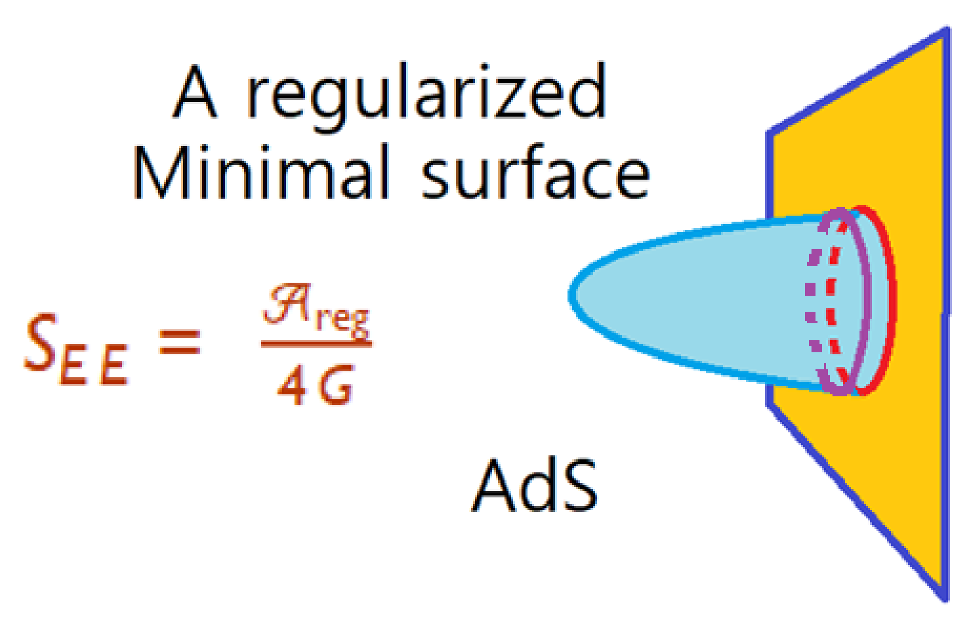
- How can we study this quantity through the AdS/CFT correspondence?
- Ryu and Takayanagi's proposal.

1. **Holographic derivation of entanglement entropy from AdS/CFT**
(1573) Shinsei Ryu, Tadashi Takayanagi (Santa Barbara, KITP). Mar 2006. 5 pp.
Published in **Phys.Rev.Lett. 96 (2006) 181602**
NSF-KITP-06-11
DOI: 10.1103/PhysRevLett.96.181602
e-Print: **hep-th/0603001** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service; AMS MathSciNet
Detailed record - Cited by 1573 records 1000+



# (Holographic) entanglement entropy

- What is a dual quantity to this entropy?
- Their answer is




- It is much easier to calculate this quantities than to do same thing by the field theory calculation.
- The first law of entanglement entropy = linearized Einstein equation. Raamsdonk(2017)
- See Raamsdonk(2017) and Sin(2017)  for nonlinear extension.

# Motivation

- So we will follow entanglement entropy through Ryu-Takayanagi's proposal.
- How can we trust the proposal?
- One can see that It can be a reliable approach for two dimensional field theories.
- In 2 dimensional CFT or FT, we have more results by a field theory calculation than the other dimensions.
- We can check Ryu-Takayanagi's formula with the field theory results.
- The proposal is good for the results in 2-dim.
- Although there are many plausible explanations supporting RT formula, we are not sure how much it is right and when it is reliable.

# Motivation

- It is very difficult to compute the entanglement entropy in the strong coupling regime by field theory calculation.
- We hope to find something which can be compared to Ryu-Takayanagi's entanglement entropy.
- If the entanglement entropy can be measured in experiments, then we may compare the RT formula to the experimental data.

# Motivation

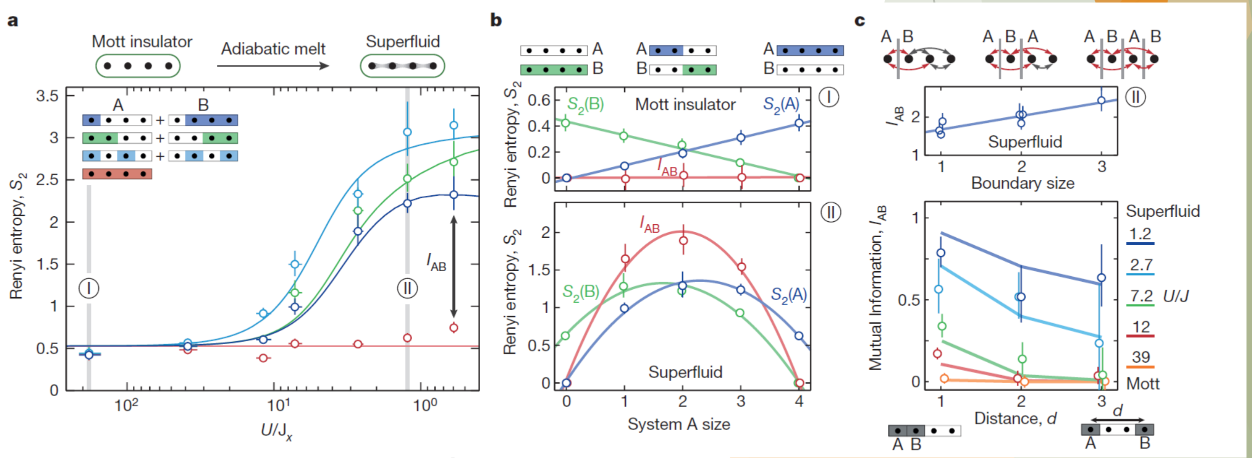
- Can we measure the entanglement entropy?
- Experimentalists are developing methods to measure the entanglement entropy or other related quantities.
- For instance, EE, Renyi entropy and mutual information for a very simple system. However their system is so far from the system for results of RT formula

nature
International journal of science

Measuring entanglement entropy in a quantum many-body system

Rajibul Islam, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli & Markus Greiner

*Nature* **528**, 77–83 (03 December 2015) | Download Citation



# Motivation

- Anyway experimentalists are developing the methods.
- Usually an entropy is not a direct observable.
- More important quantity is variations of entropy.
- If we measure entanglement entropy changes under some external sources, those may be compared to the Ryu-Takayangi's calculations.
- So we will focus on a holographic entanglement entropy response to an external source.
- It could be compared to result from some experiment near future.

# Motivation

- Since one of the basic experiments to investigate matters is measuring responses to an electric field, we choose the electric field as a source.
- Therefore, we will examine the entanglement entropy change by applying an electric field through the gauge/gravity duality.
- We will consider constant electric field for simplicity.
- Thus we will care DC currents.

- How does a system change by turning an external electric field?
- The electric field induces two kinds of currents.
- Electric current J and Heat current Q

$$\begin{pmatrix} \langle J^i \rangle \\ \langle Q^i \rangle \end{pmatrix} = \begin{pmatrix} \hat{\sigma}^{ij} & \hat{\alpha}^{ij}T \\ \hat{\bar{\alpha}}^{ij}T & \hat{\bar{\kappa}}^{ij}T \end{pmatrix} \begin{pmatrix} E_j \\ -(\nabla_j T)/T \end{pmatrix}$$

$$\sigma^{ij} \Leftrightarrow -\frac{iG^{ij}_{JJ}}{\omega}$$

$$\bar{\alpha}^{ij}T \Leftrightarrow \frac{i(\mu G^{ij}_{JJ} - G^{ij}_{TJ})}{\omega}$$

- The coefficients are given by Kubo formula : Green's Function (Two point Functions)
- These two point functions are called electric conductivity, thermoelectric conductivity and thermal conductivity.

# Background dual to a system in an external fields

- A non-dynamical electric and magnetic field in gauge/gravity duality correspond to a local bulk gauge field.

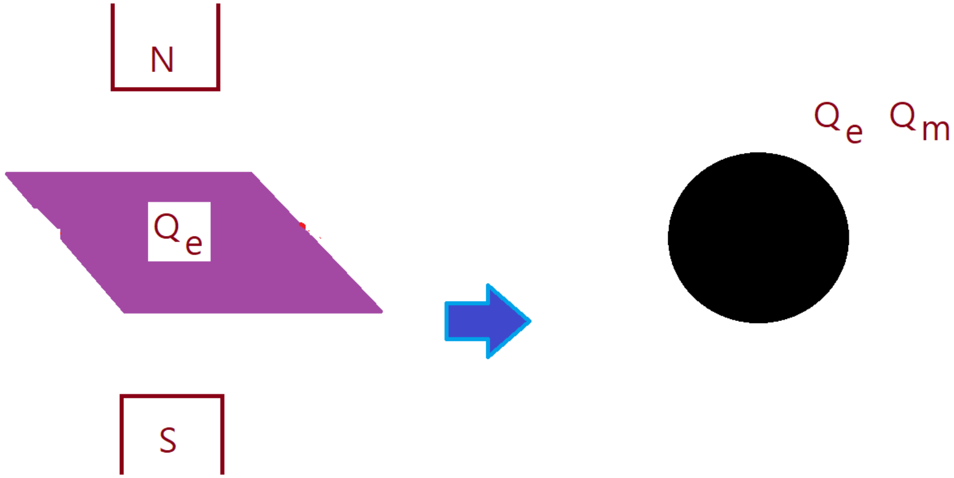
$$A_\mu(x, r) \sim A_\mu{}^b(x) + \frac{J_\mu{}^b}{r^{d-1}} + \ldots$$

$$F_{\mu\nu} = \partial_\mu A_\nu{}^b - \partial_\nu A_\mu{}^b$$

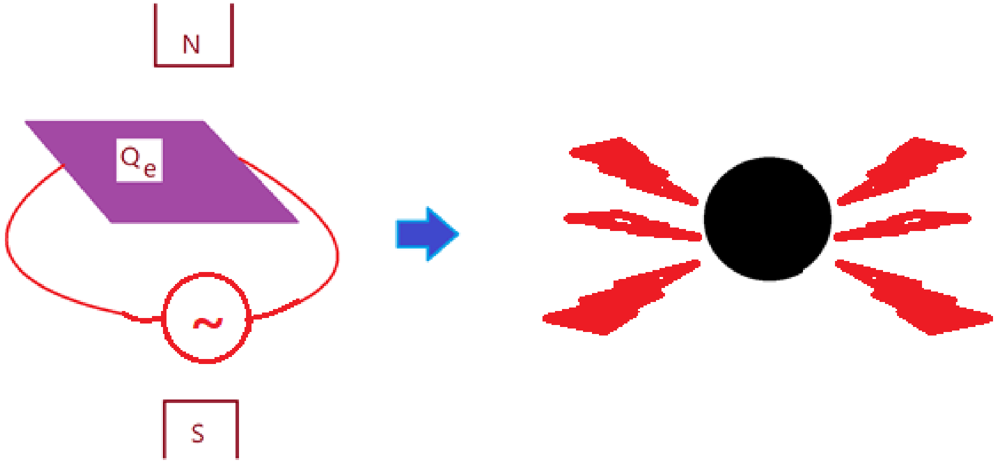
- We focus on d = 3, i.e, 2+1 dimensional CFT, because there have been a lot of works on this topic recently.

# Background dual to a system in an external fields

- Dyonic BH



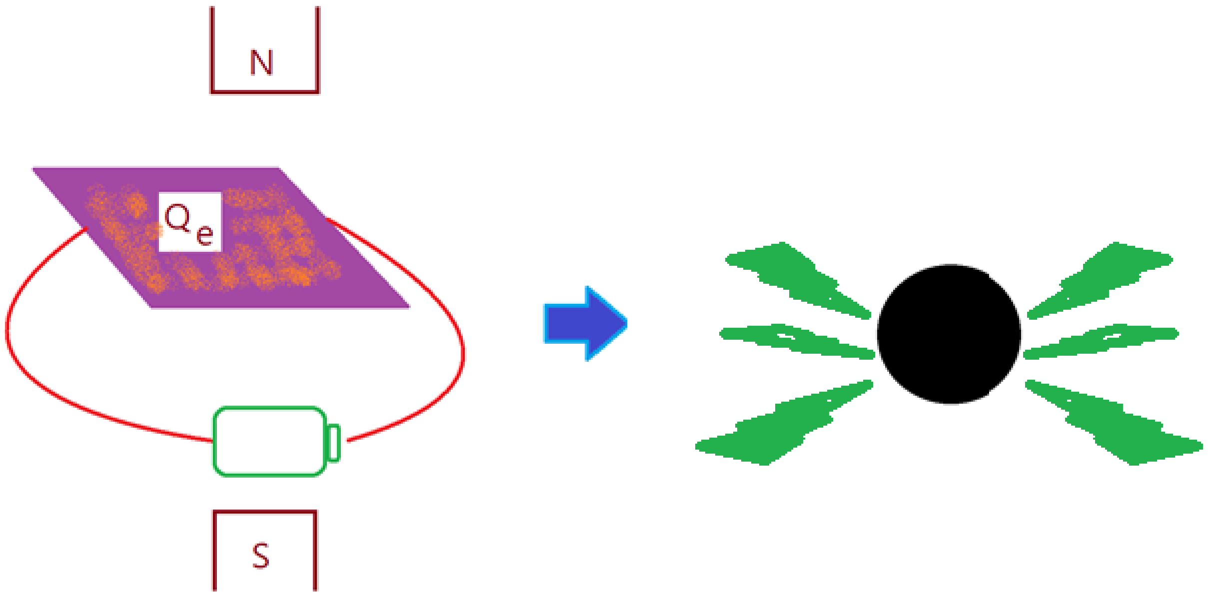

- Dyonic BH + gauge field ; weak electric field ( DC or AC )



# DC electric field for simplicity

- As a simplest gravity dual : Axion Model
- to have finite DC conductivity

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{4}F^2 - \frac{1}{2}\sum_{I=1}^{2}(\partial\chi_I)^2\right)$$



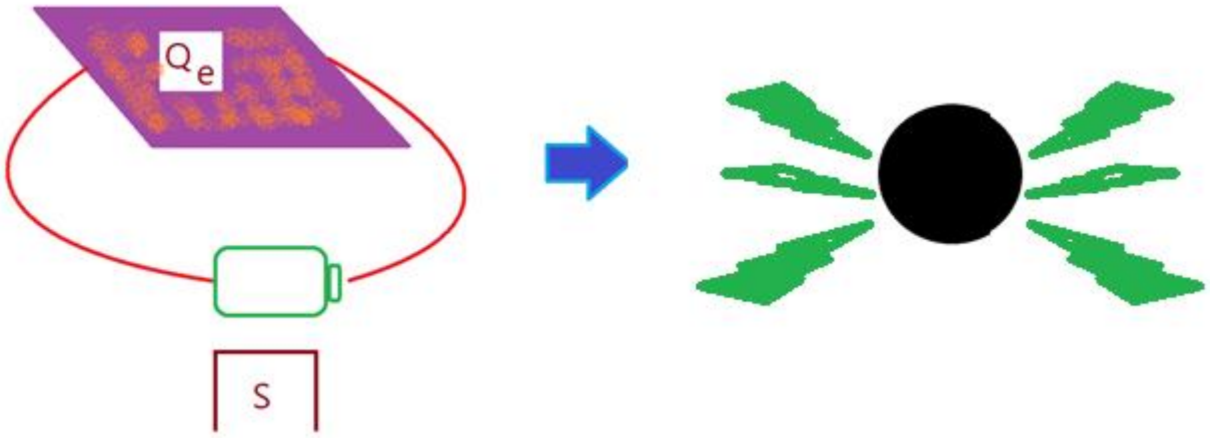
- This bulk theory admits a black brane solution.

$$ds^2 = -U(r)dt^2 + r^2\left(dx^2 + dy^2\right) + \frac{dr^2}{U(r)}$$

$$A = q\left(\frac{1}{r_h} - \frac{1}{r}\right)dt\ ,\ \ \chi_I = (\beta x, \beta y)\ \ ,$$

$$U(r) = \left(r^2 - \frac{\beta^2}{2} - \frac{M}{r} + \frac{q^2}{4r^2}\right)$$

$$\delta A_x = -E_x t + a_x(r)\ \ ,\ \ \delta g_{tx} = r^2 h_{tx}(r)\ \ ,\ \ \delta g_{rx} = r^2 h_{rx}(r)\ \ ,\ \ \delta\chi_x = \psi_x(r),$$



$$h''_{tx}(r) + \frac{4\,h'_{tx}(r)}{r} - \frac{\beta^2 h_{tx}(r)}{r^2\,U(r)} + \frac{q\,a'_x(r)}{r^4} = 0 \quad , \quad \psi'_x(r) - \beta h_{rx}(r) - \frac{qE_x}{\beta\,r^2 U(r)} = 0$$

$$a''_x(r) + \frac{U'(r)a'_x(r)}{U(r)} + \frac{qh'_{tx}(r)}{U(r)} = 0 \ ,$$

$$\psi''_x(r) + \left(\frac{U'(r)}{U(r)} + \frac{2}{r}\right)\psi'_x(r) - \beta h'_{rx}(r) - \beta\left(\frac{U'(r)}{U(r)} + \frac{2}{r}\right)h_{rx}(r) = 0 \quad ,$$

$$J^x = \lim_{r\to\infty}\mathcal{J}(r) \quad , \quad Q^x = T^{tx} - \mu J^x = \lim_{r\to\infty}\mathcal{Q}(r)$$

$$\begin{aligned} \mathcal{J}(x) &\equiv \sqrt{-g}F^{xr} = U(r)\ a'_x(r) + q\ h_{tx}(r), \\ Q(r) &\equiv U^2(r)\left(\frac{\delta g_{tx}}{U(r)}\right)' - A_t(r)\mathcal{J}(r) \quad . \end{aligned}$$

- Current and Heat current

$$J^x = \lim_{r\to r_h} \mathcal{J}(r) \quad , \quad Q^x = \lim_{r\to r_h} \mathcal{Q}(r)$$

- In-falling BC

$$\begin{aligned} a_x &\sim -\frac{E_x \log(r-r_h)}{4\pi T} + a_x^{(0)} + \mathcal{O}(r-r_h) \quad , \quad h_{tx} \sim h_{tx}^{(0)} + \mathcal{O}(r-r_h) \\ h_{rx} &\sim \frac{\mathbb{H}_{rx}}{r^2 U(r)} + h_{rx}^{(0)} + \mathcal{O}(r-r_h) \quad , \quad \psi_x \sim \psi_x^{(0)} + \mathcal{O}(r-r_h) \quad , \\ U &\sim 4\pi T(r-r_h) + \cdots \quad . \end{aligned}$$

$$\mathbb{H}_{rx} = r_h^2 h_{tx}^{(0)} \quad , \quad h_{tx}^{(0)} = -\frac{qE_x}{\beta^2 r_h^2}$$

# Contents

- Conductivities

$$\begin{pmatrix} J \\ Q \end{pmatrix} = \begin{pmatrix} \sigma & \alpha T \\ \bar{\alpha} T & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E \\ -(\nabla T)/T \end{pmatrix}$$

$$\sigma = 1 - \frac{q h_{tx}^{(0)}}{E_x} = 1 + \frac{\mu^2}{\beta^2} \quad , \quad \bar{\alpha} = -\frac{4\pi r_h^2 h_{tx}^{(0)}}{E_x} = \frac{4\pi\mu}{\beta^2} r_h,$$

$$\begin{aligned} ds^2 &= -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2(dx^2 + dy^2) \\ &\quad +2\lambda\Big(g_{tx}(r)dtdx + g_{rx}(r)drdx\Big) + \mathcal{O}\left(\lambda^2\right), \end{aligned}$$

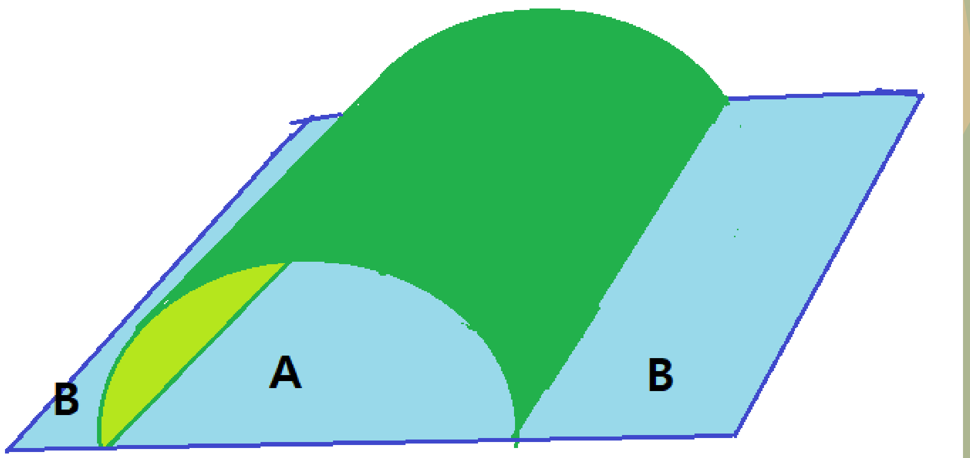
# HEE in this background

$$
\begin{aligned}
ds^2 &= -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2(dx^2 + dy^2) \\
&\quad +2\lambda\Big(g_{tx}(r)dtdx + g_{rx}(r)drdx\Big) + \mathcal{O}\left(\lambda^2\right),
\end{aligned}
$$

- Ryu-Takayanagi Formula (The Strip case)

$$t = 0 \quad , \quad r = r(\sigma_1) \quad , \quad x = \sigma_1 \quad , \quad y = \sigma_2$$

$$A = \int_{-L/2}^{L/2} dy \int_{-l/2}^{l/2} dx \sqrt{\frac{r^2 r'^2}{U(r)} + r^4 + 2\lambda r^2 g_{rx} r'}$$



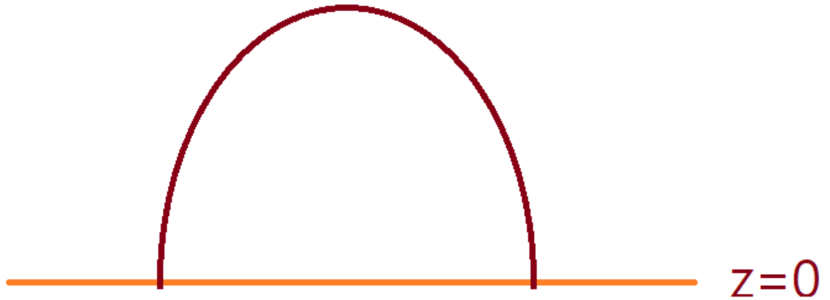
# Strip case

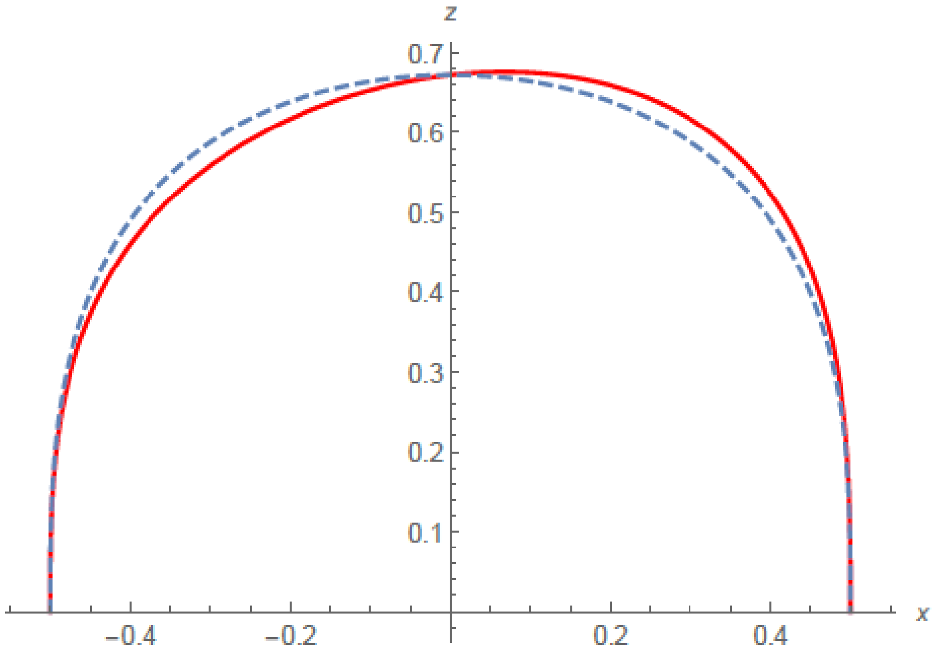
- With a convenient coordinate z

$$A = \frac{L}{l}\int_{-1/2}^{1/2} d\sigma \sqrt{\frac{z'^2}{z^6 f(z)} + \frac{1}{z^4} - 2\lambda \frac{z'}{z^4} g_{rx}(z)}$$

$$H = \frac{L}{l}\frac{1-\lambda g_{rx} z'}{z\sqrt{\frac{z'^2}{f(z)} + z^2(1-2\lambda z' g_{rx})}} = \frac{L}{l z_*^2}$$

$$\begin{aligned} z' &= \pm\frac{\sqrt{f(z)\left(z_*^4 - z^4\right)}}{z} - \lambda\,\frac{f(z) g_{rx}(z)\left(z_*^4 - z^4\right)}{z^2} + \mathcal{O}\left(\lambda^2\right) \\ &= \pm\frac{\sqrt{f(z)\left(z_*^4 - z^4\right)}}{z} + \lambda\, E_x \frac{\bar{\alpha} l^2 \left(z_*^4 - z^4\right)}{4\pi z^2} + \mathcal{O}\left(\lambda^2\right) \quad , \end{aligned}$$



# Strip case

- S = A / 4 G

$$A_{reg} = 2\int_{\epsilon}^{z_*} dz \frac{L z_*^2}{z^3\sqrt{f(z)\left(z_*^4 - z^4\right)}} \quad ,$$

- Expand z

$$z(\sigma) = z_0(\sigma) + \lambda z_1(\sigma) + \mathcal{O}\left(\lambda^2\right)$$

- Eom : Even and Odd

$$z_0'' - \frac{z_0'^2 f'(z_0)}{2f(z_0)} + 2z_0 f(z_0) + \frac{z_0'^2}{z_0} = 0$$

$$z_1'' + \frac{z_1 f'(z_0)^2 z_0'^2}{2f(z_0)^2} + 2z_0 z_1 f'(z_0) + \frac{z_0'\left(-z_1 f''(z_0) z_0' + g_{rx}(z_0) f'(z_0) z_0'^2 - 2f'(z_0) z_1'\right)}{2f(z_0)}$$
$$+ 2f(z_0)\left(z_1 - 3z_0 g_{rx}(z_0) z_0'\right) + g_{rx}'(z_0) z_0'^3 + \frac{2z_0' z_1' - g_{rx}(z_0) z_0'^3}{z_0} - \frac{z_1 z_0'^2}{z_0^2} = 0$$

# Strip case

$$A_{reg} = 2\int_{\epsilon}^{z_*} dz \frac{L z_*^2}{z^3\sqrt{f(z)\,(z_*^4 - z^4)}}$$

- Zeroth order : we obtained analytic result in some limits and numerical result.
- Analytic results in the small l limit

$$\begin{aligned} S_{EE}^{\text{strip}} &= \frac{L}{4G_N l}\Bigg[\frac{2\,l}{\epsilon} + \frac{\sqrt{2}\,\pi^2\,\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})^3} + \frac{\pi\, r_h^2\,\gamma_1\,\Gamma(\frac{1}{4})}{\sqrt{2}\,\Gamma(-\frac{1}{4})^2\,\Gamma(\frac{7}{4})} l^2 + \frac{\pi\, r_h^3 (2\,\gamma_1 - 4 - \gamma_2)\Gamma(\frac{1}{4})}{\sqrt{2}\,\Gamma(-\frac{1}{4})^3} l^3 \\ &+ \frac{r_h^4 (576\,\pi^3 (3\,\gamma_1^2 - 4\,\gamma_2)\Gamma(\frac{3}{4}) - 5\sqrt{2}\,\gamma_1^2\,\Gamma(\frac{1}{4})^7)}{92160\,\pi^2\,\Gamma(\frac{3}{4})^5} l^4 \\ &+ \frac{8\,\pi^2\, r_h^5\,\gamma_1 (2\,\gamma_1 - 4 - \gamma_2)(\pi^3\,\Gamma(-\frac{1}{4}) + 48\,\Gamma(\frac{3}{4})^5)}{3\,\Gamma(-\frac{1}{4})^7\,\Gamma(\frac{3}{4})^6} l^5 + \mathcal{O}(l^6)\Bigg] \end{aligned}$$

# Strip case

- Zero temperature result

$$S_{EE}^{\text{strip}} = \frac{L}{4G_N l}\left[\frac{2l}{\epsilon} + \frac{\sqrt{2}\,\pi^2\,\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{4})^3} + \frac{64\,\pi^2\,\beta^2}{3\,\Gamma(-\frac{1}{4})^4}\,l^2 + \frac{16\,\pi^2\,r_h(4\,r_h^2-\beta^2)}{\Gamma(-\frac{1}{4})^4}\,l^3 \right.$$
$$+\frac{8\,\pi\,(-160\,\pi^2\,\beta^4\,\Gamma(-\frac{3}{4})^2+(3\,\beta^4+8\,r_h^2\,\beta^2-48\,r_h^4)\Gamma(-\frac{1}{4})^6)}{5\,\Gamma(-\frac{1}{4})^{10}}\,l^4$$
$$\left.-\frac{8\pi^2\,r_h\,\beta^2(4r_h^2-\beta^2)(\pi^3\,\Gamma(-\frac{1}{4})+48\,\Gamma(\frac{3}{4})^5)}{\Gamma(-\frac{1}{4})^7\,\Gamma(\frac{3}{4})^5\,\Gamma(\frac{7}{4})}\,l^5 + \mathcal{O}(l^6)\right].$$

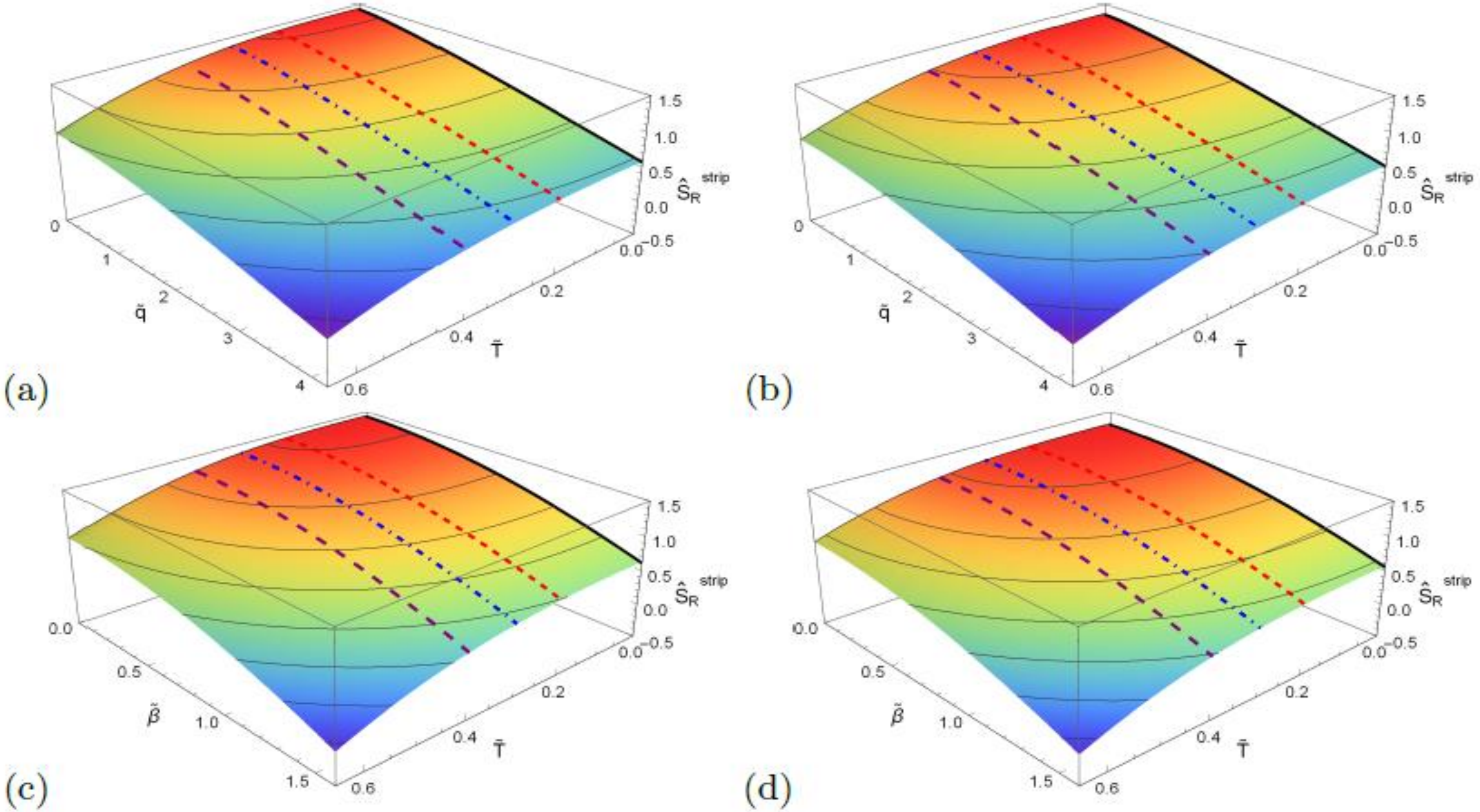
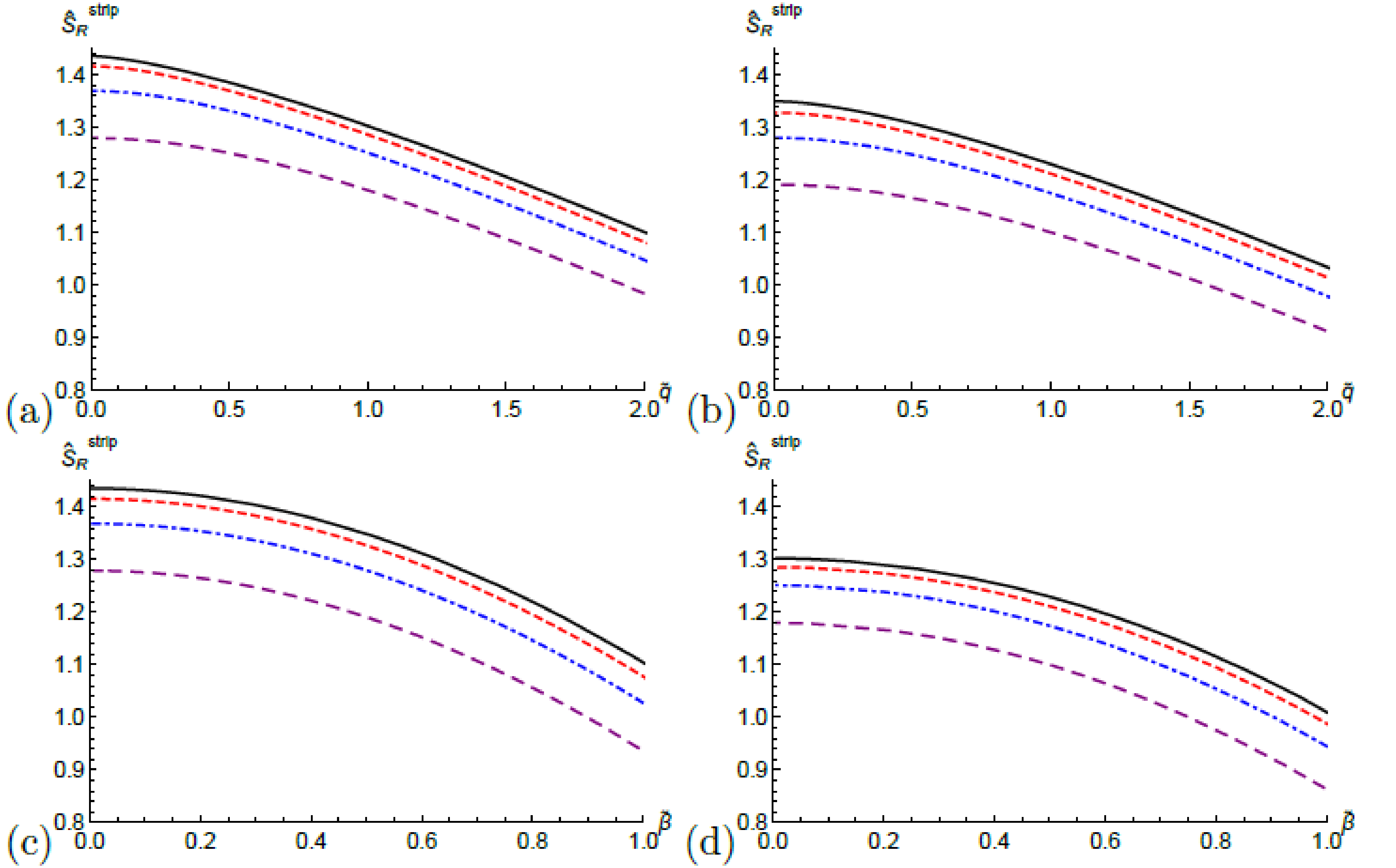



Figure 2: The refined function $\hat{S}_R^{\text{strip}}$ at $\tilde{\beta} = 0\,$(a), $\tilde{\beta} = 0.5\,$(b), $\tilde{q} = 0\,$(c) and $\tilde{q} = 1\,$(d) : The lines on the surfaces denote $\tilde{T} = 0\,$(Solid), 0.2 (Shortdashed), 0.3 (Dotdashed), 0.4 (Longdashed) and these lines are plotted in Fig. 3.

- This is consistent with Gushterov, O'Bannonb and Rodgers 2017

$$\hat{S}_R^{\text{strip}} \equiv \frac{2l}{\epsilon} - \frac{4\,G_N l}{L} S_{EE}^{\text{strip}} \quad .$$



$\hat{S}_R^{\,\mathrm{strip}}$
1.4
1.3
1.2
1.1
1.0
0.9
0.8
(a)
0.0
0.5
1.0
1.5
2.0
$\tilde{q}$
(b)
(c)
0.2
0.4
0.6
0.8
$\tilde{\beta}$
(d)

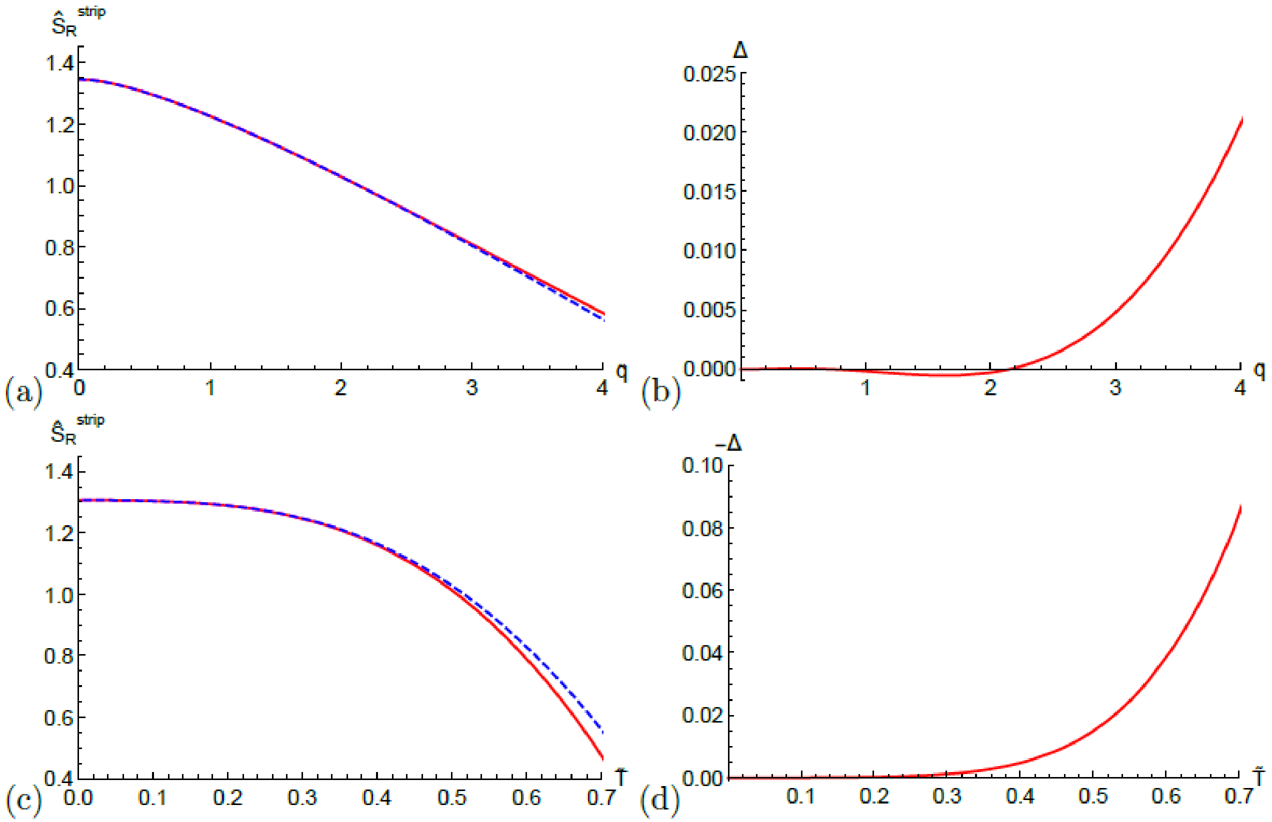



Figure 4: Comparison of the numerical calculation(Dashed) and the analytic calculation(Solid) for $\tilde{\beta} = 0.5$: (a) is for a fixed temperature $\tilde{T} = 0.1$ and (b) describes the difference $\Delta$ between two results. And (c) and (d) show the case with a fixed charge density $\tilde{q} = 0.5$. It can be seen that the absolute value of the difference $|\Delta|$ in the case of the fixed temperature becomes larger than that in the case where the charge density is fixed.

# Strip case 1st order response to E ?

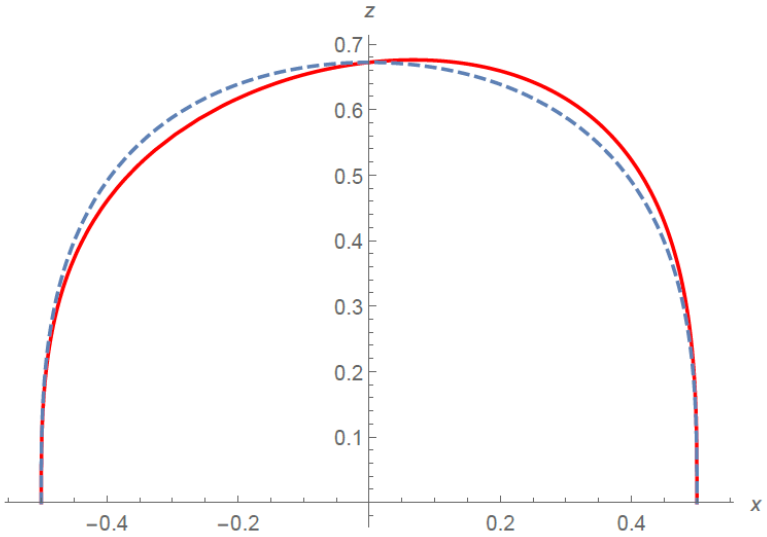
$$A_{reg} = 2\int_{\epsilon}^{z_*} dz \frac{L z_*^2}{z^3\sqrt{f(z)\left(z_*^4 - z^4\right)}}$$

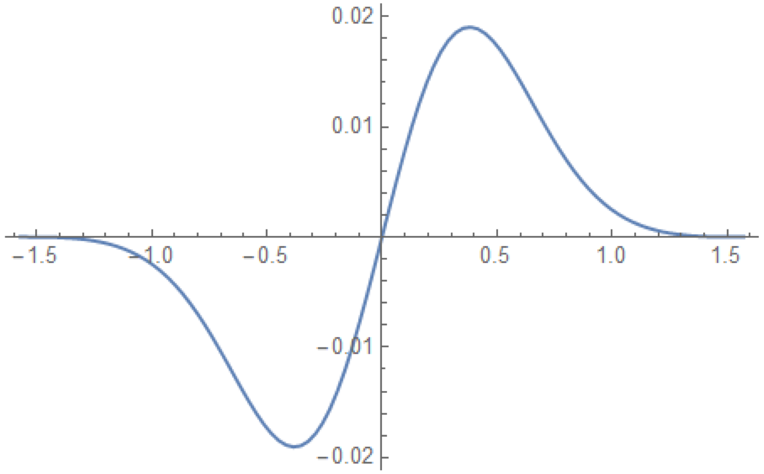
- No linear response to the electric field!
- To see the effect of entanglement entropy from electric field, we have to consider quadratic order in the electric field.
- We don't know the background.
- We cannot see the change under an electric field in this case.

- Why can't we see the linear response?
- This case is so symmetric case. The solution of first order equation is an odd function!

$$z_0'' - \frac{z_0'^2 f'(z_0)}{2f(z_0)} + 2z_0 f(z_0) + \frac{z_0'^2}{z_0} = 0$$

$$z_1'' + \frac{z_1 f'(z_0)^2 z_0'^2}{2f(z_0)^2} + 2z_0 z_1 f'(z_0) + \frac{z_0'\left(-z_1 f''(z_0) z_0' + g_{rx}(z_0) f'(z_0) z_0'^2 - 2f'(z_0) z_1'\right)}{2f(z_0)}$$
$$+ 2f(z_0)\left(z_1 - 3z_0 g_{rx}(z_0) z_0'\right) + g_{rx}'(z_0) z_0'^3 + \frac{2z_0' z_1' - g_{rx}(z_0) z_0'^3}{z_0} - \frac{z_1 z_0'^2}{z_0^2} = 0$$

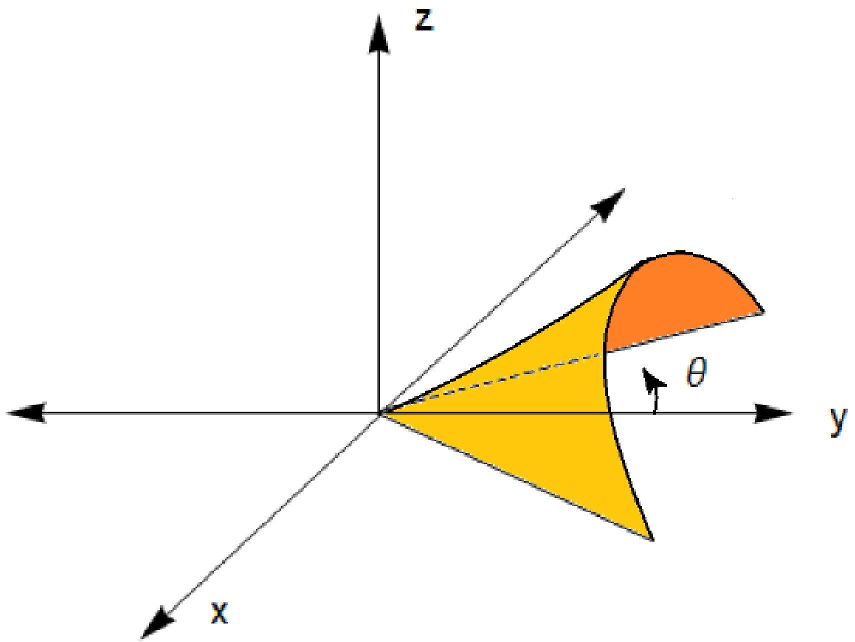


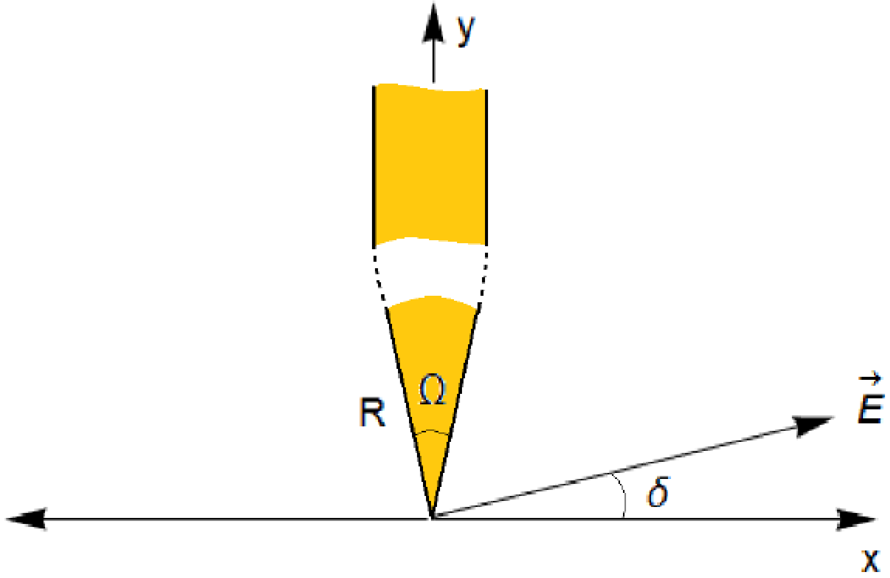



- To obtain the effect of the electric field, we may try two kinds of situations
  - finding the quadratic order background in the electric field
  - Or we can try an asymmetric case.
- Entanglement entropy with a wedge entangling region!

**Corner contributions to holographic entanglement entropy**
Pablo Bueno (Madrid, IFT), Robert C. Myers (Perimeter Inst. Theor. Phys.). May 28, 2015. 63 pp.
Published in **JHEP 1508 (2015) 068**
DOI: 10.1007/JHEP08(2015)068
e-Print: **arXiv:1505.07842 [hep-th]** | **PDF**

## Wedge Type




$$\sigma = \theta/\Omega \quad , \quad z = \Omega w \quad , \quad \tilde{\beta} = \Omega\beta \quad , \quad \tilde{M} = \Omega^3 M \quad \tilde{q} = \Omega^2 q \quad .$$

- The area

$$A = \frac{1}{\Omega}\int_{-1/2}^{1/2} d\sigma \int_0^R d\rho \left( \frac{\rho}{w^2}\sqrt{1+\frac{w'^2+\Omega^2\rho^2\dot{w}^2}{\rho^2 w^2 f(w)}} + \lambda\Omega^2 \frac{\mathbb{H}_{rx}\left(w'\cos(\delta-\Omega\sigma)-\Omega\rho\dot{w}\sin(\delta-\Omega\sigma)\right)}{w^2 f(w)\sqrt{1+\frac{w'^2+\Omega^2\rho^2\dot{w}^2}{\rho^2 w^2 f(w)}}} \right)$$

- Small angle limit

$$A = \frac{1}{\Omega}\int_{-1/2}^{1/2} d\sigma \int_0^R d\rho \left( \sum_{n=0}^{3} \mathcal{L}_{(n)}^{0th}\Omega^n + \lambda \sum_{n=2}^{3} \mathcal{L}_{(n)}^{1st}\Omega^n + \mathcal{O}\left(\Omega^6\right) \right)$$

$$\mathcal{L}_{(0)}^{0th} = \frac{\rho\sqrt{\frac{w'^2}{\rho^2}+1}}{w^2}, \quad \mathcal{L}_{(1)}^{0th} = 0, \quad \mathcal{L}_{(2)}^{0th} = \frac{\left(\beta^2 w^2 \left(w'\right)^2 + 2\rho^2\dot{w}^2\right)}{4\rho w^2\sqrt{\frac{(w')^2}{\rho^2}+1}}, \quad \mathcal{L}_{(3)}^{0th} = \frac{Mww'^2}{2\rho\sqrt{\frac{w'^2}{\rho^2}+1}}$$

and

$$\mathcal{L}_{(2)}^{1st} = \mathbb{H}_{rx}\frac{w'\cos\delta}{\sqrt{\frac{w'^2}{\rho^2}+1}} \quad , \quad \mathcal{L}_{(3)}^{1st} = \mathbb{H}_{rx}\frac{\sin\delta\left(\sigma w' - \rho\dot{w}\right)}{\sqrt{\frac{w'^2}{\rho^2}+1}}$$

- Small R → the solution can be expanded in terms of polynomials of rho.

$$
\begin{aligned}
w(\rho,\sigma) &= w_0(\rho,\sigma) + \lambda\, w_1(\rho,\sigma) \\
&= \rho\, h_{0,1}(\sigma) + \Omega^2 \left( \sum_{i=1}^{3} h_{2,i}(\sigma)\rho^i \right) + \Omega^3 h_{3,4}(\sigma)\rho^4 \\
&\quad + \lambda\, \mathbb{H}_{rx} \left( \Omega^2 g_{2,3}(\sigma)\rho^3 \cos\delta + \Omega^3 g_{3,3}(\sigma)\rho^3 \sin\delta \right) \ .
\end{aligned}
$$

- eoms

$$
\begin{aligned}
h''_{0,1} &= -\frac{2\left(\left(h'_{0,1}\right)^{2}+1\right)}{h_{0,1}} \\
h''_{2,1} &= \frac{h_{0,1}^{3}\left(2\left(h'_{0,1}\right)^{2}-1\right)-4h_{0,1}h'_{0,1}h'_{2,1}+2h_{2,1}\left(\left(h'_{0,1}\right)^{2}+1\right)}{h_{0,1}^{2}} \\
h''_{2,2} &= \frac{2\left(h_{2,2}\left(\left(h'_{0,1}\right)^{2}+1\right)-2h_{0,1}h'_{0,1}h'_{2,2}\right)}{h_{0,1}^{2}} \\
h''_{2,3} &= -\frac{\beta^{2}h_{0,1}^{3}\left(\left(h'_{0,1}\right)^{2}-2\right)+8h_{0,1}h'_{0,1}h'_{2,3}-4h_{2,3}\left(\left(h'_{0,1}\right)^{2}+1\right)}{2h_{0,1}^{2}} \\
h''_{3,4} &= \frac{Mh_{0,1}^{4}\left(4-3\left(h'_{0,1}\right)^{2}\right)-8h_{0,1}h'_{0,1}h'_{3,4}+4h_{3,4}\left(\left(h'_{0,1}\right)^{2}+1\right)}{2h_{0,1}^{2}} \\
g''_{2,3} &= -\frac{4g'_{2,3}h'_{0,1}}{h_{0,1}}+\frac{2g_{2,3}\left(\left(h'_{0,1}\right)^{2}+1\right)}{h_{0,1}^{2}}-6h_{0,1}h'_{0,1} \\
g''_{3,3} &= -\frac{4g'_{3,3}h'_{0,1}}{h_{0,1}}+\frac{2g_{3,3}\left(\left(h'_{0,1}\right)^{2}+1\right)}{h_{0,1}^{2}}+\frac{2\left(-3\sigma h_{0,1}^{3}h'_{0,1}-2h_{0,1}^{4}\left(h'_{0,1}\right)^{2}+h_{0,1}^{4}\right)}{h_{0,1}^{2}}
\end{aligned}
$$

- Regularized Area

$$\mathcal{A}_{reg} = \frac{1}{\Omega}\int_{\sigma_{min}}^{\sigma_{max}} d\sigma \int_{\alpha(\sigma)+\lambda\beta(\sigma)}^{R} d\rho \left(\sum_{n=0}^{3} \mathcal{L}_{(n)}^{0th}\Omega^n + \lambda \sum_{n=2}^{3} \mathcal{L}_{(n)}^{1st}\Omega^n\right)$$

- Linear term from the integrand and the integral region

$$\begin{aligned}
\mathcal{A}_{reg} \;&=\; \frac{1}{\Omega}\int_{\sigma_{min}^{(0)}}^{\sigma_{max}^{(0)}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \left[\left(\sum_{n=0}^{3} \mathcal{L}_{(n)}^{0th}\Omega^n\right)_{w_0+\lambda w_1} + \lambda\left(\sum_{n=2}^{3} \mathcal{L}_{(n)}^{1st}\Omega^n\right)_{w_0}\right] \\
&\quad + \frac{1}{\Omega}\left(\int_{\sigma_{min}}^{\sigma_{max}} d\sigma \int_{\alpha(\sigma)+\lambda\beta(\sigma)}^{R} d\rho - \int_{\sigma_{min}^{(0)}}^{\sigma_{max}^{(0)}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho\right)\left(\sum_{n=0}^{3} \mathcal{L}_{(n)}^{0th}\Omega^n\right)_{w_0}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}_{reg} \;\sim\;\; & \frac{1}{\Omega}\int_{\sigma^{(0)}_{min}}^{\sigma^{(0)}_{max}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho \left(\sum_{n=0}^{3} \mathcal{L}^{0th}_{(n)}\Omega^n\right)_{w_0} \\
& -\frac{\lambda\,\mathbb{H}_{rx}}{\Omega}\cos\delta \int_{-\sigma^{(0)}_{max}}^{\sigma^{(0)}_{max}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho\, \frac{\rho\,\Omega^2\left(2g_{2,3}\left(h'^2_{0,1}+1\right)-h_{0,1}g'_{2,3}h'_{0,1}\right)}{h^3_{0,1}\sqrt{h'^2_{0,1}+1}} \\
& -\frac{\lambda\,\mathbb{H}_{rx}}{\Omega}\sin\delta \int_{-\sigma^{(0)}_{max}}^{\sigma^{(0)}_{max}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho\, \frac{\rho\,\Omega^3\left(2g_{3,3}\left(h'^2_{0,1}+1\right)-h_{0,1}g'_{3,3}h'_{0,1}\right)}{h^3_{0,1}\sqrt{h'^2_{0,1}+1}} \\
& +\frac{\lambda\,\mathbb{H}_{rx}}{\Omega}\int_{-\sigma^{(0)}_{max}}^{\sigma^{(0)}_{max}} d\sigma \int_{\alpha(\sigma)}^{R} d\rho\,\rho \left(\frac{\Omega^2 h'_{0,1}\cos\delta}{\sqrt{h'^2_{0,1}+1}} + \frac{\Omega^3\left(\sigma h'_{0,1}-h_{0,1}\right)\sin\delta}{\sqrt{h'^2_{0,1}+1}}\right) \\
& +\frac{\lambda}{\Omega}\left(\sigma^{(1)}_{max}-\sigma^{(1)}_{min}\right)\left(\int_{\alpha(\sigma)}^{R} d\rho \left(\sum_{n=0}^{3}\mathcal{L}^{0th}_{(n)}\Omega^n\right)_{w_0}\right)_{\sigma=\sigma^{(0)}_{max}} \\
& -\frac{\lambda}{\Omega}\int_{-\sigma^{(0)}_{max}}^{\sigma^{(0)}_{max}} d\sigma\beta(\sigma)\left(\sum_{n=0}^{3}\mathcal{L}^{0th}_{(n)}\Omega^n\right)_{\rho=\alpha(\sigma)} \quad ,
\end{aligned}
$$

$$\delta\mathcal{A}_{reg} = -\lambda\,\mathbb{H}_{rx}\Omega^2\,\sin\delta\int_{-\sigma_{max}^{(0)}}^{\sigma_{max}^{(0)}} d\sigma\left(\mathcal{I}_1+\mathcal{I}_2\right)+\mathcal{O}\left(\Omega^3\right)\quad ,$$

where

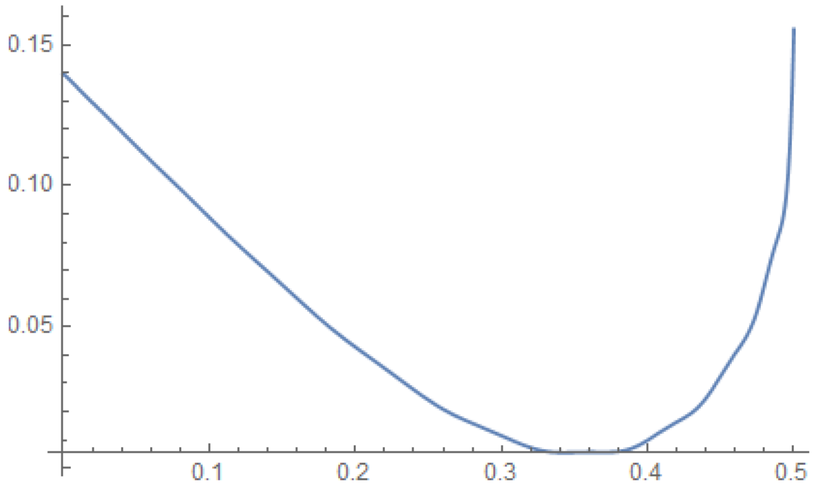
$$\begin{aligned}
\mathcal{I}_1 &= \frac{\left(R^2-\alpha(\sigma)^2\right)\left(2g_{3,3}(\sigma)\left(h_{0,1}'(\sigma)^2+1\right)-h_{0,1}(\sigma)g_{3,3}'(\sigma)h_{0,1}'(\sigma)\right)}{2h_{0,1}(\sigma)^3\sqrt{h_{0,1}'(\sigma)^2+1}} \\
\mathcal{I}_2 &= \left(R^2-\alpha(\sigma)^2\right)\left(-\frac{\lambda\left(\sigma h_{0,1}'(\sigma)-h_{0,1}(\sigma)\right)}{2\sqrt{h_{0,1}'(\sigma)^2+1}}\right)\quad ,
\end{aligned}$$

- Finally we obtain the linear term in the electric field.

$$
\begin{aligned}
&\delta A_{reg} \qquad (51)\\
&= \lambda\, R^2\, \Omega^2\, \mathbb{H}_{rx}\, \sin\delta \int_{-\sigma_{max}^{(0)}}^{\sigma_{max}^{(0)}} d\sigma\, \frac{\sigma h(\sigma)^3 h'(\sigma) - h(\sigma)^4 + h(\sigma)h'(\sigma)g'(\sigma) - 2g(\sigma)\left(h'(\sigma)^2+1\right)}{2h(\sigma)^3\sqrt{h'(\sigma)^2+1}},
\end{aligned}
$$

$$
\begin{aligned}
h'' &= -\frac{2\left(h'^2+1\right)}{h}\\
g'' &= -\frac{4g'h'}{h} + \frac{2g\left(1+h'^2\right)}{h^2} + 2h^2 - 6\sigma h h' - 4h^2h'^2
\end{aligned}
$$

- The integrand is a positive function



- The linear response of the holographic entanglement entropy is

$$\frac{\delta S_{EE}}{\delta E_x} = \frac{R^2\,\Omega^2}{4G}\mathcal{N}\,\frac{\bar{\alpha}}{4\pi}\sin\delta$$

# Discussion and Future Direction

- Response of the entanglement entropy to an external electric field can be written in terms of measurable quantity.
- The correction is proportional to the thermo-electric coefficient a measurable quantity(Wedge type HEE).
- ----------Future Directions---------
- Quadratic order in E
- Response to AC Electric field
- Response to the temperature gradient
- Spherical entangling surfaces : What happens to the F-theorem.
- Response to magnetic field (comparable to field theory calculations).
- One top-down model with momentum relaxation
  - Mass deformed ABJM with m(x) Kwon and KKK : arXiv:1806.06963 **[hep-th]** discussed by Gauntlett

- Thank you very much !