Conference Gauge/Gravity Duality 2018
Würtzburg, August 2, 2018

Holographic RG flows on Curved manifolds and F-functions.

Elias Kiritsis

CCTP/ITCP University of Crete APC, Paris

Bibliography

Ongoing work with:

Francesco Nitti, Lukas Witkowski, Jewel Ghosh (APC, Paris)

Published work in:

- arXiv:1805.01769
- ArXiv:1711.08462
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Based on earlier work:

- with Francesco Nitti and Wenliang Li ArXiv:1401.0888
- with Vassilis Niarchos ArXiv:1205.6205

Introduction

• The Wilsonian RG is controlled by first order flow equations of the form

$$\frac{dg_i}{dt} = \beta_i(g_i) \quad , \quad t = \log \mu$$

• Despite current knowledge, there are many aspects of QFT RG flows (in the most symmetric case: unitary relativistic QFTs), that are still not understood.

♠ It is not known if the end-points of RG flows in 4d are fixed points or include other exotic possibilities (limit circles or “chaotic” behavior)

♠ This is correlated with the potential symmetry of scale invariant theories: are they always conformally invariant? (CFTs)?

• In 2d, the answer to this question is ”yes”.

♠ Although in 4d this has been analyzed also recently, there are still loop-holes in the argument.

El Showk+Rychkov+Nakayama, Luty+Polchinski+Rattazzi,

Dymarsky+Komargodksi+Schwimmer+Theisen+Farnsworth+Luty+Prilepina

♠ In 2d it is a folk-theorem that the strong version of the c-theorem is expected to exclude limit cycles and other exotic behavior in Unitary Relativistic QFTs.

Zamolodchikov

• The folk-theorem between the strong version of the a-theorem and the appearance of limit cycles has at least one important loop-hole:

If the β-functions have branch singularities away from the UV fixed point, then a limit cycle can be compatible with the strong version of the a/c-theorem.

Curtright+Zachos

• If this ever happens, it can only happen "beyond perturbation theory".

C-functions and F-Functions

• In 2 and 4 dimensions we have established c-theorems and associated c-functions, that interpolate properly between UV and IR CFTs along an RG flow.

Zamolodchikov, Cardy, Komargodky+Schwimmer, ...

• In 3-dimensions, there is an F-theorem for CFTs associated with the S^3 renormalized partition function.

Jafferis, Jafferis+Klebanov+Pufu+Safdi

• But the associated partition function fails to be a monotonic F-function along the the flow.

Klebanov+Pufu+Safdi, Taylor+Woodhead

• There is an alternative "F-function": the appropriately renormalized entanglement entropy associated to an S^2 in R^3.

Myers+Sinha, Liu+Mazzei

• There is a general proof that in 3d this is always monotonic.

Casini+Huerta+Myers

The Goal

• Build an understanding of the general structure of holographic RG flows of QFTs on flat space.

• Build an understanding of the general structure of holographic RG flows of QFTs on curved spaces (spheres etc)

• Use this knowledge to revisit F-functions in 3 and more dimensions.

• Here I will present some highlights of this search.

Holographic RG flows: the setup

• For simplicity and clarity I will consider the bulk theory to contain only the metric and a single scalar (Einstein-dilaton gravity), dual to the stress tensor $T_{\mu\nu}$ and a scalar operator O of a dual QFT.

• The two derivative action (after field redefinitions) is

$$S_{bulk} = M^{d-1} \int d^{d+1}x \, \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] + S_{GH}$$

• We assume $V(\phi)$ is analytic everywhere except possibly at $\phi = \pm\infty$.

• We will consider the AdS regime: ($V < 0$ always) and look (in the beginning) for solutions with d-dimensional Poincaré invariance.

$$ds^2 = du^2 + e^{2A(u)} dx_\mu dx^\mu \quad , \quad \phi(u)$$

• The Einstein equations have three integration constants.

• The Einstein equations can be turned to first order equations using the "superpotential" (no-supersymmetry here).

$$\dot{A} = -\frac{1}{2(d-1)} W(\phi) \quad , \quad \dot{\phi} = W'(\phi) \quad , \quad dot = \frac{d}{du}$$

$$-\frac{d}{4(d-1)} W(\phi)^2 + \frac{1}{2} W(\phi)'^2 = V(\phi) \quad , \quad {}' = \frac{d}{d\phi}$$

• These equations have the same number of integration constants. In particular there is a continuous one-parameter family of $W(\phi)$.

• Given a $W(\phi)$, $A(u)$ and $\phi(u)$ can be found by integrating the first order flow equations.

• The two integration constants will be later interpreted as couplings of the dual QFT.

• The third integration constant hidden in the superpotential equation controls the vev of the operator dual to ϕ.

• Therefore:

RG flows are in one-to one correspondence with the solutions of the "superpotential equation".

$$-\frac{d}{4(d-1)}W(\phi)^2+\frac{1}{2}W(\phi)'^2=V(\phi)$$

• This is the key equation I will be addressing in most of this talk.

• Regularity of the bulk solution fixes the W-equation integration constant (uniquely in generic cases).

General properties of the superpotential

- Because of the symmetry $(W,u)\to(-W,-u)$ we can always take $W>0$.

- The superpotential equation implies

$$W(\phi)=\sqrt{-\frac{4(d-1)}{d}V(\phi)+\frac{2(d-1)}{d}W'^2}\geq\sqrt{-\frac{4(d-1)}{d}V(\phi)}\equiv B(\phi)>0$$

- The holographic "c-theorem" for all flows:

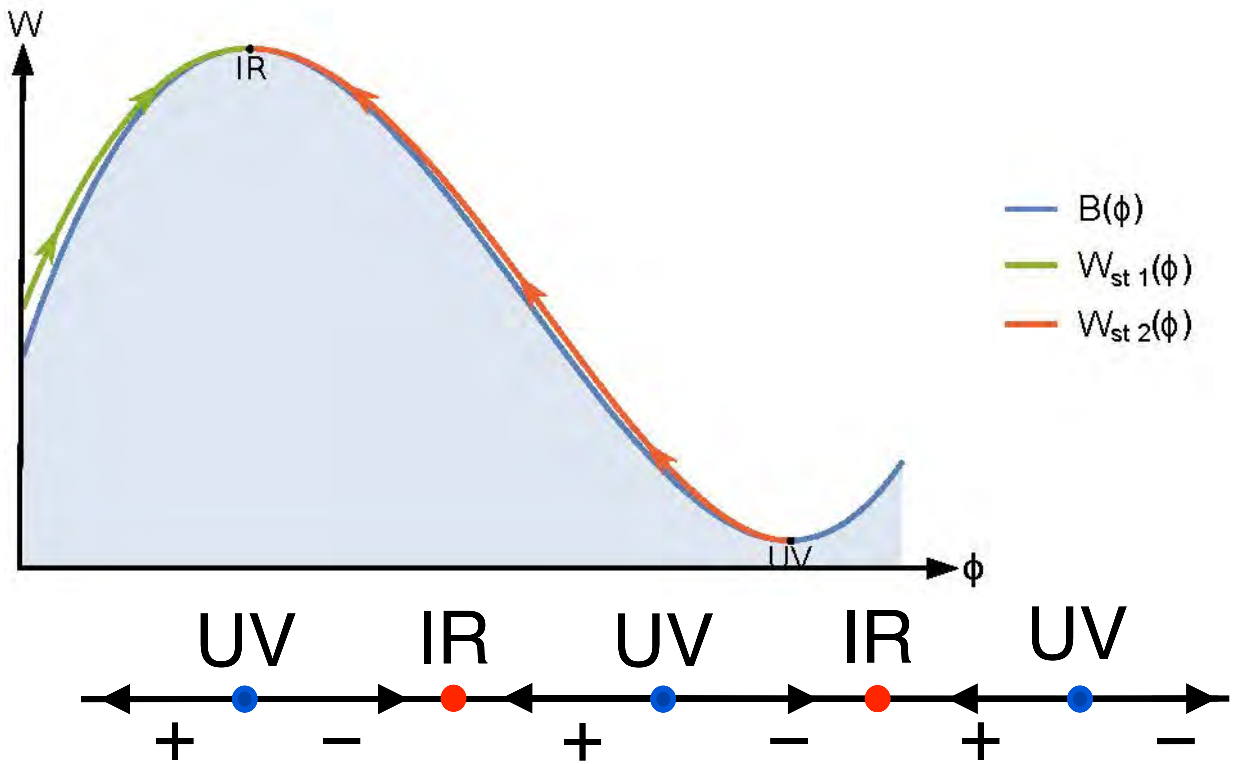
$$\frac{dW}{du}=\frac{dW}{d\phi}\;\frac{d\phi}{du}=W'^2\geq 0$$

- The only singular flows are those that end up at $\phi\to\pm\infty$.

- All regular solutions to the equations are flows from an extremum of V to another extremum of V (for finite ϕ).

The standard holographic RG flows

• The standard lore says that the maxima of the potential correspond to UV fixed points, the minima to IR fixed points, and the flow from a maximum is to the next minimum.



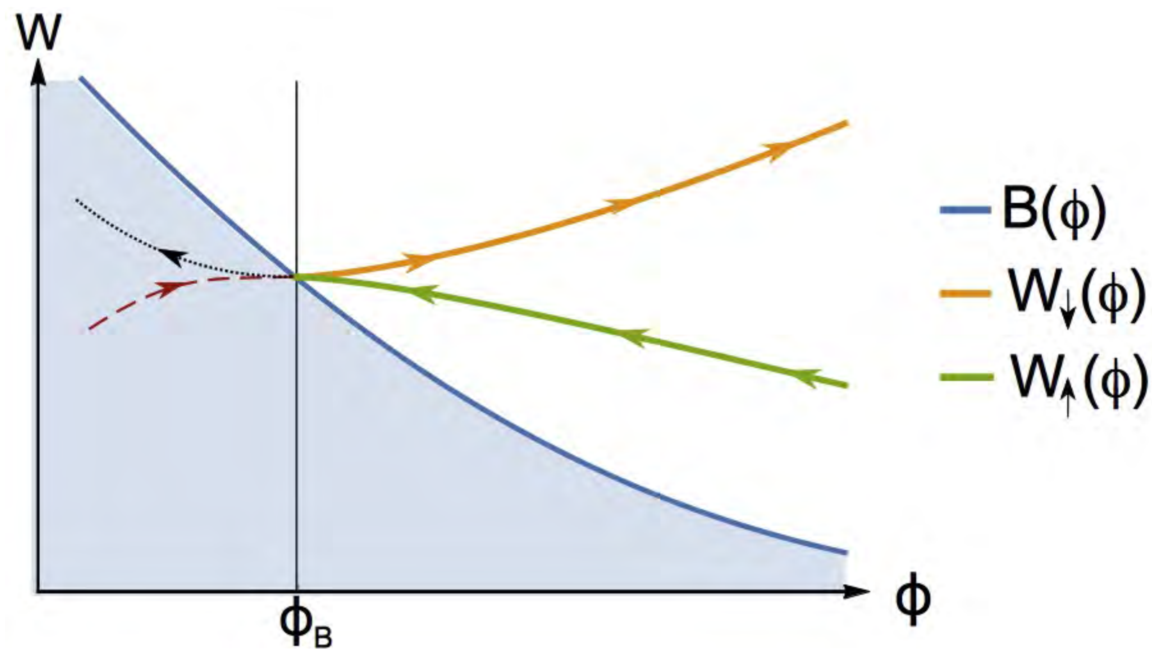
• The real story is a bit more complicated.

Bounces

- When W reaches the boundary region $B(\phi)$ at a generic point, it develops a generic non-analyticity.

$$W_{\pm}(\phi) = B(\phi_B) \pm (\phi - \phi_B)^{\frac{3}{2}} + \cdots$$

- There are two branches that arrive at such a point.



• Although W is not analytic at ϕ_B, the full solution (geometry+ϕ) is regular at the bounce.

• The only special thing that happens is that $\dot{\phi} = 0$ at the bounce.

• All bulk curvature invariants are regular at the bounce!

• All fluctuation equations of the bulk fields are regular at the bounce!

• The holographic β-function behaves as

$$\beta \equiv \frac{d\phi}{dA} = \pm\sqrt{-2d(d-1)\frac{V'(\phi_B)}{V(\phi_B)}(\phi - \phi_B) + \mathcal{O}(\phi - \phi_B)}$$

• The β-function is patch-wise defined. It has a branch cut at the position of the bounce.

• It vanishes at the bounce without the flow stopping there.

• This is non-perturbative behavior.

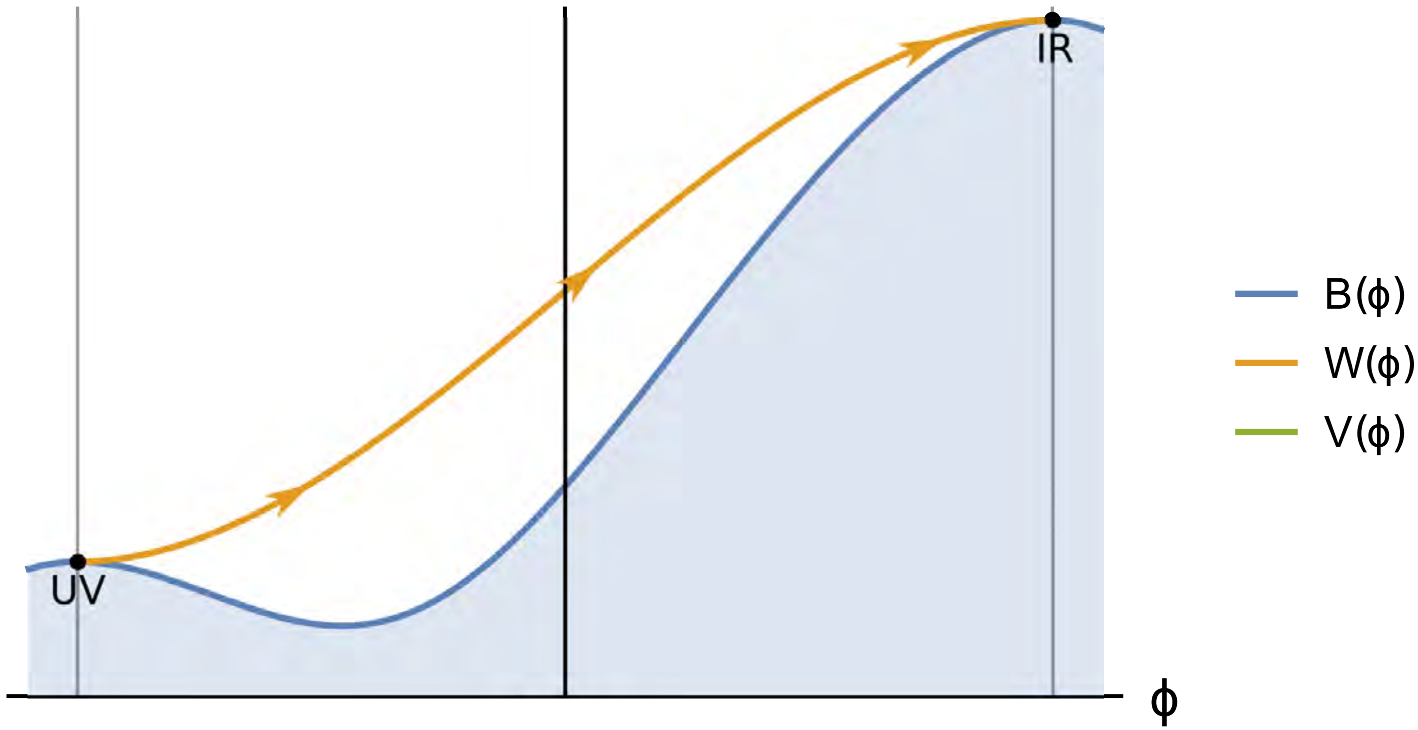
• Such behavior was conjectured that could lead to limit cycles without violation of the a-theorem.

Curtright+Zachos

• We can show that limit cycles cannot happen in theories with holographic duals (and no extra "active" dimensions).

Exotica

- Vev flow between two minima of the potential

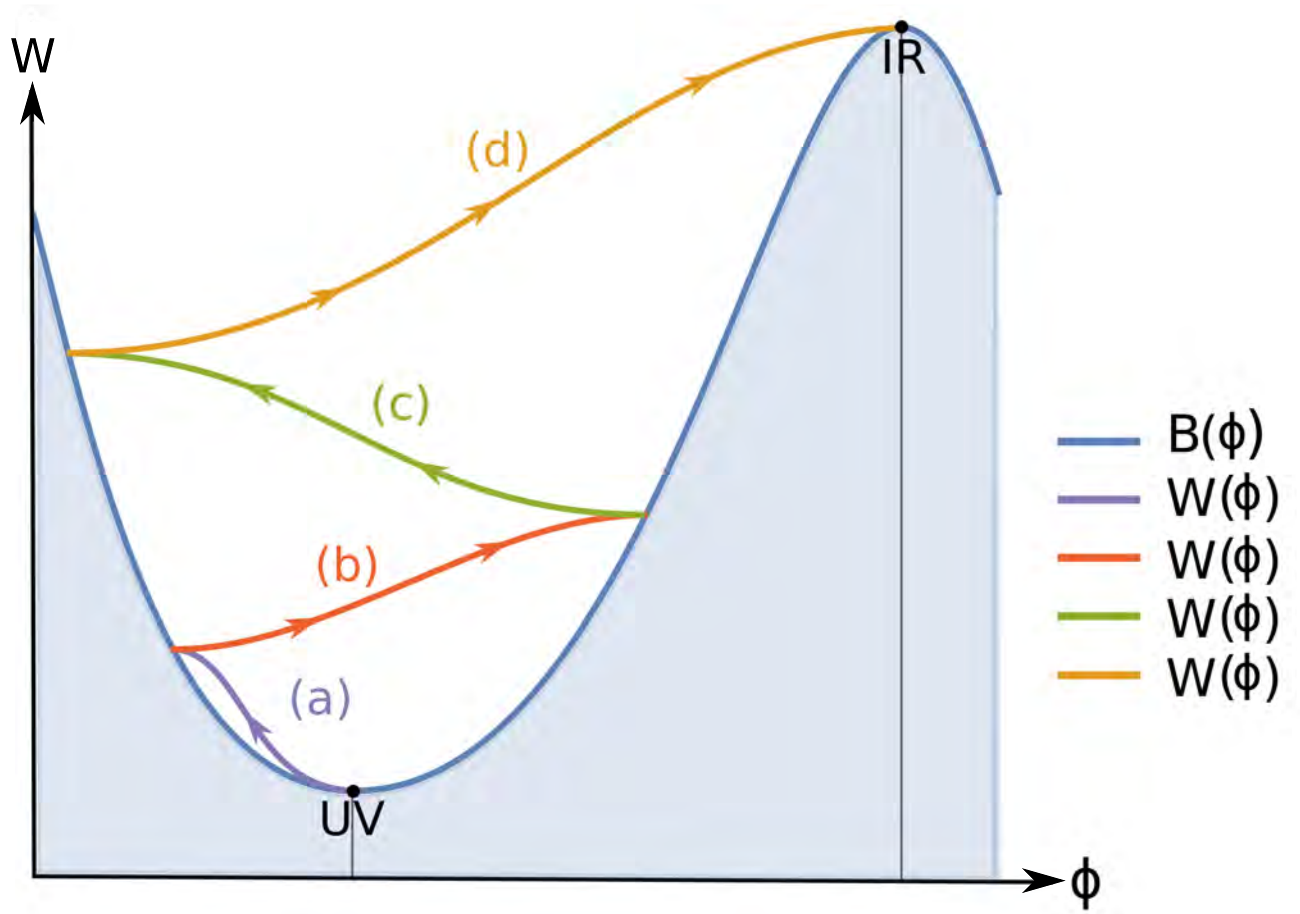


- Exists only for special potentials. It is a flow driven by the vev of an irrelevant operator.

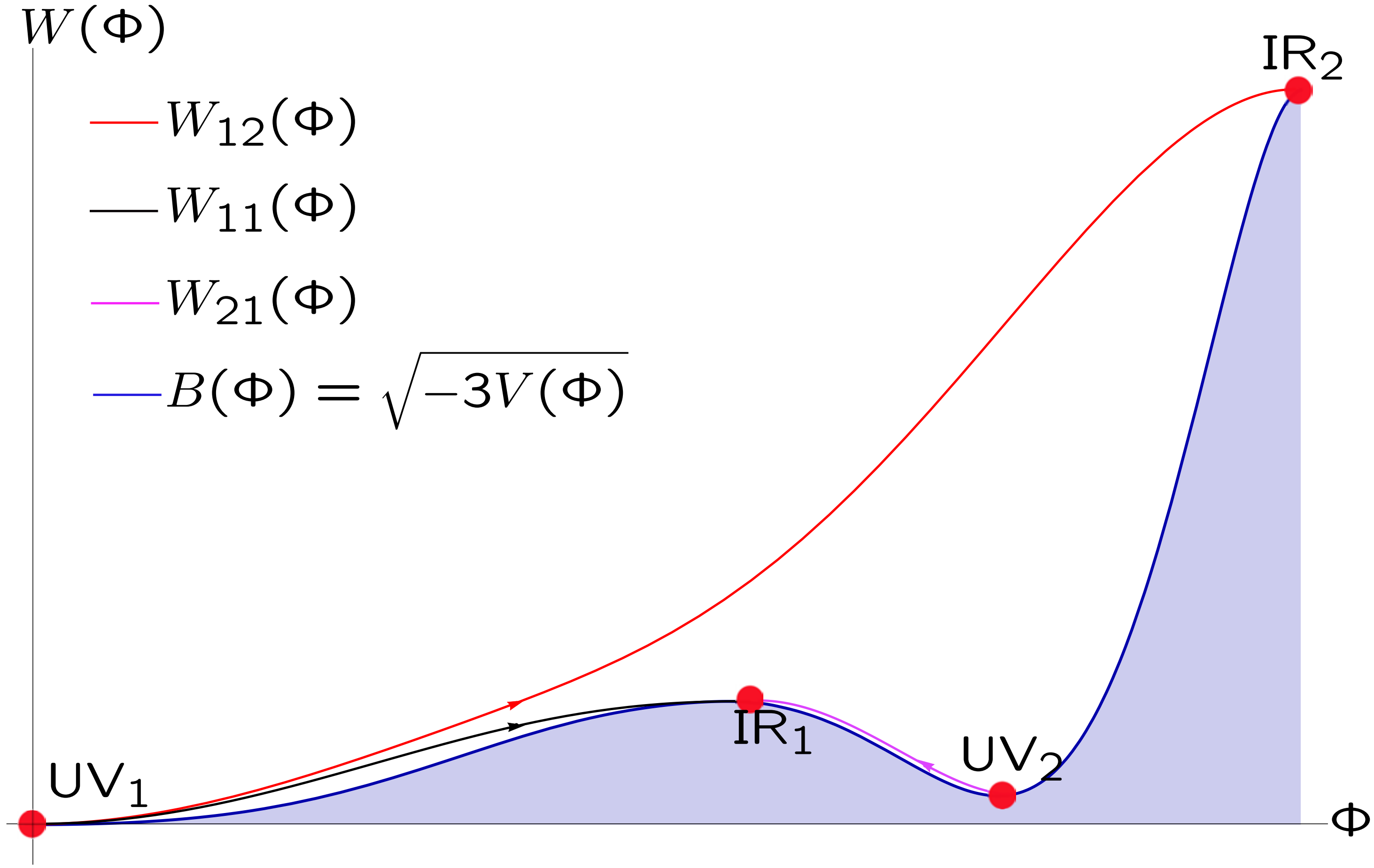
- A analogous phenomenon happens in N=1 sQCD.

Seiberg, Aharony

Regular multibounce flows



Skipping fixed points



Quantum field theories on curved manifolds

• There are many reasons to be interested in QFTs over curved manifiolds

♠ Compact manifolds like S^n are important to regularize massless/CFTs in the IR.

♠ QFT on deSitter manifolds is interesting due to the fact we live in a patch of (almost) de Sitter.

♠ The induced effective gravitational action as a function of curvature can serve as a Hartle-Hawking wave-function for three-metrics.

♠ Curvature, although UV-irrelevant, is IR relevant and can change importantly the IR structure of a given theory. We will see examples of quantum phase transitions driven by curvature.

♠ It will also turn out to be a useful tool in analysing sphere partition functions and $\mathcal{F}$-theorems.

The setup

• The holographic ansatz for the ground-state solution is

$$ds^2 = du^2 + e^{2A(u)}\zeta_{\mu\nu}dx^\mu dx^\nu \quad , \quad \phi(u)$$

• Now there are two parameters (couplings) for the solution: ϕ_0 and R_{uv}. They combine in a single dimensionless parameter:

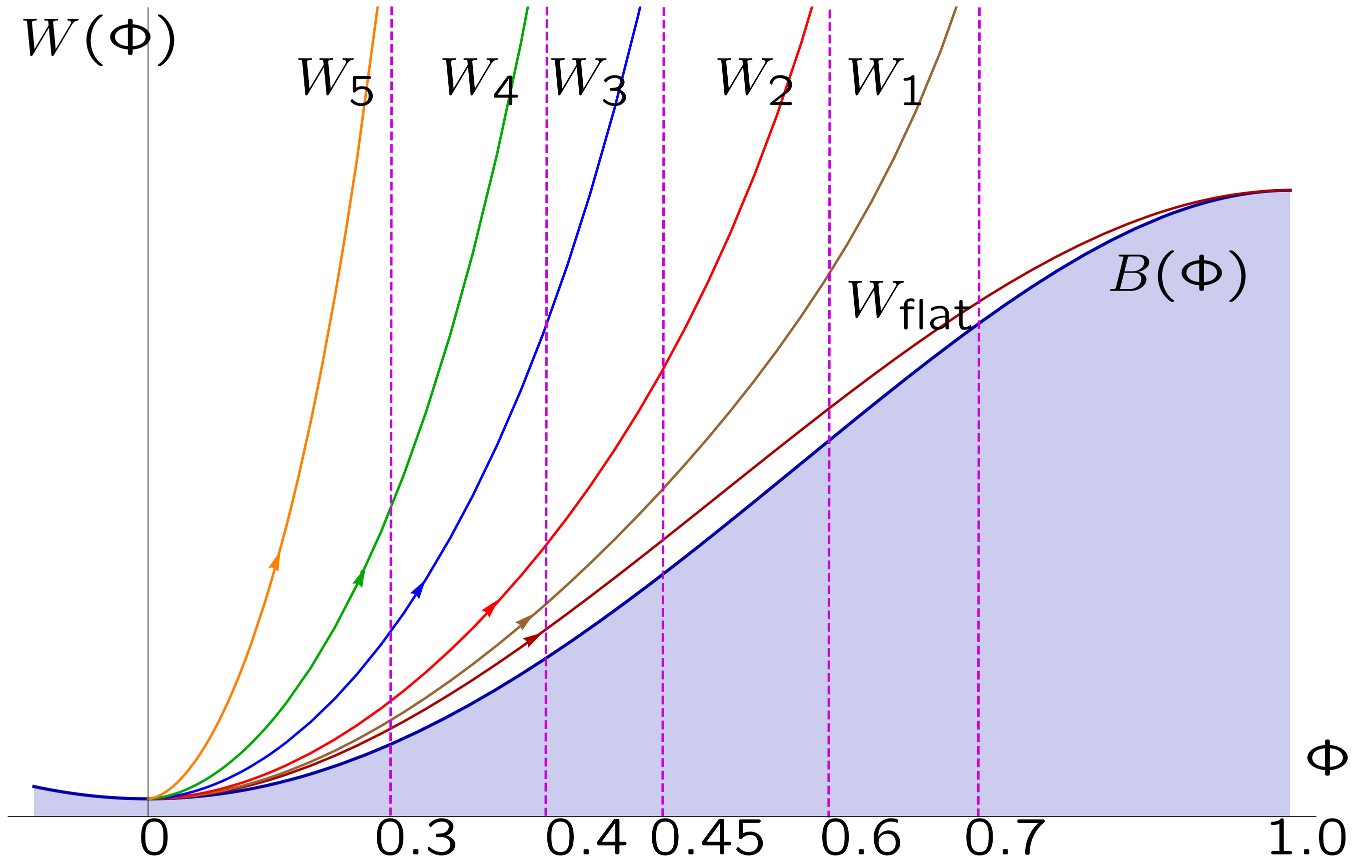
$$\mathcal{R} \equiv \frac{R_{\text{UV}}}{\phi_0^{\frac{2}{\Delta_-}}}$$

• $\mathcal{R} \to 0$ will probe the full original theory except a small IR region.

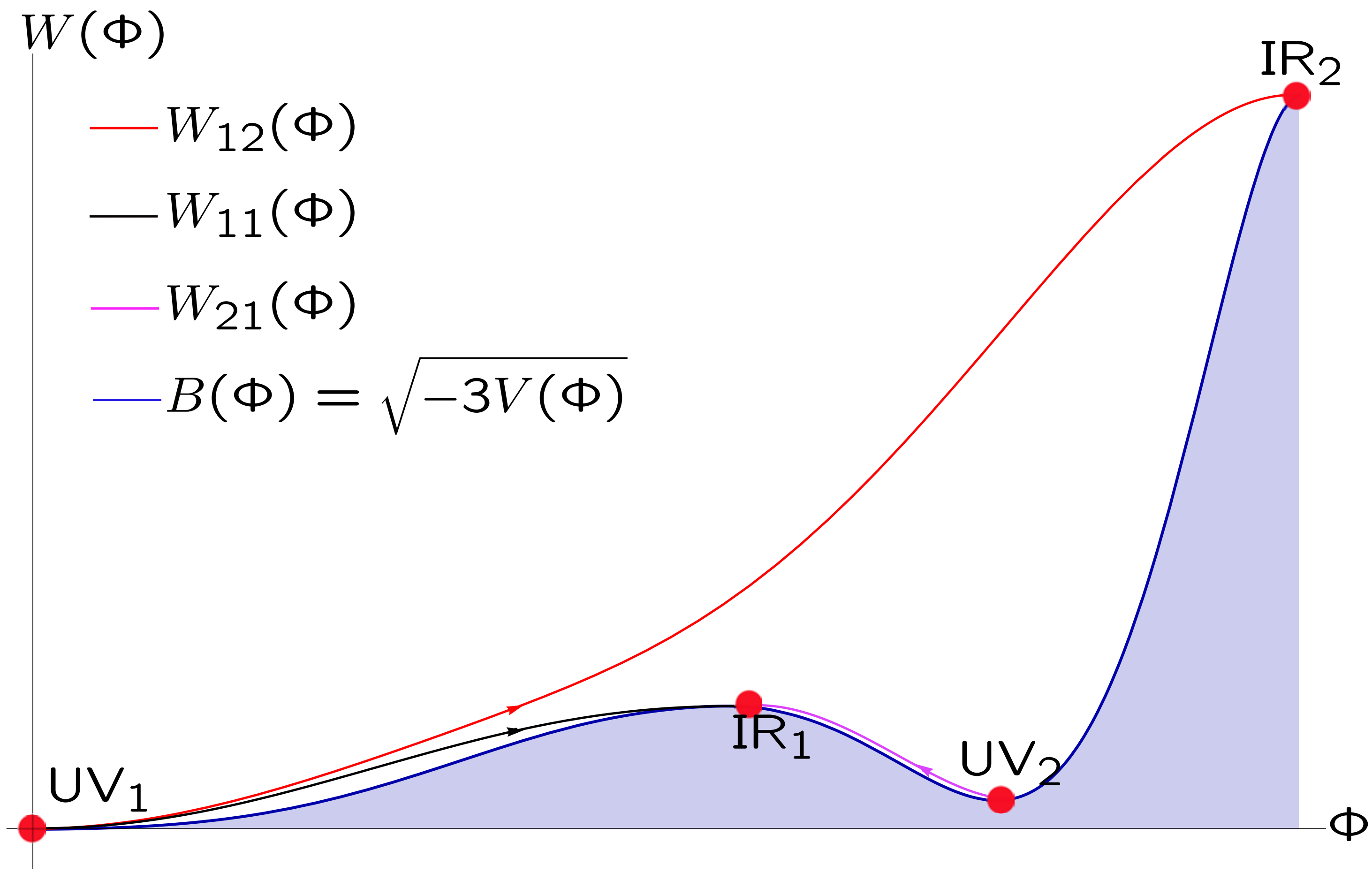
• $\mathcal{R} \to \infty$ will explore only the UV of the original theory.

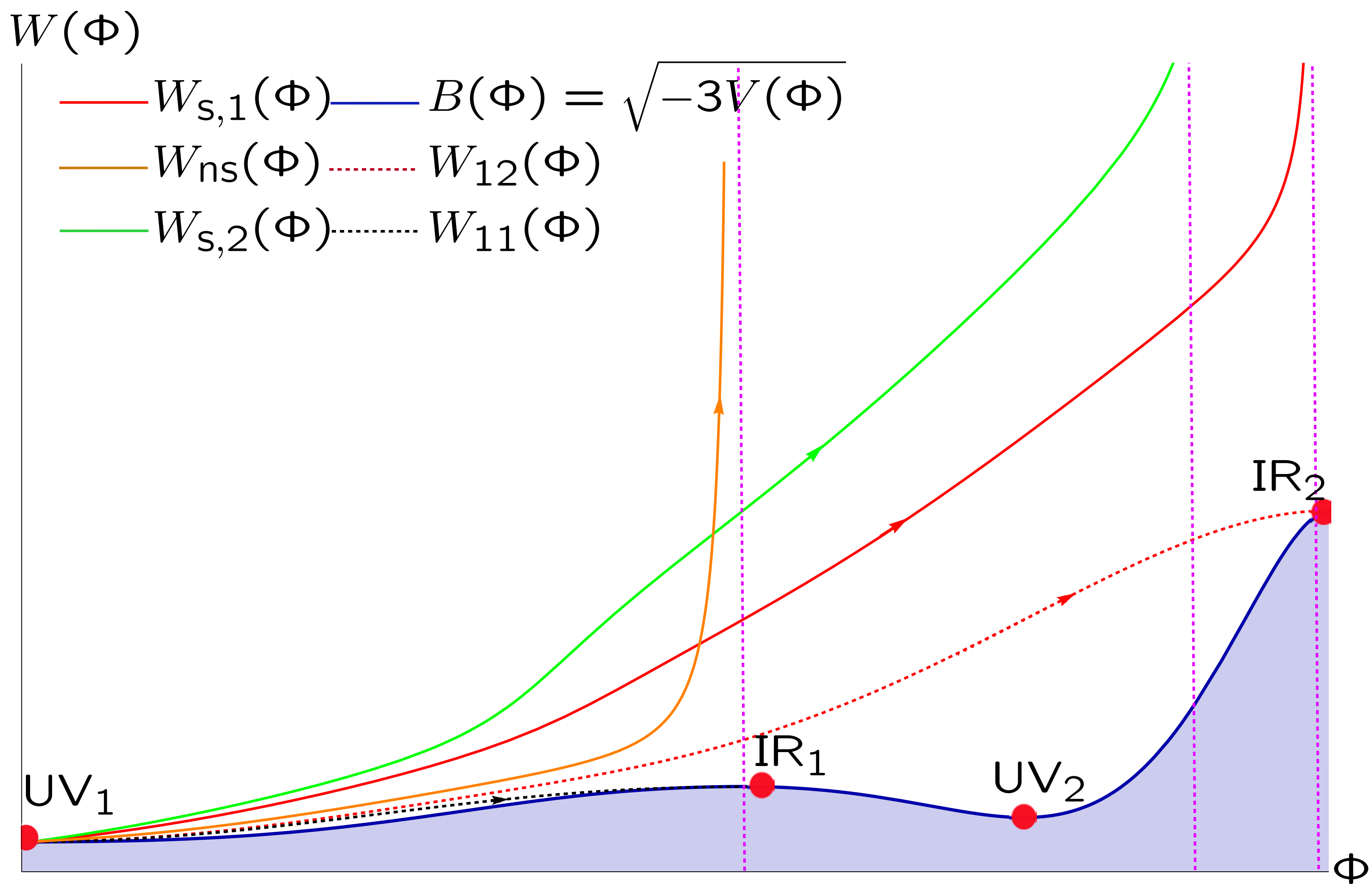
• Therefore by varying $\mathcal{R}$ we have an invariant/well-defined dimensionless number that tracks the UV flow from the UV to the IR.

The vanilla flows at finite curvature

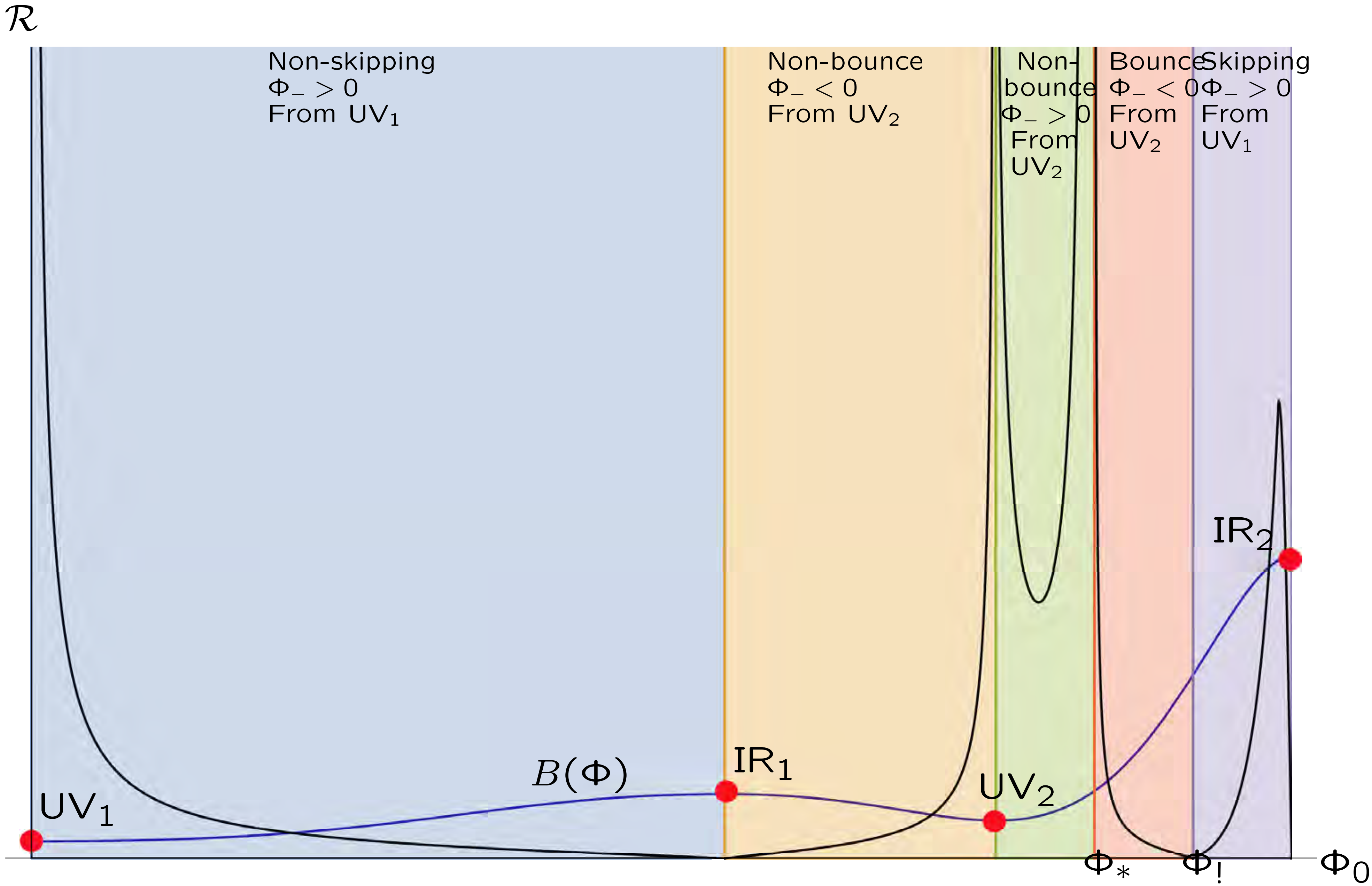


Skipping flows at finite curvature





The solid lines represent the superpotential $W(\Phi)$ corresponding to the three different solutions starting from UV_1 which exist at small positive curvature. Two of them (red and green curves) are skipping flows and the third one (orange curve) is non-skipping. For comparison, we also show the flat RG flows (dashed curves)



$\mathcal{R}$
Non-skipping
$\Phi_- > 0$
From UV_1
Non-bounce
$\Phi_- < 0$
From UV_2
Non-bounce
$\Phi_- > 0$
From UV_2
Bounce
$\Phi_- < 0$
From UV_2
Skipping
$\Phi_- > 0$
From UV_1
IR_2
$B(\Phi)$
IR_1
UV_2
UV_1
Φ_*
$\Phi_!$
Φ_0

Spontaneous breaking saddle points

• The $\mathcal{R} \to \infty$ flows associated with $\phi = \phi_*$ are a distinct branch of the theory.

• At $\phi = \phi_*$, ϕ_0 (the source) vanishes, therefore $\mathcal{R} \to \infty$ although R_{uv} =finite.

• The point $\phi = \phi_*$ (a single solution) is a one-parameter family of saddle points with $\phi_0 = 0$ but a non trivial (relevant) vev

$$\langle O \rangle = \xi_* \, R_{UV}{}^{\frac{\Delta_+}{2}}$$

• Therefore the CFT UV_2 has two saddle points at finite positive curvature R_{UV}. In one $\langle O \rangle = 0$ and in the other $\langle O \rangle \neq 0$.

Stabilisation by curvature

• The theories with $\phi_0 > 0$ and $\mathcal{R} < \mathcal{R}_*$ do not exist.

• But for $\mathcal{R} > \mathcal{R}_*$ there are two non-trivial saddle points

• This is an example of a theory that in flat space, it exists for $\phi_0 < 0$ but not for $\phi_0 > 0$.

• But the theory with $\phi_0 > 0$ exists when $\mathcal{R} > \mathcal{R}_*$.

• There is a simple example from weakly-coupled field theory that exhibits similar behavior:

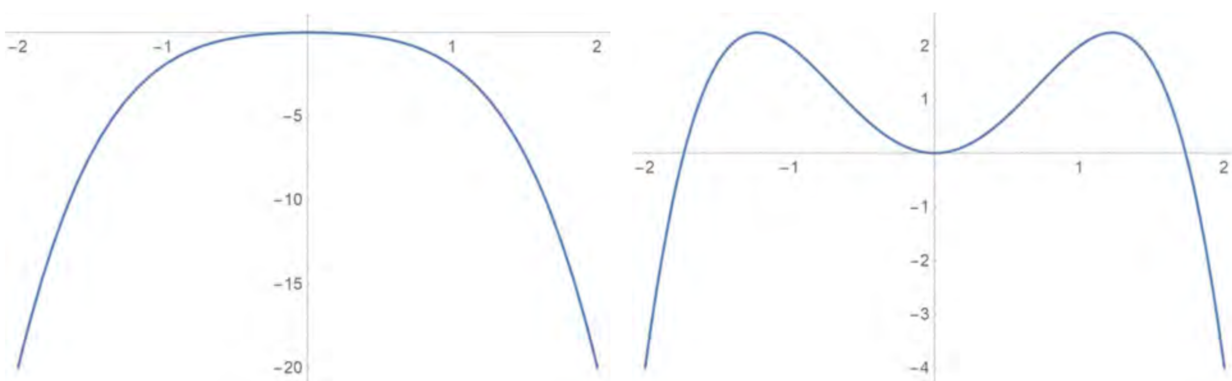
$$V_{flat}(\phi) = -\lambda\phi^4 - m^2\phi^2$$

• When $\lambda > 0$ the theory does not exist.

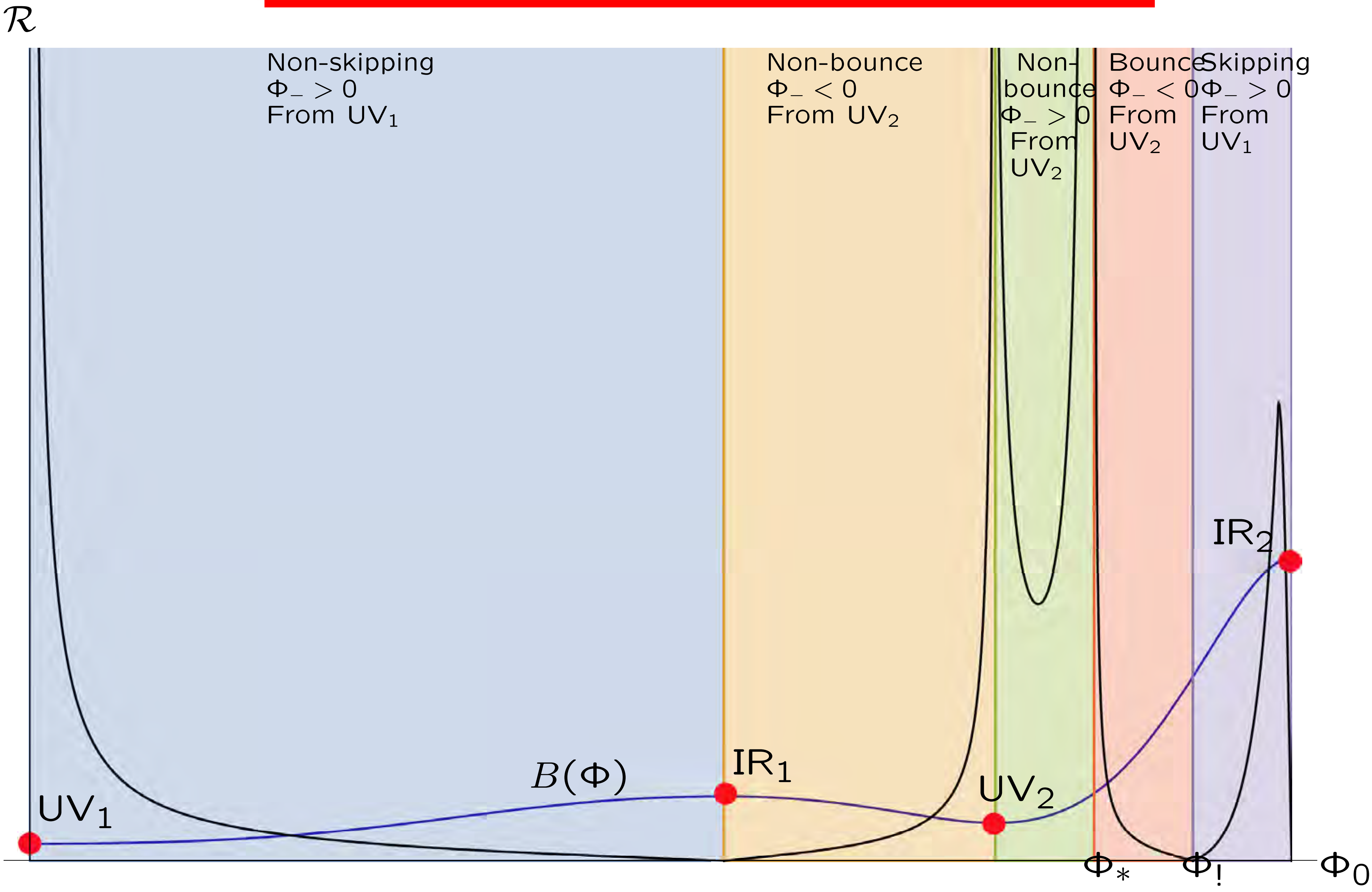
• At sufficiently high curvature

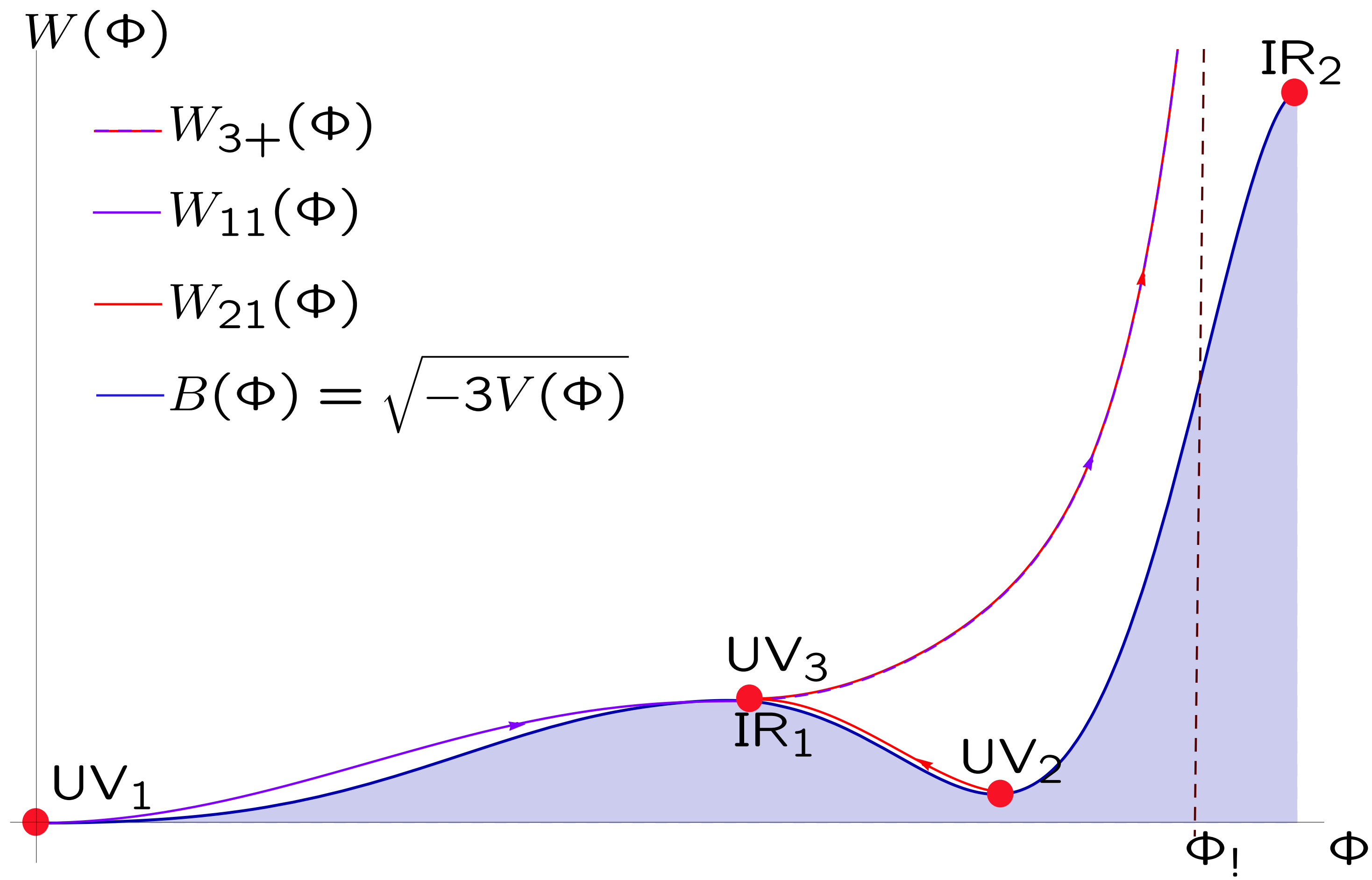
$$V_R(\phi) = -\lambda\phi^4 - m^2\phi^2 + \frac{1}{6R^2}\phi^2$$

the theory develops new extrema:



Unusual vev flows from minima





$W(\Phi)$
$W_{3+}(\Phi)$
$W_{11}(\Phi)$
$W_{21}(\Phi)$
$B(\Phi) = \sqrt{-3V(\Phi)}$
IR_2
UV_3
IR_1
UV_2
UV_1
$\Phi_!$
Φ

• $\Phi_!$ cannot be reached from either UV$_1$ or UV$_2$ but only from IR$_1$.

• The Flow from IR$_1$ to $\Phi_!$ has zero source and a vev

$$\langle O\rangle = \xi_! \; {R_{UV}}^{\frac{\Delta_+}{2}}$$

• At the IR$_1$ we have an AdS boundary.

• As $\mathcal{R} \equiv R_{\mathrm{UV}}{\phi_0}^{-\frac{2}{\Delta_-}}$, $\mathcal{R} \to 0$ when $\phi_0 \to 0$.

• This is again a one-parameter family of saddle points with different curvature where the theory is driven by the vev of an irrelevant operator.

The free energy and the entanglement entropy

- A direct computation gives for the unrenormalized free energy

$$F \equiv S_{\text{on-shell}} = M_P^{d-1}\int d^dx\sqrt{-\zeta}\left(\left[e^{dA}W\right]_{\text{UV}} + \frac{2R^{\text{uv}}}{d}\int_{\text{UV}}^{\text{IR}} du\, e^{(d-2)A(u)}\right)$$

- For positive $\mathcal{R} > 0$, this is the free energy both for S^d and dS_d.

- It can be renormalized by subtracting the $\mathcal{R} = 0$ superpotential and the coefficient of the Einstein term. This is scheme dependence (two parameters in three dimensions).

- $F^{\text{renormalized}}$ is a function of only $\mathcal{R}$ (no cutoff).

- Consider the entanglement entropy of a QFT on a deSitter space

$$ds^2 = -dt^2 + R^2\cosh^2(t/R)(d\theta^2 + \sin^2\theta\ d\Omega_{d-2}^2)$$

between two spatial hemispheres that have S^{d-2} as boundary using RT

$$S_{EE} = M_P^{d-1}\frac{2R}{d}\int d^dx\sqrt{-\zeta}\int_{\text{UV}}^{\text{IR}} du\, e^{(d-2)A(u)}\,.$$

Ben-Ami+Carmi+Smolkin

• For a CFT it is also the entanglement entropy for the flat case.

Casini+Huerta+Myers

• This is the same as the thermal entropy of the static patch in deSitter.

• For the de Sitter entanglement entropy a renormalization à la Liu-Mezzei is still UV-divergent.

QUESTION: in 3d, can we define an F-function from the free energy of a QFT on S^3?

Renormalization in 3d

$$F_{d=3}(\Lambda, \mathcal{R}) = -(M\ell)^2\Omega_3\left[\mathcal{R}^{-\frac{3}{2}}\left(4\Lambda^3(1+\mathcal{O}(\Lambda^{-2\Delta_-})) + C(\mathcal{R})\right)`+\right.$$

$$\left. + \mathcal{R}^{-\frac{1}{2}}\left(\Lambda(1+\mathcal{O}(\Lambda^{-2\Delta_-})) + B(\mathcal{R}\right) + \cdots\right] \quad , \quad \Lambda \equiv \frac{e^{A(\epsilon)}}{\ell|\phi_0|^{\frac{1}{\Delta_-}}}$$

- $B(\mathcal{R}), C(\mathcal{R})$ are the vevs of O and a (part of a) derivative of the stress tensor.

- We renormalize

$$F^{\text{renorm}}_{d=3}(\mathcal{R}|B_{ct}, C_{ct}) = -(M\ell)^2\Omega_3\left[\mathcal{R}^{-\frac{3}{2}}(C(\mathcal{R}) - C_{ct}) + \mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R}) - B_{ct})\right]$$

- Similarly the renormalized deSitter entanglement entropy is

$$S^{\text{renorm}}_{EE}(\mathcal{R}|B_{ct} = (M\ell)^2\Omega_3\mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R}) - B_{ct})$$

$\mathcal{F}$-functions

- We will use $\mathcal{R}$ as an interpolating variable between IR (0) and UV (∞).
- $\mathcal{F}$ must be UV and IR finite.
- An $\mathcal{F}$-function must satisfy

$$\lim_{\mathcal{R}\to\infty} \mathcal{F}(\mathcal{R}) \equiv \mathcal{F}_{UV} = 8\pi^2 (M\ell_{UV})^2 \quad , \quad \lim_{\mathcal{R}\to 0} \mathcal{F}(\mathcal{R}) \equiv \mathcal{F}_{IR} = 8\pi^2 (M\ell_{IR})^2$$

$$\frac{d\mathcal{F}}{d\mathcal{R}} \leq 0$$

- There are four proposals using the free energy ($D_a \equiv \frac{1}{a}\frac{\partial}{\partial \mathcal{R}} + 1$).

$$\mathcal{F}_1(\mathcal{R}) \equiv D_{1/2}\, D_{3/2}\, F(\Lambda, \mathcal{R})$$

$$\mathcal{F}_2(\mathcal{R}) \equiv D_{1/2}\, F^{\text{ren}}(\mathcal{R}|B_{ct}, C_{ct,0})$$

$$\mathcal{F}_3(\mathcal{R}) \equiv D_{3/2}\, F^{\text{ren}}(\mathcal{R}|B_{ct,0}, C_{ct}),$$

$$\mathcal{F}_4(\mathcal{R}) \equiv F^{\text{ren}}(\mathcal{R}|B_{ct,0}, C_{ct,0}).$$

$$B_{ct,0} = B(0) + C'(0) \quad , \quad C_{ct,0} = C(0) \quad , \quad \tilde{B}_{ct,0} = B(0)$$

and another two using the entanglement entropy

$$\mathcal{F}_5(\mathcal{R}) \equiv D_{1/2} S_{EE}(\Lambda, \mathcal{R})$$

$$\mathcal{F}_6(\mathcal{R}) = S_{EE}^{\text{ren}}(\mathcal{R}|B_{ct,0})$$

It turns out that

$$\mathcal{F}_6(\mathcal{R}) = \mathcal{F}_1(\mathcal{R}) \quad , \quad \mathcal{F}_5(\mathcal{R}) = \mathcal{F}_3(\mathcal{R})$$

- All $\mathcal{F}_{1,2,3,4}$ asymptote properly in the UV and IR limits.

♠ All $\mathcal{F}_{1,2,3,4}$ are monotonic in many numerical holographic examples we analyzed when $\Delta > \frac{3}{2}$.

♠ In order to work properly, when $\Delta < \frac{3}{2}$, $\mathcal{F}_{1,2,3,4}$ should be replaced by their Legendre transforms.

♠ This prescription also makes the free theories (the massive fermion and boson) to be monotonic as well.

♠ We have no general proof of monotonicity.

Outlook

- Exotic holographic flows can appear for rather generic potentials.

- The black holes associated with them have been analyzed and exhibit many of the phenomena mentioned for the finite curvature case.

  Gursoy+Kiritsis+Nitti+Silva-Pimenda, Attems+Bea+Casalderrey-Solana+Mateos+Triana+Zilhao

- Bouncing flows seem to be intermediate between regular monotonic flows and limit cycles. Are they an artifact of the large-N expansion?

- One should try to prove monotonicity of $\mathcal{F}_i$ and extend also to 5 dimensions.

- We have another proposal for an $\mathcal{F}(\Lambda)$ but this is not fully developed yet.

- Our analysis and the unusual curved solutions we find, seem to have a radical impact on the stability of AdS minima due to CdL decay processes.

THANK YOU!

UV and IR divergences of F and S_{EE}

- The unrenormalized $F(\Lambda, \mathcal{R})$ and $S_{EE}(\Lambda, \mathcal{R})$.

♠ UV divergences $\Lambda \to \infty$:

$$F(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(\Lambda + \cdots) \;\; \text{and} \;\; \mathcal{R}^{-\frac{3}{2}}(\Lambda^3 + \cdots)$$

$$S_{EE}(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(\Lambda + \cdots)$$

♠ IR divergences $\mathcal{R} \to 0$:

$$F(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}\,(B_0 + C_1) \;\; \text{and} \;\; \mathcal{R}^{-\frac{3}{2}}\,C_0$$

$$S_{EE}(\Lambda, \mathcal{R}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}\,B_0$$

where

$$C(\mathcal{R}) \simeq C_0 + C_1\mathcal{R} + \mathcal{O}(\mathcal{R}^2) \quad , \quad B(\mathcal{R}) \simeq B_0 + \mathcal{O}(\mathcal{R})$$

• The renormalized F and S_{EE}: only UR divergences, $\mathcal{R} \to 0$.

$$F^{\text{ren}}(\mathcal{R}|B_{ct}, C_{ct}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(B_0 + C_1 - B_{ct}) \quad \text{and} \mathcal{R}^{-\frac{3}{2}}(C_0 - C_{ct})$$

$$S_{EE}^{\text{ren}}(\mathcal{R}|\tilde{B}_{ct}, C_{ct}) \quad : \quad \mathcal{R}^{-\frac{1}{2}}(B_0 - \tilde{B}_{ct})$$

• We can remove UV divergences from unrenormalized functions by acting with

$$D_{3/2} \equiv \frac{2}{3}\frac{\partial}{\partial \mathcal{R}} + 1 \quad , \quad D_{1/2} \equiv 2\frac{\partial}{\partial \mathcal{R}} + 1 \quad , \quad D_{3/2}\mathcal{R}^{-\frac{3}{2}} = 0 \quad , \quad D_{1/2}\mathcal{R}^{-\frac{1}{2}} = 0$$

• We can remove IR divergences by choosing appropriately our scheme (subtractions)

$$B_{ct,0} = B(0) + C'(0) \quad , \quad C_{ct,0} = C(0) \quad , \quad \tilde{B}_{ct,0} = B(0)$$

$\mathcal{F}$-functions (II)

In terms of the two functions $B(\mathcal{R})$ and $C(\mathcal{R})$ the $\mathcal{F}$ functions can be written as

$$\frac{\mathcal{F}_1(\mathcal{R})}{(M\ell)^2\Omega_3} = -\ \frac{4}{3}\mathcal{R}^{\frac{1}{2}}(2B'(\mathcal{R}) + C''(\mathcal{R}) + \mathcal{R}\ B''(\mathcal{R}))$$

$$\frac{\mathcal{F}_2(\mathcal{R})}{(M\ell)^2\Omega_3} = -2\mathcal{R}^{-\frac{3}{2}}(-(C(\mathcal{R}) - C(0)) + \mathcal{R}C'(\mathcal{R}) + \mathcal{R}^2 B'(\mathcal{R}))$$

$$\frac{\mathcal{F}_3(\mathcal{R})}{(M\ell)^2\Omega_3} = -\frac{4}{3}\mathcal{R}^{-\frac{1}{2}}(B(\mathcal{R}) + C'(\mathcal{R}) - B(0) - C'(0)) + \mathcal{R}B'(\mathcal{R}))$$

$$\frac{\mathcal{F}_4(\mathcal{R})}{(M\ell)^2\Omega_3} = -\mathcal{R}^{-\frac{3}{2}}(C(\mathcal{R}) - C(0)) + \mathcal{R}(B(\mathcal{R}) - B(0))$$

We also have the relation

$$C'(\mathcal{R}) = \frac{1}{2}B(\mathcal{R}) - \mathcal{R}B'(\mathcal{R}).$$

Holography and "Quantum" RG

• Enter holography as a means of probing strong coupling behavior.

• Holography provides a neat description of RG Flows.

• It also gives a natural a-function and the strong version of the a-theorem holds.

♠ But...the relevant equations that are converted into RG equations are second order!

• It is known for some time that the Hamilton-Jacobi formalism in holography gives first order RG-equations.

de Boer+Verlinde^2, Skenderis+Townsend, Gursoy+Kiritsis+Nitti, Papadimitriou, Kiritsis+Li+Nitti

• This would imply that (conceptually at least) holographic RG flows are very similar to (perturbative) QFT flows.

The extrema of V

The expansion of the potential near an extremum is

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad ,$$

• The series solution of the superpotential is

$$W_\pm = 2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \cdots$$

• Near a maximum, W_- is part of a continuous family (parametrized by a vev)

• W_+ is an isolated solution.

• Near a minimum, regularity makes W_- unique.

• Near a minimum, W_+ describes a “UV fixed point”

The strategy

• Review of the holographic RG flows.

• Understanding the space of solutions.

• Standard RG flows start a maximum of the bulk potential and end at a nearby minimum.

• We find exotic holographic RG flows:

♠ "Bouncing flows" : the β-function has branch cuts.

♠ "Skipping flows" : the theory bypasses the next fixed point.

♠ "Irrelevant vev flows" : the theory flows between two minima of the bulk potential.

• Outlook

Regularity

• One key point: out of all solutions W, typically one only gives rise to a regular bulk solution. (and more generally a discrete number*).

• All others have bulk singularities and are therefore unacceptable* (holographic) classical solutions.

• This reduces the number of (continuous) integration constants from 3 to 2.

• This has a natural interpretation in the dual QFT: the theory determines it possible vevs (we exclude flat directions).

• The remaining first order equations are now the first order RG equations for the coupling and the space-time volume.

• Now we can favorably compare with QFT RG Flows.

General properties of the superpotential

- From the superpotential equation we obtain a bound:

$$W(\phi)^2 = -\frac{4(d-1)}{d}V(\phi) + \frac{2(d-1)}{d}W'^2 \geq -\frac{4(d-1)}{d}V(\phi) \equiv B^2(\phi) > 0$$

- Because of the $(u, W) \to (-u, -W)$ symmetry we can fix the flow (and sign of W) so that we flow from $u = -\infty$ (UV) to $u = \infty$ (IR). This implies that:

$$W > 0 \quad \text{always} \quad \text{so} \quad W \geq B$$

- The holographic "a-theorem":

$$\frac{dW}{du} = \frac{dW}{d\phi}\frac{d\phi}{du} = W'^2 \geq 0$$

so that the a-function any decreasing function of W always decreases along the flow, ie. W is positive and increases.

• The inequality now can be written directly in terms of W:

$$W(\phi) \geq B(\phi) \equiv \sqrt{-\frac{4(d-1)}{d}V(\phi)}$$

• The maxima of V are minima of B and the minima of V are maxima of B.

• The bulk potential provides a lower boundary for W and therefore for the associated flows.

• Regularity of the flow=regularity of the curvature and other invariants of the bulk theory:

A flow is regular iff W, V remain finite during the flow.

• V aws assumed finite for ϕ finite. The same can be proven for W.

Therefore singular flows end up at $\phi \to \pm\infty$

.

Holographic RG Flows

• A QFT with a (relevant) scalar operator $O(x)$ that drives a flow, has two parameters: the scale factor of a flat metric, and the $O(x)$ coupling constant.

• These two parameters, generically correspond to the two integration constants of the first order bulk equations.

• Since ϕ is interpreted as a running coupling and A is the log of the RG energy scale, the holographic β-function is

$$\dot{A} = -\frac{1}{2(d-1)}W(\phi) \quad , \quad \dot{\phi} = W'(\phi)$$

$$\frac{d\phi}{dA} \quad = \quad -\frac{1}{2(d-1)}\,\frac{d}{d\phi}\log W(\phi) \quad \equiv \quad \beta(\phi) \sim \frac{1}{C}\frac{d}{d\phi}C(\phi)$$

• $C \sim 1/W^{d-1}$ is the (holographic) C-function for the flow.

Girardello+Petrini+Porrati+Zaffaroni, Freedman+Gubser+Pilch+Warner

• $W(\phi)$ is the non-derivative part of the Schwinger source functional of the dual QFT =on-shell bulk action.

de Boer+Verlinde[2]

$$S_{on-shell} = \int d^d x \sqrt{\gamma}\ W(\phi) + \cdots \Big|_{u \to u_{UV}}$$

• The renormalized action is given by

$$S_{renorm} = \int d^d x \sqrt{\gamma}\ (W(\phi) - W_{ct}(\phi)) + \cdots \Big|_{u \to u_{UV}} =$$

$$= constant \int d^d x\ e^{dA(u_0) - \frac{1}{2(d-1)} \int_{\phi_{UV}}^{\phi_0} d\tilde{\phi} \frac{W'}{W}} + \cdots$$

• The statement that $\frac{dS_{renorm}}{du_0} = 0$ is equivalent to the RG invariance of the renormalized Schwinger functional.

• It is also equivalent to the RG equation for ϕ.

• We can prove that

$$T_\mu{}^\mu = \beta(\phi)\ \langle O \rangle$$

• The Legendre transform of S_{renorm} is the (quantum) effective potential for the vev of the QFT operator O.

Detour: The local RG

- The holographic RG can be generalized straightforwardly to the local RG

$$\dot{\phi} = W' - f' \; R + \frac{1}{2}\left(\frac{W}{W'}f'\right)'(\partial\phi)^2 + \left(\frac{W}{W'}f'\right)\Box\phi + \cdots$$

$$\dot{\gamma}_{\mu\nu} = -\frac{W}{d-1}\gamma_{\mu\nu} - \frac{1}{d-1}\left(f \; R + \frac{W}{2W'}f'(\partial\phi)^2\right)\gamma_{\mu\nu} +$$

$$+2f \; R_{\mu\nu} + \left(\frac{W}{W'}f' - 2f''\right)\partial_\mu\phi\partial_\nu\phi - 2f'\nabla_\mu\nabla_\nu\phi + \cdots$$

Kiritsis+Li+Nitti

- $f(\phi)$, $W(\phi)$ are solutions of

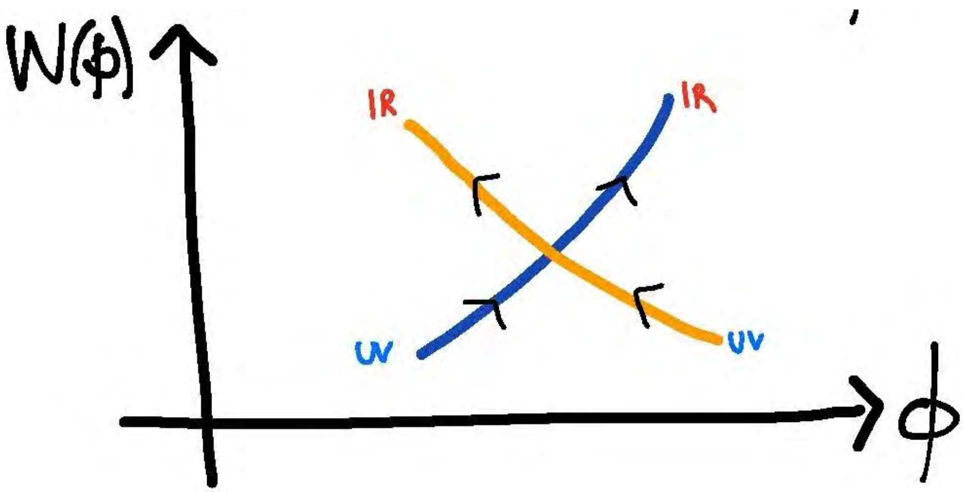
$$-\frac{d}{4(d-1)}W^2 + \frac{1}{2}W'^2 = V \quad , \quad W' \; f' - \frac{d-2}{2(d-1)}W \; f = 1$$

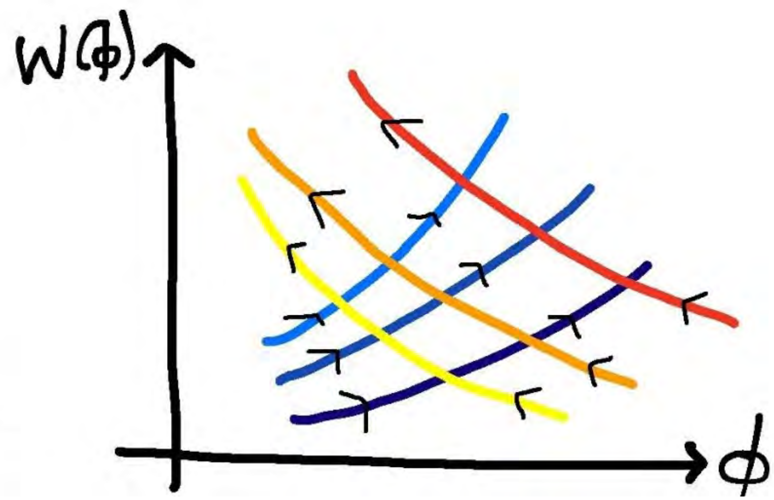
- Like in 2d σ-models we may use it to define "geometric" RG flows.

More flow rules

• At every point away from the $B(\phi)$ boundary $(W > B)$ always two solutions pass:

$$W' = \pm\sqrt{2V + \frac{d}{2(d-1)}W^2} = \pm\sqrt{\frac{d}{2(d-1)}\left(W^2 - B^2\right)}$$





The critical points of W

- On the boundary $W = B$, we obtain $W' = 0$ and only one solution exists.

- The critical $(W' = 0)$ points of W come in three kinds:

♠ $W = B$ at non-extremum of the potential (generic).

♠ Maxima of V (minima of B) (non-generic)

♠ Minima of V (maxima of B) (non-generic)

The BF bound

- The BF bound can be written as

$$\frac{4(d-1)}{d}\ \frac{V''(0)}{V(0)} \leq 1$$

- If a solution for W near $\phi = 0$ exists, then the BF bound is automatically satisfied as it can be written

$$\left(\frac{4(d-1)}{d}\frac{W''(0)}{W(0)} - 1\right)^2 \geq 0$$

- When BF is violated, although there is no (real) W, there exists a UV-regular solution for the flow: $\phi(u)$, $A(u)$.

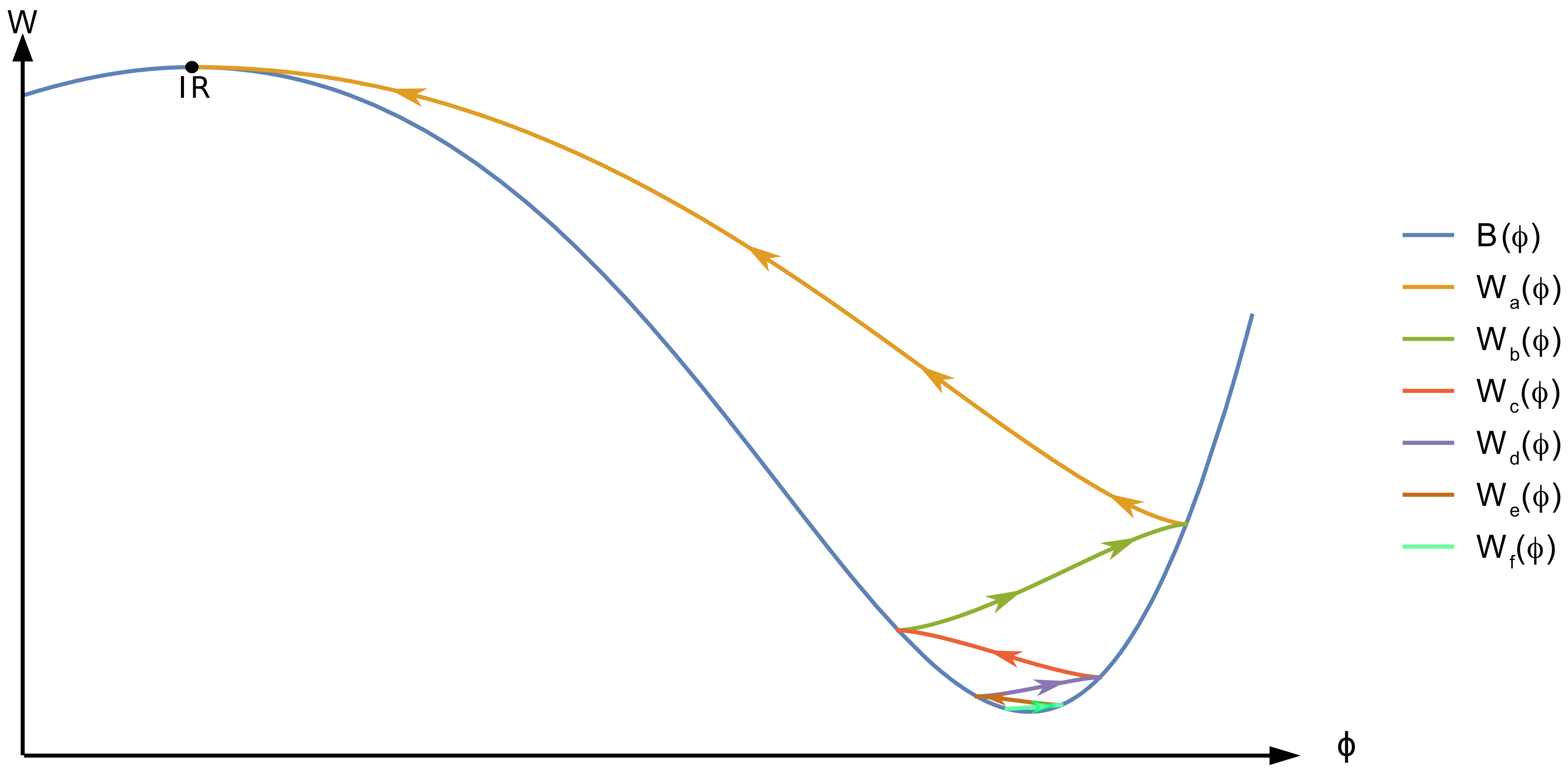
- This solution is unstable against linear perturbations (and corresponds to a non-unitary CFT).

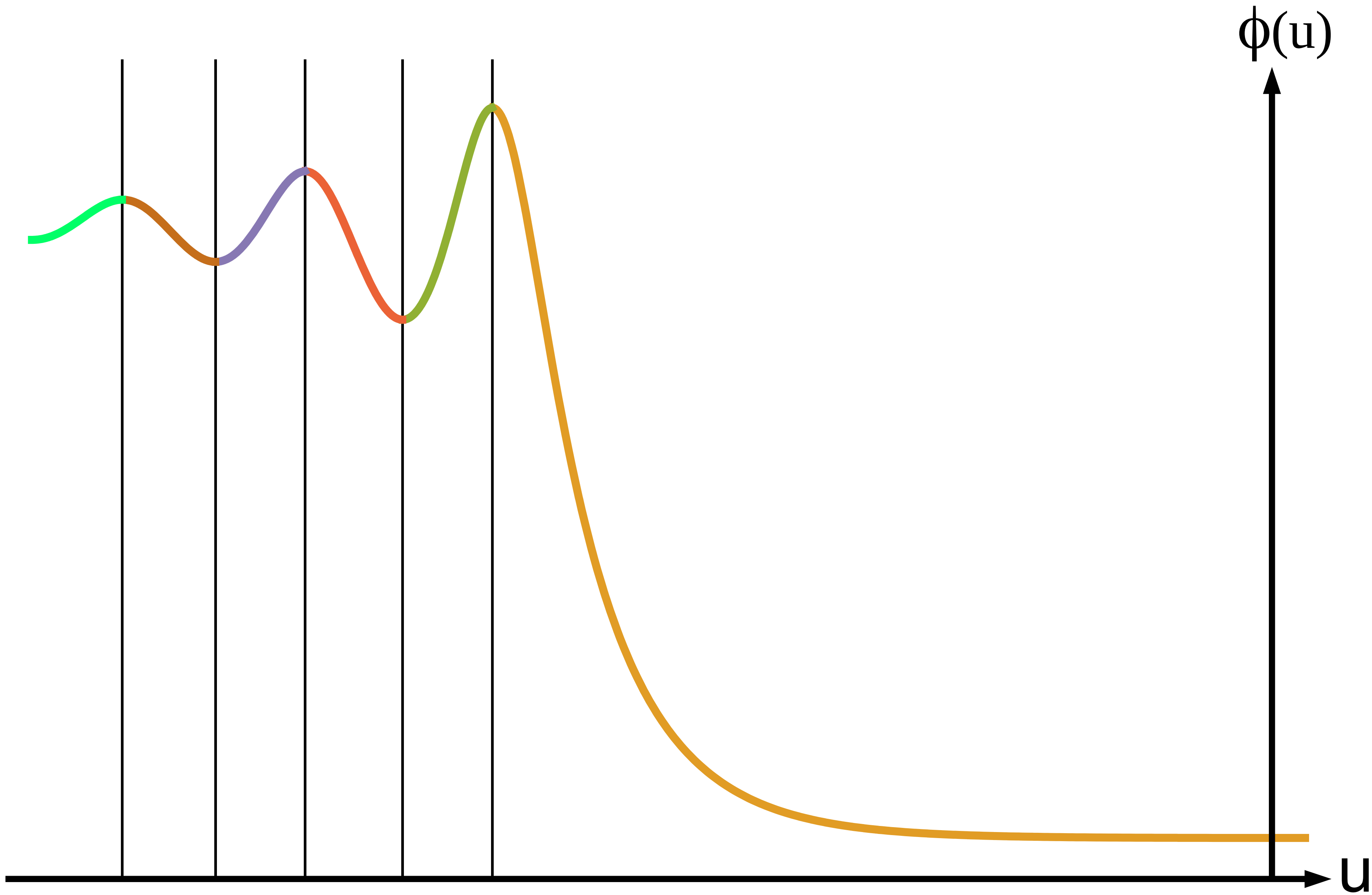
BF violating flows

- As mentioned there can be flows out of a BF-violating UV fixed point.

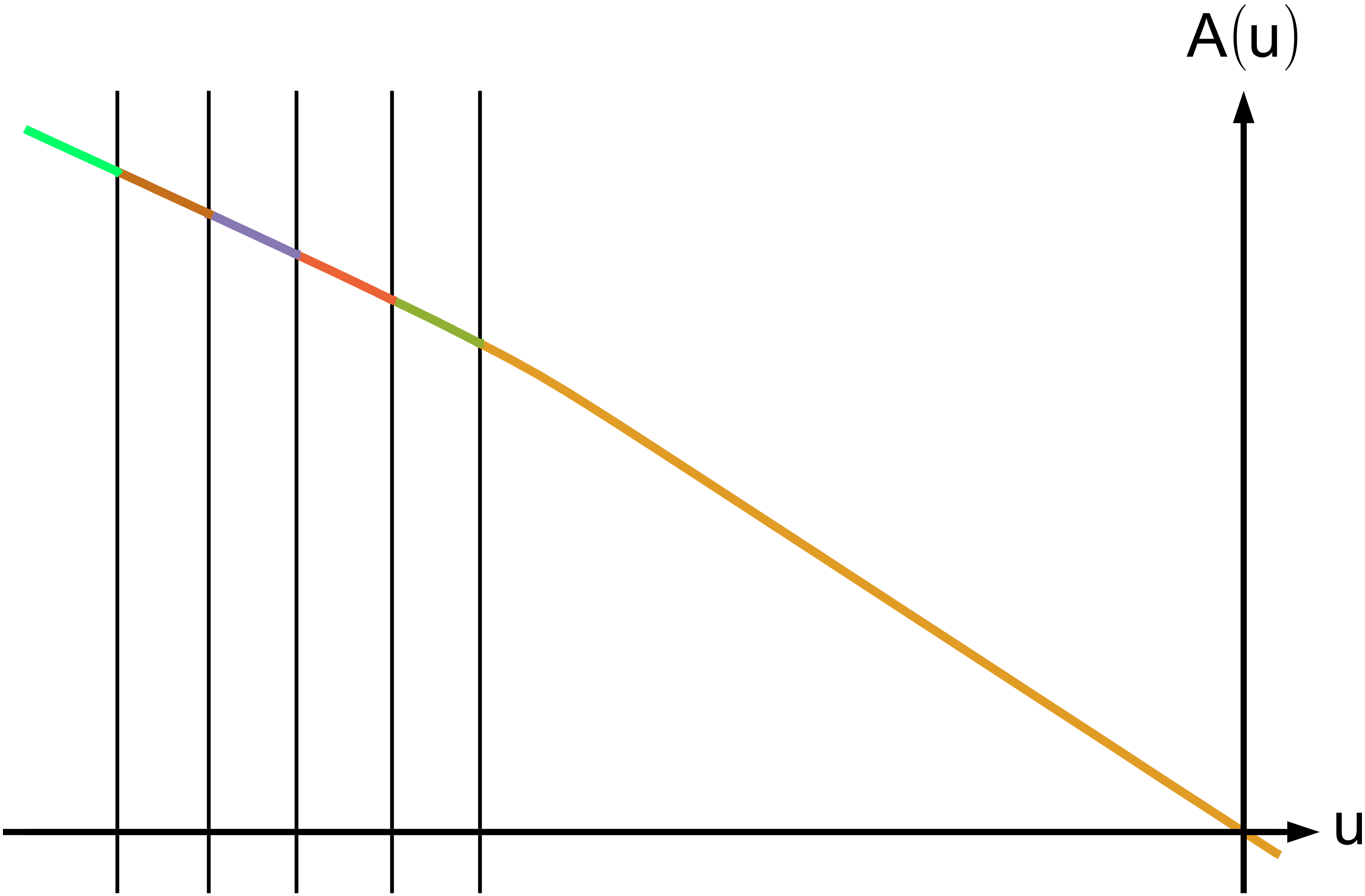
- No β-function description of such flows in the UV.

- Such flows have an infinite-cascade of bounces as one goes towards the UV.





φ(u)
u



• Although the flow is regular, it is unstable.

The maxima of V

- We will examine solutions for W near a maximum of V.

- We put the maximum at $\phi = 0$.

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad \Delta_+ \geq \Delta_- \geq 0$$

- We set (locally) $\ell = 1$ from now on.

- If $W'(0) = 0$ there are two classes of solutions:

• A continuous family of solutions (the W_- family)

$$W_- = 2(d-1) + \frac{\Delta_-}{2}\phi^2 + \cdots + C\phi^{\frac{d}{\Delta_-}}[1+\cdots] + \mathcal{O}(C^2)$$

• The solution for ϕ and A corresponding to this, is the standard UV source flow:

$$\phi(u) = \alpha\, e^{\Delta_- \, u} + \cdots + \frac{\Delta_-}{d}\, C\, e^{\Delta_+ \, u} + \cdots \quad , \quad e^A = e^{u-A_0} + \cdots \quad , \quad u \to -\infty$$

• the solution describes the UV region ($u \to -\infty$) with a perturbation by a relevant operator of dimension $\Delta_+ < d$.

• The source is α. It is not part of W.

• C determines the vev: $\langle O \rangle \sim C\ \alpha^{\frac{\Delta_+}{\Delta_-}}$.

• A single isolated solution W_+

$$W_+ = 2(d-1) + \frac{\Delta_+}{2}\phi^2 + \mathcal{O}(\phi^3) \quad , \quad \Delta_+ > \Delta_-$$

• The associated solution for ϕ, A is

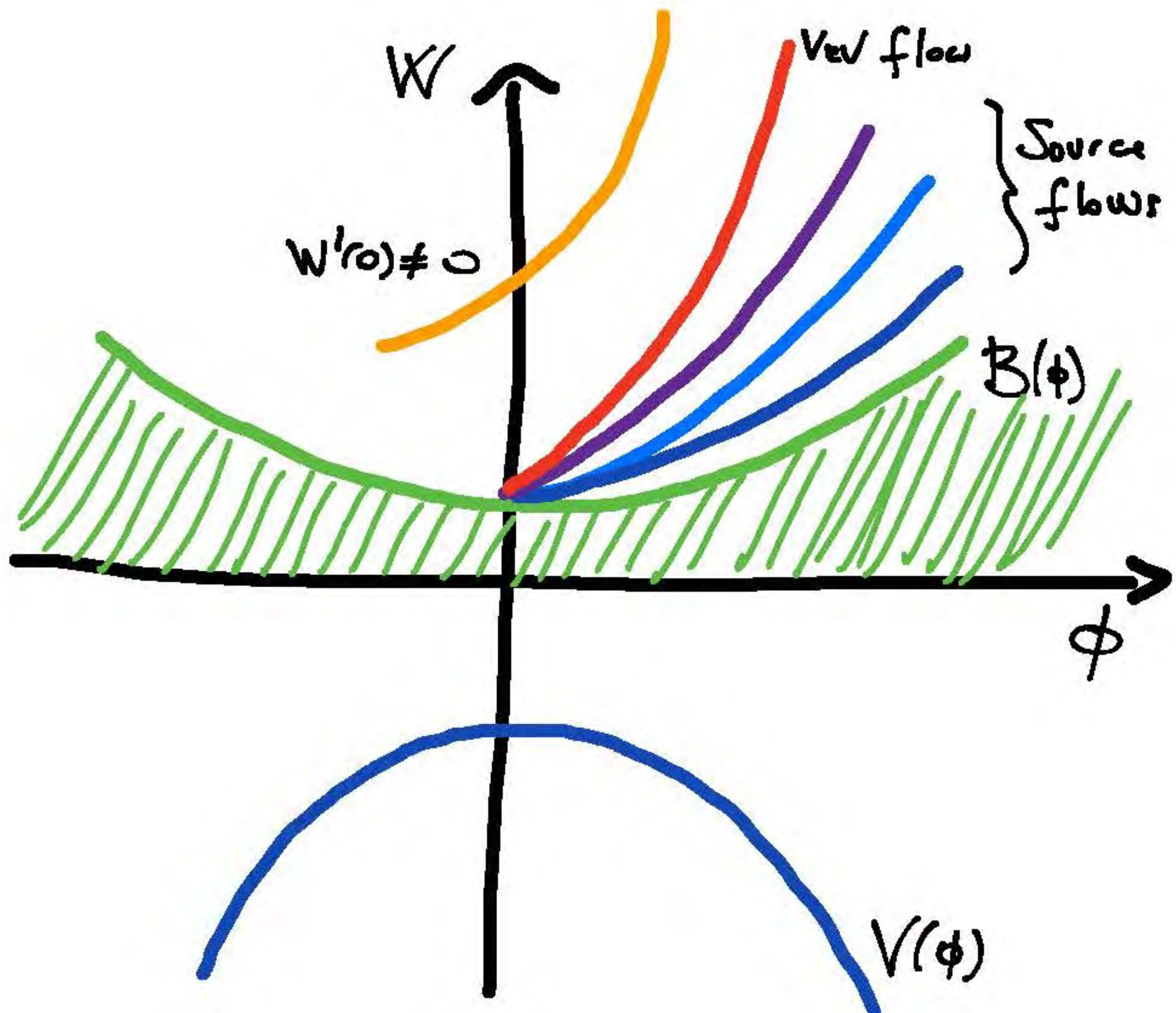
$$\phi(u) = \alpha\ e^{\Delta_+\ u} + \cdots \quad , \quad e^A = e^{-u+A_0} + \cdots$$

• This is a vev flow ie. the source is zero.

$$\langle O \rangle = (2\Delta_+ - d)\ \alpha$$

• The value of the vev is NOT determined by the superpotential equation. This is a moduli space.

• The whole class of solutions exists both from the left of $\phi = 0$ and from the right.



RETURN1 RETURN2

The minima of V

- We expand the potential near the minimum:

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right] \quad , \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$$

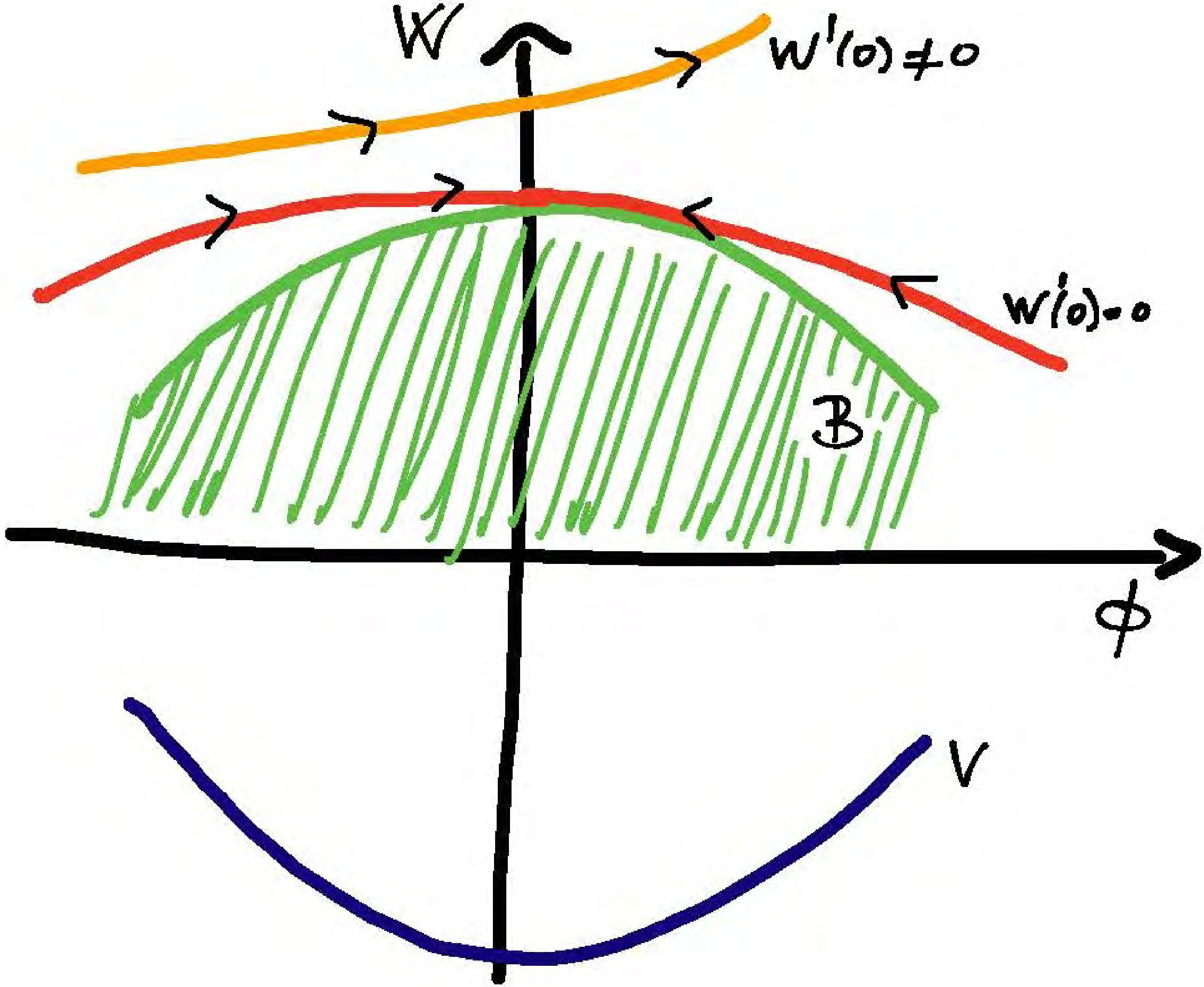
$$m^2 > 0 \quad , \quad \Delta_+ > 0 \quad , \quad \Delta_- < 0$$

- There are two isolated solutions with $W'(0) = 0$.

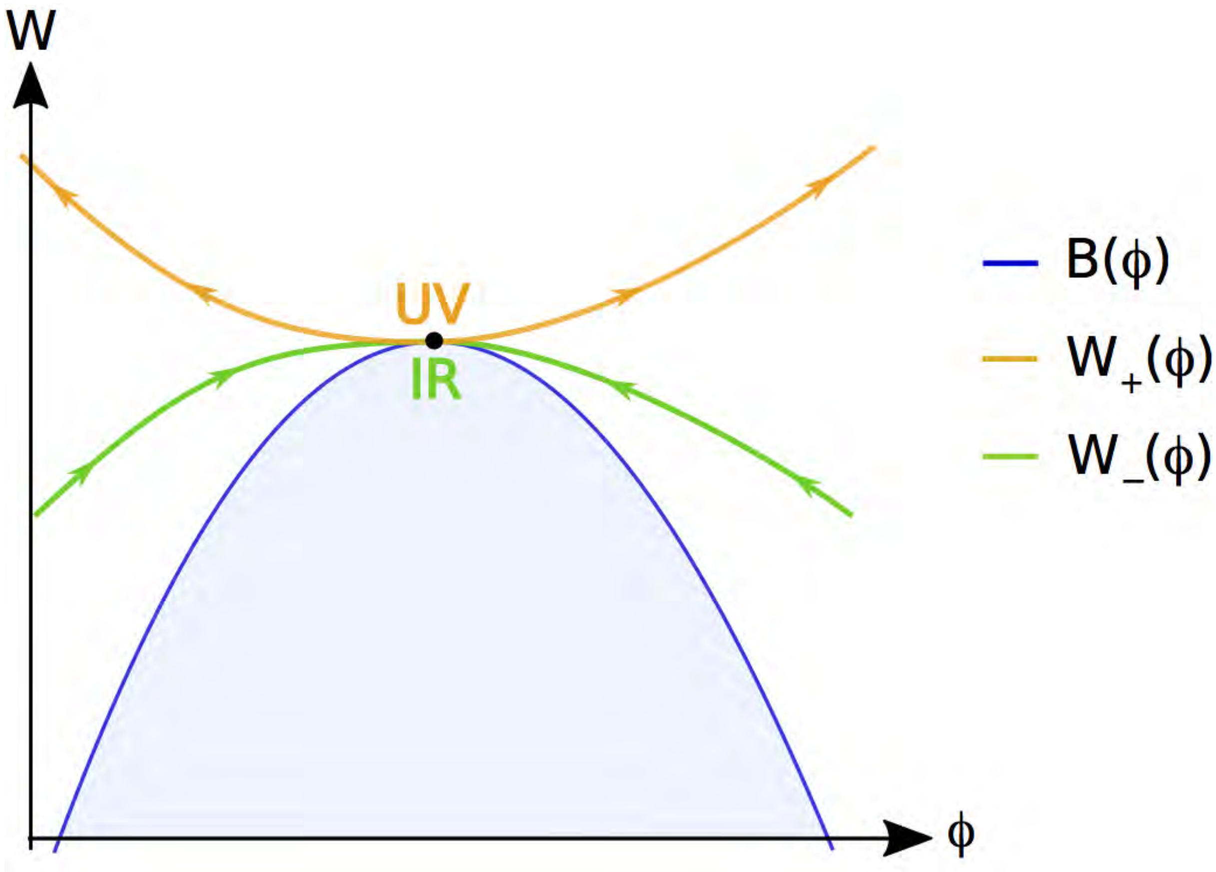
$$W_\pm(\phi) = \frac{1}{\ell}\left[2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \mathcal{O}(\phi^3)\right],$$

- No continuous parameter here as it generates a singularity.

- Although the solutions look similar, their interpretation is very different. W_+ has a local minimum while W_- has a local maximum.



W
W'(0) ≠ 0
W(0) = 0
B
φ
V



• There is again a moduli space.

♠ A W_+ solution is globally regular only in special cases.

♠ Therefore a minimum of the potential can be either an IR fixed point or a UV fixed point.

The maxima of V

- We will examine solutions for W near a maximum of V.
- We put the maximum at $\phi = 0$.
- When $V'(0) = 0$, $W''(0)$ is finite.

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right]$$

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \quad , \quad m^2\ell^2 \quad < 0 \quad , \quad \Delta_+ \geq \Delta_- \geq 0$$

- We set (locally) $\ell = 1$ from now on.
- If $W'(0) \neq 0$ there is one solution (per branch) off the critical curve,
- If $W'(0) = 0$ there are two classes of solutions:

• A continuous family of solutions (the W_- family)

$$W_- = 2(d-1) + \frac{\Delta_-}{2}\phi^2 + \cdots + C\phi^{\frac{d}{\Delta_-}}[1+\cdots] + \mathcal{O}(C^2)$$

• The solution for ϕ and A corresponding to this, is the standard UV source flow:

$$\phi(u) = \alpha\, e^{\Delta_- \, u} + \cdots + \frac{\Delta_-}{d}\, C\, e^{\Delta_+ \, u} + \cdots \quad , \quad e^A = e^{u-A_0} + \cdots \quad , \quad u \to -\infty$$

• the solution describes the UV region ($u \to -\infty$) with a perturbation by a relevant operator of dimension $\Delta_+ < d$.

• The source is α. It is not part of W.

• C determines the vev: $\langle O \rangle \sim C\ \alpha^{\frac{\Delta_+}{\Delta_-}}$.

• The near-boundary AdS is an attractor of all these solutions.

- A single isolated solution W_+ also arriving at $W(0) = B(0)$

$$W_+ = 2(d-1) + \frac{\Delta_+}{2}\phi^2 + \mathcal{O}(\phi^3) \quad , \quad \Delta_+ > \Delta_-$$

- Always $W_+'' > W_-''$.

- The associated solution for ϕ, A is

$$\phi(u) = \alpha \; e^{\Delta_+ \; u} + \cdots \quad , \quad e^A = e^{-u+A_0} + \cdots$$

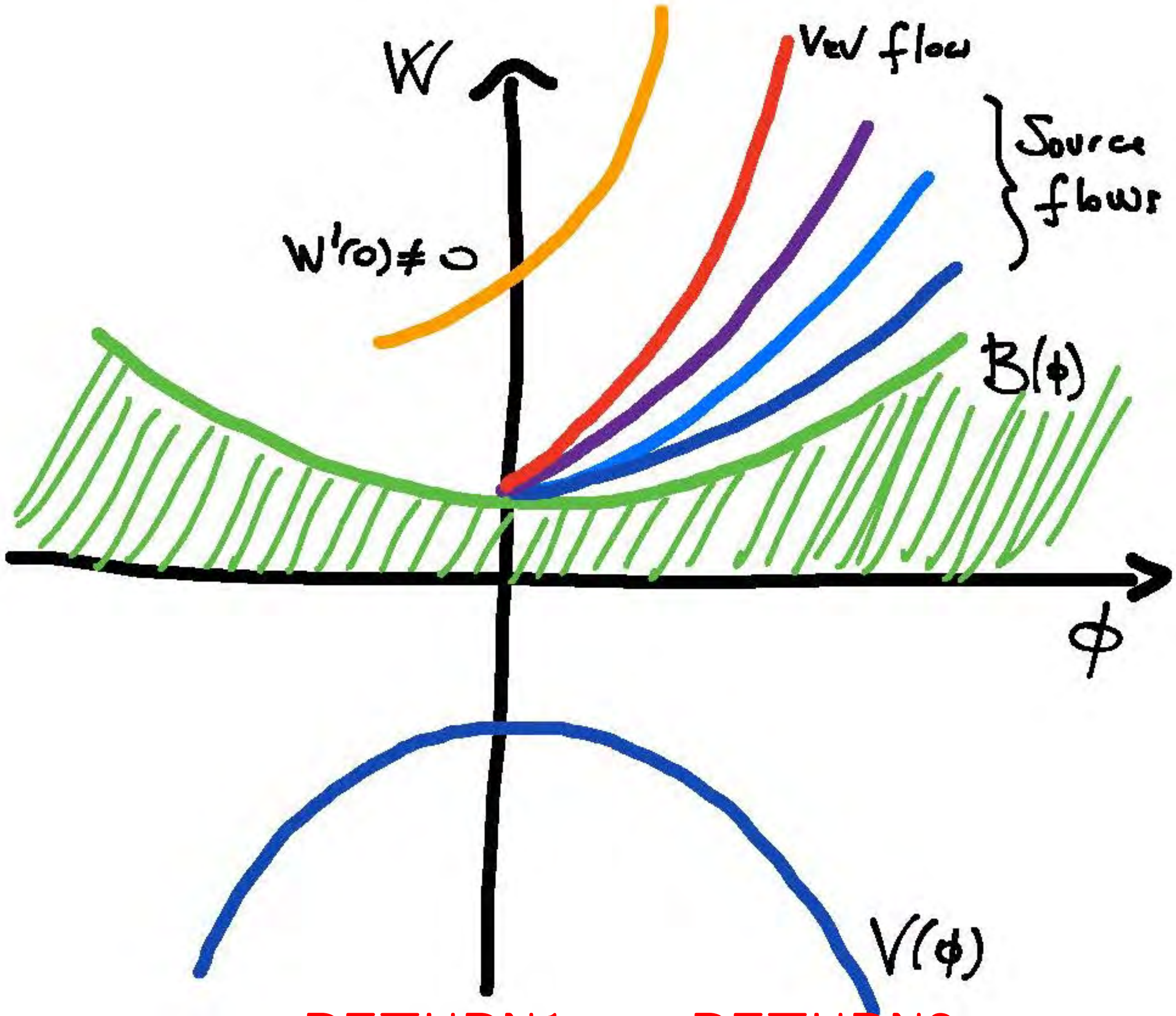
- This is a vev flow ie. the source is zero.

$$\langle O \rangle = (2\Delta_+ - d) \; \alpha$$

- The value of the vev is NOT determined by the superpotential equation.

- It can be reached in a appropriately defined limit $C \to \infty$ of the W_- family.

- The whole class of solutions exists both from the left of $\phi = 0$ and from the right.



RETURN1 RETURN2

The minima of V

- We expand the potential near the minimum:

$$V(\phi) = -\frac{1}{\ell^2}\left[d(d-1) - \frac{m^2\ell^2}{2}\phi^2 + \mathcal{O}(\phi^3)\right] \quad , \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2}$$

$$m^2 > 0 \quad , \quad \Delta_+ > 0 \quad , \quad \Delta_- < 0$$

- There are solutions with $W'(0) \neq 0$. These are solutions that do not stop at the minimum.

- There are two isolated solutions with $W'(0) = 0$.

$$W_\pm(\phi) = \frac{1}{\ell}\left[2(d-1) + \frac{\Delta_\pm}{2}\phi^2 + \mathcal{O}(\phi^3)\right],$$

- No continuous parameter here as it generates a singularity.

- Although the solutions look similar, their interpretation is very different. W_+ has a local minimum while W_- has a local maximum.

• The W_- solution:

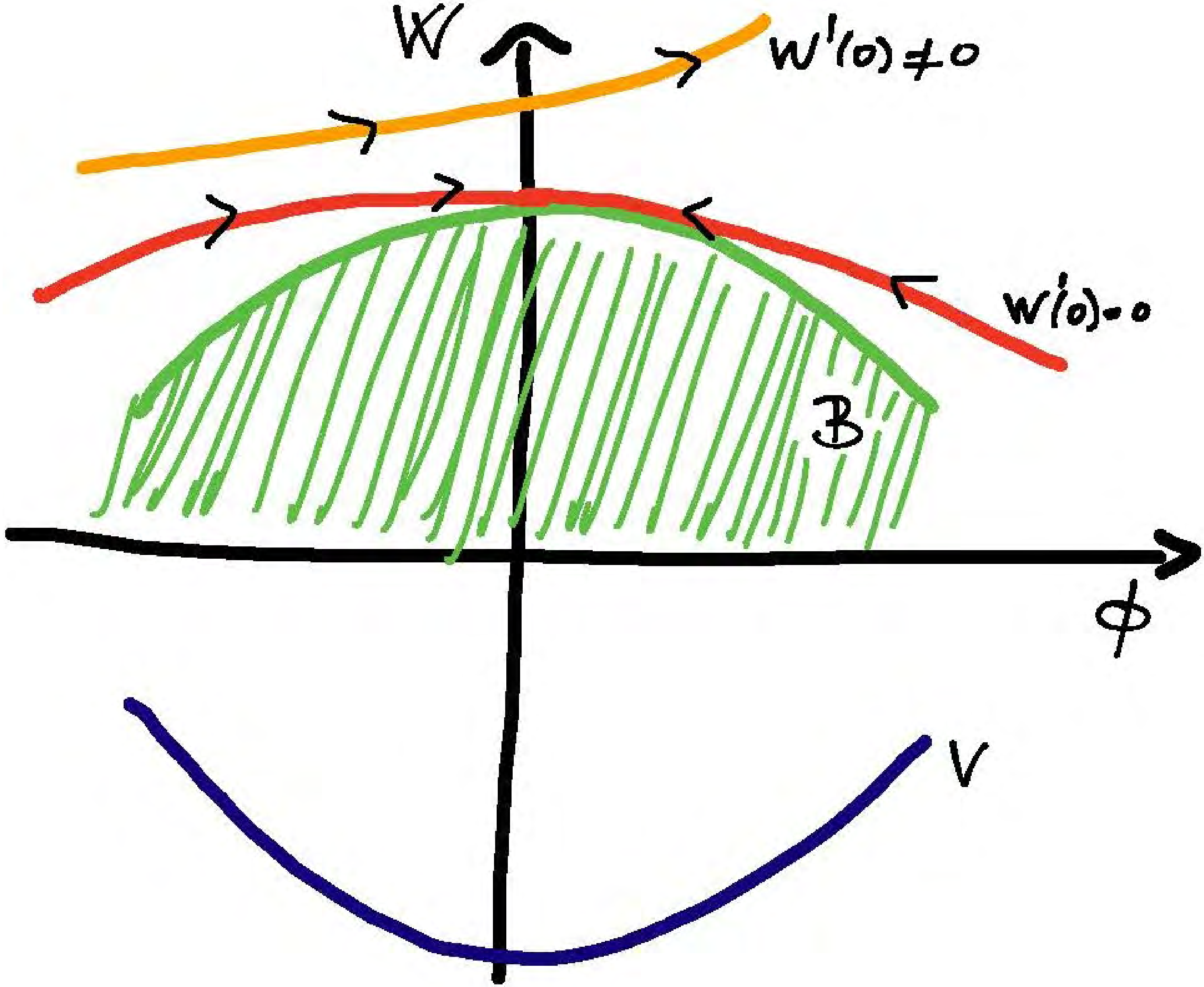
$$\phi(u) = \alpha\ e^{\Delta_-\ u} + \cdots \quad , \quad e^A = e^{-(u-u_0)} + \cdots .$$

• Since $\Delta_- < 0$, small ϕ corresponds to $u \to +\infty$ and $e^A \to 0$.

• This signal we are in the deep interior (IR) of AdS.

• The driving operator has (IR) dimension $\Delta_+ > d$ and a zero vev in the IR.

• Therefore W_- generates locally a flow that arrives at an IR fixed point.


W
W'(0) ≠ 0
W'(0) = 0
B
ϕ
V

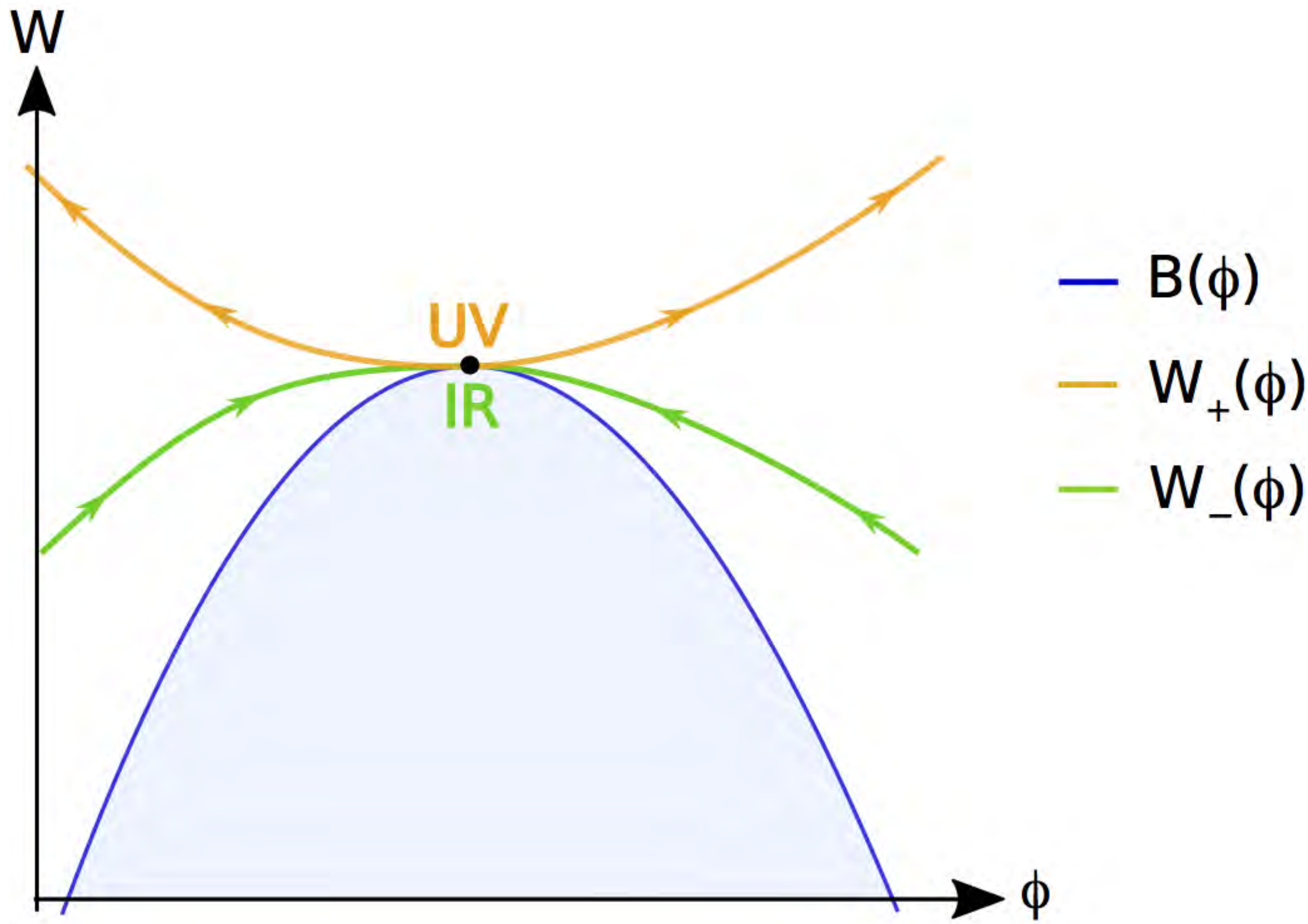
• The W_+ solution is:

$$\phi(u) = \alpha \, e^{\Delta_+ \, u} + \cdots \quad , \quad e^A = e^{-(u-u_0)} + \cdots .$$

• Since $\Delta_+ > 0$ small ϕ corresponds to $u \to -\infty$ and $e^A \to +\infty$.

• This solution described the near-boundary (UV) region of a fixed point.

• This solution is driven by the vev of an operator with (UV) dimension $\Delta_+ > d$ (irrelevant).



♠ A minimum of the potential can be either an IR fixed point or a UV fixed point.

The first order formalism

- In this case the two first order flow equations are modified:

$$\dot{A} = -\frac{1}{2(d-1)}W(\phi) \quad , \quad \dot{\phi} = S(\phi)$$

$$\frac{d}{2(d-1)}W^2 + (d-1)S^2 - dSW' = -2V \quad , \quad SS' - \frac{d}{2(d-1)}WS = V'$$

- The two superpotential equations have two integration constants.

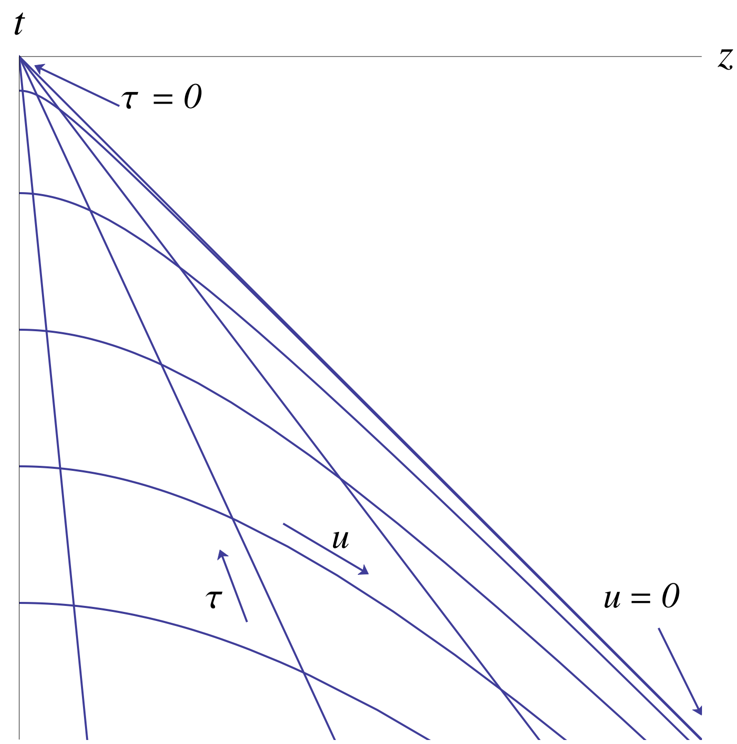
- One of them, C, is the vev of the scalar operator (as usual).

- The other is the dimensionless curvature, $\mathcal{R}$.

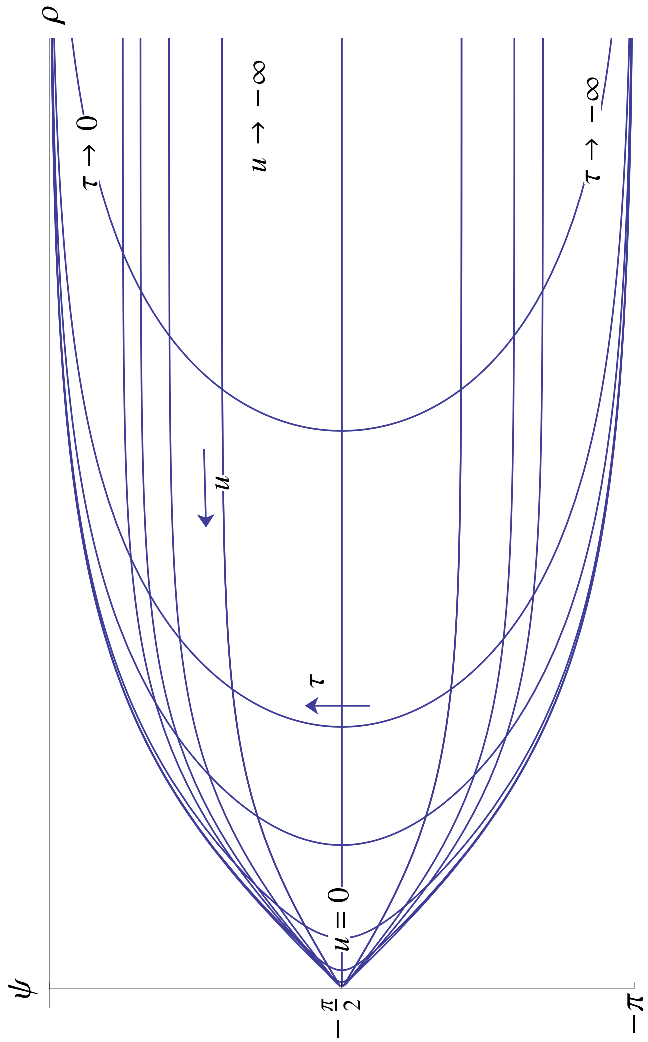
- The structure near an maximum (UV) of the potential has the "resurgent" expansion

$$W(\phi) = \sum_{m,n,r \in Z_0^+} A_{m,n,r} \, (C \, \phi^{\frac{d}{\Delta_-}})^m \, (\mathcal{R} \, \phi^{\frac{2}{\Delta_-}})^n \, \phi^r$$

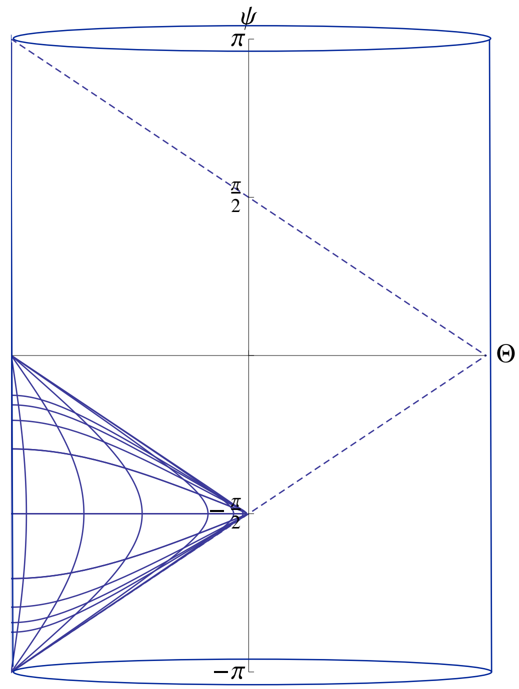
Coordinates



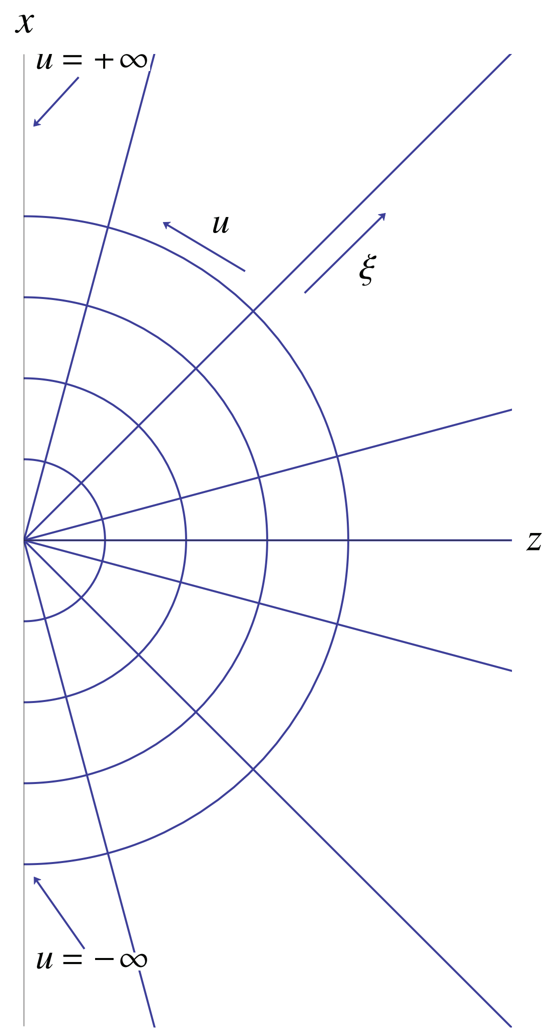
Relation between Poincaré coordinates (t, z) and dS-slicing coordinates (τ, u). Constant u curves are half straight lines all ending at the origin ($\tau \to 0^-$); Constant τ curves are branches of hyperbolas ending at $u = 0$ (null infinity on the $z = -t$ line). The boundary $z = 0$ corresponds to $u \to -\infty$.



Embedding of the dS patch in global coordinates. The flow endpoint $u = 0$ corresponds to the point $\rho = 0, \psi = -\pi/2$ in global coordinates. the AdS boundary is at $\rho = +\infty$ and it is reached along u as $u \to -\infty$, and along τ both as $\tau \to -\infty$ and as $\tau \to 0$.

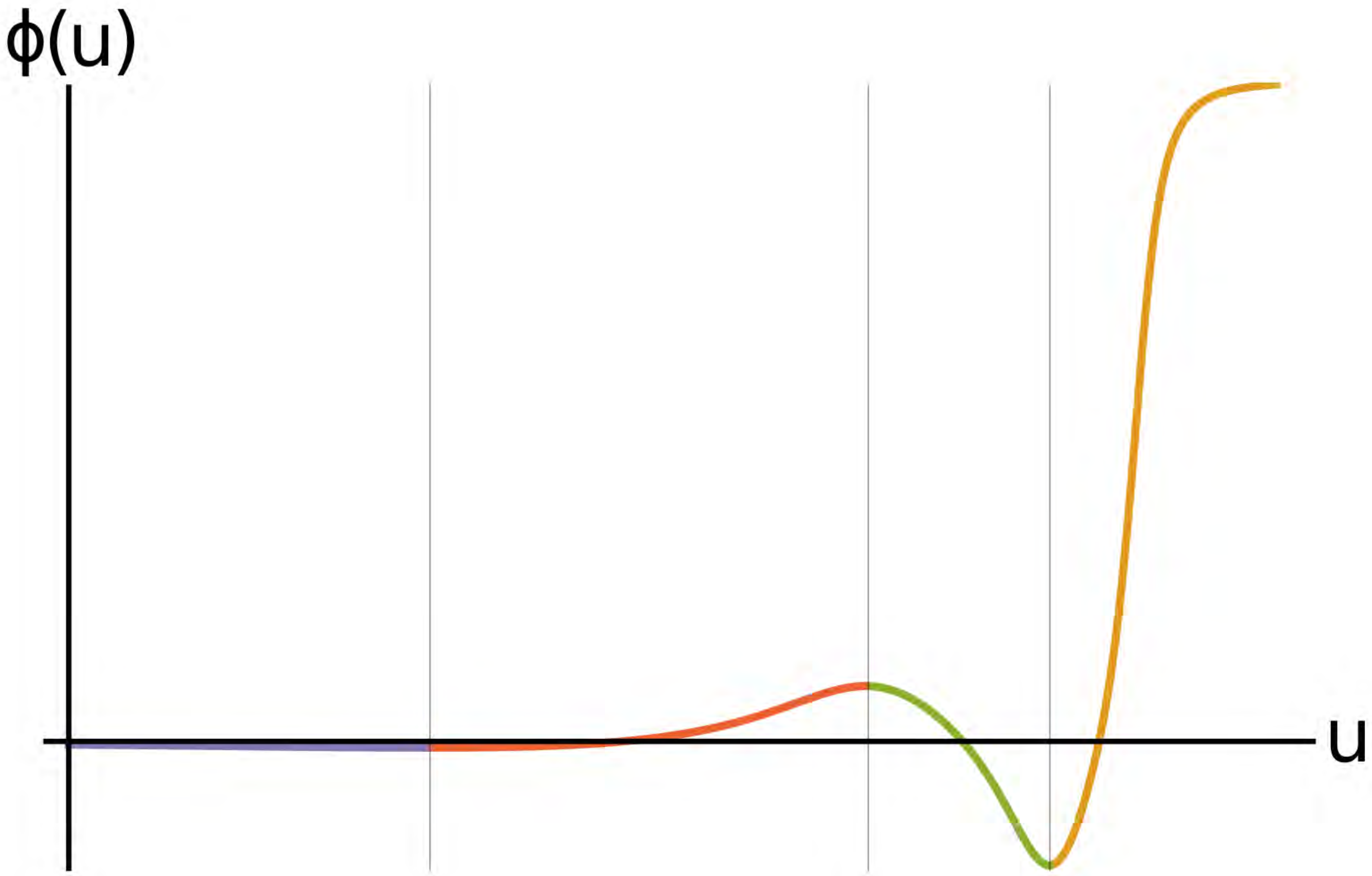


Embedding of the dS patch in global conformal coordinates, $\tan\Theta = \sinh\rho$, where each point is a $d-1$ sphere "filled" by Θ. The boundary is at $\Theta = \pi/2$. The dashed lines correspond to the Poincaré patch embedded in global conformal coordinates. The flow endpoint $u = 0$ is situated on the Poincaré horizon.



Relation between Poincaré coordinates (x, z) and AdS-slicing coordinates (ξ, u). Constant u curves are half straight lines all ending at the origin ($\xi \to 0^-$); Constant ξ curves are semicircle joining the two halves of the boundary at $u = \pm\infty$.

Bounces




A(u)
u


UV
IR

Curtright, Jin and Zachos gave an example of an RG Flow that is cyclic but respects the strong C-theorem

$$\beta_n(\phi) = (-1)^n\sqrt{1-\phi^2} \quad \to \quad \phi(A) = sin(A)$$

If we define the superpotential branches by $\beta_n = -2(d-1)W_n'/W_n$ we obtain

$$\log W_n = \frac{(2n+1)\pi + 2(-1)^n(\arcsin(\phi) + \phi\sqrt{1-\phi^2})}{8(d-1)}$$

and we can compute the potentials from $V = W'^2/2 - dW^2/4(d-1)$ to obtain $V_n(\phi)$.

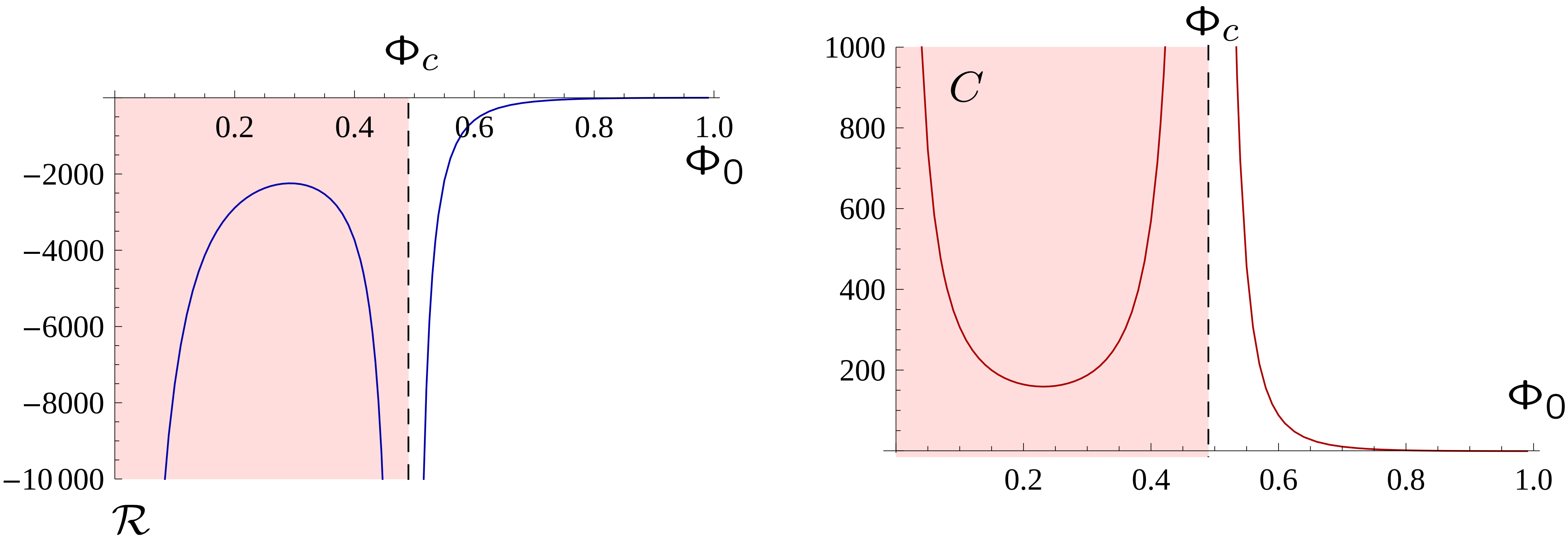
Such piece-wise potentials then satisfy

$$V_{n+2}(\phi) = e^{\frac{\pi}{2(d-1)}}\ V_n(\phi)$$

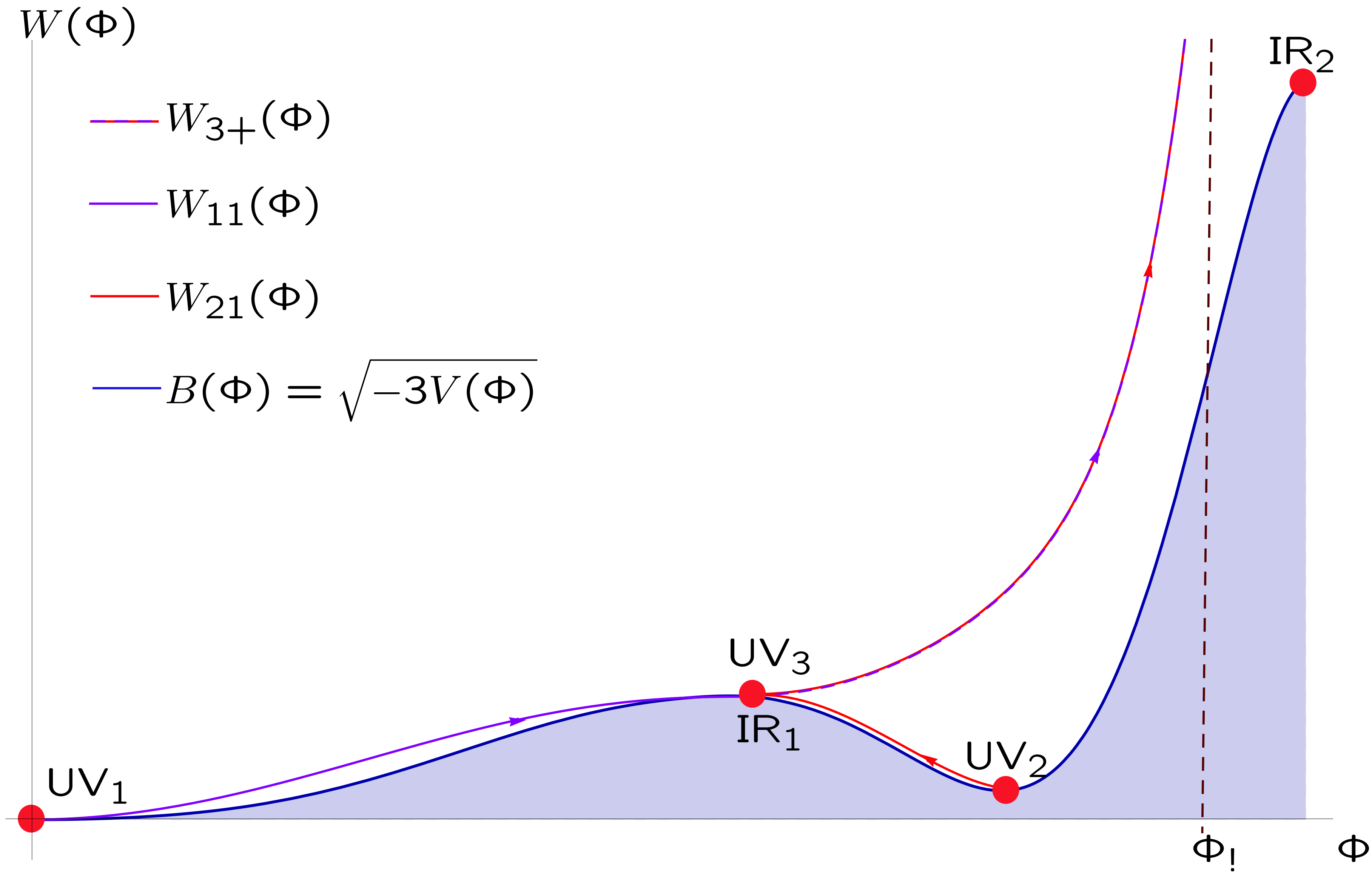
- No such potentials can arise in string theory.

- Holography can provide only "approximate" cycles.

Flows in AdS

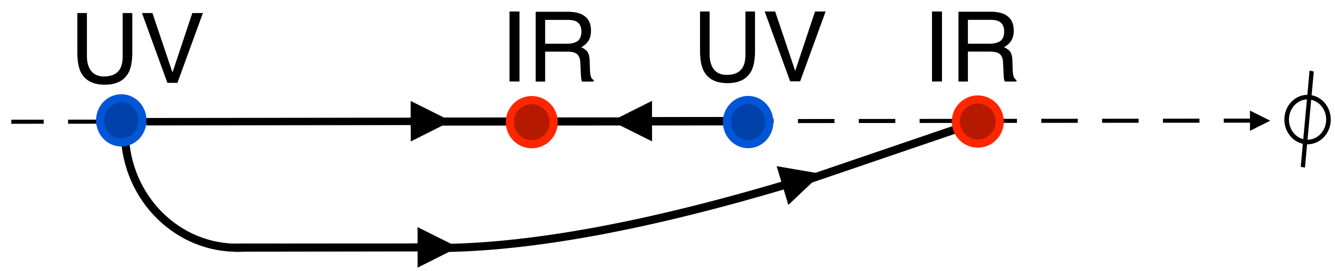


QFT on AdS$_d$: dimensionless curvature $\mathcal{R} = R^{(uv)}|\Phi_-|^{-2/\Delta_-}$ and dimensionless vev $C = \frac{\Delta_-}{d}\langle\mathcal{O}\rangle|\Phi_-|^{-\Delta_+/\Delta_-}$ vs. Φ_0 for the Mexican hat potential with $\Delta_- = 1.2$. Flows with turning points in the rose-colored region leave the UV fixed point at $\Phi = 0$ to the left before bouncing and finally ending at positive Φ_0. Flows with turning points in the white region are direct: They leave the UV fixed point at $\Phi = 0$ to the right and do not exhibit a reversal of direction. The flow with turning point Φ_c on the border between the bouncing/non-bouncing regime corresponds to a theory with vanishing source Φ_-. As a result, both $\mathcal{R}$ and C diverge at this point.



$W(\Phi)$
$W_{3+}(\Phi)$
$W_{11}(\Phi)$
$W_{21}(\Phi)$
$B(\Phi) = \sqrt{-3V(\Phi)}$
IR_2
UV_3
IR_1
UV_2
UV_1
$\Phi_!$
Φ

RG flows with IR endpoint $\Phi_0 \to \Phi_!$. When the endpoint Φ_0 approaches $\Phi_!$ flows from both UV$_1$ and UV$_2$ pass by closely to IR$_1$, passing through IR$_1$ exactly for $\Phi_0 = \Phi_!$. This is shown by the purple and red curves. Beyond IR$_1$ both these solutions coincide, which is denoted by the colored dashed curve. These have the following interpretation. The flows from UV$_1$ and UV$_2$ should not be continued beyond IR$_1$, which becomes the IR endpoint for the zero curvature flows W_{11} and W_{21}. The remaining branch (the colored dashed curve) is now an independent flow denoted by W_{3+}. This is a flow from a UV fixed point at a minimum of the potential (denoted by UV$_3$ above) to $\Phi_!$ and corresponds to a W_+ solution with fixed value $\mathcal{R} = R^{\mathsf{uv}}|\Phi_+|^{-2/\Delta_+} \neq 0$. While flows from UV$_1$ and UV$_2$ can end arbitrarily close to $\Phi_!$, the endpoint $\Phi_0 = \Phi_!$ cannot be reached from UV$_1$ or UV$_2$.



• It is not possible in this example to redefine the topology on the line so that the flow looks “normal”

• The two flows $UV_1 \to IR_1$ and $UV_1 \to IR_2$ correspond to the same source but different vev’s.

• One can calculate the free-energy difference of these two flows: the one that arrives at the IR fixed point with lowest a, is the dominant one.

Detailed plan of the presentation

- The vanilla flows 19 minutes
- Skipping flows at finite curvature 21 minutes
- Spontaneous breaking saddle points 22 minutes
- Stabilisation by curvature 24 minutes
- Unusual vev flows from minima 26 minutes
- The free energy and entanglement entropy 28 minutes
- Renormalization in 3d 29 minutes
- F-functions 31 minutes
- Outlook 32 minutes

- UV and IR divergences of F and S_{EE} 33 minutes
- $\mathcal{F}$-functions, II 34 minutes
- Holography and the Quantum RG 35 minutes
- The extrema of V 36 minutes
- The strategy 37 minutes
- Regularity 38 minutes
- General Properties of the superpotential 41 minutes
- Holographic RG Flows 45 minutes
- Detour: the local RG 47 minutes
- More flow rules 48 minutes

- The critical points of W 50 minutes
- The BF bound 51 minutes
- BF-violating flows 53 minutes
- The maxima of V 61 minutes
- The minima of V 68 minutes
- The first order formalism 70 minutes
- Coordinates 72 minutes
- Bounces 74 minutes