Pole-skipping in holographic theories

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Introduction

- Chaotic behaviour of quantum many-body systems can be characterised by

$$C(x,t) = -\langle [W(x,t), V(0,0)]^2 \rangle_T \sim e^{\lambda(t - x/v_B)}$$

λ Lyapunov exponent
v_B butterfly velocity

Larkin & Ovchinnikov
Shenker & Stanford
Kitaev
...

- Holographic theories: parameters determined by properties of the black hole horizon

$$\lambda = 2\pi T = \frac{r_0^2 f'(r_0)}{2} \qquad v_B^2 = \frac{4\pi T}{d h'(r_0)}$$

Shenker & Stanford
Roberts & Stanford
Roberts & Swingle
Blake

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + h(r) d\vec{x}^2$$

Chaos in energy density Green's function?

- The chaos parameters also show up in other contexts

  poles $\omega(k)$ of $G^R_{\varepsilon\varepsilon}(\omega, k)$ in Schwarzschild-AdS5

→ numerics indicate that the hydrodynamic pole passes through the point

$$\omega = i\lambda\,, \qquad k = i\frac{\lambda}{v_B}$$

Grozdanov, Schalm & Scopelliti

→ residue vanishes

Pole-skipping

- "Quantum hydrodynamics" proposed in which chaotic behaviour arises due to hydrodynamic degrees of freedom

Blake, Lee, Liu

- A line of poles of $G^R_{\varepsilon\varepsilon}(\omega, k)$ **must** pass through the point $\omega = i\lambda\,, \quad k = i\dfrac{\lambda}{v_B}$
- The residue vanishes at the special point → pole-skipping

- Does this happen in holographic theories? Why?
- Must be 1). due to a generic property of gravity

  2). controlled by physics near the horizon

Outline of the talk

1. Einstein's equations near the special point
2. An explicit example of pole skipping
3. Conclusions and generalisations

Retarded Green's functions in holography

- Black brane spacetime

$$ds^2 = -r^2 f(r) dv^2 + 2 dv dr + h(r) dx_i dx^i$$

- Two-point functions are controlled by small perturbations
  e.g.

$$\phi(r \to \infty) = A(\omega, k) r^0 + B(\omega, k) r^{-1} + \ldots$$

- Impose ingoing boundary conditions at the horizon to uniquely fix

$$G^R_{\mathcal{O}\mathcal{O}}(\omega, k) = \frac{B(\omega, k)}{A(\omega, k)}$$

- Normalisable, ingoing solution ⟶ pole

Gravitational perturbations

- $G^R_{\varepsilon\varepsilon}(\omega, k)$ is controlled by perturbations of the metric
- In the absence of matter:

$\delta g_{vv}(r, \omega, k)$
$\delta g_{vx}(r, \omega, k)$ all couple together (radial gauge)
$\delta g_{x^i x^i}(r, \omega, k)$

- Make an ansatz that is manifestly ingoing at the horizon r_0

$$\delta g_{\mu\nu}(r) = \delta g^{(0)}_{\mu\nu} + \delta g^{(1)}_{\mu\nu}(r - r_0) + \delta g^{(2)}_{\mu\nu}(r - r_0)^2 + \ldots$$

- Then solve the Einstein equations order-by-order to fix the coefficients.

Gravitational perturbations near the horizon

- e.g. the vv-component of Einstein's equations is

$$\left(-i\frac{d}{2}\omega h'(r_0)+k^2\right)\delta g^{(0)}_{vv}-(2\pi T+i\omega)\left[i\omega\delta g^{(0)}_{x^ix^i}+2ik\delta g^{(0)}_{vx}\right]=0$$

- The solution depends on (d+2) parameters e.g. $\delta g^{(0)}_{vx}, \delta g^{(1)}_{vv}, \delta g^{(0)}_{x^ix^i}$

- But the equation above vanishes identically when $\omega = 2\pi T i, \quad k = \frac{2\pi T i}{v_B}$

→ the solution depends on one additional parameter

→ ingoing boundary conditions do not uniquely fix the retarded Green's functions.

Green's functions close to the special point

- Move infinitesimally away from the special point

$$\omega = i\lambda + \delta\omega, \quad k = ik_0 + \delta k$$

- The solution depends on the direction you move

$$\frac{\delta\omega}{\delta k} = \frac{2k_0 \delta g_{vv}^{(0)}}{\frac{d}{2}h'(r_0)\delta g_{vv}^{(0)} - 2\pi T g_{x^i x^i}^{(0)} - 2k_0 \delta g_{vx}^{(0)}},$$

→ can find a normalisable solution by choosing the direction appropriately!

→ a pole must pass through this point.

- It is a consequence of Einstein's equation on the horizon.

The axion model in AdS4

- Gravitational action: $$S = \int d^4x\sqrt{-g}\left(\mathcal{R} + 6 - \frac{1}{2}\sum_{i=1}^{2}\partial_\mu\varphi_i\partial^\mu\varphi_i\right)$$

- Black brane solution:

Andrade & Withers

$$ds^2 = -r^2 f(r)dv^2 + 2dvdr + r^2 dx_i dx^i \qquad \varphi_i = m x^i$$

$$f(r) = 1 - \frac{m^2}{2r^2} - \left(1 - \frac{m^2}{2r_0^2}\right)\frac{r_0^3}{r^3}$$

- The free parameter m controls the breaking of translational symmetry.

→ radically alters the properties of $G^R_{\varepsilon\varepsilon}(\omega, k)$

RD & Gouteraux

- The special point in Fourier space is $$\omega = i\frac{r_0^2 f'(r_0)}{2}, \quad k = i\sqrt{r_0^3 f'(r_0)}$$

Perturbation equation in the axion model

- $G^R_{\varepsilon\varepsilon}(\omega,k)$ can be extracted from a gauge invariant perturbation

Kovtun & Starinets

$$\psi \equiv r^4 f\left\{\frac{d}{dr}\left[\frac{\delta g_{xx}+\delta g_{yy}}{r^2}\right]-\frac{i\omega}{r^4 f}(\delta g_{xx}+\delta g_{yy})-\frac{2ik}{r^2}\left(\delta g_{xr}+\frac{\delta g_{vx}}{r^2 f}\right)-2rf\left(\delta g_{rr}+\frac{2}{r^2 f}\delta g_{vr}+\frac{1}{r^4 f^2}\delta g_{vv}\right)-\frac{k^2+r^3 f'}{r^5 f}\delta g_{yy}\right\}$$
$$-\frac{(k^2+r^3 f')}{(k^2+m^2)}\frac{mr}{2}\left(\frac{m}{r^2}(\delta g_{xx}-\delta g_{yy})-2ik\delta\varphi_1\right)$$

via $$G^R_{\varepsilon\varepsilon}(\omega,k)=k^2\left(k^2+m^2\right)\frac{\psi^{(0)}(\omega,k)}{\psi^{(1)}(\omega,k)+i\omega\psi^{(0)}(\omega,k)}$$

where $\psi(r\to\infty,\omega,k)\to\psi^{(0)}(\omega,k)+\psi^{(1)}(\omega,k)r^{-1}+\dots$

- It obeys a relatively simple equation

$$\frac{d}{dr}\left[\frac{r^2 f}{(k^2+r^3 f')^2}\psi'\right]-\frac{2i\omega}{(k^2+r^3 f')^2}\psi'-\frac{(2r_0^2-m^2)3ir_0\omega+k^2(k^2+m^2)}{r^2\left(k^2+r^3 f'\right)^3}\psi=0$$

Solution at the special point

- Generically, there is a unique ingoing solution

$$\psi = a_1\eta_1 + a_2\eta_2, \qquad \eta_{1,2} = (r-r_0)^{\alpha_{1,2}} \text{ as } (r \to r_0), \qquad \alpha_1 = \frac{i\omega}{2\pi T}, \quad \alpha_2 = 0$$

- But at the special point there is not

$$\psi = a_1\eta_1 + a_2\eta_2, \qquad \eta_{1,2} = (r-r_0)^{\alpha_{1,2}} \text{ as } (r \to r_0), \qquad \alpha_1 = 0, \quad \alpha_2 = 1.$$

- Explicit solution: $\psi(r) = c_1 + c_2 \int_{r_0}^{r} \mathrm{d}r \frac{\exp\left(\frac{m^2-3r_0^2}{r_0\sqrt{3r_0^2-2m^2}}\tan^{-1}\left(\frac{2r+r_0}{\sqrt{3r_0^2-2m^2}}\right)\right)}{r\sqrt{2(r^2+rr_0+r_0^2)-m^2}}$

- An arbitrary linear combination of the two solutions is allowed

$\longrightarrow$ $G^R_{\varepsilon\varepsilon}(\omega, k)$ depends on free parameter

Solution away from the special point

- Moving away from the special point fixes $G^R_{\varepsilon\varepsilon}(\omega, k)$ uniquely.

$$k^2 = -k_0^2 + \epsilon, \qquad \omega = 2\pi T i - \frac{i}{2k_0}\frac{\delta\omega}{\delta k}\epsilon$$

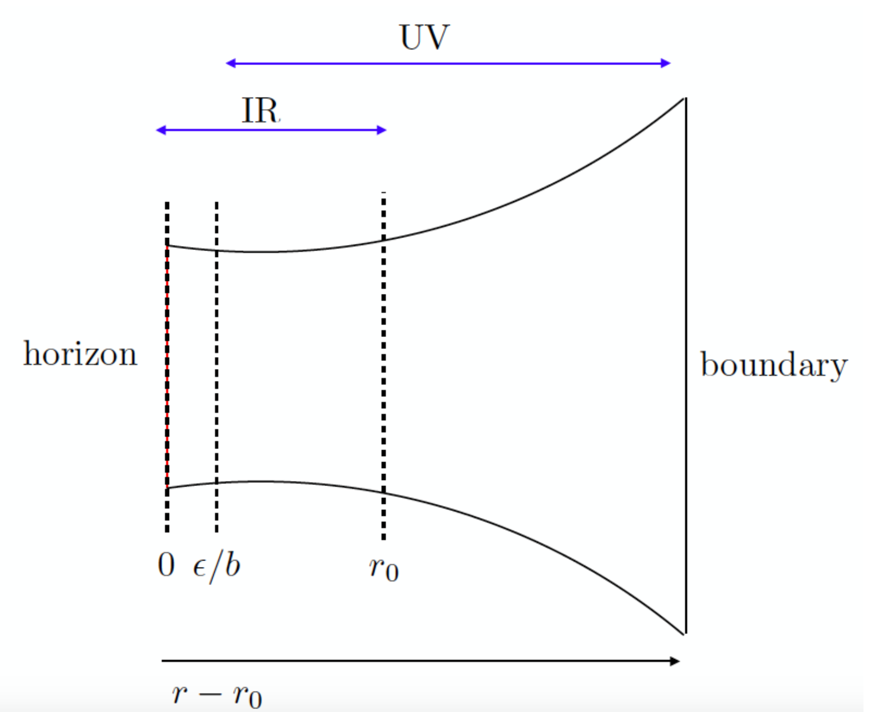
- The singular change in near-horizon boundary conditions comes from

$$k^2 + r^3 f'(r) = \epsilon + b(r - r_0) + O((r-r_0)^2), \quad b = 3r_0^2 f'(r_0) + r_0^3 f''(r_0)$$

- Fix the coefficient by using a matching calculation.

- Access the IR region by changing coordinate

$$(r - r_0) = \frac{\epsilon y}{b}$$



Results from matching

- There is a unique ingoing solution in the IR region

$$\psi_{IR} = 1 + bF\left(\frac{\delta\omega}{\delta k}\right)(r - r_0) + \dots \qquad F\left(\frac{\delta\omega}{\delta k}\right) = \frac{4\left(3r_0^2 - m^2\right)}{3\left(6r_0^2 - m^2\right)\left(2r_0^2 - m^2\right)} - \frac{v_B}{16\pi^2 T^2}\frac{\delta\omega}{\delta k}$$

- The arbitrary coefficient in the UV region is fixed by matching to this:

$$\psi_{\mathrm{UV}}(r) = 1 + \frac{b}{c}F\left(\frac{\delta\omega}{\delta k}\right)\int_{r_0}^{r} \mathrm{d}r\, \frac{\exp\left(\frac{m^2-3r_0^2}{r_0\sqrt{3r_0^2-2m^2}}\tan^{-1}\frac{(2r+r_0)}{\sqrt{3r_0^2-2m^2}}\right)}{r\sqrt{2(r^2+rr_0+r_0^2)-m^2)}}$$

$$c = \frac{\exp\left(\frac{m^2-3r_0^2}{r_0\sqrt{3r_0^2-2m^2}}\tan^{-1}\left(\frac{3r_0}{\sqrt{3r_0^2-2m^2}}\right)\right)}{r_0\sqrt{6r_0^2-m^2}}$$

- The direction $\delta\omega/\delta k$ picks out a specific linear combination of the two independent solutions.

Retarded Green's function in the axion model

- The Green's function near the special point is

$$G^R_{\varepsilon\varepsilon}(\omega, k) = C\,\frac{\delta\omega - q_z\delta k}{\delta\omega - q_p\delta k}$$

where we have explicit expressions for q_p and q_z

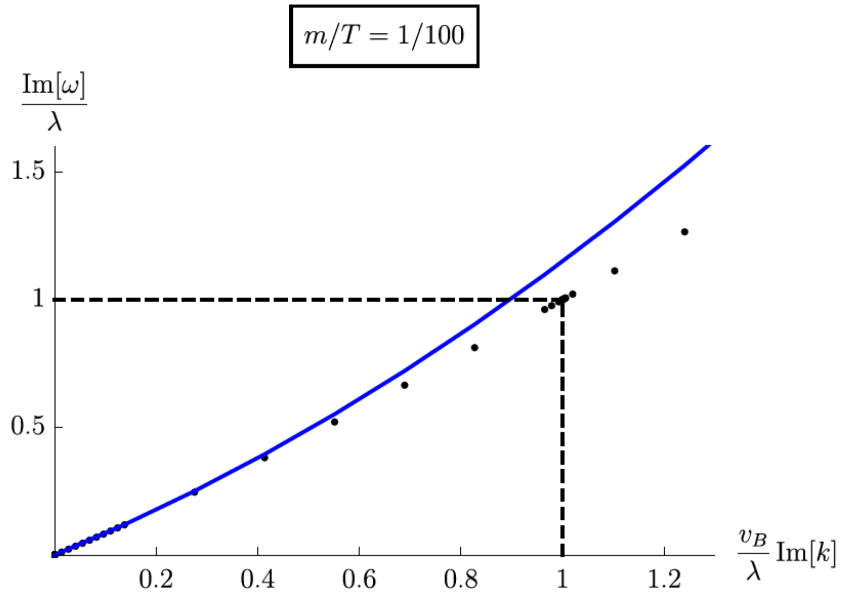
- A quantitative confirmation of our argument from the Einstein equations.

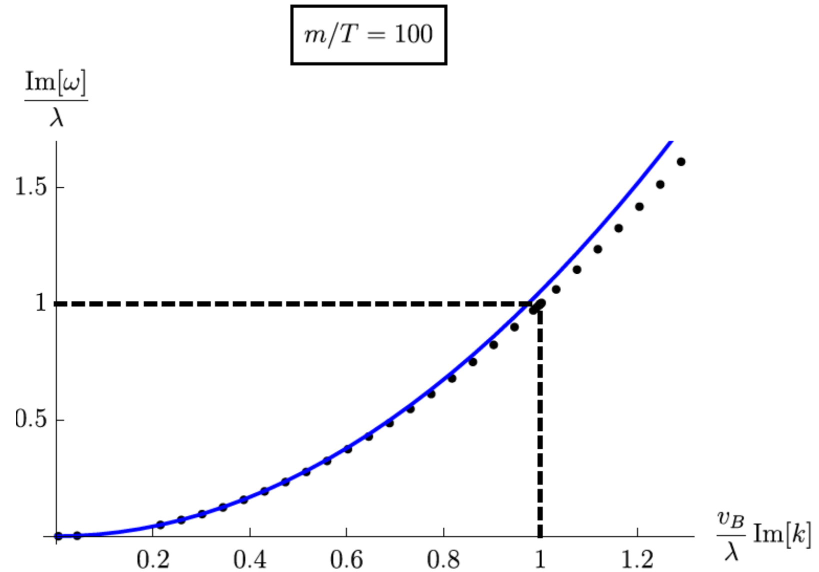
- The slope of the lines of poles passing through the special point is

$$\frac{1}{v_B}\frac{\delta\omega}{\delta k} = \frac{8(3r_0^2 - m^2)}{3(2r_0^2 - m^2)} + \frac{4(6r_0^2 - m^2)}{3(m^2 - 2r_0^2)}\frac{1}{\tilde{N}(m, r_0)}$$

$$\tilde{N}(m,r_0) = \frac{\sqrt{6r_0^2 - m^2}}{\exp\left(\frac{m^2-3r_0^2}{r_0\sqrt{3r_0^2-2m^2}}\tan^{-1}\left(\frac{3r_0}{\sqrt{3r_0^2-2m^2}}\right)\right)}\left(\int_{r_0}^{\infty} \mathrm{d}r \frac{\exp\left(\frac{m^2-3r_0^2}{r_0\sqrt{3r_0^2-2m^2}}\tan^{-1}\frac{(2r+r_0)}{\sqrt{3r_0^2-2m^2}}\right)}{r\sqrt{2(r^2+rr_0+r_0^2)-m^2}} + \frac{1}{\sqrt{2}2\pi T}\exp\left(\mathrm{sgn}(3r_0^2 - 2m^2)\frac{\pi(m^2-3r_0^2)}{2r_0\sqrt{3r_0^2-2m^2}}\right)\right)$$

Dispersion relations in the axion model





$$\omega(k) = v_s k - i\frac{k^2}{8\pi T}$$

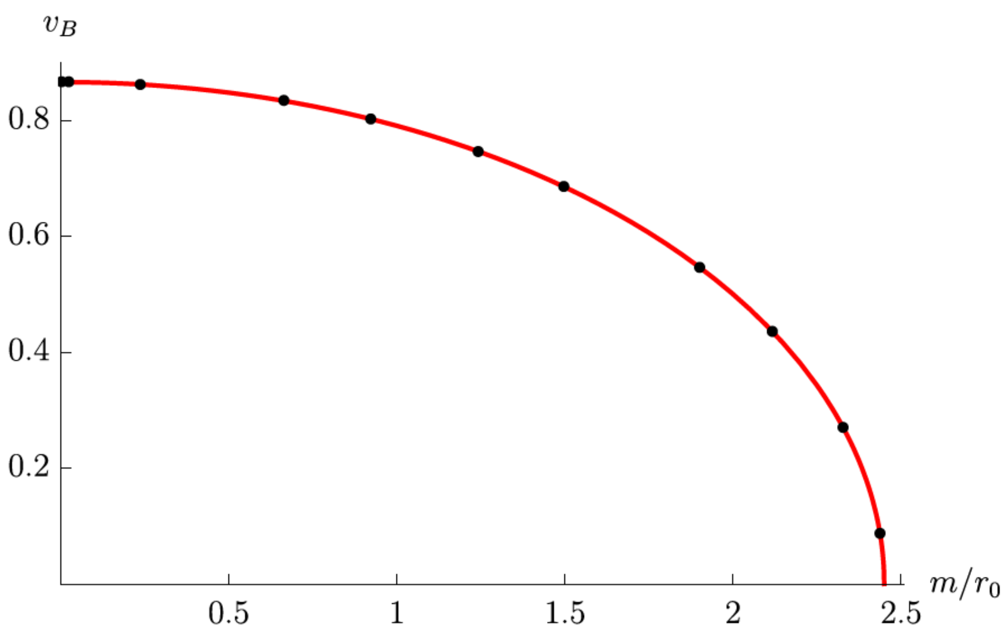
$$\omega(k) = -iD_E k^2$$

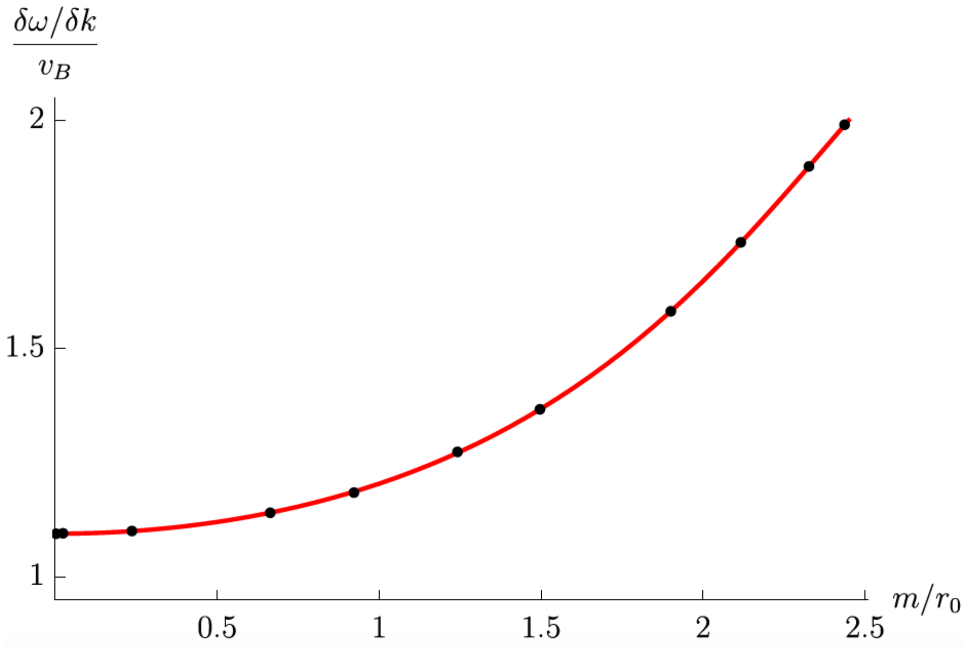
Herzog

RD & Gouteraux

- It is always a hydrodynamic pole that passes through the special point.

Numerical results in the axion model





Quantitative agreement with numerics for all m/T.

Summary

- There are striking signatures of many-body chaos in $G^R_{\varepsilon\varepsilon}(\omega, k)$ in holographic theories.

- Origin: vanishing of one of Einstein's equations on the black hole horizon

$$\left(\left(\frac{2\pi T}{v_B}\right)^2 + k^2\right)\delta g^{(0)}_{vv} = 0 \quad \text{at} \quad \omega = 2\pi T i$$

→ a pole (and a zero) passing through this point.

- Detailed verification in a particular example: AdS4 axion model.

- In this example, it is always a hydrodynamic pole that skips.

Further work

1). More general matter content e.g. Einstein-Maxwell-Dilaton theories

→ which correlators exhibit pole skipping?

→ is it always a hydrodynamic pole that skips?

2). Useful constraints on hydrodynamic transport coefficients?

3). More direct evidence for the 'quantum hydrodynamics' proposal?

4). Connection to shock wave solution governing OTOCs

5). The effects of corrections to Einstein gravity

Extra slides……