# Fine Grained Chaos in $AdS_2$ Gravity

Moshe Rozali

University of British Columbia

Wurzburg, August 2018

Based on: Felix Haehl, MR , arXiv:1712.04963, Phys. Rev. Lett. **120**, no. 12, 121601 (2018), also new paper with Felix, to appear shortly.

## Outline:

# Quantifying Chaos in Many-Body Quantum Systems

- Classical chaos has short-time manifestations (divergence of trajectories) as well as long-time ones (ergodicity). Classical ergodicity makes connection with the physics of thermalization or transport.

- In quantum chaos, short-time manifestations have to do with the recently much discussed out-of-time-ordered correlators (OTOCs), whereas long times are usually associated with spectral statistics (or ETH). Quantum-Classical correspondence holds for early times, but late time and classical limits do not commute.

- Quantum ergodicity is aimed to provide the foundation of statistical mechanics. Simple observables in a typical state "look thermal". More tricky to explain the dynamics towards the final thermal state.

- Not just an academic exercise, some many-body systems do not thermalize (MBL).

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# Quantifying Chaos in Many-Body Quantum Systems

In this talk,

- Define a series of increasingly more "fine-grained" OTOCs that probe aspects of thermalization beyond the initial scrambling.
- Calculate them in the context of $AdS_2$ and $AdS_3$ gravity.
- Show evolution of these correlators reveals longer and longer time scales, the $2k$-point function will evolve for $(k-1)t_*$, where $t_*$ is the scrambling time.

Similar ideas:

D. Roberts and B. Yoshida, "Chaos and complexity by design," [arXiv:1610.04903].

Z. W. Liu, S. Lloyd, E. Y. Zhu and H. Zhu, "Entanglement, quantum randomness, and complexity beyond scrambling," [arXiv:1703.08104].

## Out-of-Time-Order Correlators

The OTOCs are close cousins of

- Semi-classical extension of the classical divergence of trajectories and Lyapunov exponent.
- Growth of localized (or simple) operator under unitary time evolution.
- Loschmidt echo.

Those all lead to the idea of looking at growth of commutators of general Hermitian operators $[V(t), W(0)]$. The expectation value of the commutator is related to linear response, to a perturbation V as measured subsequently by W.

# Out-of-Time-Order Correlators

To quantify the "size" of the commutator look at the second moment

$$C[t] = -\langle [V(0), W(t)]^2 \rangle_\beta$$

Where the expectation value is evaluated at inverse temperature $\beta$.

The non-trivial part of this is the 2-OTOC

$$F(t) = \langle V(0) W(t) V(0) W(t) \rangle_\beta \sim \frac{1}{N} e^{\lambda t}$$

As the other operator orderings are determined by factorization at large $t$. The OTOC grows exponentially

## Comments:

$$F(t) = \langle V(0)W(t)V(0)W(t)\rangle_\beta \sim \frac{1}{N}e^{\lambda t}$$

- Note this has two operators and two "switchbacks" in both Lorentzian time and operator ordering (=Euclidean time). This is the combinatorics we seek to generalize.

- The 2-OTO starts low and experiences exponential growth until it becomes order one (where higher order corrections cause it to saturate). The time it is allowed to grow is determined by how low it can get, or how sensitive it is to the special starting point. This defines the scrambling time $t_* \sim LogN$

- our correlators are more fine-grained, in that they can get start lower, as they probe finer features of the initial state, and thus they can grow for a longer time.

- the rate of growth $\lambda$ is bounded, $\lambda_L \leq \frac{2\pi}{\beta}$

Let's review and generalize the calculation of the 2-OTO in $AdS_2$, related to the SYK model.

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# Dilaton Gravity in 1+1 Dimensions

Gravity in $1+1$ dimensions is trivial, to get something interesting we need at least a dilaton. So look at the theory with a negative cosmological constant (JT gravity)

$$I = -\frac{1}{16\pi G_N} \int \sqrt{-g}\, \phi(R + 2) + 2 \int_{bdy} \phi_{bdy}\, K$$

## Possibly coupled to matter fields.

Locally any solution is trivial, it can be mapped to $AdS_2$ and a fixed dilaton profile, in some coordinates $\phi(z) \sim \frac{1}{z}$, where z=0 is the boundary of $AdS_2$.

Importantly, this diverges so this theory only makes sense with a cutoff and the boundary condition $\phi(z = \epsilon) \sim \frac{\phi_r(u)}{\epsilon}$, for some fixed $\phi_r(u)$.

Now different solutions to the equations of motion are identical locally but could be different globally. They are characterized by the shape of the curve on which the prescribed boundary condition is satisfied.

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## Schwarzian Action

In the end, the gravitational action reduces to a boundary term, which describes the dynamics of the soft mode $t(u)$, which represents a diffeomorphism of the circle.

$$-I_{grav} = \frac{1}{\kappa^2} \int du \left[ -\frac{1}{2} \left(\frac{t''}{t'}\right)^2 + \left(\frac{t''}{t'}\right)' \right]$$

This is the Schwarzian action, which is determined by a pattern of spontaneous and explicit conformal symmetry breaking.

To compute correlators perturbatively in a black hole background, we transform $t(u) = \tan\frac{\tau(u)}{2}$, corresponding to working with temperature $\beta = 2\pi$, and expand around the saddle: $\tau(u) = u + \kappa\,\varepsilon(u)$. Our expansion parameter is $\kappa$, the gravitational constant. We are taking backreaction into account perturbatively in $\kappa$.

## Soft Mode Propagator

To leading order in $\kappa$ the Schwarzian action gives a quadratic term, and hence a propagator for the mode $\varepsilon(u)$

$$\langle \varepsilon(u)\varepsilon(0)\rangle = \frac{1}{2\pi}\left[ \frac{2\sin u - (\pi + u)}{2}(\pi + u) \right.$$
$$\left. + 2\pi\Theta(u)(u - \sin u) \right]$$

We are only interested in the part that simultaneously cares about time-ordering, and is exponentially growing in Lorentzian time. We will organize the calculation to isolate that part later.

For our purposes the propagator can be replaced by $-\Theta(u)\sin u$.

## Matter Action

We couple the gravity theory to a matter action which represents external massless particles:

$$- I_{matter} = \frac{1}{2\pi} \int du_1 du_2 \frac{t'(u_1)t'(u_2)}{(t(u_1) - t(u_2))^2} j(u_1)j(u_2)$$

where $j$ is a source for the operator whose correlator we are calculating. This reproduces, via the usual AdS/CFT. the boundary correlators of chiral primary field of dimension 1.
We write the expansion in $\kappa$ as

$$-I_{matter} = \frac{1}{2\pi} \int du_1 du_2 \frac{j(u_1)j(u_2)}{4\sin^2(\frac{u_{12}}{2})} \sum_{p \geq 0} \kappa^p \, \mathcal{B}^{(p)}(u_1, u_2) \tag{1}$$

where $u_{12} \equiv u_1 - u_2$ and $\mathcal{B}^{(0)} = 1$.

## Interaction Vertices

We need the first and second order expansions, corresponding to the way the matter sources the soft mode $\varepsilon(u)$ to orders $\kappa$ and $\kappa^2$

$$\mathcal{B}^{(1)}(u_1, u_2) = \varepsilon'(u_1) + \varepsilon'(u_2) - \frac{\varepsilon(u_1) - \varepsilon(u_2)}{\tan(\frac{u_{12}}{2})}$$

$$\mathcal{B}^{(2)}(u_1, u_2) = \frac{1}{4\sin^2(\frac{u_{12}}{2})} \Big[ (2 + \cos u_{12}) \left(\varepsilon(u_1) - \varepsilon(u_2)\right)^2$$
$$+ 4\sin^2\left(\frac{u_{12}}{2}\right) \varepsilon'(u_1)\varepsilon'(u_2)$$
$$- 2\sin u_{12} \left(\varepsilon(u_1) - \varepsilon(u_2)\right) \left(\varepsilon'(u_1) + \varepsilon'(u_2)\right) \Big]$$

We will not need self-interaction for the soft mode, though that can also be easily obtained if needed.

## Maximally Braided Correlators

To define the correlators we are interested in, we start with the observation that the braiding operation $V \, V \, W \, W \to V \, W \, V \, W$, relating time-ordered and OTOs, can be generalized to any number of pairs of operators.

The maximally-braided 2k-point function is defined to have the maximal number of braiding, e.g.

$V_1 \, V_1 \, V_2 \, V_2 \, V_3 \, V_3 \to V_1 \, V_2 \, V_1 \, V_3 \, V_2 \, V_3$

$V_1 \, V_1 \, V_2 \, V_2 \, V_3 \, V_3 \, V_4 \, V_4 \to V_1 \, V_2 \, V_1 \, V_3 \, V_2 \, V_4 \, V_3 \, V_4$

etc. Subtracting off the less than maximal braided operators results in the operator

$V_1 \, [V_2, V_3] \, [V_3, V_4] \, ... V_{2k}$

whose (properly normalized) expectation value we call the "maximally-braided" correlator.
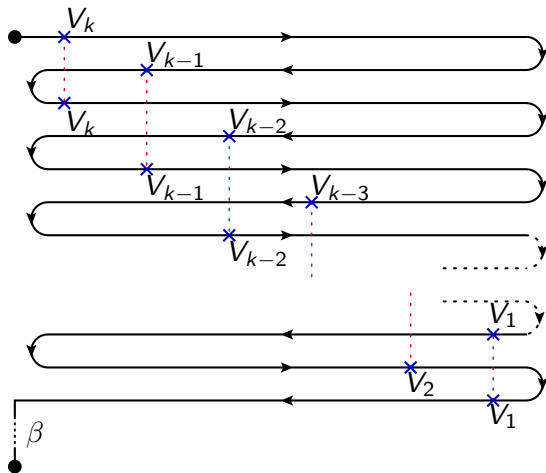
## Maximally Braided Correlators

These are easy to calculate, we just keep track of the cost of braiding for each of the ingredients of the calculation, i.e. we organize the result by the number of Euclidean theta functions.

Furthermore, in the Schwarzian theory, to leading order in $N$ each such correlator has exactly one diagram contributing:



$$\smile = \mathcal{B}^{(1)}(u_{2i-1}, u_{2i})$$

$$\smile = \mathcal{B}^{(2)}(u_{2i-1}, u_{2i})$$

$$\sim\!\!\sim = \langle \varepsilon\varepsilon \rangle$$
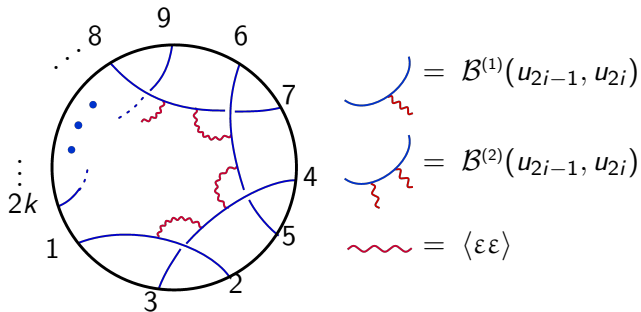
## Lorentzian Times

Once we calculate the maximally braided correlator in Euclidean space, we can put the operators at various Lorentzian times. Schematically

# Maximal OTO Configurations

One can classify all Lorentzian time configurations by OTO-number, roughly the number of "irreducible" switchbacks in Lorentzian time. We are interested in the maximal OTO number for our maximally braided correlators.

Generally a $2k$-point function depends on $2k - 1$ time differences. Maximally braided correlator is designed to have the maximal number of "soft mode" exchanges, so the maximal number of exponentially growing factors, in some time combination.

Maximal OTO time configuration aligns all these factors, a $2k$ correlator which is maximally braided and also maximal OTO number, has a very simple result, to leading order in $N$:

$$I_{2k}(t_1, t_2, ..., t_{2k}) \sim \frac{1}{N^{k-1}} e^{\lambda_L(t_{2k} - t_1)}$$

## Fine-Grained Chaos

$I_{2k}(t_1, t_2, ..., t_{2k}) \sim \frac{1}{N^{k-1}} e^{\lambda_L(t_{2k}-t_1)}$
Comments

- It is not meaningful to compare different $2k$-point functions, but one can compare the connected $I_{2k}$ to products on lower point functions. We see that $I_{2k}$ increases at a *slower* rate than the disconnected pieces.

- Correspondingly, $I_{2k}$ increase for longer time periods, $(k-1)t_*$, where $t_*$ is the scrambling time.

- the suppression factor $N^{k-1}$ is the leading order for the connected part of the 2k-point function, it corresponds to the fact that such higher point function probes more fine-grained features of the state.

## Conclusions

Possible Directions for further research

- Relation to unitary designs and frame potentials.
- Relation to other fine-grained probes, e.g the time-dependence of Renyi entropies.
- Lorentzian calculation and relation to (multiple) shock waves.
- Relation to "complexity".
- Calculation in higher dimensions (including butterfly velocities).

In this context, new paper with Felix, "Effective Field Theory of Chaotic CFTs", to appear.

# 2-OTOs in 2-dimensional CFTs

- The calculation of OTO 4-point function is described in a paper by Roberts and Stanford. It involves the identity conformal block at large central charge, and its analytic continuation.

- Also some assumptions about the dimensions of external operators and dominance of the identity block, and going beyond the range of validity of known results.

- In the end, at leading order, large c CFT is maximally chaotic, and the butterfly velocity is the speed of light.

- Generalizing to higher OTOs would involve similar steps for the higher-point conformal blocks.

Instead we use a new idea: hydrodynamic description of chaos,

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# EFT of Chaotic CFTs

Very briefly, one can write the effective field theory for "boundary gravitons" in 2D CFT with large central charge. Those soft modes are holomorphic reparametrizations. They end up having a quadratic action and matter coupling very much like the ones discussed above.

This is related to 2D induced gravity discussed by Polyakov, though crucially around the thermal state and not the vacuum.

Using the EFT to calculate the OTOCs reproduces the Roberts- Stanford result, and can be easily generalized to the fine-grained chaos correlators.

The effective action has many other uses in 2D CFT, beyond the relevance to chaos. In some sense it generates the identity conformal block at large central charge.