Modes of Entanglement in Chern-Simons Theories

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based on arXiv:1611.05460, 1705.09611, 1801.01131
with ***J. Fliss, O. Parrikar, T.L. Hughes, V. Balasubramanian, ...***

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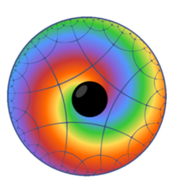

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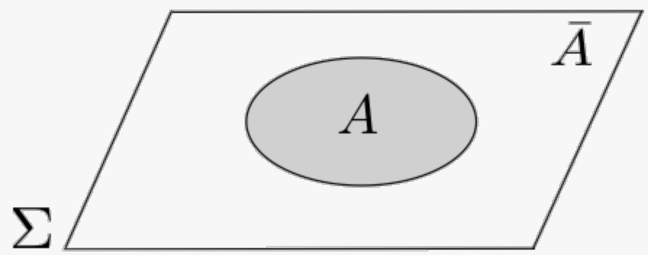
Entanglement Structure

- **of basic interest are the patterns of entanglement in any quantum theory, especially quantum field theories**
    - in gauge/gravity duality, expected to play a central role in 'bulk emergence'
    - in condensed matter physics, a primary observable especially in topological states of matter
- **entanglement inequalities, such as the positivity and monotonicity of relative entropy, play a powerful role, constraining QFTs in interesting ways**
    - recent work on establishing ANEC, QNEC is but one example

[1605.08072]
[Faulkner et al 1706.09432]

Bi-partite entanglement

- often in QFT, interested in spatial entanglement
  - standard construction presupposes $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$



- for a state $|\Psi\rangle$ on Σ, trace over degrees of freedom in $\overline{A}$

$\longrightarrow$ reduced density matrix $\hat{\rho}_A$

$$S^{(\alpha)}(A) = \frac{1}{1-\alpha} \log tr_{\mathcal{H}_A} \hat{\rho}_A^{\alpha}$$

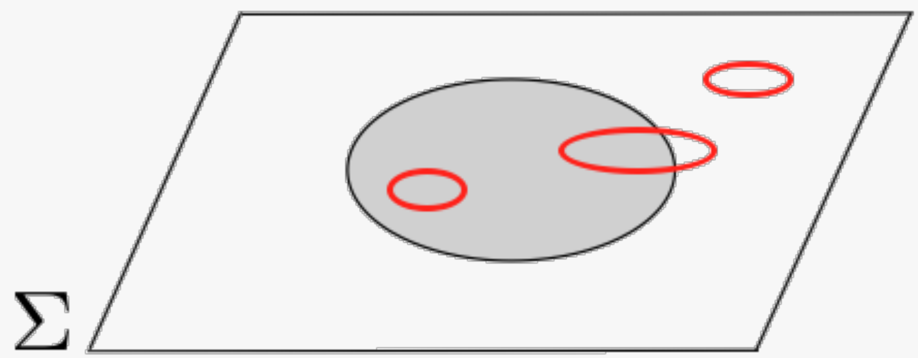
Rényi entropies

$$S_{EE}(A) = -tr_{\mathcal{H}_A} \hat{\rho}_A \log \hat{\rho}_A$$

entanglement entropy

Bi-partite entanglement

- works well for some QFTs, such as scalars and spinor fields
- it doesn't work for gauge theories, as the Hilbert space does not factorize



- observables aren't generally local
  - cutting and gluing of regions involves degrees of freedom on cut
- in 3d CS, this is particularly familiar
  - bulk is topological, but WZW on 1+1 edges

3d Chern-Simons

- **the non-factorizability of the Hilbert space is strikingly evident here**
  - think of C-S theory on 3-mfld (locally) of the form $M_3 \sim \mathbb{R} \times \Sigma$
  - path integral over half-spacetime with (space-like) boundary Σ gives a wave-functional (half-spacetime ~ solid Σ)
    - thus associate a Hilbert space $\mathcal{H}_\Sigma$ to Σ
    - the various states in $\mathcal{H}_\Sigma$ correspond to non-trivial Wilson loops
- **The simplest example is** $S^2 = D^2 \cup D^2$

$$\dim \ \mathcal{H}_{S^2} = 1 \qquad \textbf{but} \qquad \dim \ \mathcal{H}_{D^2} > 1$$

so

$$\mathcal{H}_{S^2} \subset \mathcal{H}_{D^2} \otimes \mathcal{H}_{D^2}$$

- Thus entanglement knows about *edge modes*

Entanglement Entropy in CS

- A convenient basis for the Hilbert space is obtained by inserting Wilson loops in some representation of the gauge group

- Bi-partite entanglement entropy, in the case where space is the aforementioned Riemann surface Σ, is determined by dual CFT data, principally the *modular S-matrix*

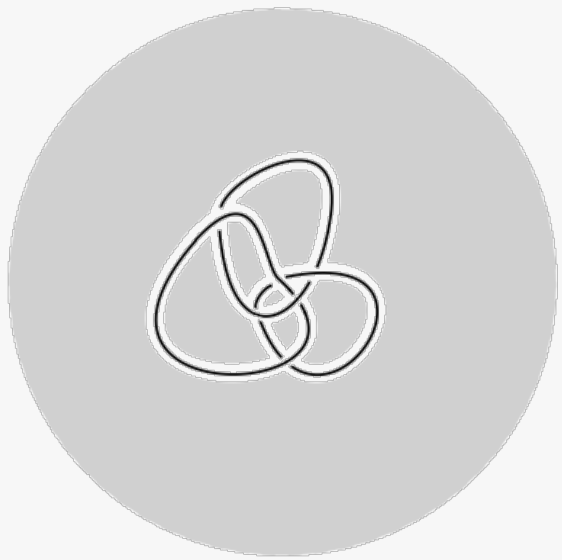
[Levin, Wen]
[Kitaev, Preskill]

- depends on the topology of the entanglement cut, as well as details of the pure state.

- Can be computed in a generic situation by the replica trick, using surgery methods. [RGL, E. Fradkin, S. Dong, S, Nowling '08]
    - These methods bypass the issues associated with edge modes
    - Isolate topological entanglement, UV divergence formally invisible

Multi-partite Entanglement

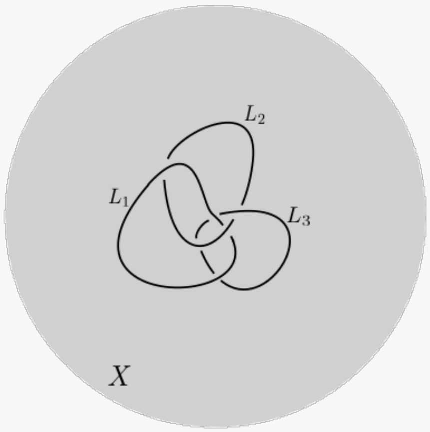
- An interesting generalization of that story is to take the 3d space-time to be a *link complement* $M_{(n)} = S^3 \backslash N(\mathcal{L}^n)$ [1611.05460, 1801.01131]



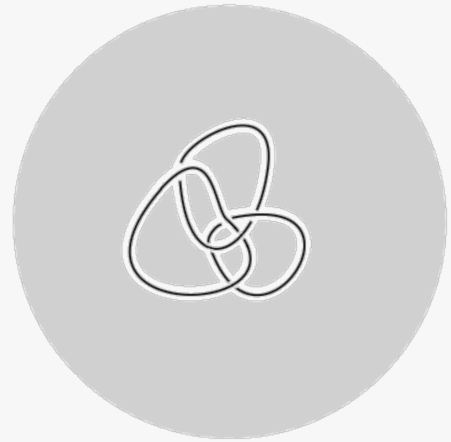
- Such a *multi-boundary* structure can be thought of as giving information about *multi-partite* entanglement.

Multi-boundary states in CS_3

- a simple way to generate such manifolds is to start with a closed 3-manifold X (e.g. S^3) and an n-component link $\mathcal{L}^n$ in X



$$\mathcal{L}^n = L_1 \cup L_2 \cup \ldots \cup L_n$$



- fatten each component into a tubular neighbourhood, yielding $N(\mathcal{L}^n)$
- the *link complement* $M_{(n)} = S^3 \backslash N(\mathcal{L}^n)$ is a 3-mfld with n-component boundary
- CS path integral on $M_{(n)}$ gives a state $\boxed{|\mathcal{L}^n\rangle \in \otimes_j \mathcal{H}(T^2)_j}$
- Study entanglement by tracing over a subset of components

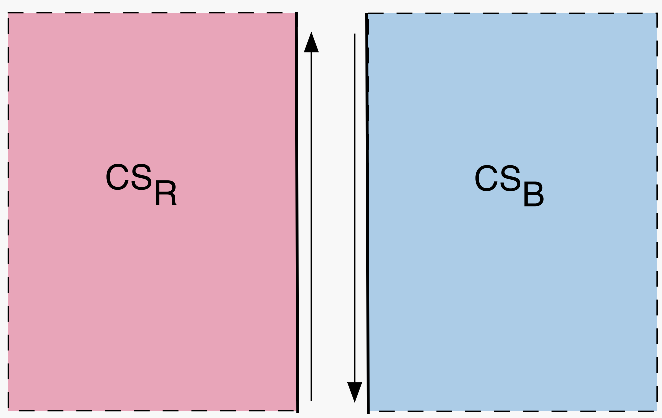
Multi-partite Entanglement

- One finds that entanglement entropy is a framing-independent link invariant.
- Completely understood for Abelian theories
  - Depends only on Gauss linking
- Case-by-case for non-Abelian theories
  - (e.g., $SU(2)_k$, $SL(2,\mathbb{C})$)
- Various entanglement measures can be used to distinguish links
- Broad classes of link states are associated with standard classes of multi-partite entanglement
  - (e.g., torus links are GHZ-like, others W-like)

Quantum entanglement
=
topological entanglement

Heterogeneous CS

- Edge modes play a more direct role in entanglement if one considers distinct CS theories that share an interface
  - The previous studies corresponds to one of these being trivial
  - Previously studied in a "coupled wire construction" [Cano, Hughes, Mulligan '14]
- When separate, the two phases would typically support chiral massless boundary modes
- As they are brought together, generically expect *gapping interactions*



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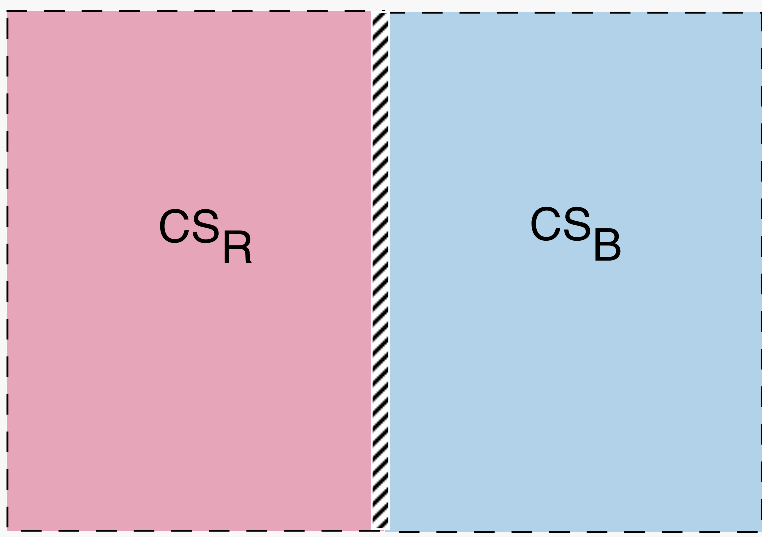
Heterogeneous CS

- In the continuum, these gapping interactions can be thought of as corresponding to (topological) interface conditions
- What is perhaps surprising is that even though the edges modes are gapped, they contribute anomalously to entanglement measures.
  - Specifically, when the entanglement cut is taken to lie along the interface



- The simplest examples are $U(1)^N$ theories with different K-matrices

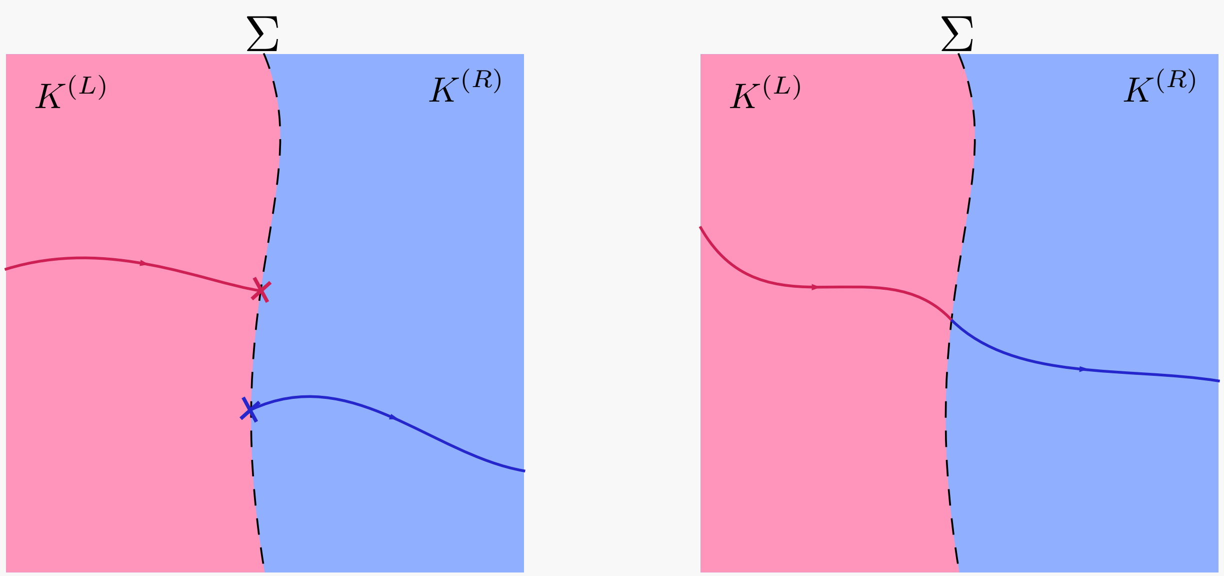
$$S_{CS} = \frac{{K^{(L)}}^{IJ}}{4\pi} \int_{\mathcal{M}_L} A_I^{(L)} \wedge dA_J^{(L)} + \frac{{K^{(R)}}^{IJ}}{4\pi} \int_{\mathcal{M}_R} A_I^{(R)} \wedge dA_J^{(R)} + S_\Sigma(A_L, A_R).$$

- The action on the interface can be taken to be topological
  - More general interface actions, involving a choice of complex structure, are believed to modify only high energy physics, at least in the case that the interface is completely gapped.
- Such interface (boundary) conditions have been studied and classified

[e.g. Kapustin, Saulina '07]

- Corresponds to instructions on how to glue the CS theories together, equivalent to gapping interactions from the pov of the interface modes
- Boundary conditions pick a 'Lagrangian subspace' of the K-matrix, so pick a polarization on the interface

- Generically an interface will support global $U(1)^N$ charges
  - In figures, depicted as Wilson lines ending on the interface
- Topological interface conditions provide an identification of the gauge group across interface, describing Wilson lines that can cross
- Thus the physics of *edge modes* is fundamental here



- **Particular linear combinations of fields remains invariant under residual $U(1)^N$; specified by vectors $v^{(L)}, v^{(R)}$**
  - These determine the structure of the gapping interactions, and interface conditions

- **A quantity relevant to entanglement is an** *effective K-matrix*

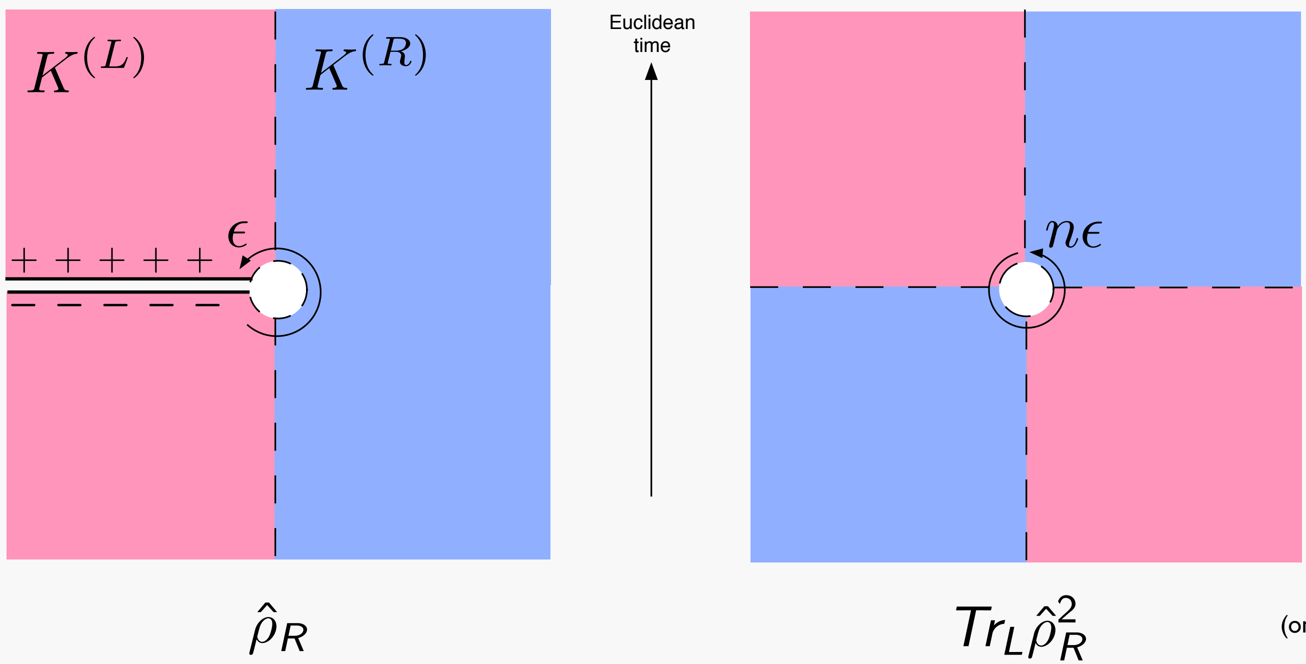
$$K_{eff} \equiv {v^{(L)}}^T \cdot K^{(L)} \cdot v^{(L)} = {v^{(R)}}^T \cdot K^{(R)} \cdot v^{(R)}$$

- **e.g., determines entanglement entropy**

$$S \sim -\frac{1}{2} \ln |\det K_{eff}|$$

Heterogeneous CS

- One way to proceed is via the replica trick
- This leads to an interesting complication — the path integrals representing Rényi entropies are "striped" with alternating phases



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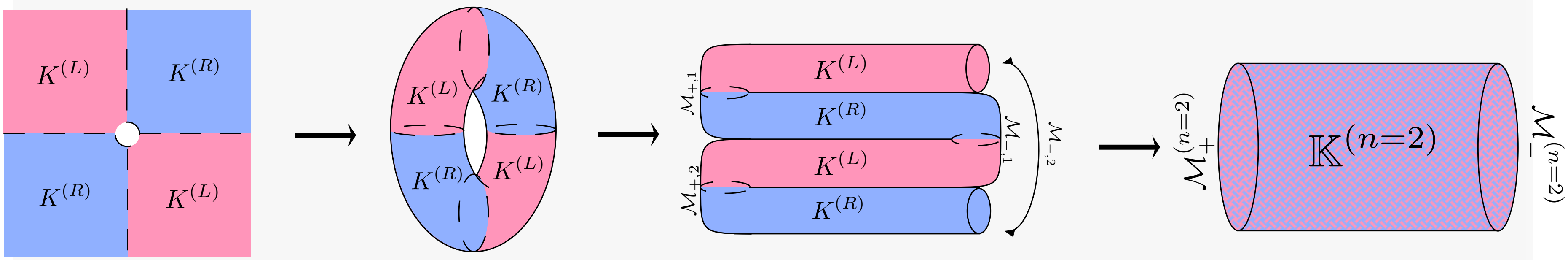
- One can visualize/organize this in several ways



- **Fold into n copies, gluing condition determined by top int conditions**
    - Single homogeneous theory with K-matrix K_{eff} on n-replicated mfld.
- **UV cutoff (around entangling cut) plays role of finite size of torus**
    - The last figure corresponds to an interpretation as a transition between boundary states — Ishibashi states emerge

WZW Interpretation

- **The Ishibashi interpretation comes about through WZW supported on regulator surface**

- **There is a complementary "extended Hilbert space" picture**
  - A technique for embedding states in a larger Hilbert space, managing edge mode states [Donnelly '11]
  - Need this to construct a reduced density matrix in gauge theories

- **E.g.,** $\mathcal{H}_{S^2} \hookrightarrow \mathcal{H}_{D^2} \otimes \mathcal{H}_{D^2}$
  - Define embedding of $\mathcal{H}_{S^2}$ by gauge invariance
  - Depends on identification of gauge transformations at shared interface

Extended Hilbert Space Interpretation

- Gauge transformations on disks generated by

$$\hat{\mathcal{Q}}_{\mathcal{H}_{D^2_L}}(\lambda) = \frac{k}{4\pi}\sum_n \lambda_n \hat{J}_n \qquad \hat{\mathcal{Q}}_{\mathcal{H}_{D^2_R}}(\lambda) = \frac{k}{4\pi}\sum_n \overline{\lambda}_n \hat{\overline{J}}_n$$

- e.g., choose λ continuous across interface — $\overline{\lambda}_n = \lambda_{-n}$

$$\hat{\mathcal{Q}}_{\mathcal{H}_{D^2_L}\otimes\mathcal{H}_{D^2_R}}(\lambda) = \frac{k}{4\pi}\sum_n \lambda_n \left(\hat{J}_n \otimes 1 + 1 \otimes \hat{\overline{J}}_{-n}\right)$$

- Then gauge invariance of state $|\psi\rangle$ reads $\hat{\mathcal{Q}}_{\mathcal{H}_{D^2_L}\otimes\mathcal{H}_{D^2_R}}|\psi\rangle = 0$

Ishibashi condition

- Tracing over $\mathcal{H}_{D^2_R}$ is trace over oscillator modes $\hat{\overline{J}}_{-n}$
- CS subregion entanglement $\leftrightarrow$ Ishibashi L-R entanglement
- When applied to heterogeneous example, reproduces ent ent.

Summary

- topological field theories such as 3d CS is an important testing ground for quantum information ideas in the context of continuum field theories
- e.g., detailed understanding of extended Hilbert space techniques in the continuum
- Would like to formulate further multi-partite notions
  - e.g., are there other states than Ishibashi states that play a similar role in multi-partite entanglement? “vertex states”