Einstein Double Field Equations

$$G_{AB} = 8\pi G T_{AB}$$

Hereafter A, B **are** $\mathbf{O}(D, D)$ **indices**

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Prologue

– GR is based on Riemannian geometry, where the only geometric and gravitational field is the Riemannian metric, $g_{\mu\nu}$. Other fields are meant to be extra matter.

– On the other hand, string theory suggests to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric:

- They form the closed string massless sector, being ubiquitous in all string theories,

$$\int \mathrm{d}^D x \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) \qquad \text{where} \qquad H = \mathrm{d}B\,.$$

This action hides $\mathbf{O}(D, D)$ symmetry of T-duality which transforms g, B, ϕ into one another. Buscher 1987

– T-duality hints at a natural augmentation to General Relativity, in which the entire closed string massless sector constitutes the fundamental gravitational multiplet and the above action corresponds to 'pure' gravity.

Double Field Theory (DFT), initiated by Siegel 1993 & Hull, Zwiebach 2009-2010, turns out to provide a concrete realization for this idea of Stringy Gravity by manifesting $\mathbf{O}(D, D)$ T-duality.

- My talk sketches the geometric construction of Stringy Gravity.
  In particular, I will introduce Einstein Double Field Equations, $G_{AB} = 8\pi G T_{AB}$, as the unifying single expression for all the equations of motion of the closed string massless sector,
  as well as Newton-Cartan, Carroll and Gomis-Ooguri gravities (non-Riemannian).

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DFT as Stringy Gravity

Notation for $\mathbf{O}(D,D)$ and $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$ Symmetries

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots, M, N, \cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$p, q, \cdots$	$\mathbf{Spin}(1,D{-}1)_L$ vector	$\eta_{pq} = \mathrm{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1,D{-}1)_L$ spinor	$C_{\alpha\beta}, \qquad (\gamma^p)^T = C\gamma^p C^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D{-}1,1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+--\cdots-)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D{-}1,1)_R$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

– Each symmetry rotates its own indices *exclusively*: spinors are $\mathbf{O}(D,D)$ singlet.

– The constant $\mathbf{O}(D,D)$ metric, $\mathcal{J}_{AB}$, decomposes the doubled coordinates into two parts,

$$x^A = (\tilde{x}_\mu, x^\nu)\,, \qquad \partial_A = (\tilde{\partial}^\mu, \partial_\nu)\,,$$

where μ, ν are D-dimensional curved indices.

– The twofold local Lorentz symmetries indicate two distinct locally inertial frames for the left-moving and the right-moving closed string sectors separately $\Rightarrow$ Unification of IIA and IIB.

- **Closed string massless sector as ‘Stringy Graviton Fields’**

  The stringy graviton fields consist of the DFT dilaton, d, and DFT metric, $\mathcal{H}_{MN}$:

$$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad\qquad \mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN}=\mathcal{J}_{KM}\,.$$

  Combining $\mathcal{J}_{MN}$ and $\mathcal{H}_{MN}$, we get a pair of symmetric projection matrices,

$$P_{MN}=P_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,\qquad P_{L}{}^{M}P_{M}{}^{N}=P_{L}{}^{N}\,,$$
$$\bar{P}_{MN}=\bar{P}_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,\qquad \bar{P}_{L}{}^{M}\bar{P}_{M}{}^{N}=\bar{P}_{L}{}^{N}\,,$$

  which are orthogonal and complete,

$$P_{L}{}^{M}\bar{P}_{M}{}^{N}=0\,,\qquad\qquad P_{M}{}^{N}+\bar{P}_{M}{}^{N}=\delta_{M}{}^{N}\,.$$

  Further, taking the “square roots" of the projectors,

$$P_{MN}=V_{M}{}^{p}V_{N}{}^{q}\eta_{pq}\,,\qquad\qquad \bar{P}_{MN}=\bar{V}_{M}{}^{\bar{p}}\bar{V}_{N}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

  we get a pair of DFT vielbeins:

$$V_{Mp}V^{M}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{M\bar{p}}\bar{V}^{M}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Mp}\bar{V}^{M}{}_{\bar{q}}=0\,,\qquad V_{M}{}^{p}V_{Np}+\bar{V}_{M}{}^{\bar{p}}\bar{V}_{N\bar{p}}=\mathcal{J}_{MN}\,.$$

Classification of DFT backgrounds, 1707.03713 with Kevin Morand

The most general form of the DFT metric, $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $\mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN} = \mathcal{J}_{KM}$**, is characterized by two non-negative integers,** $(n, \bar{n})$, $0 \leq n + \bar{n} \leq D$**:**

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} + Y_i^\mu X_\lambda^i - \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu} + X_\kappa^i Y_i^\nu - \bar{X}_\kappa^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} + 2X_{(\kappa}^i B_{\lambda)\rho} Y_i^\rho - 2\bar{X}_{(\kappa}^{\bar{\imath}} B_{\lambda)\rho} \bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

i) Symmetric and skew-symmetric fields: $H^{\mu\nu} = H^{\nu\mu}$, $K_{\mu\nu} = K_{\nu\mu}$, $B_{\mu\nu} = -B_{\nu\mu}$;

ii) Two kinds of eigenvectors having zero eigenvalue, with $i, j = 1, 2, \cdots, n$ & $\bar{\imath}, \bar{\jmath} = 1, 2, \cdots, \bar{n}$,

$$H^{\mu\nu}X_\nu^i = 0\,, \qquad H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}} = 0\,, \qquad K_{\mu\nu}Y_j^\nu = 0\,, \qquad K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^\nu = 0\,;$$

iii) Completeness relation: $H^{\mu\rho}K_{\rho\nu} + Y_i^\mu X_\nu^i + \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\nu^{\bar{\imath}} = \delta^\mu{}_\nu$.

- Orthonormality follows

$$Y_i^\mu X_\mu^j = \delta_i{}^j\,, \qquad \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\mu^{\bar{\jmath}} = \delta_{\bar{\imath}}{}^{\bar{\jmath}}\,, \qquad Y_i^\mu \bar{X}_\mu^{\bar{\jmath}} = \bar{Y}_{\bar{\imath}}^\mu X_\mu^j = 0\,.$$

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i) Symmetric and skew-symmetric fields: $\quad H^{\mu\nu}=H^{\nu\mu},\quad K_{\mu\nu}=K_{\nu\mu},\quad B_{\mu\nu}=-B_{\nu\mu}\,;$

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- It is instructive to note the $\mathbf{O}(D,D)$ invariant trace, $\mathcal{H}_{A}{}^{A}=2(n-\bar{n})$ and

$$\mathcal{H}_{AB}=\left(\begin{array}{cc}1&0\\B&1\end{array}\right)\left(\begin{array}{cc}H & Y_{i}(X^{i})^{T}-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^{T}\\ X^{i}(Y_{i})^{T}-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^{T} & K\end{array}\right)\left(\begin{array}{cc}1&-B\\0&1\end{array}\right).$$

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I. $(n, \bar{n}) = (0, 0)$ corresponds to the Riemannian case or "Generalized Geometry":

$$\mathcal{H}_{MN} \equiv \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \quad e^{-2d} \equiv \sqrt{|g|} e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

II. Generically, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X^i_\mu \, \partial_+ x^\mu(\tau, \sigma) \equiv 0 \,, \qquad \bar{X}^{\bar{\imath}}_\mu \, \partial_- x^\mu(\tau, \sigma) \equiv 0 \,.$$

- Such non-Riemannian examples include
  - $(1, 0)$ Newton-Cartan gravity ***c.f.* Obers' talk** ($\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2$, $\lim_{c \to \infty} g^{-1}$ is finite & degenerate)
  - $(1, 1)$ Gomis-Ooguri non-relativistic string **Charles Melby-Thompson, Rene Meyer, Ko, JHP 2015**
  - $(D-1, 0)$ ultra-relativistic Carroll gravity
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- **Diffeomorphisms** in Stringy Gravity are given by "generalized Lie derivative": Siegel 1993

$$\hat{\mathcal{L}}_\xi T_{A_1\cdots A_n} := \xi^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}\xi_B - \partial_B\xi_{A_i})T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n}\,,$$

where ω_T is the weight, *e.g.* $\delta e^{-2d} = \partial_B(\xi^B e^{-2d})$, $\delta V_{Ap} = \xi^B\partial_B V_{Ap} + (\partial_A\xi_B - \partial_B\xi_A)V^B{}_p$.

- For consistency, the so-called 'section condition' should be imposed: $\partial_M\partial^M = 0$.

  From $\partial_M\partial^M = 2\partial_\mu\tilde{\partial}^\mu$, the section condition can be easily solved by letting $\tilde{\partial}^\mu = 0$.

  The general solutions are then generated by the $\mathbf{O}(D, D)$ rotation of it.

- The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi_i(x) = \Phi_i(x+\Delta)\,, \qquad \Delta^M = \Phi_j\partial^M\Phi_k\,,$$

where $\Phi_i, \Phi_j, \Phi_k \in \{\, d, \mathcal{H}_{MN}, \xi^M, \partial_N d, \partial_L\mathcal{H}_{MN}, \cdots \}$, arbitrary functions appearing in DFT, and Δ^M is said to be derivative-index-valued.

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- The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi_i(x) = \Phi_i(x+\Delta)\,, \qquad \Delta^M = \Phi_j\partial^M\Phi_k\,,$$

where $\Phi_i, \Phi_j, \Phi_k \in \{\, d, \mathcal{H}_{MN}, \xi^M, \partial_N d, \partial_L\mathcal{H}_{MN}, \cdots\}$, arbitrary functions appearing in DFT, and Δ^M is said to be derivative-index-valued.

▶ 'Physics' should be invariant under such shifts of the doubled coordinates in Stringy Gravity.

Doubled-yet-gauged spacetime

JHP 1304.5946

Doubled coordinates, $x^M = (\tilde{x}_\mu, x^\nu)$, are gauged through an equivalence relation,

$$x^M \sim x^M + \Delta^M(x)\,,$$

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Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- If we solve the section condition by letting $\tilde{\partial}^\mu \equiv 0$, and further choose $\Delta^M = c_\mu \partial^M x^\mu$, we note

$$(\tilde{x}_\mu\,,\, x^\nu) \sim (\tilde{x}_\mu + c_\mu\,,\, x^\nu) \quad : \quad \tilde{x}_\mu\text{'s are gauged and } x^\nu\text{'s form a section.}$$

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- Further, if we 'gauge' $\mathrm{d}x^{M}$ explicitly by introducing a derivative-index-valued gauge potential,

$$\mathrm{d}x^{M}~~\longrightarrow~~Dx^{M}=\mathrm{d}x^{M}-\mathcal{A}^{M}\,,\qquad\qquad \mathcal{A}^{M}\partial_{M}=0\,,$$

it is possible to define $\mathbf{O}(D,D)$ & diffeomorphism covariant 'proper length' through a path integral,

$$\textbf{Proper Length}~:=-\ln\left[\int\mathcal{D}\mathcal{A}~\exp\left(-\int\sqrt{Dx^{M}Dx^{N}\mathcal{H}_{MN}}\right)\right]\,.$$

Under $\delta x^{M}=\xi^{M}$, $\delta(\mathrm{d}x^{M})=\mathrm{d}x^{N}\partial_{N}\xi^{M}\neq\mathrm{d}x^{N}(\partial_{N}\xi^{M}-\partial^{M}\xi_{N})$ *versus* $\delta(Dx^{M})=Dx^{N}(\partial_{N}\xi^{M}-\partial^{M}\xi_{N})$.

- For the (0, 0) Riemannian DFT-metric, with $\tilde{\partial}^{\mu}\equiv 0$, $\mathcal{A}^{M}=A_{\mu}\partial^{M}x^{\mu}=(A_{\mu},0)$, from

$$Dx^{M}Dx^{N}\mathcal{H}_{MN}\equiv\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}+\left(\mathrm{d}\tilde{x}_{\mu}-A_{\mu}+\mathrm{d}x^{\rho}B_{\rho\mu}\right)\left(\mathrm{d}\tilde{x}_{\nu}-A_{\nu}+\mathrm{d}x^{\sigma}B_{\sigma\nu}\right)g^{\mu\nu}$$

  after integrating out A_{μ}, the proper length reduces to the conventional one,

$$\textbf{Length}~\Longrightarrow~\int\sqrt{\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}g_{\mu\nu}(x)}\,.$$

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Doubled-yet-gauged sigma models

The definition of the proper length readily leads to ‘covariant’ actions:

I. Particle action — Ko-JHP-Suh 2016

$$\mathcal{S}_{\rm particle}=\int{\rm d}\tau\ e^{-1}\,D_{\tau}x^{M}D_{\tau}x^{N}\mathcal{H}_{MN}(x)-\tfrac{1}{4}m^{2}e$$

II. String action — Lee-JHP 2013, *c.f.* Hull 2006

$$S_{\rm string}=\tfrac{1}{4\pi\alpha'}\int{\rm d}^{2}\sigma\ -\tfrac{1}{2}\sqrt{-h}h^{ij}D_{i}x^{M}D_{j}x^{N}\mathcal{H}_{MN}(x)-\epsilon^{ij}D_{i}x^{M}\mathcal{A}_{jM}$$

With the $(0,0)$ Riemannian DFT-metric plugged, after integrating out the auxiliary fields, the above actions reduce to the conventional ones:

$$\mathcal{S}_{\rm particle}\ \Rightarrow\ \int{\rm d}\tau\ e^{-1}\,\dot{x}^{\mu}\dot{x}^{\nu}g_{\mu\nu}-\tfrac{1}{4}m^{2}e\,,$$

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III. κ-symmetric doubled-yet-gauged Green-Schwarz superstring, unifying IIA & IIB — JHP 2016

$$S_{\rm GS}=\tfrac{1}{4\pi\alpha'}\int{\rm d}^{2}\sigma\ -\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_{i}^{M}\Pi_{j}^{N}\mathcal{H}_{MN}-\epsilon^{ij}D_{i}x^{M}\left(\mathcal{A}_{jM}-i\Sigma_{jM}\right)\,,$$

where $\Pi_{i}^{M}:=D_{i}x^{M}-i\Sigma_{i}^{M}$ and $\Sigma_{i}^{M}:=\bar{\theta}\gamma^{M}\partial_{i}\theta+\bar{\theta}'\bar{\gamma}^{M}\partial_{i}\theta'$.

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On the other hand, upon the generic $(n, \bar{n})$ DFT backgrounds, the auxiliary gauge potential decomposes into three parts:

$$A_\mu = K_{\mu\rho} H^{\rho\nu} A_\nu + X_\mu^i Y_i^\nu A_\nu + \bar{X}_\mu^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu A_\nu \,.$$

- The first part appears quadratically, which leads to Gaussian integral.
- The second and third parts appear linearly, as Lagrange multipliers, to prescribe

i) **Particle freezes over the $(n + \bar{n})$ dimensions**

$$X_\mu^i \dot{x}^\mu \equiv 0 \,, \qquad \bar{X}_\mu^{\bar{\imath}} \dot{x}^\mu \equiv 0 \,.$$

Remaining orthogonal directions are described by a reduced action:

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ii) **String becomes chiral over the n dimensions and anti-chiral over the $\bar{n}$ dimensions**

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Covariant derivatives and curvatures in Stringy Gravity feature two stages:

‘semi-covariance’ and **‘complete covariantization’**.

- **Semi-covariant derivative :** Jeon-Lee-JHP 2010, 2011

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} ,$$

for which the stringy Christoffel connection can be uniquely fixed,

$$\Gamma_{CAB} = 2\big(P\partial_C P\bar{P}\big)_{[AB]} + 2\big(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\big)\partial_D P_{EC} - \tfrac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\big)\big(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\big)$$

by demanding the compatibility, $\nabla_A P_{BC} = \nabla_A \bar{P}_{BC} = \nabla_A d = 0$, and some torsionless conditions.

$*$ There are no normal coordinates where Γ_{CAB} would vanish point-wise: Equivalence Principle is broken for string (*i.e.* extended object) but recoverable for point particle.

- **Semi-covariant Riemann curvature :**

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB}\Gamma_{ECD}\right) , \qquad S_{[ABC]D} = 0 ,$$

where R_{ABCD} denotes the ordinary "field strength": $R_{CDAB} = \partial_A\Gamma_{BCD} - \partial_B\Gamma_{ACD} + \Gamma_{AC}{}^E\Gamma_{BED} - \Gamma_{BC}{}^E\Gamma_{AED}$.

By construction, it varies as 'total derivative': $\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}$.

- **Semi-covariant 'Master' derivative :**

$$\mathcal{D}_A := \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A = \nabla_A + \Phi_A + \bar{\Phi}_A .$$

The two spin connections for the $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ local Lorentz symmetries are determined in terms of the stringy Christoffel connection by requiring the compatibility with DFT vielbeins,

$$\mathcal{D}_A V_{Bp} = \nabla_A V_{Bp} + \Phi_{Ap}{}^q V_{Bq} = 0 , \qquad \mathcal{D}_A \bar{V}_{B\bar{p}} = \nabla_A \bar{V}_{B\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0 .$$

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- **Complete covariantization**
  - Tensors,

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,,$$
$$\bar{P}_C{}^D P_{A_1}{}^{B_1} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_{\bar{p}} T_{q_1 q_2 \cdots q_n} \,,$$

$$\mathcal{D}^p T_{p\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p} q_1 q_2 \cdots q_n} \,; \qquad \mathcal{D}_p \mathcal{D}^p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_{q_1 q_2 \cdots q_n} \,.$$

  - Spinors, ρ^α, $\rho'^{\bar{\alpha}}$, $\psi^\alpha_{\bar{p}}$, $\psi'^{\bar{\alpha}}_p$,

$$\gamma^p \mathcal{D}_p \rho \,, \quad \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \rho' \,, \quad \mathcal{D}_{\bar{p}} \rho \,, \quad \mathcal{D}_p \rho' \,, \quad \gamma^p \mathcal{D}_p \psi_{\bar{q}} \,, \quad \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \psi'_q \,, \quad \mathcal{D}_{\bar{p}} \psi^{\bar{p}} \,, \quad \mathcal{D}_p \psi'^p \,.$$

  - RR sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$ **O**(D,D) *covariant nilpotent operators*

$$\mathcal{D}_\pm \mathcal{C} := \gamma^p \mathcal{D}_p \mathcal{C} \pm \gamma^{(D+1)} \mathcal{D}_{\bar{p}} \mathcal{C} \bar{\gamma}^{\bar{p}} \,, \quad (\mathcal{D}_\pm)^2 = 0 \quad \Longrightarrow \quad \mathcal{F} := \mathcal{D}_+ \mathcal{C} \quad (\,\text{RR flux}\,) \,.$$

  - Yang-Mills,

$$\mathcal{F}_{p\bar{q}} := \mathcal{F}_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \qquad \text{where} \qquad \mathcal{F}_{AB} := \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i\,[\mathcal{V}_A, \mathcal{V}_B] \,.$$

  - Curvatures,

$$S_{p\bar{q}} := S_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \quad (\,\text{Ricci}\,) \,, \qquad S_{(0)} := (P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD} \quad (\,\text{scalar}\,) \,.$$

- **Complete covariantization**
  - Tensors,

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \implies \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} ,$$
$$\bar{P}_C{}^D P_{A_1}{}^{B_1} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \implies \mathcal{D}_{\bar{p}} T_{q_1 q_2 \cdots q_n} ,$$

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Assuming $(0,0)$ Riemannian background, $\{e_\mu{}^p, \bar{e}_\mu{}^{\bar{p}}, B, \phi\}$, they reduce *e.g.* to

- Generalized Geometry Hitchin 2002, Gualtieri 2004, Waldram *et al.* 2008, 2011 :

$$\begin{aligned}\mathcal{D}_{\bar{p}}T_q &= \tfrac{1}{\sqrt{2}}\left(\partial_{\bar{p}}T_q + \omega_{\bar{p}qr}T^r + \tfrac{1}{2}H_{\bar{p}qr}T^r\right) ,\\ \gamma^p\mathcal{D}_p\rho &= \tfrac{1}{\sqrt{2}}\gamma^m\left(\partial_m\rho + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\rho + \tfrac{1}{24}H_{mnp}\gamma^{np}\rho - \partial_m\phi\rho\right) .\end{aligned}$$

- With $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, H–twisted & democratic RR Bergshoeff, Kallosh, Ortín, Roest, Van Proeyen 2001 :

$$\mathcal{D}_+ \;\Rightarrow\; \mathrm{d} + H\wedge \quad , \qquad\qquad \mathcal{D}_- \;\Rightarrow\; \star\,(\mathrm{d} + H\wedge\ )\,\star \quad .$$

- The scalar curvature gives the closed string effective action :

$$\int e^{-2d}\,S_{(0)} = \int \sqrt{|g|}e^{-2\phi}\left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\right) .$$

These results show how closed string massless sector, $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, should couple to extra matters minimally, while forming (pure) Stringy Gravity.

Equipped with the semi-covariant derivatives, one can construct, e.g.

- $D=10$ Maximally Supersymmetric Double Field Theory, Jeon-Lee-JHP-Suh 2012

$$\mathcal{L}_{\rm type\,II}=e^{-2d}\Big[\tfrac{1}{8}S_{(0)}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})+i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'$$
$$-i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}}+i\bar{\psi}'^{p}\mathcal{D}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\Big]$$

which unifies IIA and IIB SUGRAs, thanks to the twofold spin groups.

- Minimal coupling to the $D=4$ Standard Model, Kangsin Choi & JHP 2015 [PRL]

$$\mathcal{L}_{\rm SM}\;=\;e^{-2d}\left[\begin{array}{l}\frac{1}{16\pi G_N}S_{(0)}\\ +\sum_{\mathcal{V}}P^{AB}\bar{P}^{CD}\mathrm{Tr}(\mathcal{F}_{AC}\mathcal{F}_{BD})+\sum_{\psi}\bar{\psi}\gamma^{a}\mathcal{D}_{a}\psi+\sum_{\psi'}\bar{\psi}'\bar{\gamma}^{\bar{a}}\mathcal{D}_{\bar{a}}\psi'\\ -\mathcal{H}^{AB}(\mathcal{D}_{A}\phi)^{\dagger}\mathcal{D}_{B}\phi-\;V(\phi)+y_d\,\bar{q}\cdot\phi\,d+y_u\,\bar{q}\cdot\tilde{\phi}\,u+y_e\,\bar{l}'\cdot\phi\,e'\end{array}\right]$$

Every single term above is completely covariant, w.r.t. O(D,D), diffeomorphisms, and twofold local Lorentz symmetries, Spin$(1,D{-}1)_L\times$ Spin$(D{-}1,1)_R$.

Derivation of the Einstein Double Field Equations

Henceforth, we consider a general action for Stringy Gravity coupled to matter fields, Υ_a,

$$\int_\Sigma e^{-2d}\Big[\,\tfrac{1}{16\pi G}S_{(0)} + L_{\rm matter}\Big]\,,$$

where $S_{(0)}$ is the stringy scalar curvature and $L_{\rm matter}$ is the matter Lagrangian equipped with the completely covariantized master derivatives, $\mathcal{D}_M$. The integral is taken over a section, Σ.

We seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a.

– Firstly, the pure Stringy Gravity term transforms, up to total derivatives ($\simeq$), as

$$\delta\left(e^{-2d}S_{(0)}\right) \simeq 4e^{-2d}\left(\bar{V}^{B\bar{q}}\delta V_B{}^p S_{p\bar{q}} - \tfrac{1}{2}\delta d\, S_{(0)}\right)$$

– Secondly, the matter Lagrangian transforms as

$$\delta\Big(e^{-2d}L_{\rm matter}\Big) \simeq e^{-2d}\left(-2\bar{V}^{A\bar{q}}\delta V_A{}^p K_{p\bar{q}} + \delta d\, T_{(0)} + \delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right)$$

where we have been naturally led to define

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\rm matter}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_A{}^p}\right)\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\left(e^{-2d}L_{\rm matter}\right)}{\delta d}\,.$$

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- Combining the two results, the variation of the action reads

$$\delta\int_\Sigma e^{-2d}\left[\tfrac{1}{16\pi G}S_{(0)}+L_{\rm matter}\right]$$
$$=\int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{8\pi G}\delta d(S_{(0)}-8\pi GT_{(0)})+\delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right]$$

Hence, the equations of motion are for now exhaustively,

$$S_{p\bar{q}}=8\pi GK_{p\bar{q}}\,,\qquad S_{(0)}=8\pi GT_{(0)}\,,\qquad \frac{\delta L_{\rm matter}}{\delta\Upsilon_a}=0\,.$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_\xi\Upsilon_a=\hat{\mathcal{L}}_\xi\Upsilon_a$ and

$$\delta_\xi d=-\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_\xi\left(e^{-2d}\right)=-\tfrac{1}{2}\mathcal{D}_A\xi^A\,,\qquad \bar{V}^{A\bar{q}}\delta_\xi V_A{}^p=\bar{V}^{A\bar{q}}\hat{\mathcal{L}}_\xi V_A{}^p=2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,.$$

Substituting these, the diffeomorphic invariance of the action implies

$$0=\int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\left\{4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)}-8\pi GT_{(0)})\right\}+\delta_\xi\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right]$$

which leads to the definitions of the off-shell conserved **stringy Einstein curvature,**

$$G_{AB}:=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,,\qquad \mathcal{D}_AG^{AB}=0\qquad\text{(off-shell)}\,,$$

JHP-Rey-Rim-Sakatani 2015

and the on-shell conserved **stringy Energy-Momentum tensor,**

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- Since G_{AB} and T_{AB} each have $D^2 + 1$ components decomposable as

$$V^A{}_p \bar{V}^B{}_{\bar{q}} G_{AB} = 2S_{p\bar{q}} \,, \qquad G^A{}_A = -DS_{(0)} \,, \qquad V^A{}_p \bar{V}^B{}_{\bar{q}} T_{AB} = 2K_{p\bar{q}} \,, \qquad T^A{}_A = -DT_{(0)} \,,$$

the equations of motion of the DFT vielbeins and dilaton can be unified into a single expression:

Einstein Double Field Equations

$$G_{AB} = 8\pi G T_{AB}$$

which is naturally consistent with the central idea that Stringy Gravity treats the entire closed string massless sector as geometrical stringy graviton fields.

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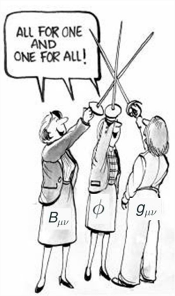
- Restricting to the $(0,0)$ Riemannian backgrounds, the EDFE decompose into

$$\begin{aligned} R_{\mu\nu} + 2\bigtriangledown_{\mu}(\partial_{\nu}\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} &= 8\pi G K_{(\mu\nu)}\,, \\ \bigtriangledown^{\rho}\left(e^{-2\phi}H_{\rho\mu\nu}\right) &= 16\pi G e^{-2\phi}K_{[\mu\nu]}\,, \\ R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &= 8\pi G T_{(0)}\,. \end{aligned}$$



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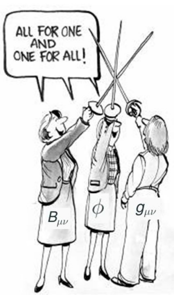

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Examples: $T_{AB} := 4\, V_{[A}{}^{p}\,\bar{V}_{B]}{}^{\bar{q}} K_{p\bar{q}} - \frac{1}{2}\mathcal{J}_{AB} T_{(0)}$

- RR sector,

$$L_{\mathrm{RR}} = \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})\,, \qquad K_{p\bar{q}} = -\tfrac{1}{4}\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,, \qquad T_{(0)} = 0\,.$$

- Spinor field,

$$L_{\psi} = \bar{\psi}\gamma^p\mathcal{D}_p\psi + m_\psi\bar{\psi}\psi\,, \qquad K_{p\bar{q}} = -\tfrac{1}{4}(\bar{\psi}\gamma_p\mathcal{D}_{\bar{q}}\psi - \mathcal{D}_{\bar{q}}\bar{\psi}\gamma_p\psi)\,, \qquad T_{(0)} \equiv 0\,.$$

- Scalar field,

$$L_{\varphi} = -\tfrac{1}{2}\mathcal{H}^{MN}\partial_M\varphi\partial_N\varphi - V(\varphi)\,, \qquad K_{p\bar{q}} = \partial_p\varphi\partial_{\bar{q}}\varphi\,, \qquad T_{(0)} = -2L_\varphi\,.$$

- Fundamental string: with $D_i y^M = \partial_i y^M - \mathcal{A}_i^M$ (doubled-yet-gauged),

$$e^{-2d}L_{\mathrm{string}} = \tfrac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \left[-\tfrac{1}{2}\sqrt{-h}h^{ij}D_iy^MD_jy^N\mathcal{H}_{MN}(y) - \epsilon^{ij}D_iy^M\mathcal{A}_{jM}\right]\delta^D\big(x - y(\sigma)\big)\,,$$

$$K_{p\bar{q}} = \tfrac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-h}h^{ij}(D_iy)_p(D_jy)_{\bar{q}}\, e^{2d(x)}\delta^D\big(x - y(\sigma)\big)\,, \qquad T_{(0)} = 0\,.$$

– More examples in our paper include Yang-Mills, point particle, superstring, *etc.*

Conclusion

- String theory may predict its own gravity, *i.e.* Stringy Gravity (DFT), rather than GR.
- Stringy Gravity may be formulated in 'doubled-yet-gauged' spacetime, unifies Riemannian SUGRA and non-Riemannian Newton-Cartan, Carroll, Gomis-Ooguri, *etc.* and deserves further explorations, *e.g.*

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Conclusion

- String theory may predict its own gravity, *i.e.* Stringy Gravity (DFT), rather than GR.

- Stringy Gravity may be formulated in 'doubled-yet-gauged' spacetime, unifies Riemannian SUGRA and non-Riemannian Newton-Cartan, Carroll, Gomis-Ooguri, *etc.* and deserves further explorations, *e.g.*

$\mathbf{O}(D,D)$ covariant holographic dual of Stringy Gravity?

Thank you

One must be prepared to follow up the consequence of theory, and feel that one just has to accept the consequences no matter where they lead.

– Paul Dirac –

Einstein Double Field Equations

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Core idea: string theory predicts its own gravity rather than GR

In General Relativity the metric $g_{\mu\nu}$ is the only geometric and gravitational field, whereas in string theory the closed-string massless sector comprises a two-form potential $B_{\mu\nu}$ and the string dilaton ϕ in addition to the metric $g_{\mu\nu}$. Furthermore, these three fields transform into each other under T-duality. This hints at a natural augmentation of GR: upon treating the whole closed string massless sector as stringy graviton fields, Double Field Theory [1, 2] may evolve into 'Stringy Gravity'. Equipped with an $\mathbf{O}(D,D)$ covariant differential geometry beyond Riemann [3], we spell out the definitions of the stringy Einstein curvature tensor and the stringy Energy-Momentum tensor. Equating them, all the equations of motion of the closed string massless sector are unified into a single expression [4],

$$G_{AB} = 8\pi G T_{AB} \tag{1}$$

which we dub the Einstein Double Field Equations.

Double Field Theory as Stringy Gravity

• **Built-in symmetries & Notation:**

– $\mathbf{O}(D,D)$ T-duality

– DFT diffeomorphisms (ordinary diffeomorphisms plus B-field gauge symmetry)

– Twofold local Lorentz symmetries, $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$

$\Rightarrow$ Two locally inertial frames exist separately for the left and the right modes.

Index	Representation	Metric (raising/lowering indices)
$A,B,\cdots,M,N,\cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$p,q,\cdots$	$\mathbf{Spin}(1,D{-}1)$ vector	$\eta_{pq}=\mathrm{diag}(-++\cdots+)$
$\alpha,\beta,\cdots$	$\mathbf{Spin}(1,D{-}1)$ spinor	$C_{\alpha\beta},\quad(\gamma^p)^T=C\gamma^pC^{-1}$
$\bar{p},\bar{q},\cdots$	$\mathbf{Spin}(D{-}1,1)$ vector	$\bar{\eta}_{\bar{p}\bar{q}}=\mathrm{diag}(+--\cdots-)$
$\bar{\alpha},\bar{\beta},\cdots$	$\mathbf{Spin}(D{-}1,1)$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}},\quad(\bar{\gamma}^{\bar{p}})^T=\bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

The $\mathbf{O}(D,D)$ metric $\mathcal{J}_{AB}$ divides doubled coordinates into two: $x^A=(\tilde{x}_\mu,x^\nu)$, $\partial_A=(\tilde{\partial}^\mu,\partial_\nu)$.

• **Doubled-yet-gauged spacetime:**

The doubled coordinates are 'gauged' through a certain equivalence relation, $x^A\sim x^A+\Delta^A$, such that each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$ [5]. This implies a section condition, $\partial_A\partial^A=0$, which can be conveniently solved by setting $\tilde{\partial}^\mu\equiv 0$.

• **Stringy graviton fields (closed-string massless sector), $\{d, V_{Mp}, \bar{V}_{N\bar{q}}\}$:**

Defining properties of the DFT-metric,

$$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad \mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN}=\mathcal{J}_{KM}\,, \tag{2}$$

set a pair of symmetric and orthogonal projectors,

$$P_{MN}=P_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,\qquad P_L{}^MP_M{}^N=P_L{}^N\,,$$
$$\bar{P}_{MN}=\bar{P}_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,\qquad \bar{P}_L{}^M\bar{P}_M{}^N=\bar{P}_L{}^N\,,\qquad P_L{}^M\bar{P}_M{}^N=0\,.$$

Further, taking the "square roots" of the projectors, we acquire a pair of DFT vielbeins,

$$P_{MN}=V_M{}^pV_N{}^q\eta_{pq}\,,\qquad \bar{P}_{MN}=\bar{V}_M{}^{\bar{p}}\bar{V}_N{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

satisfying their own defining properties,

$$V_{Mp}V^M{}_q=\eta_{pq}\,,\quad \bar{V}_{M\bar{p}}\bar{V}^M{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\quad V_{Mp}\bar{V}^M{}_{\bar{q}}=0\,,\quad V_M{}^pV_{Np}+\bar{V}_M{}^{\bar{p}}\bar{V}_{N\bar{p}}=\mathcal{J}_{MN}\,.$$

The most general solutions to (2) can be classified by two non-negative integers $(n,\bar{n})$ [6],

$$\mathcal{H}_{MN}=\begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^\mu X_\lambda^i-\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_\kappa^iY_i^\nu-\bar{X}_\kappa^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

where $1\leq i\leq n$, $1\leq\bar{\imath},i\leq\bar{n}$ and

$$H^{\mu\nu}X^i_\nu=0\,,\quad H^{\mu\nu}\bar{X}^{\bar{\imath}}_\nu=0\,,\quad K_{\mu\nu}Y_i^\nu=0\,,\quad K_{\mu\nu}\bar{Y}_{\bar{\imath}}^\nu=0\,,\quad H^{\mu\rho}K_{\rho\nu}+Y_i^\mu X_\nu^i+\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\nu^{\bar{\imath}}=\delta^\mu{}_\nu\,.$$

Strings become chiral and anti-chiral over n and $\bar{n}$ directions: $X^i_\mu\partial_+x^\mu=0$, $\bar{X}^{\bar{\imath}}_\mu\partial_-x^\mu=0$. Examples include $(0,0)$ Riemannian geometry as $K_{\mu\nu}=g_{\mu\nu}$, $H^{\mu\nu}=g^{\mu\nu}$, $(1,1)$ Gomis-Ooguri non-relativistic background, $(1,0)$ Newton-Cartan gravity, and $(D-1,0)$ Carroll gravity.

• **Covariant derivative:**

The 'master' covariant derivative, $\mathcal{D}_A=\partial_A+\Gamma_A+\Phi_A+\Phi_A$, is characterized by compatibility:

$$\mathcal{D}_Ad=\mathcal{D}_AV_{Bp}=\mathcal{D}_A\bar{V}_{B\bar{p}}=0\,,\qquad \mathcal{D}_A\mathcal{J}_{BC}=\mathcal{D}_A\eta_{pq}=\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=\mathcal{D}_AC_{\alpha\beta}=\mathcal{D}_A\bar{C}_{\bar{\alpha}\bar{\beta}}=0\,.$$

The stringy Christoffel symbols are [3]

$$\begin{aligned}\Gamma_{CAB}=&\,2\left(P\partial_CP\bar{P}\right)_{[AB]}+2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E-P_{[A}{}^DP_{B]}{}^E\right)\partial_DP_{EC}\\ &-4\left(\tfrac{1}{P_M{}^M-1}P_{C[A}P_{B]}{}^D+\tfrac{1}{\bar{P}_M{}^M-1}\bar{P}_{C[A}\bar{P}_{B]}{}^D\right)\left(\partial_Dd+(P\partial^EP\bar{P})_{[ED]}\right)\,,\end{aligned}$$

and the spin connections are $\Phi_{Apq}=V^B{}_p(\partial_AV_{Bq}+\Gamma_{AB}{}^CV_{Cq})$, $\bar{\Phi}_{A\bar{p}\bar{q}}=\bar{V}^B{}_{\bar{p}}(\partial_A\bar{V}_{B\bar{q}}+\Gamma_{AB}{}^C\bar{V}_{C\bar{q}})$.
In Stringy Gravity, there are no normal coordinates where Γ_{CAB} would vanish point-wise: the Equivalence Principle holds for point particles but is generically broken for strings (i.e. extended objects).

• **Scalar and 'Ricci' curvatures:**

The semi-covariant Riemann curvature in Stringy Gravity is defined by

$$S_{ABCD}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^E{}_{AB}\Gamma_{ECD}\right)\,,$$

where $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{ACE}\Gamma_B{}^E{}_D-\Gamma_{BCE}\Gamma_A{}^E{}_D$ (the "field strength" of Γ_{CAB}). The completely covariant 'Ricci' and scalar curvatures are, with $S_{AB}=S_{ACB}{}^C$,

$$S_{p\bar{q}}:=V^A{}_p\bar{V}^B{}_{\bar{q}}S_{AB}\,,\qquad S_{(0)}:=\left(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{CD}\right)S_{ABCD}\,.$$

While $e^{-2d}S_{(0)}$ corresponds to the original DFT Lagrangian density [1, 2], or the 'pure' Stringy Gravity, the master covariant derivative fixes its minimal coupling to extra matter fields, *e.g.* type II maximally supersymmetric DFT [7] or the Standard Model [8].

Derivation of Einstein Double Field Equations

Variation of the action for Stringy Gravity coupled to generic matter fields, Υ_a, gives

$$\begin{aligned}&\delta\int e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)}+L_{\mathrm{matter}}\Big]\\ =&\int e^{-2d}\Big[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{8\pi G}\delta d(S_{(0)}-8\pi GT_{(0)})+\delta\Upsilon_a\frac{\delta L_{\mathrm{matter}}}{\delta\Upsilon_a}\Big]\\ =&\int e^{-2d}\Big[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\{G_{AB}-8\pi GT_{AB}\}+(\hat{\mathcal{L}}_\xi\Upsilon_a)\frac{\delta L_{\mathrm{matter}}}{\delta\Upsilon_a}\Big]\,,\end{aligned}$$

where the second line is for generic variations and the third line is specifically for diffeomorphic transformations. We are naturally led to define

$$K_{p\bar{q}}:=\frac{1}{2}\left(V_{Ap}\frac{\delta L_{\mathrm{matter}}}{\delta\bar{V}_A{}^{\bar{q}}}-\bar{V}_{A\bar{q}}\frac{\delta L_{\mathrm{matter}}}{\delta V_A{}^p}\right)\,,\qquad T_{(0)}:=e^{2d}\times\frac{\delta\left(e^{-2d}L_{\mathrm{matter}}\right)}{\delta d}\,,$$

and subsequently the stringy *Einstein curvature*, G_{AB}, and *Energy Momentum tensor*, T_{AB},

$$G_{AB}=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,,\qquad \mathcal{D}_AG^{AB}=0\qquad(\text{off-shell})\,,$$
$$T_{AB}:=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,,\qquad \mathcal{D}_AT^{AB}=0\qquad(\text{on-shell})\,.$$

The equations of motion of the stringy graviton fields are thus unified into a single expression, the *Einstein Double Field Equations* (1). Note that $G_A{}^A=-DS_{(0)}$, $T_A{}^A=-DT_{(0)}$.

Restricting to the $(0,0)$ Riemannian background, the Einstein Double Field Equations reduce to

$$\begin{aligned}R_{\mu\nu}+2\nabla_\mu(\partial_\nu\phi)-\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}&=8\pi GK_{(\mu\nu)}\,,\\ \nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right)&=16\pi Ge^{-2\phi}K_{[\mu\nu]}\,,\\ R+4\Box\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}&=8\pi GT_{(0)}\,,\end{aligned}$$

which imply the conservation law, $\mathcal{D}_AT^{AB}=0$, given explicitly by

$$\nabla^\mu K_{(\mu\nu)}-2\partial^\mu\phi\,K_{(\mu\nu)}+\tfrac{1}{2}H_\nu{}^{\lambda\mu}K_{[\lambda\mu]}-\tfrac{1}{2}\partial_\nu T_{(0)}=0\,,\qquad \nabla^\mu\left(e^{-2\phi}K_{[\mu\nu]}\right)=0\,.$$

The Einstein Double Field Equations also govern the dynamics of other non-Riemannian cases, $(n,\bar{n})\neq(0,0)$, where the Riemannian metric, $g_{\mu\nu}$, cannot be defined.

Examples

– *Pure Stringy Gravity with cosmological constant:*

$$\tfrac{1}{16\pi G}e^{-2d}\left(S_{(0)}-2\Lambda_{\mathrm{DFT}}\right)\,,\qquad K_{p\bar{q}}=0\,,\qquad T_{(0)}=\tfrac{1}{4\pi G}\Lambda_{\mathrm{DFT}}\,.$$

– *RR sector, given by a* $\mathbf{Spin}(1,9)\times\mathbf{Spin}(9,1)$ *bi-spinorial potential,* $\mathcal{C}^\alpha{}_{\bar{\alpha}}$*:*

$$L_{\mathrm{RR}}=\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})\,,\qquad K_{p\bar{q}}=-\tfrac{1}{4}\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,,\qquad T_{(0)}=0\,,$$

where $\mathcal{F}=\mathcal{D}_+\mathcal{C}=\gamma^p\mathcal{D}_p\mathcal{C}+\gamma^{(11)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}}$ is the RR flux set by an $\mathbf{O}(D,D)$ covariant "H-twisted" cohomology, $(\mathcal{D}_+)^2=0$, and $\bar{\mathcal{F}}=\bar{C}^{-1}\mathcal{F}^TC$ is its charge conjugate [7].

– *Spinor field:* $L_\psi=\bar{\psi}\gamma^p\mathcal{D}_p\psi+m_\psi\bar{\psi}\psi$, $K_{p\bar{q}}=-\tfrac{1}{4}(\bar{\psi}\gamma_p\mathcal{D}_{\bar{q}}\psi-\mathcal{D}_{\bar{q}}\bar{\psi}\gamma_p\psi)$, $T_{(0)}=0$.

– *Green-Schwarz superstring (κ-symmetric):*

$$e^{-2d}L_{\mathrm{string}}=\tfrac{1}{4\pi\alpha'}\int\mathrm{d}^2\sigma\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_i^M\Pi_j^N\mathcal{H}_{MN}-\epsilon^{ij}D_iy^M(\mathcal{A}_{jM}-i\Sigma_{jM})\right]\delta^D(x-y(\sigma))\,,$$
$$K_{p\bar{q}}(x)=\tfrac{1}{4\pi\alpha'}\int\mathrm{d}^2\sigma\sqrt{-h}h^{ij}(\Pi_i^MV_{Mp})(\Pi_j^N\bar{V}_{N\bar{q}})e^{2d}\delta^D(x-y(\sigma))\,,\qquad T_{(0)}=0\,,$$

where $\Sigma_i^M=\bar{\theta}\gamma^M\partial_i\theta+\bar{\theta}'\bar{\gamma}^M\partial_i\theta'$ and $\Pi_i^M=\partial_iy^M-\mathcal{A}_i^M-i\Sigma_i^M$ (doubled-yet-gauged) [9].

Gravitational effect

The regular spherical solution to the $D=4$ Einstein Double Field Equations shows that Stringy Gravity modifies GR (Schwarzschild geometry), in particular at "short" dimensionless scales, R/MG, *i.e.* distance normalized by mass times Newton constant. This might shed new light upon the dark matter/energy problems, as they arise essentially from "short distance" observations. Furthermore, it would be intriguing to view the B-field and DFT dilaton d as 'dark gravitons', since they decouple from the geodesic motion of point particles, which should be defined in string frame [10].

	Electron ($R\simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_\odot G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M\propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^9	1.0×10^8	1.5×10^6	$\sim10^5$	0^+

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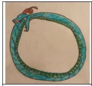
Gravitational effect

- The regular spherical solution to the $D = 4$ Einstein Double Field Equations shows that Stringy Gravity modifies GR (Schwarzschild geometry), in particular at “short" dimensionless scales, R/MG, *i.e.* distance normalized by mass times Newton constant.

  This might shed new light upon the dark matter/energy problems, as they arise essentially from “short distance" observations:

	Electron ($R \simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_{\odot}G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M \propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim 10^5$	0^+

- Furthermore, it would be intriguing to view the B-field and DFT dilaton d as ‘dark gravitons’, since they decouple from the geodesic motion of point particles, which should be defined in string frame.