Master Thesis

BRST Quantization of Massive Spin-2 Particles in the Stückelberg Formalism

Julius-Maximilians-Universität Würzburg

submitted by

Raimond Abt
from Sonthofen

Field of Study: Mathematical Physics

Supervisor: Prof. Dr. Thorsten Ohl

Theoretical Physics II
Theoretical Elementary Particle Physics

Würzburg, September 2016

Abstract

This thesis deals with the quantization of spin-2 particles by using the BRST formalism. First the necessary notations and mathematical concepts are introduced, where the main focus lies on the introduction of the BRST formalism. After this, spin-2 particles are discussed. From the classical point of view, they can be described by the Fierz-Pauli action, which is proven here. Subsequently, Stückelberg fields are used to introduce gauge invariances. These allow, together with suitable gauge fixings, to perform a BRST quantization for the massive case. The resulting vector space of physical states is isometric isomorphic to the space that is generated by the classical polarizations. This is proven by using the quartet mechanism. An analogous result arises for the massless case.

After this, a method is presented that allows to relate the algebraic structure of the BRST transformations of the involved fields in the massive case to the one in the massless case. Furthermore, some possible external sources for spin-2 particles are derived in such a way that they are compatible with the BRST formalism. Some Slavnov-Taylor identities are also derived and in addition it is shown that the introduced gauge invariances do not uniquely determine the resulting action. Eventually, the relation between massless spin-2 fields and gravity is derived.

Zusammenfassung

Die vorliegende Arbeit behandelt die Quantisierung von spin-2 Teilchen unter Verwendung des BRST-Formalismus. Nach einer Einführung in die verwendeten Notationen und mathematischen Konzepte, wobei der Schwerpunkt bei der Einführung des BRST-Formalismus angesiedelt ist, werden spin-2 Teilchen diskutiert. Diese lassen sich klassisch durch die Fierz-Pauli-Wirkung beschreiben, was hier bewiesen wird. Anschließend werden Stückelbergfelder verwendet, um Eichinvarianzen einzuführen. Diese können zusammen mit geeigneten Eichfixierungen benutzt werden, um eine BRST-Quantisierung für den massiven Fall durchzuführen. Der resultierende Vektorraum der physikalischen Zustände ist isometrisch isomorph zum Vektorraum, welcher durch die klassischen Polarisationen generiert wird. Dies wird mittels des Quartettmechanismus bewiesen. Ein analoges Resultat ergibt sich für den masselosen Fall.

Anschließend wird eine Methode vorgestellt, die es erlaubt die algebraische Struktur der BRST-Transformationen der beteiligten Felder im massiven mit jener im masselosen Fall in Bezug zu stellen. Weiterhin werden mögliche äußere Quellen für spin-2 Teilchen hergeleitet, welche mit dem BRST-Formalismus verträglich sind. Des Weiteren werden einige Slavnov-Taylor Identitäten hergeleitet und gezeigt, dass die eingeführten Eichinvarianzen die daraus resultierende Wirkung nicht eindeutig festlegen. Schließlich wird die Beziehung zwischen masselosen spin-2 Feldern und Gravitation hergeleitet.

Contents

Chapter 1

Introduction

One of the most important concepts of theoretical physics are gauge theories. In classical field theories gauge transformations originate from physical redundancies of a given field. To be more precise, it is possible that not every configuration of a given field resembles a different physical state. Gauge transformations turn field configurations into other field configurations that represent the same state. Therefore, they can be performed without any worries that the physical setup might be altered by them. This is a very powerful tool for solving the equations of motion for the fields of interest. By fixing the gauge, one can introduce additional differential equations, which the fields are supposed to adhere to, without restricting the set of the physical states that can be described by the fields. The additional differential equations eventually allow to simplify the equations of motion.

If the dynamics of a certain field is supposed to be given by an action, this action is usually constructed in such a way that it is invariant under gauge transformations, since this ensures that the corresponding equations of motion are gauge invariant as well.

This strategy plays a crucial role in particle physics. One can expand the action that describes the particles of ordinary matter (i.e. fermions) in such a way that it is invariant under certain gauge transformations. This requires the introduction of gauge bosons. Prominent examples are the photon, the W- and Z-bosons and the gluons. The corresponding new terms in the action in particular describe the electromagnetic, weak and strong interactions respectively. So it is possible to introduce interactions simply by demanding that the corresponding action has to be gauge invariant. Consequently, the theory of elementary particles is basically a gauge theory.

These statements clearly show that it is important to have a good understanding of quantized gauge theories. However, the quantization of them turns out to be quite subtle, since the vector space that is generated by the quantized fields usually contains unphysical states. For example, it is possible that this space contains states with negative norm. For abelian gauge theories, like $U(1)$ in the photonic case, there is a chance to identify a subspace as the actual physical space by introducing additional supplementary conditions, like the Gupta-Bleuler condition. For the non-abelian case this is a more subtle problem, since it is not so easy to find a subspace that is invariant under time evolution with this strategy, as pointed out in [3].

A very elegant way to solve this dilemma is given by the *BRST formalism* (Becchi, Rouet, Stora and Tyutin). It was originally discovered as a by-product of the gauge fixing method of Faddeev, Popov and De Witt and offers a very general strategy for a covariant quantization of gauge theories. This *BRST quantization* comes with an explicit form of the space of physical states and therefore allows to make very general statements about the properties it should have.

Furthermore, it offers a method that allows to derive identities among Green's functions in a very rigorous way. These identities are called *Slavnov-Taylor identities*. There is a very large quantity of literature that deals with the BRST formalism and its applications. Concerning this work, [3], [7], [8], [9], [10] [12], [13] and [17] shall be mentioned explicitly.

The BRST formalism is often applied to quantize spin-1 fields, since all the gauge bosons that are observed in nature are particles with a spin of 1. For massless spin-1 fields the formalism can be applied without any problems, but in the massive case the fields usually do not offer gauge transformations. However, the existence of gauge transformations is vital in order to perform the BRST quantization. Consequently, one has to to adapt the action that describes the particles of interest in such a way that it is invariant under certain gauge transformations.

Such adaptations are usually performed by applying the *Higgs mechanism*. However, this approach requires an actual modification of the theory. A new field, the *Higgs field*, has to be introduced, which interacts with the fields of interest. The masses of the fields are then interpreted as a consequence of these interactions and the spontaneous breakdown of symmetries. An introduction to these concepts can be found in [11].

It is possible to avoid this modification of the theory and still obtain the desired gauge invariance by using the so-called *Stückelberg trick* instead of the Higgs mechanism. This formalism also introduces new fields, but they do not carry any additional degrees of freedom. Nevertheless, they can be used to introduce gauge transformations for massive spin-1 fields as well. These gauge transformations then allow to apply the BRST formalism. This strategy for a suitable BRST quantization of massive spin-1 fields is pursued in [7] and [13], for example.

However, the BRST formalism is not restricted to the spin-1 case. It is also possible to treat spin-2 fields quantum-mechanically by using this formalism. In order to achieve an advanced understanding of the method of BRST quanization, such cases with spins of higher order certainly should not be neglected. Therefore, the BRST quantization of spin-2 fields is discussed in this thesis.

In the literature, spin-2 particles are often treated in the context of gravity, as it is done for example in [5]. The reason for that is the fact that the particles that cause gravitational interactions are usually assumed to be massless spin-2 particles. So, from the physical point of view, it is quite reasonable to work with spin-2 particles in this context. However, this restriction is not necessary. It is also possible to treat spin-2 fields just as a certain type of particle that gets its dynamics from a certain action, the so-called *Fierz-Pauli action*. This approach is pursued in [6], for example. In this thesis, spin-2 particles are treated in this way, too.

Just like in the spin-1 case, Stückelberg fields need to be introduced in order to obtain matching gauge transformations for massive spin-2 particles. This is done in [5], for example. The resulting adapted Fierz-Pauli action allows a BRST quantization of spin-2 fields. The main goal of this work is to understand how the vector space of quantized spin-2 fields relates to the polarizations of their classical analogs both in the massive and in the massless case. Furthermore, possible couplings to external sources, which are compatible with the BRST formalism, are derived.

Chapter 2

Mathematical Concepts

Before the actual mathematical analysis of spin-2 particles can be performed, it is necessary to introduce the corresponding mathematical concepts. Especially the BRST formalism offers a rigorousness and elegance in its formulation that deserves a closer look. However, the general concepts of quantum field theory will not be presented here, since they are well-known. An introduction to them can be found for example in [3], [12] and [16]. Furthermore, some results from classical Lagrangian and Hamiltonian mechanics are needed to justify the Lagrangian that is supposed to describe spin-2 particles. But first some notations, which will be used in the following chapters, have to be presented.

2.1 Notations and Conventions

Initially, it is important to mention that all functions of any kind will be considered to be sufficiently smooth, such that all the appearing derivatives are well-defined. The *Dirac delta distribution* on $\mathbb{R}^D$, with $D \in \mathbb{N}$, is usually denoted as

$$\delta^{(D)}(x). \tag{2.1.1}$$

In this thesis, the dimension of the argument of the distribution will always be clear from the context. Therefore, it will not be displayed explicitly, i.e. the convention

$$\delta^{(D)}(x) = \delta(x) \tag{2.1.2}$$

is used.

The complex conjugate of some parameter c is represented by c^*. The hermitian conjugate of an operator a is denoted as $a^\dagger$.

Furthermore, note that $g_{\mu\nu}$ will always denote the metric tensor $\eta_{\mu\nu}$ for a flat spacetime, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1.3}$$

since this is the standard practice in particle physics. The only exception occurs in Section 7.3. Furthermore, the Einstein summation convention will be used. Greek indices run from 0 to 3 in this context. Latin indices, usually i, j, k, run from 1 to 3 in this context and refer to spatial coordinates, if nothing different is mentioned.

Coordinates are written as lowercase Latin letters, usually x in position and p in momentum space. The μ-th component of a coordinate x is denoted as x^μ, i.e. coordinates are identified with contravariant vectors in position and momentum space. If only the spatial elements of a coordinate shall be considered, this is marked by using the corresponding bold letter. For example $\mathbf{x}$ is the spatial part of x. In some cases, x^0 will be denoted as t. A product of the form xp always refers to the contraction of the two coordinates via the metric tensor, i.e.

$$xp = x^\mu g_{\mu\nu} p^\nu = x_\mu p^\mu \quad \text{and in particular} \quad p^2 = p_\mu p^\mu. \tag{2.1.4}$$

In order to distinguish tensor fields from coordinates and to keep track of the rank of those fields, their indices are stated explicitly. For example, A_μ represents the whole tensor $A_\mu dx^\mu$ as well as its μ-th component. What of the two cases is meant by A_μ will be either irrelevant or clear from the context. A shift of the index $A_\mu \longmapsto A^\mu$ is always performed by the contraction with the metric tensor:

$$A^\mu = A_\nu g^{\mu\nu}. \tag{2.1.5}$$

The trace of any tensor $K_{\mu\nu}$ is abbreviated with K, i.e.

$$K_{\mu\nu} g^{\mu\nu} = K^\mu{}_\mu = K. \tag{2.1.6}$$

In order to symmetrize a tensor, the notation

$$K_{(\mu\nu)} = \frac{1}{2}(K_{\mu\nu} + K_{\nu\mu}) \tag{2.1.7}$$

is used. Furthermore, no difference between column and row vectors will be made. In the rare cases where a vector has to be explicitly written in its components, it will be written as column vector. If it is mentioned in the text, it will be written as row vector in order to avoid an unnecessary waste of space.

The time derivative $\dot{f}$ of a function f refers to the derivative with respect to x^0, i.e.

$$\dot{f} = \frac{d}{dt} f = \frac{d}{dx^0} f. \tag{2.1.8}$$

A *functional* $F[\phi_a]$ is a mapping that maps fields or a set of fields ϕ_a to scalars or scalar functions, i.e. functions of the form

$$f : \phi_a(x) \longmapsto f(x, \phi_a(x), \partial_r \phi_a(x), \partial_r \partial_l \phi_a(x), ...). \tag{2.1.9}$$

In this work these functions and scalars will always be real valued. The index a shall be contained in a finite set of indices.

Let F be of the form

$$F[\phi_a] = \int_M d^D x f(x, \phi_a(x), \partial_r \phi_a(x), \partial_r \partial_l \phi_a(x), ...), \tag{2.1.10}$$

where f is a scalar function in x, ϕ_a and its derivatives of arbitrary order and M an open subset of $\mathbb{R}^D$. The *functional left derivative*

$$\frac{\delta_L F[\phi_a]}{\delta \phi_a} : x \longmapsto \frac{\delta_L F[\phi_a]}{\delta \phi_a(x)} \tag{2.1.11}$$

of F is defined as the function that satisfies the relation

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} F[\phi_a + \epsilon y_a] = \int_M d^D x \sum_b y_b(x) \frac{\delta_L F[\phi_a]}{\delta \phi_b(x)}, \tag{2.1.12}$$

for any field y_a that vanishes at the boundary of M together with all its derivatives. In the case of an unbounded M, y_b is supposed to fall off sufficiently fast at infinity. The left hand side of this definition is basically the derivative of F at ϕ_a in direction y_a. Note that the functional derivative is supposed to be the same type of function as the fields ϕ_a. So if the ϕ_a are for example symmetric tensors, the functional derivative of F shall be chosen in such a way that it is a symmetric tensor as well.

By using the special form of F, one can construct its functional derivative explicitly:

$$\begin{aligned} &\frac{d}{d\epsilon}\Big|_{\epsilon=0} F[\phi_a + \epsilon y_a] \\ &= \int_M d^D x \sum_a \Big(y_a \frac{\partial_L f}{\partial \phi_a} + \sum_{r=1}^{D} \partial_r y_a \frac{\partial_L f}{\partial \partial_r \phi_a} + \sum_{r,l=1}^{D} \partial_r \partial_l y_a \frac{\partial_L f}{\partial \partial_r \partial_l \phi_a} + ... \Big) \\ &= \int_M d^D x \sum_a \Big(y_a \frac{\partial_L f}{\partial \phi_a} - \sum_{r=1}^{D} y_a \partial_r \frac{\partial_L f}{\partial \partial_r \phi_a} + \sum_{r,l=1}^{D} y_a \partial_r \partial_l \frac{\partial_L f}{\partial \partial_r \partial_l \phi_a} + ... \Big), \end{aligned} \tag{2.1.13}$$

where the method of partial integration has been used. Consequently, one gets

$$\frac{\delta_L F}{\delta \phi_a} = \frac{\partial_L f}{\partial \phi_a} - \sum_{r=1}^{D} \partial_r \frac{\partial_L f}{\partial \partial_r \phi_a} + \sum_{r,l=1}^{D} \partial_r \partial_l \frac{\partial_L f}{\partial \partial_r \partial_l \phi_a} + ... \quad . \tag{2.1.14}$$

The partial left derivatives $\frac{\partial_L}{\partial \phi_a}$ are defined in an analogous way as the functional left derivative: Let $g(\phi_a)$ be a function of a field ϕ_a, then $\frac{\partial_L}{\partial \phi_a} g$ is the function that satisfies

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} g(\phi_a + \epsilon y_a) = \sum_b y_b \frac{\partial_L}{\partial \phi_b} g(\phi_a) \tag{2.1.15}$$

and is of the same type as the fields ϕ_a. The left derivatives that differentiate f with respect to higher derivatives of ϕ_a are defined in the same way.

The notation *left derivative* refers to the fact that the test function y_a is always multiplied from the left to the corresponding derivatives. In an analogous way one can introduce a *right derivative*. The distinction between left and right derivatives is irrelevant in the bosonic case, but for fermionic fields (such as Faddeev-Popov ghosts), which anticommute, a difference in the sign between left and right derivatives can occur. By keeping (2.1.12) in mind, it is easy to verify that for a ϕ_a^τ, which depends on a scalar parameter τ, the chain rule

$$\frac{d}{d\tau} F[\phi_a^\tau] = \int_M d^D x \sum_b \frac{d\phi_b^\tau}{d\tau} \frac{\delta_L F[\phi_a^\tau]}{\delta \phi_b^\tau} \tag{2.1.16}$$

holds.

The function $f(x, \phi_a(x), \partial_r\phi_a(x), ...)$ itself is also a functional $f[\phi_a](x)$. It is possible to introduce the functional left derivative for such functions as well. One wants to obtain a mapping that satisfies

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} f[\phi_a + \epsilon y_a](x) = \Big(\int_M d^D x' \sum_a y_a(x') \frac{\delta_L f}{\delta\phi_a(x')}\Big)(x), \tag{2.1.17}$$

for all $x \in M$. So $\frac{\delta_L f}{\delta\phi_a}$ has to depend on two spacetime coordinates. Since this will allow to formulate some expressions in a very compact way in Section 2.4, the notation

$$\frac{\delta_L f}{\delta\phi_a(x')}[\phi_a](x) \tag{2.1.18}$$

will be used for

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} f[\phi_a + \epsilon y_a](x) = \int_M d^D x' \sum_b y_b(x') \frac{\delta_L f}{\delta\phi_b(x')}[\phi_a](x). \tag{2.1.19}$$

The explicit form of $\frac{\delta_L f}{\delta\phi_a}$ can be obtained by writing f as an integral

$$f[\phi_a](x) = \int_M d^D x' \delta(x - x') f[\phi_a](x'). \tag{2.1.20}$$

This implies

$$\begin{aligned} &\frac{d}{d\epsilon}\Big|_{\epsilon=0} f[\phi_a + \epsilon y_a](x) = \int_M d^D x' \delta(x - x') \frac{d}{d\epsilon}\Big|_{\epsilon=0} f[\phi_a + \epsilon y_a](x') \\ &= \int_M d^D x' \delta(x - x') \sum_b \Big(y_b(x') \frac{\partial_L f}{\partial\phi_b}[\phi_a](x') + \sum_{r=1}^{D} \frac{\partial y_b(x')}{\partial x'^r} \frac{\partial_L f}{\partial\partial_r\phi_b}[\phi_a](x') + ...\Big) \\ &= \int_M d^D x' \sum_b y_b(x') \Big(\delta(x - x') \frac{\partial_L f}{\partial\phi_b}[\phi_a](x') - \sum_{r=1}^{D} \frac{\partial}{\partial x'^r}\Big(\delta(x - x') \frac{\partial_L f}{\partial\partial_r\phi_b}[\phi_a](x')\Big) \\ &\quad + ...\Big) \end{aligned} \tag{2.1.21}$$

and therefore

$$\frac{\delta_L f}{\delta\phi_a(x')}[\phi_a](x) = \delta(x - x') \frac{\partial_L f}{\partial\phi_a}[\phi_a](x') - \sum_{r=1}^{D} \frac{\partial}{\partial x'^r}\Big(\delta(x - x') \frac{\partial_L f}{\partial\partial_r\phi_a}[\phi_a](x')\Big) + ... \quad . \tag{2.1.22}$$

Note that $\frac{\delta_L f}{\delta\phi_a}$ is a distribution, since it contains delta distributions.

Remark The definition of the functional derivative of f, as presented above, somehow appears to be ill-defined from a mathematical point of view, since it contains derivatives of the delta distribution. The more elegant way to introduce such a concept would be to define $\frac{\delta_L f}{\delta\phi_a}$ simply as the linear operator that is given by

$$\frac{\delta_L f[\phi_a]}{\delta\phi_a} : y_a \longmapsto \frac{d}{d\epsilon}\Big|_{\epsilon=0} f[\phi_a + \epsilon y_a]. \tag{2.1.23}$$

In particular, this would eliminate the difference between left and right functional derivatives. This and the subtleties with the delta distribution originate from the desire to identify the functional derivative with a distribution that relates to the derivative of f in direction y_a in the way described in (2.1.19). If one interprets the integral over the product of two functionals as a scalar product, this closely resembles the relation between derivative and gradient. The reason why $\frac{\delta_L f}{\delta\phi_a}$ is introduced like this is due to the fact that it turns out to be quite useful for the discussion of the BRST transformations in Section 2.4 and the use of Poisson brackets in Section 3.1 to have an actual expression of this object in form of a distribution.

Furthermore, one finds

$$\begin{aligned}\int_M d^Dx' \sum_b y_b(x') \frac{\delta_L F}{\delta\phi_b(x')}[\phi_a] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F[\phi_a+\epsilon y_a] = \int_M d^Dx \frac{d}{d\epsilon}\Big|_{\epsilon=0} f[\phi_a+\epsilon y_a](x) \\ = \int_M d^Dx \int_M d^Dx' \sum_b y_b(x') \frac{\delta_L f}{\delta\phi_b(x')}[\phi_a](x)\end{aligned} \tag{2.1.24}$$

and therefore

$$\frac{\delta_L F}{\delta\phi_a(x')}[\phi_a] = \int_M d^Dx \frac{\delta_L f}{\delta\phi_a(x')}[\phi_a](x), \tag{2.1.25}$$

i.e. the functional derivative commutes with the integration. This allows to formulate a chain rule for the functional derivative.

Theorem 2.1.1 *Let F be a functional of the form* (2.1.10) *and g_a functionals of the form* (2.1.9). *Then the chain rule*

$$\frac{\delta_L F[g_a[\phi_a]]}{\delta\phi_a(x)} = \int_M d^Dx' \sum_b \frac{\delta_L g_b[\phi_a](x')}{\delta\phi_a(x)} \frac{\delta_L F[\psi_a]}{\delta\psi_b(x')}\Big|_{\psi_a=g_a[\phi_a]} \tag{2.1.26}$$

holds.

Proof: Let y_a be a field that vanishes at the boundary of M together with all its derivatives. Obviously, the functional derivative of g_a, and all its derivatives, in direction of y_a vanishes at the boundary of M, i.e.

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} g_a[\phi_a+\epsilon y_a]\Big|_{\partial M} = 0, \quad \frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_r g_a[\phi_a+\epsilon y_a]\Big|_{\partial M} = 0, \dots \quad . \tag{2.1.27}$$

This can be used together with the method of partial integration to get

$$\begin{aligned}\frac{d}{d\epsilon}\Big|_{\epsilon=0} F[g_a[\phi_a+\epsilon y_a]] &= \int_M d^Dx' \frac{d}{d\epsilon}\Big|_{\epsilon=0} f(x', g_a[\phi_a+\epsilon y_a](x'), \partial_r g_a[\phi_a+\epsilon y_a](x'), ...) \\ &= \int_M d^Dx' \sum_b \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} g_b[\phi_a+\epsilon y_a](x') \frac{\partial_L f[\psi_a](x')}{\partial\psi_b} \\ &\quad + \sum_{r=1}^{D} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \partial_r g_b[\phi_a+\epsilon y_a](x') \frac{\partial_L f[\psi_a](x')}{\partial\partial_r\psi_b} + ...\Big)\Big|_{\psi_a=g_a[\phi_a]} \\ &= \int_M d^Dx' \sum_b \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_b[\phi_a+\epsilon y_a](x') \frac{\delta_L F[\psi_a]}{\delta\psi_b(x')}\Big|_{\psi_a=g_a[\phi_a]}\end{aligned}$$

$$= \int_M d^D x' \int_M d^D x \sum_{b,c} y_c(x) \frac{\delta_L g_b[\phi_a](x')}{\delta \phi_c(x)} \frac{\delta_L F[\psi_a]}{\delta \psi_b(x')} \Big|_{\psi_a = g_a[\phi_a]}$$
$$= \int_M d^D x \sum_c y_c(x) \int_M d^D x' \sum_b \frac{\delta_L g_b[\phi_a](x')}{\delta \phi_c(x)} \frac{\delta_L F[\psi_a]}{\delta \psi_b(x')} \Big|_{\psi_a = g_a[\phi_a]}, \quad (2.1.28)$$

which completes the proof. □

As a final remark regarding functionals, note that the functionals in this thesis will always considered to be functions of ϕ_a and its derivatives or integrals over such functions. Therefore, the functional left derivative is defined for any functional that occurs in this work.

Let ϕ be a quantized field that describes an exterior particle with mass m in position space. Then one can introduce creation and annihilation operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ in momentum space to write ϕ as

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \left\{ a(\mathbf{p}) e^{ipx} + a^\dagger(\mathbf{p}) e^{-ipx} \right\} \Big|_{p^0 = E_\mathbf{p}}, \quad (2.1.29)$$

with $E_\mathbf{p} = \sqrt{|\mathbf{p}|^2 + m^2}$. In order to keep such expressions as short as possible, the abbreviation

$$\widetilde{dp} = \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \Big|_{p^0 = E_\mathbf{p}} \quad (2.1.30)$$

shall be introduced.

For two operators A and B the notation

$$[A, B]_\pm = AB \pm BA \quad (2.1.31)$$

is used, i.e. $[\cdot, \cdot]_-$ is the *commutator* and $[\cdot, \cdot]_+$ the *anticommutator*.

2.2 Lagrangian and Hamiltonian Densities

The Lagrangian and Hamiltonian formalism are both well-known concepts of theoretical physics and therefore will not be introduced. Nevertheless, some notations and useful transformations for Lagrangian and Hamiltonian densities shall be illustrated.

A Lagrangian density $\mathcal{L}$ that depends on a set of fields ϕ_a and their first order derivatives $\partial_\mu \phi_a$ always comes from a certain action, which is assumed to have the form

$$S[\phi_a] = \int_{t_1}^{t_2} dt \int d^3\mathbf{x} \mathcal{L}(\phi_a, \partial_\mu \phi_a). \quad (2.2.1)$$

Here $[t_1, t_2]$ is a given time interval. The fields do not necessarily need to be scalars but can also be 4-vectors or tensors of higher order. In this subsection, the fields ϕ_a are regarded to be bosonic. This means that the functional and partial left derivatives are the same operation as the corresponding right derivatives. The integral of $\mathcal{L}$ over the spatial components is usually interpreted as a Lagrangian function

$$L[\phi_a] = \int d^3\mathbf{x}\mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{2.2.2}$$

which relates to the Lagrangian density in a natural way.

The requirement that $\frac{\delta_L}{\delta \phi_a} S$ disappears gives the Euler-Lagrange equations. Therefore, an arbitrary functional $F[\phi_a]$ with $\frac{\delta_L}{\delta \phi_a} F = 0$ for all field configurations of ϕ_a can be added to $S[\phi_a]$ without changing the Euler-Lagrange equations. If F is the integral of a function, this gives a possible transformation of $\mathcal{L}$ that does not change the dynamics. One special case for such functions are total time derivatives.

Theorem 2.2.1 *Let f be a sufficiently smooth function of x, ϕ_a and its space and time derivatives of arbitrary order and $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ a Lagrangian density. Then $\mathcal{L} + \frac{d}{dt} f$ gives the same Euler-Lagrange equations as $\mathcal{L}$.*

PROOF: It is sufficient to show that

$$\frac{\delta_L}{\delta \phi_a} \int_{t_1}^{t_2} dt \int d^3\mathbf{x} \frac{d}{dt} f[\phi_a](x) = 0 \tag{2.2.3}$$

holds for a given time interval $[t_1, t_2]$. This can be done by a direct calculation. Let y_a be an arbitrary field that disappears at the boundary of $[t_1, t_2] \times \mathbb{R}^3$, together with all its derivatives. Then the variation of ϕ_a according to y_a yields

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{t_1}^{t_2} dt \int d^3\mathbf{x} \frac{d}{dt} f[\phi_a + \epsilon y_a](x) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int d^3\mathbf{x} f[\phi_a + \epsilon y_a](x)\Big|_{t=t_1}^{t_2}. \tag{2.2.4}$$

So the result is just the difference of two integrals over subsets of the boundary of $[t_1, t_2] \times \mathbb{R}^3$. Therefore, the y_a vanish in the remaining integrals and with them the ϵ dependency. Consequently, one gets

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{t_1}^{t_2} dt \int d^3\mathbf{x} \frac{d}{dt} f[\phi_a + \epsilon y_a](x) = 0 \tag{2.2.5}$$

for any y_a, which implies (2.2.3). □

By performing a Legendre transformation on $\mathcal{L}$ according to $\partial_0 \phi_a = \dot{\phi}_a$, a Hamiltonian density $\mathcal{H}$ can be constructed from $\mathcal{L}$. The momenta then have the form

$$\pi^a = \frac{\delta_L L}{\delta \dot{\phi}_a}. \tag{2.2.6}$$

The resulting Hamiltonian density can be related to a Hamiltonian function by integration:

$$H[\phi_a, \pi^a] = \int d^3\mathbf{x}\mathcal{H}[\phi_a, \pi^a]. \tag{2.2.7}$$

Usually the Hamiltonian density does not depend on the spatial derivatives of the momenta but in a more general context this is possible. The equations of motion then read

$$\frac{\delta_L H}{\delta \phi_a} = -\dot{\pi}^a \quad \text{and} \quad \frac{\delta_L H}{\delta \pi^a} = \dot{\phi}_a. \tag{2.2.8}$$

Note that the corresponding functional derivatives now correspond to integrals over space, not spacetime. This means that the time parameter has to be regarded as an additional scalar parameter on which the functions y_a that are used to calculate the functional derivative do not depend.

The method of adding a total time derivative to the Lagrangian density without changing the physics leads to a possibility to derive such a transformation for the Hamiltonian density simply by Legendre-transforming $\widetilde{\mathcal{L}} = \mathcal{L} + \frac{d}{dt}f$. In order to do so one has to assume that f is a function of x, ϕ_a and only its spatial derivatives of arbitrary order. f is not allowed to depend on time derivatives of any order. Otherwise the resulting Lagrangian $\widetilde{\mathcal{L}}$ could depend on time derivatives of second or higher order, which would not allow to perform an appropriate Legendre transformation. This additional restriction leads to the momenta[1]

$$\widetilde{\pi}^a = \pi^a + \frac{\delta_L F[\phi_a]}{\delta\phi_a}, \tag{2.2.9}$$

with

$$F[\phi_a] = \int d^3\mathbf{x} f[\phi_a], \tag{2.2.10}$$

and to a new Hamiltonian density

$$\widetilde{\mathcal{H}}[\phi_a, \widetilde{\pi}^a] = \mathcal{H}\Big[\phi_a, \widetilde{\pi}^a - \frac{\delta_L F[\phi_a]}{\delta\phi_a}\Big] - f_t[\phi_a] \tag{2.2.11}$$

where f_t denotes the derivative of f according to the explicit time dependence. The fact that this transformation delivers the correct dynamics can be proven without the transformation back to the Lagrange formalism.

Theorem 2.2.2 *Let $H[\phi_a, \pi^a]$ be a Hamiltonian function, which corresponds to a Hamiltonian density $\mathcal{H}[\phi_a, \pi^a]$, and f a function of x, ϕ_a and its spatial derivatives of arbitrary order and possibly additional fields that do not get their dynamics from H. Furthermore, let $F_t = \frac{d}{dt}F - \int d^3\mathbf{x} \sum_b \dot{\phi}_b \frac{\delta_L F[\phi_a]}{\delta\phi_b}$ with $F[\phi_a] = \int d^3\mathbf{x} f[\phi_a]$. Then the Hamiltonian*

$$\widetilde{H}[\phi_a, \widetilde{\pi}^a] = H\Big[\phi_a, \widetilde{\pi}^a - \frac{\delta_L F[\phi_a]}{\delta\phi_a}\Big] - F_t[\phi_a] \tag{2.2.12}$$

gives the dynamics for the transformed momenta $\widetilde{\pi}^a = \pi^a + \frac{\delta_L F[\phi_a]}{\delta\phi_a}$.

Proof: It is sufficient to show that $\frac{\delta_L}{\delta\widetilde{\pi}^a}\widetilde{H} = \dot{\phi}_a$ and $\frac{\delta_L}{\delta\phi_a}\widetilde{H} = -\dot{\widetilde{\pi}}^a$ hold. The first equality is trivial. The later can be shown by a direct calculation. In order for the statement to be true, one explicitly has to use the fact that all fields are considered to be bosonic. This implies

$$\dot{\phi}_b \frac{\delta_L F[\phi_a]}{\delta\phi_b} = \frac{\delta_L F[\phi_a]}{\delta\phi_b}\dot{\phi}_b. \tag{2.2.13}$$

[1]This is an immediate consequence of (2.1.16), with $\tau = t$.

By using the chain rule of Theorem 2.1.1, the canonical equations (2.2.8) and the easy to verify relation $\frac{\delta_L}{\delta\phi_a}\dot{\phi}_b = 0$, one finds

$$\begin{aligned}
\frac{\delta_L \widetilde{H}[\phi_a, \widetilde{\pi}^a]}{\delta\phi_a(\mathbf{x})} =& \Big(\frac{\delta_L H[\phi_a, \pi^a]}{\delta\phi_a(\mathbf{x})} + \int d^3\mathbf{x}' \sum_b \frac{\delta_L}{\delta\phi_a(\mathbf{x})}\Big(\widetilde{\pi}^b(\mathbf{x}') - \frac{\delta_L F[\phi_a]}{\delta\phi_b(\mathbf{x}')}\Big) \\
&\times \frac{\delta_L H[\phi_a, \pi^a]}{\delta\pi^b(\mathbf{x}')}\Big)\Big|_{\pi^a = \widetilde{\pi}^a - \frac{\delta_L F}{\delta\phi_a}} - \frac{\delta_L F_t[\phi_a]}{\delta\phi_a(\mathbf{x})} \\
=& -\dot{\pi}^a(\mathbf{x})\big|_{\pi^a = \widetilde{\pi}^a - \frac{\delta_L F}{\delta\phi_a}} - \int d^3\mathbf{x}' \sum_b \frac{\delta_L}{\delta\phi_a(\mathbf{x})} \frac{\delta_L F[\phi_a]}{\delta\phi_b(\mathbf{x}')}\dot{\phi}_b(\mathbf{x}') - \frac{\delta_L F_t[\phi_a]}{\delta\phi_a(\mathbf{x})} \qquad (2.2.14) \\
=& -\dot{\widetilde{\pi}}^a(\mathbf{x}) + \frac{d}{dt}\frac{\delta_L F[\phi_a]}{\delta\phi_a(\mathbf{x})} - \frac{\delta_L}{\delta\phi_a(\mathbf{x})}\Big(\int d^3\mathbf{x}' \sum_b \frac{\delta_L F[\phi_a]}{\delta\phi_b(\mathbf{x}')}\dot{\phi}_b(\mathbf{x}') + F_t[\phi_a]\Big) \\
=& -\dot{\widetilde{\pi}}^a(\mathbf{x}) + \frac{d}{dt}\frac{\delta_L F[\phi_a]}{\delta\phi_a(\mathbf{x})} - \frac{\delta_L}{\delta\phi_a(\mathbf{x})}\frac{d}{dt}F[\phi_a] = -\dot{\widetilde{\pi}}^a(\mathbf{x}).
\end{aligned}$$

Note that in this calculation the time dependence of the fields is implicit. Furthermore, it has been used that $\frac{d}{dt}$ and $\frac{\delta_L}{\delta\phi_a}$ commute. Since the time in this context is just an additional scalar parameter, which has nothing to do with the functional derivative, this is easy to verify. □

Remark The freedom that f can also depend on additional fields appears to be somehow unnecessary, since this could also be put into the explicit time dependence[2]. However, such a dependence will become important for some results for the spin-2 fields. Therefore, it is mentioned explicitly.

By keeping (2.1.16) in mind, one can easily see that F_t is just the integral of f_t over space. This leads to the desired result (2.2.11).

Remark The momenta π^a are usually introduced as $\frac{\partial_L \mathcal{L}}{\partial\dot{\phi}_a}$ (see for example [9], [10] and [12]). This is clearly an equivalent definition if $\mathcal{L}$ only depends on ϕ_a and $\partial_\mu\phi_a$. However, the modified $\widetilde{\mathcal{L}}$ depends on spatial derivatives of ϕ_a and $\dot{\phi}_a$ of arbitrary order. The definition of the momenta as it is presented here is still valid for this more general case. The proof that this leads to canonical equations that are equivalent to the Euler-Lagrange equations works in a completely analogous way as for the dynamics of classical point particles. One only needs to use the fact that the Euler-Lagrange equations can also be formulated as

$$\frac{\delta_L L}{\delta\phi_a} - \frac{d}{dt}\frac{\delta_L L}{\delta\dot{\phi}_a} = 0. \qquad (2.2.15)$$

Here the functional derivative of course corresponds to an integral over space once more. This property can be shown by a straightforward calculation.

[2] Explicit time dependence in this context means that the function does not just depend on time via ϕ_a.

2.3 Spin Representations of the Poincaré Group

One important part of this thesis is to understand why the Fierz-Pauli action actually describes spin-2 particles. Therefore, some aspects of the theory of irreducible unitary representations of the group of *inhomogeneous proper orthochronous Lorentz transformations* $ISO(3,1)^{\uparrow}$, which is called *Poincaré group* in this thesis, need to be reviewed. The results presented in this section are taken from [1], which deals with this subject in a more detailed way.

Elementary particles are mathematically described as irreducible, unitary representations of $ISO(3,1)^{\uparrow}$. Thus a method to explicitly construct such representations would be extremely useful. In fact such a method exists. To understand the idea behind it, consider the Lie algebra $\mathfrak{iso}(3,1)$. It is spanned by generators P^μ and $J^{\mu\nu}$ that obey the commutation relations

$$i[J^{\mu\nu}, J^{\rho\sigma}]_- = g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\sigma\mu}J^{\rho\nu} + g^{\sigma\nu}J^{\rho\mu} \tag{2.3.1}$$

$$i[P^\mu, J^{\rho\sigma}]_- = g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho \tag{2.3.2}$$

$$i[P^\mu, P^\nu]_- = 0. \tag{2.3.3}$$

Therefore, the negative square of the momentum $-P_\mu P^\mu$ is a quadratic and the square of the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma \tag{2.3.4}$$

a quartic Casimir operator. Since they are proportional to the unit in an irreducible representation, their value can be used to classify such a representation. More precisely, an unitary irreducible representation with

$$-P_\mu P^\mu = m^2\mathbb{1} \tag{2.3.5}$$

and

$$W_\mu W^\mu = m^2 s(s+1)\mathbb{1} \tag{2.3.6}$$

describes a particle with mass $m > 0$ and spin s. For $m = 0$, things become more subtle, since $W_\mu W^\mu$ can not be used to get a spin in this case. Nevertheless, since the momentum operators P^μ commute, they have common eigenvectors $|p, A\rangle$ with

$$P^\mu|p, A\rangle = p^\mu|p, A\rangle, \tag{2.3.7}$$

for any $m \geq 0$. Here A denotes possible other degrees of freedom. These $|p, A\rangle$ can be used to express any vector $|\Psi\rangle$ as linear combination of them. Furthermore, $p^2 = -m^2$ and $P_\mu P^\mu|\Psi\rangle = -m^2|\Psi\rangle$ follow directly from (2.3.5). So the vectors in such an irreducible representation have to satisfy

$$(P_\mu P^\mu + m^2\mathbb{1})|\Psi\rangle = 0, \tag{2.3.8}$$

which is the well-known Klein-Gordon equation. Thus a first step towards a construction of irreducible representations has been done. For the following, let $U(\Lambda, a)$ be the unitary operator, representing the element $(\Lambda, a) \in ISO(3,1)^{\uparrow}$, where Λ is an element of the *proper orthochronous Lorentz group* $SO(3,1)^{\uparrow}$ and $a \in \mathbb{R}^4$ represents an arbitrary spacetime translation. For any two such elements (Λ, a) and (Λ', a') the relation

$$U(\Lambda, a)U(\Lambda', a') = U(\Lambda\Lambda', a + \Lambda a') \tag{2.3.9}$$

holds. By looking at an infinitesimal translation (see [1]),

$$U(\mathbb{1}, a)|p, A\rangle = e^{ipa}|p, A\rangle \tag{2.3.10}$$

can be derived. Furthermore, by using some properties of $SO(3,1)^{\uparrow}$ (see [1]),

$$U(\Lambda)|p, A\rangle = \sum_{A'} C_{A'A}(\Lambda, p)|\Lambda p, A'\rangle, \tag{2.3.11}$$

where $U(\Lambda) = U(\Lambda, 0)$, can be obtained, i.e. $U(\Lambda)|p, A\rangle$ is an eigenvector of the momentum operators with eigenvalues $(\Lambda p)^{\mu}$. Furthermore, note that a $\Lambda \in SO(3,1)^{\uparrow}$ conserves the sign of p^0 for any p with $p^2 \leq 0$. Since an irreducible representation is supposed to be constructed, all vectors except 0 are cyclic. So in particular, (2.3.11) shows that all the eigenvalues p^0 of P^0 have the same sign. The matrix $C_{A'A}(\Lambda, p)$ basically defines $U(\Lambda)$. Thus determining the $C_{A'A}(\Lambda, p)$ means determining the corresponding irreducible unitary representation of the Poincaré group, since the translations only contribute a phase factor. This problem can be reduced to identifying a similar matrix of so-called *little groups* by the *method of induced representations*.

To do so, consider a fixed and non-vanishing momentum $\hat{k}$ with $\hat{k}^2 = -m^2$. Then for any momentum p, for which p^0 has the same sign as $\hat{k}^0$ and with $p^2 = -m^2$, there is a standard proper orthochronous Lorentz transformation $L(p)$, such that

$$L(p)\hat{k} = p \tag{2.3.12}$$

holds. This can be used for the following definition of the $|p, A\rangle$:

$$|p, A\rangle = N(p)U(L(p))\left|\hat{k}, A\right\rangle, \tag{2.3.13}$$

where $N(p)$ is just a numerical normalization factor. The $L(p)$ are of course not uniquely determined. In fact it is straightforward to show that two such transformations $L(p)$ and $\widetilde{L}(p)$ always relate to each other via

$$\widetilde{L}(p) = L(p)W, \tag{2.3.14}$$

where $W \in SO(3,1)^{\uparrow}$ is some transformation that keeps $\hat{k}$ invariant. These W obviously form a group, the so-called *little group* corresponding to $\hat{k}$. From (2.3.11) it immediately follows that there is a matrix $D^{\hat{k}}_{A'A}(W)$ such that

$$U(W)\left|\hat{k}, A\right\rangle = \sum_{A'} D^{\hat{k}}_{A'A}(W)\left|\hat{k}, A'\right\rangle \tag{2.3.15}$$

and

$$D^{\hat{k}}_{A'A}(W'W) = \sum_{B} D^{\hat{k}}_{A'B}(W')D^{\hat{k}}_{BA}(W), \tag{2.3.16}$$

for another element W' of the little group, hold. Thus the $D^{\hat{k}}_{A'A}(W)$ give a representation of the little group. Furthermore, it is easy to see that $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ is an element of the little group. By inserting some units (see [1]), one obtains

$$U(\Lambda)|p, A\rangle = \frac{N(p)}{N(\Lambda p)} \sum_{A'} D^{\hat{k}}_{A'A}(W(\Lambda, p))|\Lambda p, A'\rangle. \tag{2.3.17}$$

So in order to find an irreducible unitary representation of the Poincaré group, it is sufficient to find an irreducible unitary representation of the little group corresponding to a chosen $\hat{k}$.

For this it is clearly necessary to identify the little group. In the massive case one can choose $\hat{k} = (m, 0, 0, 0)$. The little group of $\hat{k}$ obviously is just the group of rotations in space, i.e. $SO(3) \subset SO(3,1)^{\uparrow}$. The massless case however does not offer such a trivial identification. By choosing $\hat{k} = (k^0, 0, 0, k^0)$, it is clear that its little group contains spatial rotations around the z-axis, i.e. $SO(2) \subset SO(3,1)^{\uparrow}$. Further investigations show that it is in fact *the two-dimensional Euclidean group* $ISO(2)$. A proof of this can be found in [16].

In a quite similar way as for $ISO(3,1)^{\uparrow}$, the method of induced representations can be applied to $ISO(2)$. By taking a look at the commutation relations of the generators π_1, π_2 and J_3 of $\mathfrak{iso}(2)$,

$$i[\pi_1, J_3]_- = \pi_2, \quad i[\pi_1, J_3]_- = \pi_2 \quad \text{and} \quad [\pi_1, \pi_2]_- = 0, \tag{2.3.18}$$

one can see that $\left|\hat{k}, A\right\rangle$ can be written in terms of a common eigenbasis $\left|\hat{k}, \xi, a\right\rangle$ of the translation generators π_n, where $\pi_n\left|\hat{k}, \xi, a\right\rangle = \xi_n\left|\hat{k}, \xi, a\right\rangle$. Just like in the previous case, the problem reduces to finding an irreducible unitary representation of the group of transformations which keep a specific ξ invariant. The short little group of $\xi = 0$, which obviously is $SO(2)$, corresponds to so-called *helicity* representations. They are the representations that are interesting for this work.

Now, after reducing the problem to finding irreducible unitary representations of special orthogonal groups, a procedure for constructing an irreducible unitary representation of the Poincaré group can be formulated. This is known as the *Bargmann-Wigner program* (see [1]). The idea behind it is to introduce manifestly covariant equations[3] whose solutions transform according to a certain unitary irreducible representation of the inhomogeneous Lorentz group $IO(3,1)$. One of these equations has to be the Klein-Gordon equation. Clearly, these solutions only carry a reducible unitary representation of the Poincaré group. However, this method allows to formulate the concepts of time reversal and parity symmetry, which are usually required in field theories. The actual irreducible representation of $ISO(3,1)^{\uparrow}$ is then identified with the positive-energy solutions, i.e. the subspace on which $p^0 > 0$ holds for all eigenvalues p^0 of P^0. Since representations of $IO(3,1)$ shall be constructed, one has now to work with the (short) little group of this group. For $m^2 > 0$ the little group is the orthogonal group $O(3)$, while the short little group for $m^2 = 0$ is $O(2)$. For them, the following general procedure can be applied.

i.) Choose an irreducible unitary representation of the (short) little group.

ii.) Introduce a wave function on Minkowski space whose values are contained in a representation R of the Lorentz group $O(3,1)$. This representation does not necessarily have to be unitary, but its restriction to the (little) group has to contain the representation of *i.)*. That means, there has to be a subspace of R, such that the application of the little group on it delivers the representation of *i.)*.

iii.) Introduce a system of linear covariant equations, such that the solutions of these evaluated at a fixed momentum carry the unitary representation of *i.)*.

[3] Differential equations in position space or algebraic ones in momentum space.

To understand this concept, consider for example a solution $p \longmapsto \sum_a \Phi_a(p)|a\rangle$, where the $|a\rangle$ shall form some basis of R. Because of the Klein-Gordon equation, $\Phi_a(p)$ vanishes for all p with $p^2 \neq -m^2$. These solutions shall be interpreted as momentum space representation of some abstract vector space. On this space one can use (2.3.17) as definition of $U(\Lambda)$, where Λ is now allowed to be any homogeneous Lorentz transformation. The parameter A then clearly labels a basis of the subspace that is spanned by the $\sum_a \Phi_a(\hat{k})|a\rangle$, which carries the irreducible representation of the little group from step *i.)*. This ensures that this representation is in fact irreducible.

Still representations of the (short) little and Lorentz group need to be found. There is a very general formalism that allows to explicitly construct such representations as subspaces of representations of the *general linear group* $GL(4)$. The method uses Young diagrams to find tensorial irreducible representations of $GL(4)$ with certain index symmetries. This is basically a generalization of the decomposition of $V \otimes V$ into the spaces of symmetric and antisymmetric tensors, which carry irreducible representations of $GL(4)$ if the vector space V carries such a representation. The general formulation of this method can be found in [1]. Here only the resulting equations for particles with integer spin s shall be presented.

Just as during the introduction of the (short) little groups, which is given above, one has to treat the massive and the massless case separately.

The representations that are identified with massive spin-s particles are totally symmetric traceless tensor fields $\Phi_{\mu_1,\ldots,\mu_s}$ (i.e. any contraction of two indices vanishes), which satisfy the Klein-Gordon equation

$$(p^2 + m^2)\Phi_{\mu_1,\ldots,\mu_s}(p) = 0 \tag{2.3.19}$$

and furthermore the transversality condition

$$p^{\mu_1}\Phi_{\mu_1,\ldots,\mu_s}(p) = 0. \tag{2.3.20}$$

The massive spin-s field carries an irreducible representation of $O(3)$. There is an explicit formula for the dimension of irreducible, symmetric tensorial representations $V_s^{O(D)}$ with rank s of any $O(D)$ (see [1]):

$$dim(V_s^{O(D)}) = \frac{(D+2s-2)(D+s-3)!}{s!(D-2)!} \tag{2.3.21}$$

Therefore, massive spin-s particles have dimension $2s+1$.

The massless case can be described by using the so-called *field strength tensor* $\mathcal{K}_{\mu_1\nu_1|\ldots|\mu_s\nu_s}$, which has to satisfy the following conditions:

i.) $\mathcal{K}_{\mu_1\nu_1|\ldots|\mu_s\nu_s}$ is antisymmetric in every pair of indices (μ_i, ν_i).

ii.) The antisymmetrization of $\mathcal{K}_{\mu_1\nu_1|\ldots|\mu_s\nu_s}$ in every set of three indices vanishes.

iii.) Any contraction of two indices in $\mathcal{K}_{\mu_1\nu_1|\ldots|\mu_s\nu_s}$ vanishes.

iv.) The equations

$$p_\rho\mathcal{K}_{\mu_1\nu_1|\ldots|\mu_i\nu_i|\ldots|\mu_s\nu_s}(p) + p_{\mu_i}\mathcal{K}_{\mu_1\nu_1|\ldots|\nu_i\rho|\ldots|\mu_s\nu_s}(p) + p_{\nu_i}\mathcal{K}_{\mu_1\nu_1|\ldots|\rho\mu_i|\ldots|\mu_s\nu_s}(p) = 0 \tag{2.3.22}$$

$$p^{\mu}\mathcal{K}_{\mu_1\nu_1|...|\mu\nu_i|...|\mu_s\nu_s}(p) = 0 \tag{2.3.23}$$

hold for all $i = 1, ..., s$.

The fourth condition obviously implies

$$p^2\mathcal{K}_{\mu_1\nu_1|...|\mu_s\nu_s}(p) = 0, \tag{2.3.24}$$

i.e. the massless Klein-Gordon equation. One way to construct such a field strength is by introducing a totally symmetric gauge field $\phi_{\nu_1...\nu_s}$ and writing $\mathcal{K}_{\mu_1\nu_1|...|\mu_s\nu_s}(p)$ as antisymmetrization of $p_{\mu_1}...p_{\mu_s}\phi_{\nu_1...\nu_s}(p)$ with respect to exchanging μ_i and ν_i, i.e.

$$\begin{aligned}\mathcal{K}_{\mu_1\nu_1|...|\mu_s\nu_s}(p) =&p_{\mu_1}p_{\mu_2}...p_{\mu_s}\phi_{\nu_1\nu_2...\nu_s}(p) - p_{\nu_1}p_{\mu_2}...p_{\mu_s}\phi_{\mu_1\nu_2...\nu_s}(p)\\ &+ p_{\nu_1}p_{\nu_2}p_{\mu_3}...p_{\mu_s}\phi_{\mu_1\mu_2\nu_3...\nu_s}(p) - ... - p_{\mu_1}...p_{\mu_{s-1}}p_{\nu_s}\phi_{\nu_1...\nu_{s-1}\mu_s}(p)\\ =& - \mathcal{K}_{\mu_1\nu_1|...|\nu_i\mu_i|...|\mu_s\nu_s}(p),\end{aligned} \tag{2.3.25}$$

for all $i = 1, ..., s$. This ensures that *i.)* and *ii.)* are always fulfilled. Clearly, there is more than one configuration of the gauge field that produces the same field strength. So not every configuration corresponds to a different physical state. One can show (see [1]) that this strategy in fact delivers two physical degrees of freedom for any s.

It is quite easy to see that this construction leads to $\mathcal{K}_{\mu\nu}(p) = p_\mu\phi_\nu(p) - p_\nu\phi_\mu(p)$ for $s = 1$, i.e. the electromagnetic field tensor. Condition *iv.)* then gives the well-known Euler-Lagrange equations for the electromagnetic 4-potential. For $s = 2$ the only nontrivial equation that follows from *iii.)* is

$$\mathcal{K}^{\mu}{}_{\nu_1|\mu\nu_2}(p) = p_\mu p^\mu\phi_{\nu_1\nu_2}(p) - p_{\nu_1}p^\mu\phi_{\mu\nu_2}(p) - p_{\nu_2}p^\mu\phi_{\nu_1\mu}(p) + p_{\nu_1}p_{\nu_2}\phi^{\mu}{}_{\mu}(p) = 0. \tag{2.3.26}$$

Furthermore, it is easy to check that this already guarantees that *iv.)* is fulfilled. Therefore, massless spin-2 particles can be characterized by a symmetric gauge field that satisfies (2.3.26). By switching to position space via a Fourier transformation, they alternatively can be described as a symmetric tensor field $h_{\mu\nu}$ that solves

$$\Box h_{\mu\nu} - \partial^\lambda\partial_\mu h_{\lambda\nu} - \partial^\lambda\partial_\nu h_{\mu\lambda} + \partial_\mu\partial_\nu h^{\lambda}{}_{\lambda} = 0. \tag{2.3.27}$$

2.4 The BRST Formalism

This section focuses on the introduction of the *BRST formalism*, which is the center piece of the subject of this thesis. It is a very rigorous method to treat gauge theories quantum mechanically and offers a strategy to deal with the subtleties of quantizing such in an algebraically quite illuminating way.

2.4.1 Subtleties of Quantizing Gauge Theories

In order to grasp the value of the BRST formalism, one has to understand the problems that occur during the quantization procedure of gauge theories. Therefore, a short review of the quantization of the $U(1)$ gauge theory, which is used to model photons, is given as an example. It is taken from [3], where the subject is treated in a far more detailed way. The main focus of the following presentation lies on the subtleties of the quantization.

Take the classical action for the $U(1)$ gauge field A_μ

$$S[A_\mu] = \int d^4x \mathcal{L} = \int d^4x \Big\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Big\}, \tag{2.4.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This action offers the well-known invariance under the gauge transformations

$$A_\mu \longmapsto A_\mu + \partial_\mu \Lambda \tag{2.4.2}$$

of A_μ. Λ is supposed to be an arbitrary function that falls off sufficiently fast at infinity. In order to perform the procedure of canonical quantization for this theory, one has to introduce the conjugate momenta

$$\Sigma^\mu = \frac{\delta_L L}{\delta \dot{A}_\mu}, \tag{2.4.3}$$

that allow to formulate the canonical commutation relations

$$[A_\mu(x), \Sigma_\nu(y)]_-\big|_{x^0=y^0} = i g_{\mu\nu} \delta(\mathbf{x} - \mathbf{y}). \tag{2.4.4}$$

They offer a way to turn the functions from the classical theory into corresponding operators for a quantum theory (see [3] or [9]). But already at this point there is a problem. It turns out that $\Sigma^0 = 0$ holds. So there is no way to introduce the canonical commutation relation $[A_0(x), \Sigma_0(y)]_-\big|_{x^0=y^0} = -i\delta(\mathbf{x} - \mathbf{y})$. The usual way to avoid this dilemma is to introduce a gauge fixing, which shall be set to

$$\mathcal{G} = \partial_\mu A^\mu \tag{2.4.5}$$

in this little example. By performing a gauge fixing, $S[A_\mu]$ can be replaced by the gauge fixed action

$$S_{GF}[A_\mu, \bar{\zeta}, \zeta] = S[A_\mu] + \int d^4x \Big\{ -\frac{1}{2\alpha} \mathcal{G}^2 + \bar{\zeta} \Box \zeta \Big\}, \tag{2.4.6}$$

where $\bar{\zeta}$ and ζ is the Faddeev-Popov anti-ghost and ghost corresponding to $\mathcal{G}$, respectively. The way from S to S_{GF} is a well-known procedure and shall not be presented here in detail. A good reference, which even deals with the more general case of non-abelian gauge fields, is [17]. In order to keep the equations as simple as possible, the parameter α shall be set to 1 in this section. This choice is usually referred to as *Feynman gauge* (see [12]) and leads to

$$S_{GF}[A_\mu, \bar{\zeta}, \zeta] = \int d^4x \Big\{ -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \bar{\zeta} \Box \zeta \Big\} \tag{2.4.7}$$

and therefore to the gauge fixed Lagrangian

$$\mathcal{L}_{GF} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu \bar{\zeta} \partial^\mu \zeta. \tag{2.4.8}$$

Now the introduction of canonical momenta is possible. One finds

$$\Sigma^\mu = \dot{A}^\mu. \tag{2.4.9}$$

However, this does not solve all the problems that occur during the quantization process. Clearly, A_μ can not describe an independent physical degree of freedom for each μ, since that would imply a total number of four degrees of freedom. But a massless spin-1 particle only has two. One degree of freedom can be eliminated by only working with states $|\Psi\rangle$ that satisfy the so-called *Gupta-Bleuler supplementary condition* (see [3])

$$\partial^\mu A_\mu^{(+)}|\Psi\rangle = 0, \tag{2.4.10}$$

where

$$A_\mu^{(+)} = \int \widetilde{dp} \sum_{r=1}^{4} \epsilon_\mu^r(\mathbf{p}) a_{\mathbf{p}}^r e^{ipx}. \tag{2.4.11}$$

Here the $\epsilon_\mu^r(\mathbf{p})$ denote a basis of polarization vectors and $a_{\mathbf{p}}^r$ the corresponding annihilation operators (see Section 4.1.2 for more details). So instead of working with the full vector space $\mathcal{V}$ of the theory, one has to look at the subspace $\mathcal{V}'$ that contains only vectors that satisfy (2.4.10) and no ghosts, since they are unphysical. In order for this method to work, one has to ensure that $\mathcal{V}'$ is invariant under time evolution.

Another obstacle is the fact that $\mathcal{V}$ turns out to contain elements $|\omega\rangle$ with non-positive norm, i.e. $\langle\omega|\omega\rangle \leq 0$. But this is quite a dilemma, since that makes the probability interpretation of the inner product impossible, not to mention that $\mathcal{V}$ is not even a Hilbert space. Vectors with $\langle\omega|\omega\rangle = 0$ but $|\omega\rangle \neq 0$ also show up in $\mathcal{V}'$. So on $\mathcal{V}'$ the probability interpretation works, but this space is still no Hilbert space. However, these critical elements turn out to be orthogonal to any other state in $\mathcal{V}'$ and therefore can be modded out by introducing equivalence classes $[|\Psi\rangle]$ with

$$|\Psi'\rangle \sim |\Psi\rangle \quad \Leftrightarrow \quad \langle\Psi''|\Big(|\Psi\rangle - |\Psi'\rangle\Big) = 0 \tag{2.4.12}$$

for any $|\Psi\rangle, |\Psi'\rangle, |\Psi''\rangle \in \mathcal{V}'$. The completion of the space of these equivalence classes is identified with the space that describes the actual physical polarizations, i.e. the wanted Hilbert space. It comes by construction with the desired positive definite inner product.

The most important result of this example is the fact that it is necessary to deal with unphysical states in order to quantize the gauge field appropriately. But this does not seem to be too surprising after all, since even in the classical case A_μ is not the physical quantity, but $F_{\mu\nu}$ is. So the problems seem to originate from the fact that there are transformations, namely the gauge transformations, that change the A_μ but not the physical system. This of course is in general the case for any gauge field. Therefore, one would expect that it is never so easy to quantize a gauge field appropriately and that several manual adjustments and reformulations of the definition of the actual physical space need to be made, similar to the case presented here. So a general formalism that allows to treat all these subtleties in a systematic approach would be very desirable. This is what the BRST formalism does.

2.4.2 The BRST Transformations for Gauge Fields

It is based on a very well hidden additional invariance of Lagrangians that describe gauge theories. To motivate the following more general procedure, consider once more the action (2.4.7). By introducing a constant Graßmann number θ, one can formulate a new set of transformations for the fields:

$$A_\mu \longmapsto A_\mu + \delta_\theta A_\mu = A_\mu + \theta\partial_\mu\zeta, \quad \bar{\zeta} \longmapsto \bar{\zeta} + \delta_\theta\bar{\zeta} = \bar{\zeta} + \theta\partial_\mu A^\mu, \quad \zeta \longmapsto \zeta + \delta_\theta\zeta = \zeta. \tag{2.4.13}$$

By using $\theta^2 = 0$, one can easily show that S_{GF} is invariant under this transformation. The existence of such an additional invariance is the key to formulating a consistent quantization for gauge fields. It is possible to generalize this so-called *BRST transformation* δ_θ for any gauge theory.

The formulation and motivation of the BRST transformation and quantization, as presented in the following subsections, is in most instances taken from [17], where the subject is treated in a more general way. However, it also contains some elements of [9] and [10]. Furthermore, some properties of the formalism, like the proofs of Theorem 2.4.1 and Theorem 2.4.2 are discussed more carefully than it is done in the literature.

In order to quantize massive spin-2 particles, it is necessary to work with fields that are invariant under several independent gauge transformations. So this has to be considered in an appropriate introduction of the formalism as well.

The generalization of the BRST invariance from the $U(1)$ case requires several modifications of the action that is supposed to describe the fields of interest. It is possible to formulate a classical version of the BRST formalism by using constraints (see for example [10]), but since only the quantum formulation of the BRST transformations is relevant for this work, this approach shall be omitted. Instead, the generating functional is adapted in such a way that the BRST invariance can be exploited to derive an appropriate quantization.

To do so, consider a set of gauge fields ϕ_a that get their dynamics from an action $S[\phi_a]$. This action shall be invariant under infinitesimal gauge transformations of the form

$$\phi_a \longmapsto \phi_a + \xi^M \delta_M \phi_a. \tag{2.4.14}$$

Here the *De Witt notation* (see [17]) is used: The indices a and M are allowed to include spacetime coordinates and also discrete labels. Over equal upper and lower indices shall be summarized, by convention. In the case of spacetime coordinates the sum has to be an integral. For example, in this notation the gauge transformation of A_μ can be written as

$$A_{\mu x} \longmapsto A_{\mu x} + \Lambda^y \delta_y A_{\mu x}, \tag{2.4.15}$$

with

$$\delta_y A_{\mu x} = -\frac{\partial}{\partial y^\mu}\delta(x-y), \quad A_{\mu x} = A_\mu(x) \quad \text{and} \quad \Lambda^y = \Lambda(y). \tag{2.4.16}$$

This results in

$$\Lambda^y \delta_y A_{\mu x} = -\int d^4y \Lambda(y) \frac{\partial}{\partial y^\mu}\delta(x-y) = \int d^4y \frac{\partial}{\partial y^\mu}\Lambda(y)\delta(x-y) = \partial_\mu \Lambda(x) \tag{2.4.17}$$

and therefore gives the gauge transformation, as it is presented in (2.4.2).

So ξ^M resembles the parametrization of the gauge transformation, while $\delta_M \phi_a$ determines how ξ^M is used to transform ϕ_a. The index M in particular contains discrete indices that label the sets of gauge transformations that are considered. Note that $\xi^M \delta_M \phi_a$ in particular is assumed to be linear in ξ^M.

The operator δ_M can be introduced for arbitrary functionals $F[\phi_a]$, simply by setting

$$\xi^M \delta_M F[\phi_a] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F[\phi_a + \epsilon \xi^M \delta_M \phi_a] = \xi^M \delta_M \phi_b \frac{\delta_L}{\delta \phi_b} F[\phi_a]. \tag{2.4.18}$$

As usual, gauge fixing functionals $\mathcal{G}_M[\phi_a]$, i.e. functions which are allowed to depend on ϕ_a and its derivatives of higher order, can be introduced. They restrict the integration over all field configurations in the generating functional to the ones that satisfy $\mathcal{G}_M[\phi_a] = \chi_M$, where χ_M is a given auxiliary field. This can be obtained by replacing the generating functional

$$Z[j^a] = \int [d\phi_a] e^{iS[\phi_a] - i\phi_a j^a}, \tag{2.4.19}$$

where $[d\phi_a] = \prod_a d\phi_a$ denotes the path integral measure, by the functional

$$Z'[j^a, \chi_M] = \int [d\phi_a] \delta(\mathcal{G}_M - \chi_M) \det(\delta_N \mathcal{G}_M) e^{iS[\phi_a] - i\phi_a j^a}. \tag{2.4.20}$$

The fields χ_M can be integrated out in the following way to obtain the gauge fixed generating functional[4]

$$\begin{aligned} Z_{GF}[j^a] &= \int [d\chi_M] e^{-\frac{i}{2\alpha}\chi^M \chi_M} Z'(j^a, \chi_M) \\ &= \int [d\phi_a] \det(\delta_N \mathcal{G}_M) e^{-\frac{i}{2\alpha}\mathcal{G}^M \mathcal{G}_M} e^{iS[\phi_a] - i\phi_a j^a}, \end{aligned} \tag{2.4.21}$$

where α is some arbitrary finite constant. Note that $\chi^M \chi_M$ usually is supposed to resemble the sum (integral) of $\chi_M \chi_M$ over M. However, it is possible that M also contains the indices of a 4-vector, i.e. μ, or the like. In this case, in order to formulate a Lorentz-invariant quantity, the raising of the corresponding part of the index M is done by contracting with the metric tensor, as usual. All the following cases where a index is raised or lowered shall be treated in the same way.

By rewriting $e^{-\frac{i}{2\alpha}\mathcal{G}^M \mathcal{G}_M}$ as Fourier integral[5] (see [17])

$$e^{-\frac{i}{2\alpha}\mathcal{G}^M \mathcal{G}_M} \propto \int [dB^M] e^{i\frac{\alpha}{2} B^M B_M + iB^M \mathcal{G}_M} \tag{2.4.22}$$

and $\det(\delta_N \mathcal{G}_M)$ as integral over Graßmann fields (see [17]), i.e. Faddeev-Popov anti-ghosts and ghosts

$$\det(\delta_N \mathcal{G}_M) \propto \int [d\bar{\eta}^M][d\eta^M] e^{i\bar{\eta}^M \eta^N \delta_N \mathcal{G}_M}, \tag{2.4.23}$$

one finds a generating functional that allows to formulate suitable BRST transformations:

$$Z_{BRST}[j^a] = \int [d\phi_a][d\bar{\eta}^M][d\eta^M][dB^M] e^{iS_{BRST}[\phi_a, B^M, \bar{\eta}^M, \eta^M] - i\phi_a j^a}, \tag{2.4.24}$$

with the new action

$$S_{BRST}[\phi_a, B^M, \bar{\eta}^M, \eta^M] = S[\phi_a] + B^M \mathcal{G}_M + \frac{\alpha}{2} B^M B_M + \bar{\eta}^M \eta^N \delta_N \mathcal{G}_M. \tag{2.4.25}$$

The auxiliary fields B^M are called *Nakanishi-Lautrup fields* (see [17]). Their role will become clear later on.

[4] This construction of $Z_{GF}[j^a]$ from $Z[j^a]$ is a straightforward generalization of the case treated in [10], where ϕ_a is a 4-vector.

[5] The proportionality symbol $\propto$, which is used in the following relations, shall indicate that the corresponding objects differ only by field-independent multiplicative factors.

Now consider for an arbitrary functional $F[\phi_a, B^M, \bar{\eta}^M, \eta^M]$ the transformation

$$\begin{aligned} F[\phi_a, B^M, \bar{\eta}^M, \eta^M] \longmapsto & F[\phi_a + \delta_\theta\phi_a, B^M + \delta_\theta B^M, \bar{\eta}^M + \delta_\theta\bar{\eta}^M, \eta^M + \delta_\theta\eta^M] \\ =& F[\phi_a, B^M, \bar{\eta}^M, \eta^M] + \delta_\theta F[\phi_a, B^M, \bar{\eta}^M, \eta^M], \end{aligned} \tag{2.4.26}$$

with

$$\delta_\theta\phi_a = \theta\eta^M\delta_M\phi_a, \tag{2.4.27}$$

$$\delta_\theta\bar{\eta}^M = -\theta B^M, \tag{2.4.28}$$

$$\delta_\theta\eta^M = -\frac{1}{2}\theta f^M{}_{NR}\eta^N\eta^R, \tag{2.4.29}$$

$$\delta_\theta B^M = 0, \tag{2.4.30}$$

and the structure constants $f^M{}_{NR}$, defined by

$$[\delta_N, \delta_R]_- = f^M{}_{NR}\delta_M. \tag{2.4.31}$$

The mapping $F \longmapsto \delta_\theta F$ is called *BRST transformation*.

The action S_{BRST} is invariant under the transformation from above, i.e. $\delta_\theta S_{BRST} = 0$. To prove this, first note that $\delta_\theta\phi_a$ has just the same structure as a gauge transformation. So the invariance of $S[\phi_a]$ is trivial. Furthermore, the BRST transformation of a given functional $F[\phi_a, B^M, \bar{\eta}^M, \eta^M]$ always has the form of an infinitesimal transformation, i.e.

$$\begin{aligned} \delta_\theta F[\phi_a, B^M, \bar{\eta}^M, \eta^M] =& \theta\eta^M\delta_M F[\phi_a, B^M, \bar{\eta}^M, \eta^M] - \theta B^M \frac{\delta_L}{\delta\bar{\eta}^M} F[\phi_a, B^M, \bar{\eta}^M, \eta^M] \\ & - \frac{1}{2}\theta f^M{}_{NR}\eta^N\eta^R \frac{\delta_L}{\delta\eta^M} F[\phi_a, B^M, \bar{\eta}^M, \eta^M]. \end{aligned} \tag{2.4.32}$$

This is easy to see by writing $F[\phi_a, B^M, \bar{\eta}^M, \eta^M]$ as a power series and exploiting $\theta^2 = 0$. Therefore, one gets

$$\begin{aligned} \delta_\theta S_{BRST} =& B^M\delta_\theta\mathcal{G}_M + \delta_\theta\bar{\eta}^M\eta^N\delta_N\mathcal{G}_M + \bar{\eta}^M\delta_\theta\eta^N\delta_N\mathcal{G}_M + \bar{\eta}^M\eta^N\delta_\theta\delta_N\mathcal{G}_M \\ =& \theta B^M\eta^N\delta_N\mathcal{G}_M - \theta B^M\eta^N\delta_N\mathcal{G}_M + \frac{1}{2}\theta f^N{}_{RO}\bar{\eta}^M\eta^R\eta^O\delta_N\mathcal{G}_M \\ & + \theta\bar{\eta}^M\eta^N\eta^R\delta_R\delta_N\mathcal{G}_M \\ =& \theta\Big(\frac{1}{2}\bar{\eta}^M\eta^R\eta^O[\delta_R, \delta_O]_-\mathcal{G}_M + \bar{\eta}^M\eta^N\eta^R\delta_R\delta_N\mathcal{G}_M\Big) \\ =& \theta\Big(- \bar{\eta}^M\eta^R\eta^O\delta_O\delta_R\mathcal{G}_M + \bar{\eta}^M\eta^N\eta^R\delta_R\delta_N\mathcal{G}_M\Big) = 0, \end{aligned} \tag{2.4.33}$$

where $\theta^2 = 0$ and the anticommutativity of all ghost fields with θ and each other have been used.

The gauge transformations that are used in this thesis commute with each other, i.e. $[\delta_M, \delta_N]_- = 0$. Therefore, the structure constants will be set to zero from now on.

Remark Before the discussion of the BRST formalism is continued, a warning needs to be given. During the derivation of Z_{BRST}, a gauge fixing $\mathcal{G}_M$ was introduced for each gauge transformation δ_M. In order for this method to work, it is necessary to ensure that all the gauge transformations δ_M are independent from each other. For example, one could introduce a second gauge transformation

$$\delta'_y = \delta_y \tag{2.4.34}$$

for the $U(1)$ gauge field A_μ. Clearly, the two operators δ_y and δ'_y correspond to the same transformation, so fixing the gauge for both independently can not be correct. There are more subtle cases, where this problem of gauge fixings that are dependent from each other is not that obvious. The most prominent example occurs during the quantization of antisymmetric tensor fields. This is discussed in [3] and [17].

So in order to ensure that the presented procedure works, one has to check whether all the occurring gauge transformations are independent from each other, i.e.

$$\xi^M \delta_M = \xi'^M \delta_M \quad \Leftrightarrow \quad \xi^M = \xi'^M \tag{2.4.35}$$

or equivalently, by using the linearity of δ_M

$$\xi^M \delta_M = 0 \quad \Leftrightarrow \quad \xi^M = 0. \tag{2.4.36}$$

For spin-2 particles this turns out to be true, so the BRST procedure works without further difficulties.

2.4.3 The Slavnov Operator and BRST Charge

This formulation of BRST transformations differs from the one presented for the $U(1)$ gauge field, where no Nakanishi-Lautrup fields showed up. How these two formulations relate to each other is explained in Section 2.4.4. However, the more general BRST transformations now come with another operator, which relates to them in a quite natural way. This is the so-called *Slavnov operator* s, which is defined in [17] by

$$\delta_\theta = \theta\, \mathrm{s}\,. \tag{2.4.37}$$

By using the relation (2.4.32) with ${f^M}_{NL} = 0$, one can easily verify that

$$\mathrm{s} = \eta^M \delta_M \phi_a \frac{\delta_L}{\delta \phi_a} - B^M \frac{\delta_L}{\delta \bar{\eta}^M} \tag{2.4.38}$$

holds. This expression can be used to prove the following important theorem.

Theorem 2.4.1 *The Slavnov operator is nilpotent, i.e.* $\mathrm{s}^2 = 0$.

PROOF: This statement can be proven by a straightforward calculation:

$$
\begin{aligned}
\mathrm{s}^2 =&\Big(\eta^M\delta_M\phi_a\frac{\delta_L}{\delta\phi_a}-B^M\frac{\delta_L}{\delta\bar\eta^M}\Big)\Big(\eta^N\delta_N\phi_b\frac{\delta_L}{\delta\phi_b}-B^N\frac{\delta_L}{\delta\bar\eta^N}\Big)\\
=&\eta^M\eta^N\delta_M\phi_a\Big(\frac{\delta_L\delta_N\phi_b}{\delta\phi_a}+\delta_N\phi_b\frac{\delta_L}{\delta\phi_a}\Big)\frac{\delta_L}{\delta\phi_b}\\
&-\eta^M B^N\delta_M\phi_a\frac{\delta_L}{\delta\phi_a}\frac{\delta_L}{\delta\bar\eta^N}-B^M\delta_N\phi_b\frac{\delta_L}{\delta\bar\eta^M}\eta^N\frac{\delta_L}{\delta\phi_b}+B^MB^N\frac{\delta_L}{\delta\bar\eta^M}\frac{\delta_L}{\delta\bar\eta^N}\\
=&\eta^M\eta^N\delta_M\phi_a\frac{\delta_L\delta_N\phi_b}{\delta\phi_a}\frac{\delta_L}{\delta\phi_b}=\frac{1}{2}\eta^M\eta^N\Big(\delta_M\phi_a\frac{\delta_L\delta_N\phi_b}{\delta\phi_a}-\delta_N\phi_a\frac{\delta_L\delta_M\phi_b}{\delta\phi_a}\Big)\frac{\delta_L}{\delta\phi_b}\\
=&\frac{1}{2}\eta^M\eta^N\big([\delta_M,\delta_N]_-\phi_b\big)\frac{\delta_L}{\delta\phi_b}=0,
\end{aligned}
\tag{2.4.39}
$$

where the anticommutativity of the ghosts, which implies $\frac{\delta_L}{\delta\bar\eta^M}\eta^N=-\eta^N\frac{\delta_L}{\delta\bar\eta^M}$ and $\frac{\delta_L}{\delta\bar\eta^M}\frac{\delta_L}{\delta\bar\eta^N}=-\frac{\delta_L}{\delta\bar\eta^N}\frac{\delta_L}{\delta\bar\eta^M}$, has been exploited[6]. □

Clearly, S_{BRST} is also invariant under s. Furthermore, it offers a very elegant way to write S_{BRST}:

$$S_{BRST}=S-\mathrm{s}\,\big(\bar\eta^M\big(\mathcal{G}_M+\frac{\alpha}{2}B_M\big)\big). \tag{2.4.40}$$

In this form the invariance of S_{BRST} is trivial. It is just a consequence of the nilpotency of s and the gauge invariance of S.

The invariance under (2.4.26) comes with a conserved Noether current and therefore with a conserved charge Ω, the so-called *BRST charge*. At the level of operators this self-adjoint operator generates the BRST transformations via

$$i\delta_\theta\Phi=i\theta\,\mathrm{s}\,\Phi=\theta[\Omega,\Phi]_\pm, \tag{2.4.41}$$

i.e. $i\,\mathrm{s}=[\Omega,\cdot]_\pm$. Here the anticommutator is used for fermionic fields, while the commutator refers to bosonic ones. Φ stands for any of the operators ϕ_a, $\bar\eta^M$ and η^M.

It is sufficient for this work to assume that the Lagrangian $\mathcal{L}_{BRST}$, that corresponds to S_{BRST}, only depends on ϕ_a, $\bar\eta^M$, η^M and their first order derivatives and has a polynomial structure in them. To get an explicit expression for Ω, one has to find a $\tilde J^\lambda$ such that $\mathrm{s}\,\mathcal{L}_{BRST}=\partial_\lambda\tilde J^\lambda$ holds[7]. The Noether current that corresponds to δ_θ is given by

$$J^\lambda=\tilde J^\lambda-\sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_\lambda\Phi}. \tag{2.4.42}$$

Consequently, the BRST charge reads

$$\Omega=\int d^3\mathbf{x}J^0=\int d^3\mathbf{x}\Big\{\tilde J^0-\sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_0\Phi}\Big\}. \tag{2.4.43}$$

[6] In order to ensure that the Slavnov operator is nilpotent for non-vanishing structure constants, one needs to introduce an additional consistency condition for them. This is treated in [17].

[7] Note that (2.4.26), by construction, always has the form of an infinitesimal transformation, which is parametrized by the Graßmann variable θ. So the ansatz that is pursued here in order to find a Noether current is certainly correct. A derivation of the general form of Noether currents can be found in [12].

For simplicity, one can assume that $\tilde{J}$ also has a polynomial structure in the fields and their derivatives. Obviously, it is always possible to find such a $\tilde{J}^\lambda$.

Now let Ψ be an arbitrary fermionic or bosonic polynomial in ϕ_a, $\bar{\eta}^M$ and η^M and their derivatives. By using the fact that s turns a fermionic field into a bosonic one and vice versa, which follows directly from (2.4.27) to (2.4.30), it is easy to see that

$$0 = -\,\mathrm{s}^2\,\Psi = [\Omega, [\Omega, \Psi]_\pm]_\mp = [\Omega^2, \Psi]_- \qquad (2.4.44)$$

holds. Again, the commutator corresponds to bosonic and the anticommutator to fermionic polynomials. In order for this relation to be true for any Ψ, Ω^2 has to be proportional to the unit, i.e.

$$\Omega^2 = \alpha \mathbb{1}. \qquad (2.4.45)$$

However, it turns out that α has to be 0. This statement is subject of the next theorem.

Theorem 2.4.2 *The BRST charge is nilpotent, i.e. $\Omega^2 = 0$.*

PROOF: In order to proof this statement, the so-called *Faddeev-Popov ghost charge* Q_G has to be introduced[8]. It is defined via the following commutation relations:

$$[iQ_G, \bar{\eta}^M]_- = -\bar{\eta}^M, \quad [iQ_G, \phi_a]_- = [iQ_G, B^M]_- = 0 \quad \text{and} \quad [iQ_G, \eta^M]_- = \eta^M. \qquad (2.4.46)$$

The eigenvalues of iQ_G are called the *(Faddeev-Popov) ghost number* of the corresponding eigenvectors. By keeping (2.4.27) to (2.4.30) in mind, it is easy to understand that the BRST charge always raises the Faddeev-Popov ghost number by one, i.e. if Ψ_n is a fermionic or bosonic polynomial in the fields and their derivatives with $[iQ_G, \Psi_n]_- = n\Psi_n$, where $n \in \mathbb{Z}$, then

$$[iQ_G, [\Omega, \Psi_n]_\pm]_- = (n+1)[\Omega, \Psi_n]_\pm \qquad (2.4.47)$$

holds. The center piece of this proof is to show that the BRST charge has the Faddeev-Popov ghost number 1, i.e. $[iQ_G, \Omega]_- = \Omega$. To do so, first consider $\tilde{J}^\lambda$. It is possible to split $\tilde{J}^\lambda$ into two parts:

$$\tilde{J}^\lambda = \tilde{J}^\lambda_1 + R^\lambda, \qquad (2.4.48)$$

where $\tilde{J}^\lambda_1$ denotes the part of $\tilde{J}^\lambda$ with ghost number 1 and R^λ its remaining parts, i.e. the sum of all the monomials that appear in $\tilde{J}^\lambda$ and have a ghost number that is not one. Obviously, it is not possible to change the ghost number of a monomial by taking derivatives. So $\partial_\lambda R^\lambda$ also contains no monomials with ghost number 1. The Lagrangian $\mathcal{L}_{BRST}$ clearly has a vanishing ghost number. Consequently, $\mathrm{s}\,\mathcal{L}_{BRST} = -i[\Omega, \mathcal{L}_{BRST}]_-$ has the ghost number 1. It is quite obvious that polynomials with different ghost numbers can not cancel each other. Therefore, one obtains

$$\partial_\lambda R^\lambda = 0. \qquad (2.4.49)$$

Otherwise, $\mathrm{s}\,\mathcal{L}_{BRST} = \partial_\lambda \tilde{J}^\lambda$ would not have the ghost number 1. Consequently, one gets

$$\partial_0 R^0 = -\partial_k R^k \qquad (2.4.50)$$

and therefore

$$R^0(t, \mathbf{x}) = -\int_{-\infty}^{t} dt' \partial_k R^k(t', \mathbf{x}). \qquad (2.4.51)$$

[8] The concept of the ghost charge is taken from [9].

So only $\tilde{J}_1^0$ contributes to Ω:

$$\begin{aligned}\Omega &= \int d^3\mathbf{x}\Big\{\tilde{J}_1^0 - \int_{-\infty}^{t} dt'\partial_k R^k(t',\mathbf{x}) - \sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_0\Phi}\Big\}\\ &= \int d^3\mathbf{x}\Big\{\tilde{J}_1^0 - \sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_0\Phi}\Big\} - \int_{-\infty}^{t} dt'\int d^3\mathbf{x}\partial_k R^k(t',\mathbf{x}) \qquad (2.4.52)\\ &= \int d^3\mathbf{x}\Big\{\tilde{J}_1^0 - \sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_0\Phi}\Big\},\end{aligned}$$

since R^k vanishes at infinity. Now the only thing that needs to be shown is that $\sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_0\Phi}$ has the ghost number one. But this is quite trivial, since $\mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{BRST}}{\partial\partial_0\Phi}$ has ghost number 1 for any Φ, as can easily be verified. Therefore, Ω is the integral over a polynomial in the fields and their derivatives with ghost number 1. This clearly implies

$$[iQ_G,\Omega]_- = \Omega, \qquad (2.4.53)$$

and therefore, together with $\Omega^2 = \alpha\mathbb{1}$, gives

$$0 = [iQ_G,\Omega^2]_- = 2\Omega^2. \qquad (2.4.54)$$

So $\Omega^2 = 0$ has been derived. □

Remark The Faddeev-Popov ghost charge Q_G plays a crucial role in the algebraic structure of the BRST formalism. This is treated in [9] in more detail, from where the relation (2.4.53) has been taken.

2.4.4 The Physical Sector

Now, after introducing the BRST charge, it is finally possible to identify the states which are regarded to be physical. For this, consider $\mathcal{V}$ to be the overall vector space of the theory. Clearly, $\mathcal{V}$ contains states, such as the ones with non-vanishing ghost number, which have no physical meaning. So the physical states of the system must be contained in a subspace of $\mathcal{V}$, which clearly has to be invariant under time evolution. Furthermore, all the matrix elements $\langle\Psi|S_{BRST}|\Psi'\rangle$ for two physical states $|\Psi\rangle$ and $|\Psi'\rangle$ have to be independent of the special choice of the gauge fixing $\mathcal{G}_M$, in order for this formalism to make sense. In other words, if $\mathcal{G}'_M$ is another gauge fixing, one must find

$$\langle\Psi|\big(S - \mathrm{s}\,\big(\bar{\eta}^M\big(\mathcal{G}_M + \frac{\alpha}{2}\bar{\eta}^M B_M\big)\big)\big)|\Psi'\rangle = \langle\Psi|\big(S - \mathrm{s}\,\big(\bar{\eta}^M\big(\mathcal{G}'_M + \frac{\alpha}{2}\bar{\eta}^M B_M\big)\big)\big)|\Psi'\rangle. \qquad (2.4.55)$$

This implies

$$0 = \langle\Psi|\,\mathrm{s}\,\big(\bar{\eta}^M\big(\mathcal{G}_M - \mathcal{G}'_M\big)\big)|\Psi'\rangle = -i\langle\Psi|[\Omega,\bar{\eta}^M\big(\mathcal{G}_M - \mathcal{G}'_M\big)]_-|\Psi'\rangle. \qquad (2.4.56)$$

In order for this to be true for any $\mathcal{G}$ and $\mathcal{G}'$, the relation (see [17])

$$\langle\Psi|\Omega = \Omega|\Psi'\rangle = 0 \qquad (2.4.57)$$

is necessary for all physical states. Exploiting the hermiticity of Ω, it follows that it is sufficient to demand that $|\Psi\rangle$ and $|\Psi'\rangle$ are contained in the kernel of Ω, i.e. $|\Psi\rangle, |\Psi'\rangle \in \ker\Omega$. So the physical sector can be identified with $\ker\Omega$, which is of course invariant under time evolution, since Ω is a conserved charge, i.e. commutes with the Hamiltonian operator. Furthermore, the vacuum state $|0\rangle$ clearly is physical and therefore must be contained in $\ker\Omega$. However, this can not be the desired vector space with the positive definite inner product that gives the usual description of physical states. There are at least states with vanishing norm included, namely the elements of the image of Ω, $\operatorname{im}\Omega$. This follows directly from the nilpotency of Ω. To see this, let $|\varsigma\rangle \in \mathcal{V}$, then $\Omega|\varsigma\rangle \in \ker\Omega$ since $\Omega^2|\varsigma\rangle = 0$ and furthermore

$$\langle\varsigma|\Omega^\dagger\Omega|\varsigma\rangle = \langle\varsigma|\Omega^2|\varsigma\rangle = 0. \tag{2.4.58}$$

But this result is just a special case of the more general relation

$$\langle\Psi|\Omega|\varsigma\rangle = 0 \tag{2.4.59}$$

for any $|\Psi\rangle \in \ker\Omega$, which means

$$\langle\Psi|\omega\rangle = 0 \tag{2.4.60}$$

holds for any $|\omega\rangle \in \operatorname{im}\Omega$. So a state $|\Psi\rangle + |\omega\rangle$ has the same matrix element with any other state in $\ker\Omega$ as $|\Psi\rangle$. Consequently, the two are physically indistinguishable, i.e. equivalent. Therefore, a good candidate for the actual physical space, which includes only distinguishable states, would be the so-called *cohomology of* Ω

$$\mathfrak{V} = \ker\Omega/\operatorname{im}\Omega. \tag{2.4.61}$$

The discussion above ensures that the inner product

$$\langle[|\Psi\rangle]|\,[|\Psi'\rangle]\rangle = \langle\Psi|\,\Psi'\rangle \tag{2.4.62}$$

of two equivalence classes $[|\Psi\rangle]$ and $[|\Psi'\rangle]$ in $\mathfrak{V}$, with representatives $|\Psi\rangle$, $|\Psi'\rangle \in \ker\Omega$, is well-defined.

The question, if this inner product in fact turns $\mathfrak{V}$ into a Hilbert space, is a nontrivial problem. Note that the most important part is to ensure that the inner product is positive definite, since then one can always consider the completion

$$\mathfrak{H} = \overline{\ker\Omega/\operatorname{im}\Omega} \tag{2.4.63}$$

as the actual physical Hilbert space. A general discussion of the positive definiteness can be found in [8] and [9]. The fact that all elements of $\operatorname{im}\Omega$ are equivalent to zero clearly is a step in the right direction. The positive definiteness will be considered to be fulfilled for the rest of this chapter. For the gauge theories that are discussed in this thesis it will be proven explicitly.

An essential property, which has to be fulfilled in order to obtain the positive definiteness, is that the only elements in $\ker\Omega$ with non-positive norm are the ones of $\operatorname{im}\Omega$. Otherwise, there would be elements in $\mathfrak{V}$ which are unequal to zero but have no positive norm, which would contradict the desired positive definiteness of the cohomology.

In order to work with creation and annihilation operators instead of the abstract cohomology of Ω, it is helpful to identify a subspace $\mathcal{V}_{phys} \subset \ker\Omega$ that is isometric isomorphic to the cohomology. If one is able to find creation operators $a_{\mathbf{p}}^{1\dagger}, ..., a_{\mathbf{p}}^{m\dagger}$, for example operators that correspond to different polarizations of the discussed particles, that (anti)commute with Ω, but can not be written as (anti)commutators of Ω with some other operators, the space generated by these creation operators is a good candidate for such a $\mathcal{V}_{phys}$. Clearly, it is contained in $\ker\Omega$, since all the basis vectors $a_{\mathbf{p}_{l_1}}^{l_1\dagger}...a_{\mathbf{p}_{l_k}}^{l_k\dagger}|0\rangle$ are contained in $\ker\Omega$:

$$\Omega a_{\mathbf{p}_{l_1}}^{l_1\dagger}...a_{\mathbf{p}_{l_k}}^{l_k\dagger}|0\rangle = \pm a_{\mathbf{p}_{l_1}}^{l_1\dagger}...a_{\mathbf{p}_{l_k}}^{l_k\dagger}\Omega|0\rangle = 0, \tag{2.4.64}$$

where $|0\rangle \in \ker\Omega$ has been exploited. Furthermore, an operator $b_{\mathbf{p}}^{\dagger}$ that can be written as (anti)commutator of Ω and another operator $\hat{a}_{\mathbf{p}}^{\dagger}$ can not be a generator for $\mathcal{V}_{phys}$, because this would mean that the element

$$b_{\mathbf{p}}^{\dagger}|0\rangle = [\Omega, \hat{a}_{\mathbf{p}}^{\dagger}]_{\pm}|0\rangle = \Omega\hat{a}_{\mathbf{p}}^{\dagger}|0\rangle \in \operatorname{im}\Omega \tag{2.4.65}$$

of $\mathcal{V}_{phys}$ has zero norm and is not zero itself. The assumption that $\mathfrak{V}$ has a positive inner product therefore does not allow the existence of such a vector in $\mathcal{V}_{phys}$. In fact this strategy to find $\mathcal{V}_{phys}$ works for the cases presented in this thesis.

Remark If one is only interested in the exterior fields, which is for example sufficient for a non-interacting theory, the auxiliary fields B^M can be integrated out, i.e. the Euler-Lagrange equations for B^M, which are

$$B^M = -\frac{1}{\alpha}\mathcal{G}^M, \tag{2.4.66}$$

can be applied to S_{BRST} to obtain the usual gauge fixed action

$$S_{GF} = S - \frac{1}{2\alpha}\mathcal{G}^M\mathcal{G}_M + \bar{\eta}^M\eta^N\delta_N\mathcal{G}_M. \tag{2.4.67}$$

The BRST transformation of $\bar{\eta}^M$ then becomes

$$\delta_\theta\bar{\eta}^M = \frac{1}{\alpha}\theta\mathcal{G}^M. \tag{2.4.68}$$

This gives just the transformations from the beginning for the $U(1)$ case[9]. S_{GF} is then invariant under the resulting transformation that corresponds to δ_θ, but in order to ensure the nilpotency of s, one has to use the Euler-Lagrange equations for $\bar{\eta}^M$, i.e. $\frac{\delta_L S_{GF}}{\delta\bar{\eta}^N} = 0$. This follows from the fact that s becomes

$$\mathrm{s} = \eta^M\delta_M\phi_a\frac{\delta_L}{\delta\phi_a} + \frac{1}{\alpha}\mathcal{G}^M\frac{\delta_L}{\delta\bar{\eta}^M} \tag{2.4.69}$$

in this case. By performing the same calculation as in the proof of Theorem 2.4.1, this then leads to

$$\begin{aligned}\mathrm{s}^2 &= \frac{1}{\alpha}\eta^M\delta_M\phi_a\frac{\delta_L\mathcal{G}^N}{\delta\phi_a}\frac{\delta_L}{\delta\bar{\eta}^N} = \frac{1}{\alpha}\eta^M\delta_M\mathcal{G}^N\frac{\delta_L}{\delta\bar{\eta}^N} = \frac{1}{\alpha}\frac{\delta_L}{\delta\bar{\eta}^N}\left(\bar{\eta}^R\eta^M\delta_M\mathcal{G}_R\right)\frac{\delta_L}{\delta\bar{\eta}_N} \\ &= \frac{1}{\alpha}\frac{\delta_L S_{GF}}{\delta\bar{\eta}^N}\frac{\delta_L}{\delta\bar{\eta}_N}.\end{aligned} \tag{2.4.70}$$

[9] The BRST formalism for $U(1)$ is presented in [17] by using this strategy.

The application of Euler-Lagrange equations is of course no restriction for the exterior fields. Under this additional requirement, the nilpotency of Ω and all its consequences are also guaranteed, since the only properties necessary to prove $\Omega^2 = 0$ are the nilpotency of s and the fact that Ω raises the ghost number by one, which of course also remains valid. This procedure is applied for the cases treated in this work, since only non-interacting fields are reviewed.

Furthermore, this leads to[10]

$$\langle \Psi | \mathcal{G}^M | \Psi' \rangle = -\alpha i \langle \Psi | [\Omega, \bar{\eta}^M]_+ | \Psi' \rangle = 0, \tag{2.4.71}$$

for any $|\Psi\rangle, |\Psi'\rangle \in \ker \Omega$. This is the operator version of the classical gauge fixing condition $\mathcal{G}_M = 0$. For the example, discussed at the beginning of the chapter, this is basically the way how the Gupta-Bleuler condition shows up in the BRST formalism. How the actual condition (2.4.10) comes into place will be discussed later on in the context of massless spin-2 particles. It is only mentioned here to clarify that the BRST formalism is in fact a more rigorous version of the manual quantization approach that has been performed at the beginning of this section.

2.4.5 Operators in the BRST Formalism

The last sections derived a very elegant formulation of physical states for gauge theories. They are elements of the abstract space $\mathfrak{H}$, which is the completion of the cohomology $\mathfrak{V}$ of a given nilpotent and hermitian operator. This operator is defined on a vector space $\mathcal{V}$ with an inner product which is neither positive nor definite. This construction of $\mathfrak{H}$ is known as *BRST quantization*. In order to learn more about the implications of this picture, the examination of operators for the BRST formalism is a good point to start with.

To formulate operators on $\mathfrak{V}$ that then can be extended to operators on $\mathfrak{H}$, the most intuitive way is to look at $\mathcal{V}$ first. In particular, the field operators Φ_a, that shall resemble any of the fields ϕ_a, $\bar{\eta}^M$ and η^M, are defined on this space. They can be used to formulate a special type of operators, so-called *smeared field operators*. These are the elements of the polynomial algebra $\mathcal{F}$ generated by operators of the form

$$\Phi_{i_1 x_1} ... \Phi_{i_n x_n} f^{x_1 ... x_n} \tag{2.4.72}$$

with a test function f that decreases sufficiently fast. Here the index a of Φ_a is explicitly expressed as multi-index ix with a spacetime coordinate x and an additional index i from some discrete index set. Let $\mathcal{O}$ be a given bounded and open subset of Minkowski space. The subalgebra of $\mathcal{F}$ that is generated by monomials of the form (2.4.72) with test functions that have compact support on $\overbrace{\mathcal{O} \times ... \times \mathcal{O}}^{n}$ is denoted by $\mathcal{F}(\mathcal{O})$. Its elements are called *smeared local field operators*.

In this context the definition of BRST transformations for smeared field operators can be motivated. For this let R^B be such an operator that consists only of bosonic monomials of the form (2.4.72). Then its BRST transformation is defined as

$$\delta_\theta R^B = \theta [i\Omega, R^B]_- . \tag{2.4.73}$$

[10]The equation (2.4.71) is motivated from [9], where similar relations are presented.

In an analogous way, the BRST transformation of an operator R^F that consists only of fermionic monomials is defined by

$$\delta_\theta R^F = \theta[i\Omega, R^F]_+. \tag{2.4.74}$$

An arbitrary smeared operator R can always be decomposed into bosonic and fermionic parts, i.e. $R = R^B + R^F$. Its BRST transformation is then given by

$$i\delta_\theta R = \theta[\Omega, R]_\pm = \theta[\Omega, R^B]_- + \theta[\Omega, R^F]_+, \tag{2.4.75}$$

where $[\cdot,\cdot]_\pm$ shall indicate to take a commutator for the bosonic and an anticommutator for the fermionic part of R. This definition of BRST transformations is clearly consistent with its counterpart (2.4.32) for functionals[11].

In order to bring operators on $\mathcal{V}$ in touch with operators on $\mathfrak{V}$, one needs to know how an operator R on $\mathcal{V}$, which does not necessarily have to be smeared, can be transformed into an operator $\hat{R}$ on $\mathfrak{V}$. The most intuitive way to do this would be to use the definition

$$\hat{R}[|\Psi\rangle] = [R|\Psi\rangle], \tag{2.4.76}$$

where $[|\Psi\rangle]$ is one of the equivalence classes in $\mathfrak{V}$ and $|\Psi\rangle \in \ker\Omega$ a particular representative of it. But this definition only makes sense when the following two properties are fulfilled.

First $R|\Psi\rangle$ has to be contained in $\ker\Omega$ for any representative of any equivalence class, i.e.

$$R\ker\Omega \subseteq \ker\Omega. \tag{2.4.77}$$

Otherwise, one would not be able to find an equivalence class that contains $R|\Psi\rangle$. Second, the definition of $\hat{R}$ has to be independent of the special choice of the representative, which means that $R|\Psi\rangle$ has to be in the same equivalence class as $R(|\Psi\rangle + |\omega\rangle)$ for any $|\omega\rangle \in \operatorname{im}\Omega$. More precisely, for any $|\omega\rangle \in \operatorname{im}\Omega$ there must be another $|\widetilde{\omega}\rangle \in \operatorname{im}\Omega$ so that the relation

$$R|\Psi\rangle + R|\omega\rangle = R|\Psi\rangle + |\widetilde{\omega}\rangle \tag{2.4.78}$$

holds. This is equivalent to the statement

$$R\operatorname{im}\Omega \subseteq \operatorname{im}\Omega. \tag{2.4.79}$$

Clearly, not every operator R satisfies these two conditions, but to get an operator $\hat{R}$ on the actual physical space $\mathfrak{H}$ from R they are unavoidable. So they can be regarded as additional properties that R has to satisfy in order to have an actual physical meaning. Such operators will be called *physical operators* from now on[12]. The algebraic structure that comes with Ω can be used to prove the more or less powerful implications of (2.4.77) and (2.4.79) that are

[11] These definitions of smeared field operators and their BRST transformations have been taken, up to some notational conventions, from [9].

[12] This definition of physical operators is motivated from the *observable operators* in [8] and [9]. These are defined in a different, but equivalent way. The notation has been changed from observable to physical, since observables are usually identified with operators that are hermitian. This restriction is not necessary for the following discussion.

taken from [9] and given in the following theorems[13]. But before they can be formulated, two additional assumptions for $\mathcal{V}$, which are also taken from [9], have to be mentioned.

First the inner product on $\mathcal{V}$ is assumed to be non-degenerate, i.e. if the inner product of a given $|\Psi\rangle \in \mathcal{V}$ with any other element in $\mathcal{V}$ vanishes, then $|\Psi\rangle$ is zero. This is basically no real restriction. If $\mathcal{V}$ would contain such elements that are non-trivial, they would clearly form a subspace $\mathcal{V}_0 \subset \mathcal{V}$. Therefore, in an analogous way as the transition from $\mathcal{V}$ to $\mathfrak{V}$ was motivated, one could abandon $\mathcal{V}$ and work with $\mathcal{V}/\mathcal{V}_0$ instead, which has a non-degenerate inner product, by construction.

Secondly the vacuum state, whose existence has already been assumed implicitly, is the only vector that is invariant under translations and is cyclic with respect to $\mathcal{F}$, i.e.

$$\mathcal{V} = \overline{\mathcal{F}|0\rangle}. \tag{2.4.80}$$

This is one of the general postulates of relativistic quantum field theory. It gives $\mathcal{V}$ a more decent topological structure.

Theorem 2.4.3 *Let R be an arbitrary operator on $\mathcal{V}$, then following statements are equivalent:*

i.) $R \ker\Omega \subseteq \ker\Omega$ *and* $R^\dagger \ker\Omega \subseteq \ker\Omega$.

ii.) $R\,\mathrm{im}\,\Omega \subseteq \mathrm{im}\,\Omega$ *and* $R^\dagger\,\mathrm{im}\,\Omega \subseteq \mathrm{im}\,\Omega$.

iii.) For any $|\Psi_1\rangle$, $|\Psi_2\rangle \in \ker\Omega$ *and* $|\omega_1\rangle$, $|\omega_2\rangle \in \mathrm{im}\,\Omega$ *the relation*

$$\big(\langle\Psi_1| + \langle\omega_1|\big)R\big(|\Psi_2\rangle + |\omega_2\rangle\big) = \langle\Psi_1|R|\Psi_2\rangle \tag{2.4.81}$$

holds.

iv.) R is a physical operator, i.e. $R\ker\Omega \subseteq \ker\Omega$ *and* $R\,\mathrm{im}\,\Omega \subseteq \mathrm{im}\,\Omega$ *hold.*

PROOF: The statement can be shown by proving *i.)* $\Rightarrow$ *ii.)* $\Rightarrow$ *iii.)* $\Rightarrow$ *i.)* and *iii.)* $\Leftrightarrow$ *iv.)*.

i.) $\Rightarrow$ *ii.)*: Let $|\omega\rangle \in \mathrm{im}\,\Omega$, then $|\omega\rangle$ is in particular contained in $\ker\Omega$. So *i.)* implies $R^\dagger R|\omega\rangle \in \ker\Omega$ and therefore

$$\langle\omega|R^\dagger R|\omega\rangle = 0, \tag{2.4.82}$$

since $|\omega\rangle \in \mathrm{im}\,\Omega$. But this means that $R|\omega\rangle$, which is also an element of $\ker\Omega$, is a vector with vanishing norm and therefore has to be contained in $\mathrm{im}\,\Omega$. So $R\,\mathrm{im}\,\Omega \subseteq \mathrm{im}\,\Omega$ is proven. $R^\dagger\,\mathrm{im}\,\Omega \subseteq \mathrm{im}\,\Omega$ can be proven in just the same way.

ii.) $\Rightarrow$ *iii.)*: Since $R|\omega_2\rangle$ and $R^\dagger|\omega_1\rangle$ are assumed to be contained in $\mathrm{im}\,\Omega$, one gets

$$\big(\langle\Psi_1| + \langle\omega_1|\big)R|\omega_2\rangle = \langle\omega_1|R|\Psi_2\rangle = 0 \tag{2.4.83}$$

and therefore *iii.)*.

[13] They can also be found in [8]. Furthermore, note that the proof of Theorem 2.4.3 is presented here in a more detailed way than it is done in [8] or [9] and that its item *iv.)* is not mentioned there.

iii.) $\Rightarrow$ *i.)*: Let $|\varsigma\rangle \in \mathcal{V}$ and $|\Psi\rangle \in \ker\Omega$. Then $\Omega|\varsigma\rangle$ is contained in $\operatorname{im}\Omega$. Therefore, *iii.)* implies, together with the hermiticity of Ω,

$$\langle\varsigma|\Omega R|\Psi\rangle = 0. \tag{2.4.84}$$

Since $|\varsigma\rangle$ was chosen to be an arbitrary element of $\mathcal{V}$ and the inner product on $\mathcal{V}$ is assumed to be non-degenerate, one gets

$$\Omega R|\Psi\rangle = 0. \tag{2.4.85}$$

That means $R|\Psi\rangle \in \ker\Omega$ for any $|\Psi\rangle \in \ker\Omega$, which implies $R\ker\Omega \subseteq \ker\Omega$. The remaining statement $R^\dagger\ker\Omega \subseteq \ker\Omega$ can be proven in a completely analogous way. One only has to look at the complex conjugate of (2.4.81) to find

$$\big(\langle\Psi_2| + \langle\omega_2|\big)R^\dagger\big(|\Psi_1\rangle + |\omega_1\rangle\big) = \langle\Psi_2|R^\dagger|\Psi_1\rangle. \tag{2.4.86}$$

So if R fulfills *iii.)*, $R^\dagger$ does it as well and the argumentation from above can be performed in a completely analogous way.

iii.) $\Leftrightarrow$ *iv.)*: The direction *iii.)* $\Rightarrow$ *iv.)* is already guaranteed by *iii.)* $\Rightarrow$ *i.)* and *iii.)* $\Rightarrow$ *ii.)*. So the only thing left to show is *iv.)* $\Rightarrow$ *iii.)*.

Since $R\big(|\Psi_2\rangle + |\omega_2\rangle\big)$ is assumed to be contained in $\ker\Omega$ and $|\omega_1\rangle \in \operatorname{im}\Omega$, one gets

$$\langle\omega_1|R\big(|\Psi_2\rangle + |\omega_2\rangle\big) = 0. \tag{2.4.87}$$

Furthermore, $R|\omega_2\rangle$ is assured to be contained in $\operatorname{im}\Omega$, this gives

$$\langle\Psi_1|R|\omega_2\rangle = 0. \tag{2.4.88}$$

These two properties clearly imply *iii.)*. □

A direct consequence of this theorem is that the physical operators form a subalgebra of the operators, i.e. if R_1 and R_2 are physical, then, along with any linear combination of them, R_1R_2, $R_1^\dagger$ and $R_2^\dagger$ are, too.

Theorem 2.4.4 *Let $R \in \mathcal{F}(\mathcal{O})$ be a smeared local operator on a bounded open spacetime region $\mathcal{O}$. Then $[\Omega, R]_\pm = 0$ holds if and only if $\Omega R|0\rangle = 0$ holds.*

PROOF: $\Rightarrow$: Let $[\Omega, R]_\pm = 0$ and R^B the bosonic and R^F the fermionic part of R. Then one finds

$$0 = [\Omega, R]_\pm|0\rangle = \big(\Omega R^B - R^B\Omega + \Omega R^F + R^F\Omega\big)|0\rangle = \Omega R|0\rangle, \tag{2.4.89}$$

since $\Omega|0\rangle = 0$.

$\Leftarrow$: Let $\Omega R|0\rangle = 0$. By keeping in mind that R is just a sum of monomials of the form (2.4.72), one can easily verify that $[\Omega, R]_\pm$ is also a smeared local operator, i.e. $[\Omega, R]_\pm \in \mathcal{F}(\mathcal{O})$. So, by using $\Omega|0\rangle = 0$ once more, one finds

$$0 = \Omega R|0\rangle = [\Omega, R]_\pm|0\rangle. \tag{2.4.90}$$

Therefore, $[\Omega, R]_\pm$ is a smeared local operator that annihilates the vacuum. There is a theorem that states that such an operator has to vanish itself. This is sometimes called the *separating property of the vacuum* (see [9]). To prove it, a more detailed examination of the structure of $\mathcal{V}$ is necessary that does not depend on the BRST formalism, but only on the general postulates of relativistic quantum field theory and therefore will not be performed here. In particular, the proof requires the cyclicity of $|0\rangle$ according to $\mathcal{F}$ and can be found in [9]. □

Theorem 2.4.5 *Let $R \in \mathcal{F}(\mathcal{O})$ be a smeared local operator on a bounded open spacetime region $\mathcal{O}$. Then R is physical if and only if it is invariant under BRST transformations, i.e.*

$$[\Omega, R]_{\pm} = 0. \tag{2.4.91}$$

Proof: Let R^B be the bosonic and R^F be the fermionic part of R. Then it is easy to see, since Ω turns bosonic fields into fermionic ones and vice versa, that $[\Omega, R^B]_-$ is a fermionic smeared local field operator and $[\Omega, R^F]_+$ a bosonic one. So $[\Omega, R]_{\pm} = [\Omega, R^B]_- + [\Omega, R^F]_+$ does vanish if and only if $[\Omega, R^B]_-$ and $[\Omega, R^F]_+$ do. Therefore and since R only appears linearly in all the mentioned relations, it is sufficient to proof the theorem for purely bosonic and fermionic operators. So let R be either purely bosonic or fermionic from now on.

$\Leftarrow$: Let $[\Omega, R]_{\pm} = 0$, then clearly $[\Omega, R^\dagger]_{\pm} = 0$ holds, too. This implies

$$\Omega R \ker \Omega = \mp R \Omega \ker \Omega = \{0\} \tag{2.4.92}$$

and

$$\Omega R^\dagger \ker \Omega = \mp R^\dagger \Omega \ker \Omega = \{0\} \tag{2.4.93}$$

and therefore $R \ker \Omega \subseteq \ker \Omega$ and $R^\dagger \ker \Omega \subseteq \ker \Omega$. This is according to Theorem 2.4.3 equivalent to the physicality of R.

$\Rightarrow$: Let $R \ker \Omega \subseteq \ker \Omega$, then it follows

$$\Omega R|0\rangle = 0 \tag{2.4.94}$$

and therefore, because of Theorem 2.4.4, $[\Omega, R]_{\pm} = 0$. □

The last theorem shows that a smeared local operator R has a physical meaning if and only if it commutes with the BRST charge, or equivalently $\delta_\theta R = 0$, i.e. R is BRST-invariant. This is also what one would have expected from the physical point of view. The BRST transformation for the fields ϕ_a basically is just a gauge transformation that is parametrized by $\theta\eta^M$. And since a physical operator should be gauge invariant, its BRST invariance would be a logical consequence. However, it speaks for the BRST formalism that this condition does not has to be demanded in some way, but is an immediate consequence of the structure of $\mathfrak{V}$.

Remark Note that for smeared local operators R a stronger version of Theorem 2.4.3 can be shown. In the proof of Theorem 2.4.5 only the property $R \ker \Omega \subseteq \ker \Omega$ has been used to show $[\Omega, R]_{\pm} = 0$. $\Omega \operatorname{im} \Omega \subseteq \operatorname{im} \Omega$, which is also guaranteed for a physical operator, has not been required. Consequently, the last three theorems show the equivalence of the statements

i.) One, and therefore all of the statements from Theorem 2.4.3 hold for R.

ii.) R is BRST-invariant.

iii.) $R|0\rangle \in \ker \Omega$.

iv.) $R \ker \Omega \subseteq \ker \Omega$.

Theorem 2.4.3 also helps to understand how physical operators R, which do not necessarily have to be smeared, can be transformed to operators $\widetilde{R}$ that act on a subspace $\mathcal{V}_{phys} \subset \ker\Omega$ that is isomorphic to $\mathfrak{V}$, in a way that is compatible with the isomorphism. This means, if $\mathcal{I}$ is an isomorphism between $\mathfrak{V}$ and $\mathcal{V}_{phys}$ one would like to find

$$\widetilde{R}\mathcal{I} = \mathcal{I}\hat{R}, \tag{2.4.95}$$

because that leads to $\widetilde{R}_1\widetilde{R}_2 = \widetilde{R_1R_2}$ for all physical operators R_1 and R_2. So all the commutation relations would be the same as for $\hat{R}_1$ and $\hat{R}_2$ and consequently also the same as for R_1 and R_2.

Since $\mathcal{V}_{phys}$ is supposed to be interpreted as a subspace of $\ker\Omega$, it appears only natural to demand that the inner product that originates from the isomorphy is just the one on $\mathcal{V}$ from the beginning, i.e. $\mathcal{I}$ is considered to be an isometric isomorphism. This leads to the following important result.

Theorem 2.4.6 *Let $\mathcal{V}_{phys} \subset \ker\Omega$ be a vector space that is isometric isomorphic to $\mathfrak{V}$ and $|\Psi\rangle \in \ker\Omega$. If there is a $|\psi\rangle \in \mathcal{V}_{phys}$ such that $[|\Psi\rangle] = [|\psi\rangle]$ holds, i.e. if the decomposition $|\Psi\rangle = |\psi\rangle + |\omega\rangle$ with $|\psi\rangle \in \mathcal{V}_{phys}$ and $|\omega\rangle \in \operatorname{im}\Omega$ exists, then it is unique.*

Proof: Since $\mathfrak{V}$ has a positive definite inner product, the one of $\mathcal{V}_{phys}$ has that property, too. Since it is the same as on $\mathcal{V}$,

$$\mathcal{V}_{phys} \cap \operatorname{im}\Omega = \{0\} \tag{2.4.96}$$

is the consequence. Now assume that there is another decomposition $|\Psi\rangle = |\psi'\rangle + |\omega'\rangle$ of $|\Psi\rangle$ with $|\psi'\rangle \in \mathcal{V}_{phys}$ and $|\omega'\rangle \in \operatorname{im}\Omega$. Then one gets

$$|\psi\rangle + |\omega\rangle = |\psi'\rangle + |\omega'\rangle \quad \Leftrightarrow \quad |\psi\rangle - |\psi'\rangle = |\omega'\rangle - |\omega\rangle \tag{2.4.97}$$

and therefore $(|\psi\rangle - |\psi'\rangle) \in \mathcal{V}_{phys} \cap \operatorname{im}\Omega$. This implies

$$|\psi\rangle - |\psi'\rangle = 0 \tag{2.4.98}$$

and therefore $|\psi\rangle = |\psi'\rangle$ and $|\omega\rangle = |\omega'\rangle$, which proves the statement. □

Note that for a given $\mathcal{V}_{phys}$ the decomposition that is described in Theorem 2.4.6 does not necessarily exist for every $|\Psi\rangle \in \ker\Omega$. To see this, consider an arbitrary $\mathcal{V}_{phys}$. Since this is usually an infinite dimensional vector space, one can construct an actual subspace $\mathcal{V}'_{phys} \subset \mathcal{V}_{phys}$ that is isometric isomorphic to $\mathcal{V}_{phys}$. Consequently, $\mathcal{V}'_{phys}$ is also isometric isomorphic to $\mathfrak{V}$.

But for a $|\psi\rangle \in \mathcal{V}_{phys}\backslash\mathcal{V}'_{phys}$ there is no $|\psi'\rangle \in \mathcal{V}'_{phys}$ such that $[|\psi\rangle] = [|\psi'\rangle]$ holds. The reason for this is the fact that $|\psi'\rangle$ would also be contained in $\mathcal{V}_{phys}$. So Theorem 2.4.6 implies $|\psi\rangle = |\psi'\rangle$, which can not be true because of $|\psi\rangle \in \mathcal{V}_{phys}\backslash\mathcal{V}'_{phys}$ and $|\psi'\rangle \in \mathcal{V}'_{phys}$. However, one can always find a vector space that is isometric isomorphic to $\mathfrak{V}$ and for that such a decomposition is possible for all $|\Psi\rangle \in \ker\Omega$.

Theorem 2.4.7 *There is always a subspace $\mathcal{V}_{phys} \subset \ker\Omega$ that is isometric isomorphic to $\mathfrak{V}$ and contains for every $|\Psi\rangle \in \ker\Omega$ a $|\psi\rangle$ such that $[|\Psi\rangle] = [|\psi\rangle]$ holds.*

PROOF: Let $\mathcal{B}$ be a basis of $\mathfrak{V}$. Select for every element of $\mathcal{B}$ an element $|\psi_\alpha\rangle$, such that

$$\{[|\psi_\alpha\rangle]\}_{\alpha\in I} = \mathcal{B} \tag{2.4.99}$$

holds for a suitable set of indices I. The *axiom of choice* guarantees that this is always possible. These $|\psi_\alpha\rangle$ now can be used to define a mapping $\mathcal{I}$ between $\mathfrak{V}$ and $\text{span}\{|\psi_\alpha\rangle\}_{\alpha\in I} \subset \ker\Omega$. To do so, one simply has to set

$$\mathcal{I}[|\psi_\alpha\rangle] = |\psi_\alpha\rangle \tag{2.4.100}$$

to define it for the basis $\mathcal{B}$ and demand its linearity. By construction this mapping is linear and surjective. Furthermore, one can easily show that it is also an isometry. This guarantees that it is injective as well. The inverse of $\mathcal{I}$ is given by

$$\mathcal{I}^{-1}|\psi\rangle = [|\psi\rangle] \tag{2.4.101}$$

for all $|\psi\rangle \in \text{span}\{|\psi_\alpha\rangle\}_{\alpha\in I}$. So $\mathcal{I}$ is an isometric isomorphism between $\mathfrak{V}$ and $\text{span}\{|\psi_\alpha\rangle\}_{\alpha\in I}$. By setting $\text{span}\{|\psi_\alpha\rangle\}_{\alpha\in I} = \mathcal{V}_{phys}$ one clearly finds for every $|\Psi\rangle \in \ker\Omega$ a $|\psi\rangle \in \mathcal{V}_{phys}$ such that $[|\Psi\rangle] = [|\psi\rangle]$ holds. Namely $|\psi\rangle = \mathcal{I}[|\Psi\rangle]$. □

In the following $\mathcal{V}_{phys}$ will always assumed to be a vector space such that for every $|\Psi\rangle \in \ker\Omega$ there is a $|\psi\rangle \in \mathcal{V}_{phys}$ for which the relation $[|\Psi\rangle] = [|\psi\rangle]$ holds. According to Theorem 2.4.7 such a space always exists and because of Theorem 2.4.6, this $|\psi\rangle$ is then uniquely determined. The reason for this choice of $\mathcal{V}_{phys}$ is the following theorem.

Theorem 2.4.8 *Let $\mathfrak{X} \subset \ker\Omega$ be a vector space such that for all $|\Psi\rangle \in \ker\Omega$ there is a unique $|\psi\rangle \in \mathfrak{X}$ with $[|\Psi\rangle] = [|\psi\rangle]$. Then $\mathfrak{X}$ is isometric isomorphic to $\mathfrak{V}$.*

PROOF: According to the assumptions, the mapping

$$\mathcal{I} : \mathfrak{V} \longrightarrow \mathfrak{X}, \quad [|\Psi\rangle] \longmapsto |\psi\rangle \tag{2.4.102}$$

is well-defined, bijective and obviously linear. Furthermore, one gets

$$\langle \mathcal{I}[|\Psi\rangle] | \mathcal{I}[|\Psi'\rangle] \rangle = \langle [|\Psi\rangle] | [|\Psi'\rangle] \rangle \tag{2.4.103}$$

for all $|\Psi\rangle, |\Psi'\rangle \in \ker\Omega$, which follows directly from the definition of the inner product on $\mathfrak{V}$. So $\mathcal{I}$ is an isometry. Therefore, it is an isometric isomorphism between $\mathfrak{V}$ an $\mathfrak{X}$. □

Remark This theorem offers a strategy to check whether $\mathfrak{V}$ comes in fact with a positive definite inner product. One only has to identify a subspace $\mathfrak{X} \subset \ker\Omega$, such that for all $|\Psi\rangle \in \ker\Omega$ there is a unique $|\psi\rangle \in \mathfrak{X}$ with $[|\Psi\rangle] = [|\psi\rangle]$. This ensures that $\mathfrak{X}$ is isometric isomorphic to $\mathfrak{V}$ and therefore $\mathfrak{V}$ has a positive definite inner product, if and only if $\mathfrak{X}$ has one.

The assumption that there is a unique $|\psi\rangle \in \mathcal{V}_{phys}$ for every $|\Psi\rangle \in \ker\Omega$ with $[|\Psi\rangle] = [|\psi\rangle]$, suggests to take the mapping

$$\mathcal{I} : \mathfrak{V} \longrightarrow \mathcal{V}_{phys}, \quad [|\Psi\rangle] \longmapsto |\psi\rangle \tag{2.4.104}$$

as isomorphism. Clearly, $\mathcal{I}$ is also an isometry. It comes with the projection

$$P^{(0)} : \ker\Omega \longrightarrow \mathcal{V}_{phys}, \quad |\Psi\rangle \longmapsto |\psi\rangle \tag{2.4.105}$$

as a shortcut from $\ker\Omega$ to $\mathcal{V}_{phys}$. Now let R be an arbitrary physical operator on $\mathcal{V}$. Then one finds

$$\widetilde{R}|\psi\rangle = \mathcal{I}\hat{R}[|\psi\rangle] = \mathcal{I}[R|\psi\rangle] = P^{(0)}R|\psi\rangle \tag{2.4.106}$$

for any $|\psi\rangle \in \mathcal{V}_{phys}$. So $\widetilde{R} = P^{(0)}R$ is the consequence. By construction it is clear that this mapping must be compatible with the multiplication of operators, as described above. But since $P^{(0)}$ is a projector, this does not seem to be easy to understand. Therefore, it is proven explicitly.

Theorem 2.4.9 *Let R_1 and R_2 be two physical operators on $\mathcal{V}$. Then $P^{(0)}R_1P^{(0)}R_2 = P^{(0)}R_1R_2$ holds on* $\ker\Omega$.

Proof: Let $|\Psi\rangle \in \ker\Omega$. Then $R_2|\Psi\rangle$ is also contained in $\ker\Omega$, which means that there is a $|\omega\rangle \in \operatorname{im}\Omega$ such that

$$R_2|\Psi\rangle = P^{(0)}R_2|\Psi\rangle + |\omega\rangle \tag{2.4.107}$$

holds. Since $P^{(0)}\operatorname{im}\Omega = \{0\}$ and $R_1|\omega\rangle \in \operatorname{im}\Omega$, one finds

$$P^{(0)}R_1R_2|\Psi\rangle = P^{(0)}R_1P^{(0)}R_2|\Psi\rangle, \tag{2.4.108}$$

which completes the proof. □

Note that this theorem gives an even stronger statement than necessary in order for this definition of $\widetilde{R}$ to work. It would be sufficient that $P^{(0)}R_1P^{(0)}R_2 = P^{(0)}R_1R_2$ holds on $\mathcal{V}_{phys}$. But it is even true on all of $\ker\Omega$.

Earlier in this section it has been shown that every physical operator on $\mathcal{V}$ can be turned into an operator on $\mathfrak{V}$. In fact $\mathcal{V}_{phys}$ can be used to show that the opposite is also true. For every operator $\mathcal{R}$ on $\mathfrak{V}$ there is a physical operator R on $\mathcal{V}$ that relates to $\mathcal{R}$ via

$$\mathcal{R} = \hat{R}. \tag{2.4.109}$$

This is easy to see: By defining $R = \mathcal{I}\mathcal{R}\mathcal{I}^{-1}$ on $\mathcal{V}_{phys}$ and $R = 0$ on $\mathcal{V}_{phys}^{\perp}$, one obtains an operator that keeps $\ker\Omega$ and $\operatorname{im}\Omega$ invariant, since $\operatorname{im}\Omega \subset \mathcal{V}_{phys}^{\perp}$ and obviously relates to $\mathcal{R}$ via (2.4.109), as can be shown by a simple calculation.

So one can work with $\mathcal{V}_{phys}$ instead of $\mathfrak{V}$ without loss of generality. The only thing necessary is to identify a $\mathcal{V}_{phys}$. For all operators on $\mathfrak{V}$ there is a physical operator on $\mathcal{V}$. These operators can be applied to the elements of $\mathcal{V}_{phys}$ without any restrictions. One only has to project back onto $\mathcal{V}_{phys}$ after all transformations have been performed.

As a final result for this section, note that operators (smeared or not) of the form

$$R = [\Omega, R']_- + [\Omega, R'']_+ \tag{2.4.110}$$

where R' and R'' are other operators, clearly map $\ker\Omega$ to $\operatorname{im}\Omega$. Therefore, they are certainly physical. But since $P^{(0)}R$ vanishes, they are just manifestations of the mapping identical to zero[14]. As a consequence, they can be neglected. Furthermore, all operators of the form

$$R = R_1...R_m\big([\Omega, R']_- + [\Omega, R'']_+\big)R_{m+1}...R_n, \tag{2.4.111}$$

with some other physical operators $R_1, ..., R_n$ can be neglected, since $P^{(0)}R = 0$, which follows directly from Theorem 2.4.9.

[14]This is also mentioned in [9], however it is justified in a similar but different way.

2.5 The Faddeev-Popov Determinant

During the adaptation of the generating functional in the last section the formulation of the determinant of $\delta_N \mathcal{G}_M$ as a path integral over ghost fields, known as the *Faddeev-Popov determinant*, has been performed without any comment. If $\mathcal{G}_M$ is a Lorentz scalar, this method is a well-known trick, which is discussed for example in [12] and [18]. But for spin-2 particles a vectorial gauge fixing shows up. Consequently, the indices M and N also contain vector indices μ and ν, i.e. there is a matrix of the form $\delta_{\nu y} \mathcal{G}_{\mu x}$. The method how the formulation of the determinant as path integral can be motivated for such a problem is basically the same as in the scalar case. To do so, one has to consider a set of anticommuting Graßmann numbers and formulate a concept of integration for them. This formulation, as performed here, is taken from [18], with a few different notations. If one is interested in a more detailed discussion of Graßmann variables, this is a good reference to start with.

Consider a Graßmann variable θ. Since $\theta^2 = 0$, the power series expansion of any analytic function f in such a variable breaks down after the first order, i.e. any analytic function in θ is of the form

$$f(\theta) = a + b\theta, \tag{2.5.1}$$

where a and b are complex constants. So the structure of analytic functions of a Graßmann number is very simple and it is possible to introduce a concept of integration simply by setting

$$\int d\theta \big\{ a + b\theta \big\} = b. \tag{2.5.2}$$

This integral comes with the substitution rule (see [16])

$$\int d\theta' = \frac{1}{c} \int d\theta, \tag{2.5.3}$$

where $\theta' = c\theta$ for some complex number c.

These concepts can be generalized for an arbitrary number of n Graßmann variables θ^i with

$$\theta^i \theta^j = -\theta^j \theta^i \tag{2.5.4}$$

for $i, j = 1, ..., n$. One can introduce a complex conjugation on such sets of variables by setting

$$(\theta^i \theta^j)^* = \theta^{j*} \theta^{i*} = -\theta^{i*} \theta^{j*}, \tag{2.5.5}$$

as pointed out in [12]. A power series of these fields clearly contains only terms where each θ^i shows up at most one time. So an analytic function has the form

$$f(\theta^1, ..., \theta^n) = a + b_1\theta^1 + b_2\theta^2 + ... + b_n\theta^n + b_{12}\theta^1\theta^2 + ... + b_{12...n}\theta^1\theta^2...\theta^n \tag{2.5.6}$$

with complex coefficients a, b_1,..., $b_{12...n}$. The only term of order n in this power series clearly is $b_{12...n}\theta^1\theta^2...\theta^n$. So, analogous to the case of one Graßmann variable, one can define the integral over f via

$$\int d\theta^1 d\theta^2 ... d\theta^n f(\theta^1, ..., \theta^n) = \epsilon^{nn-1...1} b_{12...n}. \tag{2.5.7}$$

Here $\epsilon^{i_1...i_n}$ is a Levi-Civita symbol. Note that $\epsilon^{nn-1...1} b_{12...n}$ is just the coefficient of $\theta^n\theta^{n-1}...\theta^1$ in the series expansion of f. So the idea behind this definition is to ignore all the terms with order less than n and rearrange the θ^i in the term of order n such that the most inner integral, i.e. the one for θ^n, can be performed just as in the case of $n = 1$, then the second most inner integral can be performed and so on:

$$\begin{aligned} &\int d\theta^1 d\theta^2 ... d\theta^n f(\theta^1, ..., \theta^n) = \int d\theta^1 d\theta^2 ... d\theta^n b_{12...n} \theta^1 ... \theta^n \\ &= \int d\theta^1 d\theta^2 ... d\theta^n b_{12...n} \epsilon^{nn-1...1} \theta^n \theta^{n-1} ... \theta^1 = \epsilon^{nn-1...1} b_{12...n} \int d\theta^1 d\theta^2 ... d\theta^{n-1} \theta^{n-1} ... \theta^1 \\ &= ... = \epsilon^{nn-1...1} b_{12...n}. \end{aligned} \tag{2.5.8}$$

Note that it is possible to generalize the substitution rule (2.5.3). To do so, consider an invertible $n \times n$ matrix m_{ij} of ordinary (i.e. commuting) numbers. Then one finds (see [16]),

$$\int d\theta^{1\prime} ... d\theta^{n\prime} = \frac{1}{\det(m_{ij})} \int d\theta^1 ... d\theta^n, \tag{2.5.9}$$

where $\theta^{i\prime} = \sum_{j=1}^n m_{ij} \theta^j$ for $i = 1, ..., n$. Furthermore, it is easy to verify that

$$\int d\theta^{i_1} ... d\theta^{i_n} = \epsilon^{i_1 ... i_n} \int d\theta^1 ... d\theta^n \tag{2.5.10}$$

holds.

The definition of the integral now can be used to show that the determinant of any complex $n \times n$ matrix a_{ij} can be written as

$$\det(a_{ij}) = \int \prod_{i=1}^n d\theta^i d\hat{\theta}^i \exp\Big(\sum_{i,j=1}^n \hat{\theta}^i \theta^j a_{ij}\Big), \tag{2.5.11}$$

where the θ^i and $\hat{\theta}^i$ form a set of $2n$ Graßmann variables and $\prod_{i=1}^n d\theta^i d\hat{\theta}^i$ is an abbreviation for $d\theta^1 d\hat{\theta}^1 d\theta^2 d\hat{\theta}^2 ... d\theta^n d\hat{\theta}^n$. The proof of this can be found in [18]. This very nice result is the center piece of the motivation to reformulate determinants as path integrals of ghost fields for the generating functional in the scalar case. However, the argumentation for the non-scalar case works in just the same way. Consider a set of $8n$ Graßmann variables $\theta^{\mu i}$, $\hat{\theta}^{\mu i}$ with $i = 1, ..., n$ and $\mu = 0, ..., 3$, as usual. Furthermore, let $a_{\mu i, \nu j}$ be the matrix elements of a linear mapping in these variables. The trick how this apparently more complicated case can be reduced to the previous one is simply to introduce a new set of indices $I, J = 1, ..., 4n$ and to use them to rename the Graßmann variables and matrix components:

$$\Theta^1 = \theta^{01}, \quad \Theta^2 = \theta^{11}, ..., \quad \Theta_4 = \theta^{31}, ..., \quad \Theta^{4n} = \theta^{3n}, \tag{2.5.12}$$

$$\hat{\Theta}^1 = \hat{\theta}^{01}, \quad \hat{\Theta}^2 = \hat{\theta}^{11}, ..., \quad \hat{\Theta}^4 = \hat{\theta}^{31}, ..., \quad \hat{\Theta}^{4n} = \hat{\theta}^{3n} \quad \text{and} \tag{2.5.13}$$

$$A_{11} = a_{01,01}, \quad A_{21} = a_{11,01}, ..., \quad A_{41} = a_{31,01}, ..., \quad A_{4n4n} = a_{3n,3n}. \tag{2.5.14}$$

This then leads to

$$\begin{aligned} \det(A_{IJ}) &= \int \prod_{I=1}^{4n} d\Theta^I d\hat{\Theta}^I \exp\Big(\sum_{I,J=1}^{4n} \hat{\Theta}^I \Theta^J A_{IJ}\Big) \\ &= \int \prod_{i=1}^{n} \prod_{\mu=0}^{3} d\theta^{\mu i} d\hat{\theta}^{\mu i} \exp\Big(\sum_{i,j=1}^{n} \sum_{\mu,\nu=0}^{3} \hat{\theta}^{\mu i} \theta^{\nu j} a_{\mu i, \nu j}\Big). \end{aligned} \tag{2.5.15}$$

And since $\det(A_{IJ}) = \det(a_{\mu i,\nu j})$ clearly holds, it follows

$$\det(a_{\mu i,\nu j}) = \int \prod_{i=1}^{n}\prod_{\mu=0}^{3} d\theta^{\mu i} d\hat{\theta}^{\mu i} \exp\Big(\sum_{i,j=1}^{n}\sum_{\mu,\nu=0}^{3} \hat{\theta}^{\mu i}\theta^{\nu j} a_{\mu i,\nu j}\Big). \tag{2.5.16}$$

At this point the Graßmann variables can be considered to be real, i.e. $\theta^{\mu i} = \theta^{\mu i *}$ and $\hat{\theta}^{\mu i} = \hat{\theta}^{\mu i *}$. By substituting $\hat{\theta}^{\mu j} = i\bar{\theta}^{\mu j}$ one can formulate the integral by using new variables $\bar{\theta}^{\mu j}$ with

$$\bar{\theta}^{\mu j *} = -\bar{\theta}^{\mu j}. \tag{2.5.17}$$

This is an important feature, since it implies

$$(\bar{\theta}^{\mu i}\theta^{\nu j})^* = -\theta^{\nu j}\bar{\theta}^{\mu i} = \bar{\theta}^{\mu i}\theta^{\nu j}, \tag{2.5.18}$$

which will guarantee that the action S_{BRST} is real valued. By performing this substitution and using (2.5.10) together with the Einstein summation convention, one finds

$$\det(a_{\mu i,\nu j}) \propto \int \prod_{i=1}^{n}\prod_{\mu=0}^{3} d\bar{\theta}^{\mu i} \prod_{j=1}^{n}\prod_{\nu=0}^{3} d\theta^{\nu j} \exp\Big(i\sum_{i,j=1}^{n} \bar{\theta}^{\mu i}\theta^{\nu j} a_{\mu i,\nu j}\Big). \tag{2.5.19}$$

The path integral transition $i, j \longrightarrow x, y \in \mathbb{R}^4$, $\bar{\theta}^{\mu i}, \theta^{\nu j} \longrightarrow \bar{\eta}^{\mu x}, \eta^{\nu y}$ then gives

$$\det(\delta_{\nu y}\mathcal{G}_{\mu x}) \propto \int \prod_{x}\prod_{\mu=0}^{3} d\bar{\eta}^{\mu x} \prod_{y}\prod_{\nu=0}^{3} d\eta^{\nu y} e^{i\bar{\eta}^{\mu x}\eta^{\nu y}\delta_{\nu y}\mathcal{G}_{\mu x}} \tag{2.5.20}$$

for the matrix $\delta_{\nu y}\mathcal{G}_{\mu x}$. This is just the relation (2.4.23) for this special choice of $\delta_N\mathcal{G}_M$. Furthermore, at the level of operators, the relation $\bar{\theta}^{\mu i *} = -\bar{\theta}^{\mu i}$ causes the antihermiticity of the operator $\bar{\eta}^{\mu x}$, i.e. $\bar{\eta}^{\mu x \dagger} = -\bar{\eta}^{\mu x}$.

2.6 The Stückelberg Trick

The BRST formalism clearly can only be introduced for gauge theories. But not every action that is supposed to be used to formulate a quantum theory comes with a gauge invariance. As an example consider the action

$$S[A_\mu] = \int d^4x\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\Big\}. \tag{2.6.1}$$

The corresponding Euler-Lagrange equations are

$$\frac{\delta_L S}{\delta A_\mu} = \Box A^\mu - \partial^\mu\partial_\nu A^\nu - m^2 A^\mu = 0. \tag{2.6.2}$$

By contracting them with ∂_μ, one finds $\partial_\mu A^\mu = 0$ for $m \neq 0$. This obviously shows that the Euler-Lagrange equations are equivalent to the set of equations

$$(\Box - m^2)A_\mu = 0 \quad \text{and} \quad \partial_\mu A^\mu = 0. \tag{2.6.3}$$

According to Section 2.3 S therefore describes a spin-1 field of mass m. The mass term $\int d^4x \frac{m^2}{2} A_\mu A^\mu$ is not invariant under the gauge transformations, known from the massless case. Therefore, the action S itself is also not gauge invariant. So the BRST formalism can not be applied in order to quantize massive spin-1 fields. However, the problems that occur when one is trying to canonically quantize the massless theory are also present here. Again, it is not possible to introduce canonical momenta for all of the field components A_μ and even worse, this time there is no gauge invariance, so one can not introduce a gauge fixing in order to solve this dilemma. One way to quantize A_μ is to restore the gauge invariance from the massless case.

As already mentioned in the introduction, the most prominent way to obtain a concept of gauge transformations for massive fields is to apply the Higgs mechanism. But since this would require an actual modification of the theory, this approach will not be pursued here. Instead of that, the *Stückelberg trick* shall be applied. The way as it is presented here is taken from [5]. It is quite pragmatic and focuses only on the goal to restore the gauge invariance. There are more profound ways to formulate it. If one is interested in a description, how the Stückelberg trick was originally motivated, [13] is a good paper to start with. It can also be formulated in the context of path integrals, which is done in [14]. In [7] it is applied to introduce a gauge invariance for massive spin-1 fields, which then is used for a BRST quantization of them.

The idea behind the Stückelberg trick is to introduce a new scalar field $\hat{\phi}$, the so-called *Stückelberg field*, and to replace $S[A_\mu]$ by a new action

$$S'[A_\mu, \hat{\phi}] = S[A_\mu + \partial_\mu \hat{\phi}], \tag{2.6.4}$$

i.e. A_μ is just replaced by $A_\mu + \partial_\mu \hat{\phi}$ in S. Note that the transition $A_\mu \longrightarrow A_\mu + \partial_\mu \hat{\phi}$ has just the form of a gauge transformation of A_μ. This new action S' clearly is invariant under the gauge transformation

$$A_\mu \longmapsto A_\mu + \partial_\mu \Lambda, \quad \hat{\phi} \longmapsto \hat{\phi} - \Lambda. \tag{2.6.5}$$

This invariance has nothing to do with the special form of S but only with the way, how S' is obtained from S, i.e. (2.6.4). Furthermore, note that the gauge transformations of $\hat{\phi}$ indicate that it is just a redundant, unphysical degree of freedom. All its field configurations can be turned into each other by performing a suitable gauge transformation and therefore can be regarded as equivalent, from the physical point of view. By fixing the gauge $\hat{\phi} = 0$ or more precisely, by performing a gauge transformation with $\Lambda = \hat{\phi}$, one can restore S from S'. So in this context S is interpreted as a special gauge fixed case that comes from a more general gauge theory, described by S'.

Note that the Higgs mechanism also allows to restore the original action of the massive particle by choosing a sufficient gauge, the so-called *unitarity gauge* (see [11]). However, there is a significant difference between the Higgs mechanism and the Stückelberg trick: The Higgs mechanism starts with a gauge field that is considered to be massless and introduces a new field that interacts with the former. This new field is constructed in such a way that it has a non-vanishing vacuum expectation value. The interactions, in combination with this vacuum expectation value, cause terms in the action that look like a mass term in a certain gauge[15].

[15] An introduction to the Higgs mechanism can be found in [11]

The Stückelberg trick on the other hand starts with a field that is assumed to be massive from the very beginning of the discussion and then forces the corresponding action to be gauge invariant by introducing additional, redundant degrees of freedom.

Nevertheless, it is important to mention that it is also possible to motivate the Stückelberg trick from the Higgs mechanism. For this, one has to interpret the Stückelberg field as the phase of a Higgs field. This quite intriguing approach is presented in [13].

The Stückelberg trick obviously can be generalized to turn any action for any set of fields ϕ_a into a gauge invariant action, simply by replacing ϕ_a with $\phi_a + \hat{\phi}^M \delta_M \phi_a$. The resulting action then is invariant under the gauge transformations

$$\phi_a \longmapsto \phi_a + \xi^M \delta_M \phi_a, \quad \hat{\phi}^M \longmapsto \hat{\phi}^M - \xi^M \tag{2.6.6}$$

and the Stückelberg field $\hat{\phi}^M$ clearly provides no additional physical degree of freedom, once more. The fields that are supposed to be quantized in this thesis are either massless and the corresponding action already carries a gauge invariance or the massive generalization of such fields. So the choice of the gauge transformations for the latter can be motivated from the massless case. This offers an interesting way to smoothly pass from the massive to the massless case. To understand this, consider the vector field A_μ once more. In the massive case it comes with three degrees of freedom, since it describes a massive spin-1 field, but for $m = 0$ there are only two. So the transition from massive to massless spin-1 fields can not be smooth. However, the Stückelberg trick helps to fix this problem. To see this, first the substitution $\hat{\phi} = \frac{1}{m}\phi$ has to be performed. This leads to the action

$$S_{Stb}[A_\mu, \phi] = S'[A_\mu, \frac{1}{m}\phi] = \int d^4x \Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - mA_\mu\partial^\mu\phi \Big\}, \tag{2.6.7}$$

which comes with the gauge transformations

$$A_\mu \longmapsto A_\mu + \partial_\mu \Lambda, \quad \phi \longmapsto \phi - m\Lambda. \tag{2.6.8}$$

For the massive case, this action describes three physical degrees of freedom, which are all manifested in A_μ. In the limit $m \longrightarrow 0$, S_{Stb} describes a massless spin-1 field, which has two degrees of freedom, and an additional decoupled scalar field, that carries one degree of freedom. The field ϕ does no longer change under gauge transformations, therefore this degree of freedom can now be regarded as an actual, physical one. So the total number of degrees of freedom is conserved in the limit of a massless field, which is necessary for the limit to be smooth[16]. For spin-2 fields, as discussed below, an analogous method can be applied to conserve the total number of degrees of freedom.

Another interesting aspect of the Stückelberg formalism is the fact that it does not destroy the invariance of a theory under a set of gauge transformations $\tilde{\delta}_N$ as long as they commute with the transformations δ_M that shall be introduced, $\hat{\phi}^M$ is invariant under them and $\phi_a + \hat{\phi}^M \delta_M \phi_a$ transforms like ϕ_a under them, i.e.

$$[\delta_M, \tilde{\delta}_N]_- = 0, \quad \tilde{\delta}_N \hat{\phi}^M = 0 \quad \text{and} \quad \tilde{\delta}_N(\phi_a + \hat{\phi}^M \delta_M \phi_a) = \tilde{\delta}_N \phi_a \big|_{\phi_a = \phi_a + \hat{\phi}^M \delta_M \phi_a}. \tag{2.6.9}$$

[16] This strategy, that allows to preserve the total number of physical degrees of freedom, is also taken from [5].

This can be seen quite easily. Let $S[\phi_a] = S[\phi_a + \tilde{\xi}^N \tilde{\delta}_N \phi_a]$ be the action of interest. Then one finds

$$\begin{aligned} S'[\phi_a + \tilde{\xi}^N \tilde{\delta}_N \phi_a, \hat{\phi}^M + \tilde{\xi}^N \tilde{\delta}_N \hat{\phi}^M] =& S'[\phi_a + \tilde{\xi}^N \tilde{\delta}_N \phi_a, \hat{\phi}^M] \\ =& S[\phi_a + \tilde{\xi}^N \tilde{\delta}_N \phi_a + \hat{\phi}^M \delta_M \phi_a + \tilde{\xi}^N \hat{\phi}^M \delta_M \tilde{\delta}_N \phi_a] \\ =& S[\phi_a + \hat{\phi}^M \delta_M \phi_a + \tilde{\xi}^N \tilde{\delta}_N (\phi_a + \hat{\phi}^M \delta_M \phi_a)] \\ =& S[\phi_a + \tilde{\xi}^N \tilde{\delta}_N \phi_a]\big|_{\phi_a = \phi_a + \hat{\phi}^M \delta_M \phi_a} \\ =& S[\phi_a + \hat{\phi}^M \delta_M \phi_a] = S'[\phi_a, \hat{\phi}^M]. \end{aligned} \tag{2.6.10}$$

So as long as one ensures that the new gauge transformations from the Stückelberg formalism commute with the old ones that might already exist, these old ones are not expanded to transform $\hat{\phi}^M$ and the modified argument of S behaves like the original ϕ_a, the Stückelberg trick does not destroy any existing gauge structure.

Chapter 3

Spin-2 Particles

Now, since the necessary mathematical tools have been introduced, it is time to construct an action that can be used to formulate quantized spin-2 particles. To do so, one starts with the action that describes classical spin-2 fields. However, it turns out that this action does not carry a gauge invariance in the massive case. Therefore, one has to introduce a such with the help of Stückelberg fields in order to apply the BRST formalism.

3.1 The Fierz-Pauli Action

To describe classical spin-2 particles, the Fierz-Pauli action

$$\begin{aligned} S[h_{\mu\nu}] = \int d^4x \mathcal{L} = \int d^4x \Big\{ -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} \\ - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2)\Big\} \end{aligned} \tag{3.1.1}$$

can be used. Here $h_{\mu\nu}$ is a symmetric tensor and $h = h^\mu{}_\mu$. Note that $S\big|_{m=0}$ is invariant under gauge transformations of the form

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{3.1.2}$$

for any vector field ξ_μ that falls off sufficiently fast at infinity. This can be shown by a straightforward calculation and the use of the method of partial integration. This will play a crucial role in the following chapters.

There are two possibilities to justify that S in fact represents spin-2 particles. The first is to perform a Legendre transformation in the spatial parts h_{ij} of the field. This gives a Hamiltonian that allows to count the degrees of freedom. The corresponding discussion is taken from [5]. It requires some results from a more detailed mathematical analysis of gauge theories and constraints. They will be used without a proof in this thesis. For a detailed mathematical discussion of gauge theories in this context, the reference [4] is highly recommended.

To apply this method of counting the degrees of freedom, the transformation

$$\mathcal{L} \longmapsto \widetilde{\mathcal{L}} = \mathcal{L} + \frac{df}{dt}, \tag{3.1.3}$$

with

$$f(h_{\mu\nu}, \partial_i h_{\mu\nu}) = (h_{00} + h^i{}_i)\partial^j h_{j0}, \tag{3.1.4}$$

of the Lagrangian has to be performed. The transformation clearly breaks the Lorentz invariance of the Lagrangian. This is acceptable, since this transformation is only done to get a decent Hamiltonian, which of course also does not carry the Lorentz invariance. The fact that the transformation (3.1.3) does not change the physical system follows from Theorem 2.2.1. The resulting momenta have the form

$$\widetilde{\pi}^{ij} = \frac{\delta_L \widetilde{L}}{\delta \dot{h}_{ij}} = \dot{h}^{ij} - \dot{h}^k{}_k \delta^{ij} + 2\partial^{(i} h^{j)0} + 2\partial^k h_{k0}\delta^{ij}, \tag{3.1.5}$$

where $\widetilde{L}$ is the Lagrangian function corresponding to $\widetilde{\mathcal{L}}$. Contracting with g^{ij} implies

$$\widetilde{\pi}^i{}_i = -2\dot{h}^i{}_i + 4\partial^i h_{i0}, \tag{3.1.6}$$

which can be used to get

$$\dot{h}_{ij} = \widetilde{\pi}_{ij} - \frac{1}{2}\widetilde{\pi}^k{}_k \delta_{ij} + 2\partial_{(i} h_{j)0}. \tag{3.1.7}$$

By performing some partial integrations corresponding to spatial derivatives, this leads to

$$\int d^4x \widetilde{\mathcal{L}} = \int d^4x \Big\{ \dot{h}_{ij}\widetilde{\pi}^{ij} - \mathcal{H}_0 + 2h_{0i}\partial_j \widetilde{\pi}^{ij} - m^2 h_{i0}h^{i0} + h_{00}(\partial^i\partial_i h^j{}_j - \partial^i\partial^j h_{ij} - m^2 h^i{}_i) \Big\}, \tag{3.1.8}$$

where

$$\begin{aligned} \mathcal{H}_0 =& \frac{1}{2}\widetilde{\pi}_{ij}\widetilde{\pi}^{ij} - \frac{1}{4}(\widetilde{\pi}^i{}_i)^2 + \frac{1}{2}\partial_k h_{ij}\partial^k h^{ij} - \partial_i h_{jk}\partial^j h^{ik} \\ &+ \partial_i h^{ij}\partial_j h^k{}_k - \frac{1}{2}\partial^i h^j{}_j \partial_i h^k{}_k + \frac{1}{2}m^2(h_{ij}h^{ij} - (h^i{}_i)^2). \end{aligned} \tag{3.1.9}$$

Note that, since the Legendre transformation was performed only for the spatial field components, the canonical equations will not give the dynamics for all components of $h_{\mu\nu}$. To do a Legendre transformation for all field components, a gauge fixing has to be introduced. This is done in Section 3.3 after some additional modifications of $\mathcal{L}$. Nevertheless, the procedure presented here is mathematically correct and can be regarded just as a helpful way to formulate the action S, such that the counting of the degrees of freedom becomes easier.

Remark It is also possible to derive (3.1.8) by Legendre transforming $\mathcal{L}$ and applying Theorem 2.2.2. This Legendre transformation leads to the momenta

$$\pi^{ij} = \dot{h}^{ij} - \dot{h}^k{}_k \delta^{ij} + 2\partial^{(i} h^{j)0} + \partial^k h_{k0}\delta^{ij} \tag{3.1.10}$$

and therefore gives, by using the same trick as before,

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}(\pi^k{}_k + \partial^k h_{k0})\delta_{ij} + 2\partial_{(i} h_{j)0}. \tag{3.1.11}$$

This results in the Hamiltonian

$$\begin{aligned}\mathcal{H} = \dot{h}_{ij}\pi^{ij} - \mathcal{L} =& \frac{1}{2}\pi_{ij}(\pi^{ij} - 4\partial^i h^{j0}) - \frac{1}{4}\pi^i{}_i(\pi^j{}_j + 2\partial^j h_{j0}) \\ &+ \frac{1}{2}\partial_i h_{jk}(\partial^i h^{jk} - 2\partial^k h^{ij}) - \partial_i h^{ij}\partial_j h_{00} \\ &+ \frac{1}{2}\partial^i h^j{}_j(2\partial^k h_{ki} + 2\partial_i h_{00} - 2\dot{h}_{0i} - \partial_i h^k{}_k) \\ &+ \partial^i h_{i0}(\dot{h}_{00} - \frac{3}{4}\partial^j h_{j0}) + 2\partial^i h_{j0}\partial^j h_{i0} - \partial^i h_{00}\dot{h}_{i0} \\ &+ \frac{1}{2}m^2(h_{ij}h^{ij} + 2h_{00}h^i{}_i - (h^i{}_i)^2 + 2h^{i0}h_{i0}).\end{aligned} \tag{3.1.12}$$

Finally, by using f as defined in (3.1.4) and Theorem 2.2.2, one gets the Hamiltonian

$$\begin{aligned}\widetilde{\mathcal{H}} =& \frac{1}{2}\widetilde{\pi}_{ij}(\widetilde{\pi}^{ij} - 4\partial^i h^{j0}) - \frac{1}{4}(\widetilde{\pi}^i{}_i)^2 \\ &+ \frac{1}{2}\partial_i h_{jk}(\partial^i h^{jk} - 2\partial^k h^{ij}) - \partial_i h^{ij}\partial_j h_{00} \\ &+ \frac{1}{2}\partial^i h^j{}_j(2\partial^k h_{ki} + 2\partial_i h_{00} - 2\dot{h}_{0i} - \partial_i h^k{}_k) \\ &- \partial^i \dot{h}_{i0}(h^j{}_j + h_{00}) - \partial^i h_{00}\dot{h}_{i0} - 2(\partial_i h_{j0}\partial^j h^{i0} + (\partial^i h_{i0})^2) \\ &+ \frac{1}{2}m^2(h_{ij}h^{ij} + 2h_{00}h^i{}_i - (h^i{}_i)^2 + 2h_{i0}h^{i0}),\end{aligned} \tag{3.1.13}$$

which differs only by some partial integrations corresponding to spatial derivatives from

$$\mathcal{H}_0 - (2h_{0i}\partial_j\widetilde{\pi}^{ij} - m^2 h_{i0}h^{i0} + h_{00}(\partial^i\partial_i h^j{}_j - \partial^i\partial^j h_{ij} - m^2 h^i{}_i)), \tag{3.1.14}$$

i.e. the Hamiltonian that belongs to $\widetilde{\mathcal{L}}$. Note that the dynamics of the fields $h_{0\mu}$ are not given by the canonical equations. Therefore, they have to be regarded as additional exterior fields during the transformation from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.

To count the degrees of freedom, first consider $m = 0$. Then the fields $h_{0\mu}$ in (3.1.8) can be interpreted as Lagrange multipliers that imply the four constraints

$$\chi^j = \partial_i\widetilde{\pi}^{ij} = 0 \quad \text{and} \quad \chi^4 = \partial^i\partial_i h^j{}_j - \partial^i\partial^j h_{ij} = 0. \tag{3.1.15}$$

One can show by a straightforward calculation that the Poisson bracket of every pair of them vanishes, i.e.

$$\{\chi^a(t,\mathbf{x}), \chi^b(t,\mathbf{y})\} = \int d^3\mathbf{x}' \left\{ \frac{\delta_L\chi^a(t,\mathbf{x})}{\delta h_{ij}(\mathbf{x}')}\frac{\delta_L\chi^b(t,\mathbf{y})}{\delta\widetilde{\pi}^{ij}(\mathbf{x}')} - \frac{\delta_L\chi^a(t,\mathbf{x})}{\delta\widetilde{\pi}^{ij}(\mathbf{x}')}\frac{\delta_L\chi^b(t,\mathbf{y})}{\delta h_{ij}(\mathbf{x}')} \right\} = 0 \tag{3.1.16}$$

for all $a, b = 1, ..., 4$. So in particular, they vanish on the surface on which $\chi^a = 0$ holds for all $a = 1, ..., 4$. Therefore, these constraints are so-called *first class constraints*. Furthermore, it is easy to see that they are all independent from each other, which means no constraint $\chi^a = 0$ is a consequence of the other ones, i.e. $\chi^b = 0$ for all $b \neq a$. In addition, one can easily show that

$$\begin{aligned}\{H_0(t), \chi^4(t,\mathbf{x})\} =& \int d^3\mathbf{x}' \left\{ \frac{\delta_L H_0(t)}{\delta h_{ij}(\mathbf{x}')}\frac{\delta_L\chi^4(t,\mathbf{x})}{\delta\widetilde{\pi}^{ij}(\mathbf{x}')} - \frac{\delta_L H_0(t)}{\delta\widetilde{\pi}^{ij}(\mathbf{x}')}\frac{\delta_L\chi^4(t,\mathbf{x})}{\delta h_{ij}(\mathbf{x}')} \right\} \\ =& \partial_i\partial_j\widetilde{\pi}^{ij}(t,\mathbf{x})\end{aligned} \tag{3.1.17}$$

and

$$\{H_0(t), \chi^j(t, \mathbf{x})\} = 0 \tag{3.1.18}$$

hold. H_0 is the Hamiltonian function that corresponds to $\mathcal{H}_0$. So on the surface on which $\chi^a = 0$ holds for all $a = 1, ..., 4$ those Poisson brackets vanish as well. This means that H_0 is a *first class Hamiltonian.* This property and the ones for the Poisson brackets of the constraints show that (3.1.8) describes a so-called *first class gauge system.* So there are four gauge invariances, which are generated by the four constraints.

Now the counting of the degrees of freedom is quite simple: The four constraints restrict the dynamics on the 12-dimensional phase space that is spanned on each space point by the symmetric fields $\widetilde{\pi}^{ij}$ and h_{ij} to an eight-dimensional surface. Furthermore, the four gauge invariances imply that the gauge orbits are four-dimensional. Therefore, the quotient of the surface by the orbits, i.e. the set of equivalence classes that corresponds to the physical degrees of freedom, is four dimensional. So there are two polarizations and their conjugate momenta, which is just the desired number of degrees of freedom for massless spin-2 particles.

The massive case is a little more subtle since the h_{0i} now appear quadratically and therefore can no longer be interpreted as Lagrange multipliers. Instead they are fields that obey the algebraic equations of motion

$$h_{0i} = -\frac{1}{m^2}\partial^j \pi_{ij}, \tag{3.1.19}$$

which is an immediate result of the usual variation principle formalism. Applying this to S suggests to introduce another Hamiltonian

$$\mathcal{H}'_0 = \mathcal{H}_0 + \frac{1}{m^2}\partial^j \pi_{ij} \partial_k \pi^{ik}. \tag{3.1.20}$$

This allows to write S as

$$S = \int d^4x \Big\{ \dot{h}_{ij}\widetilde{\pi}^{ij} - \mathcal{H}'_0 + h_{00}(\partial^i\partial_i h^j{}_j - \partial^i\partial^j h_{ij} - m^2 h^i{}_i) \Big\}. \tag{3.1.21}$$

h_{00} again is a Lagrange multiplier. It gives the constraint

$$\chi = -\partial^i\partial_i h^j{}_j + \partial^i\partial^j h_{ij} + m^2 h^i{}_i = 0. \tag{3.1.22}$$

The Hamiltonian is now no longer first class, since the Poisson bracket

$$\{H'_0, \chi\} = \frac{1}{2}m^2 \widetilde{\pi}^i{}_i + \partial_i\partial_j \widetilde{\pi}^{ij} \tag{3.1.23}$$

does not vanish on the surface on which $\chi = 0$ holds. So one finds a *secondary constraint*:

$$\chi' = \{H'_0, \chi\} \stackrel{!}{=} 0. \tag{3.1.24}$$

This set of constraints is secondary class, i.e. not first class, since

$$\{\chi(t, \mathbf{x}), \chi'(t, \mathbf{y})\} = \frac{3}{2}m^4 \delta(\mathbf{x} - \mathbf{y}). \tag{3.1.25}$$

This means that there is no additional gauge freedom. So the two constraints restrict the dynamics to a 10-dimensional surface in the 12-dimensional phase space, i.e. there are five degrees of freedom and their conjugate momenta, which is the case for massive spin-2 particles.

The second way to verify that the Fierz-Pauli action actually describes spin-2 particles is to check whether the solutions of the Euler-Lagrange equations are just the fields presented in Section 2.3. That means the fields have to be symmetric, which is guaranteed by construction and the Euler-Lagrange equations must be equivalent to

$$h = 0, \quad (\Box - m^2)h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0 \tag{3.1.26}$$

for the massive and

$$\Box h_{\mu\nu} - \partial^\lambda \partial_\mu h_{\lambda\nu} - \partial^\lambda \partial_\nu h_{\lambda\mu} + \partial_\mu \partial_\nu h = 0 \tag{3.1.27}$$

for the massless case. But this is quite easy to check: For an arbitrary mass, the Euler-Lagrange equations read

$$\Box h_{\mu\nu} - \partial^\lambda \partial_\mu h_{\lambda\nu} - \partial^\lambda \partial_\nu h_{\lambda\mu} + \partial_\mu \partial_\nu h + (\partial_\lambda \partial_\rho h^{\lambda\rho} - \Box h)g_{\mu\nu} - m^2(h_{\mu\nu} - hg_{\mu\nu}) = 0. \tag{3.1.28}$$

For the massive case the proof of the equivalence of (3.1.26) and (3.1.28) is taken from [5]. By acting with ∂^μ on the Euler-Lagrange equations, one gets

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0. \tag{3.1.29}$$

This can be applied to the Euler-Lagrange equations to get

$$\Box h_{\mu\nu} - \partial_\mu \partial_\nu h - m^2(h_{\mu\nu} - hg_{\mu\nu}) = 0. \tag{3.1.30}$$

The contraction of this relation with $g^{\mu\nu}$ then implies $h = 0$, which turns (3.1.29) into $\partial^\mu h_{\mu\nu} = 0$ and (3.1.30) into $(\Box - m^2)h_{\mu\nu} = 0$. So (3.1.28) implies (3.1.26). The other direction can be shown by applying (3.1.26) to (3.1.28).

The massless case can be treated in a very similar way. Set $m = 0$ in (3.1.28) and contract it with $g^{\mu\nu}$ to get

$$\Box h - \partial_\lambda \partial_\rho h^{\lambda\rho} = 0. \tag{3.1.31}$$

By plugging this back into (3.1.28), one gets (3.1.27). Contracting this with $g^{\mu\nu}$ once more gives (3.1.31) and therefore $(\partial_\lambda \partial_\rho h^{\lambda\rho} - \Box h)g_{\mu\nu}$, which is 0, can be added to (3.1.27) to regain (3.1.28). Thus the two sets of equations are equivalent.

The massive spin-2 fields, i.e. the solutions of (3.1.26), can easily be found. In particular, they are solutions of the massive Klein-Gordon equation. So they can be written as an integral over the momentum space in the following way[1]:

$$h_{\mu\nu}(x) = \int \widetilde{dp} \Big\{ \tilde{h}_{\mu\nu}(\mathbf{p}) e^{ipx} + \tilde{h}^*_{\mu\nu}(\mathbf{p}) e^{-ipx} \Big\}. \tag{3.1.32}$$

The remaining two equations from (3.1.26) obviously imply

$$p^\mu \tilde{h}_{\mu\nu}(\mathbf{p}) = 0 \quad \text{and} \quad \tilde{h}(\mathbf{p}) = 0. \tag{3.1.33}$$

[1]The following discussion is taken from [5].

The solutions of these equations span a vector space. One can choose an orthonormal basis $\hat{\varepsilon}^r_{\mu\nu}(\mathbf{p})$ of this space, i.e. the space of the classical spin-2 polarizations. By doing so it is possible to find suitable functions $n^r_{\mathbf{p}}$ for every spin-2 field configuration $h_{\mu\nu}$, such that it can be expressed as a linear combination of the form

$$h_{\mu\nu}(x) = \int \widetilde{dp} \sum_r \Big\{ n^r_{\mathbf{p}} \hat{\varepsilon}^r_{\mu\nu}(\mathbf{p}) e^{ipx} + n^{r*}_{\mathbf{p}} \hat{\varepsilon}^{r*}_{\mu\nu}(\mathbf{p}) e^{-ipx} \Big\}. \tag{3.1.34}$$

A more detailed discussion of the structure this basis can be chosen to have is given in [5].

So a naive quantization approach would be to replace all the $n^r_{\mathbf{p}}$ with annihilation and their complex conjugates with the corresponding creation operators. The result would be an operator that generates fields that carry the polarizations of a classical spin-2 field. This is just what one would expect from an appropriate quantization of spin-2 fields. Therefore, in order for the BRST quantization to be physically meaningful, it must be possible to identify the resulting physical vector space with the space that is generated by the classical spin-2 polarizations.

3.2 The Stückelberg Trick in the Massive Case

As mentioned before, the gauge invariance of the Fierz-Pauli action for a finite mass has to be restored from the massless case in order to perform the BRST procedure. This restoration can be done by using the Stückelberg trick to introduce new fields that carry no additional degrees of freedom. The application of the trick, as it is presented here, is taken from [5]. There it is performed in order to solve a different quite interesting problem, known as the *vDVZ-discontinuity* (van Dam, Veltman, Zakharov). The massless Fierz-Pauli action is used in this paper to formulate a linearization of gravity (see Section 7.3 for more details) and a corresponding generalization by adding the mass term. It turns out that some physical predictions like the light bending angle in the massless case are not compatible with the predictions resulting from the massive case by taking the limit $m \longrightarrow 0$, as one would hope.

That the transition from the massive to the massless case is not that simple can already be seen by counting the physical degrees of freedom. While for the massive case there are five, the massless case offers only two degrees of freedom. So it should not be surprising that the massless particles can not just be treated as a special case of the massive ones.

In order to formulate spin-2 particles in a way that conserves the total number of degrees of freedom, Stückelberg fields that remain unphysical as long as $m \neq 0$ holds, but turn physical and offer the missing three degrees of freedom for $m = 0$, are introduced.

The first step to obtain this quite desirable connection between the massive and the massless case is to introduce a vectorial Stückelberg field $\hat{A}_\mu$ by replacing $h_{\mu\nu}$ with $h_{\mu\nu}+\partial_\mu\hat{A}_\nu+\partial_\nu\hat{A}_\mu$ in the Fierz-Pauli action. By exploiting the gauge invariance of the massless case and performing some partial integrations, one obtains the action

$$\begin{aligned} S'[h_{\mu\nu}, \hat{A}_\mu] = \int d^4x \Big\{ & \mathcal{L}_{m=0}(h_{\mu\nu}, \partial_\lambda h_{\mu\nu}) - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \\ & - \frac{1}{2} m^2 \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} - 2m^2 (h_{\mu\nu} \partial^\mu \hat{A}^\nu - h \partial_\mu \hat{A}^\mu) \Big\}, \end{aligned} \tag{3.2.1}$$

where $\mathcal{L}_{m=0} = \mathcal{L}\big|_{m=0}$ and $\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu$. In order to prevent the terms of the Stückelberg field to vanish for $m = 0$, $\hat{A}_\mu$ needs to be rescaled by

$$\hat{A}_\mu = \frac{1}{m} A_\mu. \tag{3.2.2}$$

This leads to the action

$$\begin{aligned} S''[h_{\mu\nu}, A_\mu] = \int d^4x \Big\{ &\mathcal{L}_{m=0}(h_{\mu\nu}, \partial_\lambda h_{\mu\nu}) - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \\ &- \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2m(h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) \Big\}, \end{aligned} \tag{3.2.3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is easy to check that S'' is invariant under the gauge transformations

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad A_\mu \longmapsto A_\mu - m\xi_\mu. \tag{3.2.4}$$

For $m = 0$, S'' simply describes a massless spin-2 field and a massless spin-1 field. The total number of degrees of freedom therefore obviously is four. So there still is one degree of freedom missing. This can be changed by performing the Stückelberg trick once more, but this time on A_μ. By replacing A_μ with $A_\mu + \partial_\mu \hat{\phi}$, where $\hat{\phi}$ is a scalar field, one can translate the gauge invariance under $A_\mu \longmapsto A_\mu + \partial_\mu \Lambda$ from the massless case to the case of arbitrary mass. Λ is supposed to be an arbitrary function that falls off sufficiently fast at infinity. The result is the action

$$\begin{aligned} S'''[h_{\mu\nu}, A_\mu, \hat{\phi}] = \int d^4x \Big\{ &\mathcal{L}_{m=0}(h_{\mu\nu}, \partial_\lambda h_{\mu\nu}) - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \\ &- 2m(h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) - 2m(h_{\mu\nu} \partial^\mu \partial^\nu \hat{\phi} - h \Box \hat{\phi}) \Big\}, \end{aligned} \tag{3.2.5}$$

which is invariant under the two distinct gauge transformations

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad A_\mu \longmapsto A_\mu - m\xi_\mu, \quad \hat{\phi} \longmapsto \hat{\phi} \quad \text{and} \tag{3.2.6}$$

$$h_{\mu\nu} \longmapsto h_{\mu\nu}, \quad A_\mu \longmapsto A_\mu + \partial_\mu \Lambda, \quad \hat{\phi} \longmapsto \hat{\phi} - \Lambda. \tag{3.2.7}$$

Again, in order to prevent the additional scalar field terms from vanishing for $m = 0$, a rescaling of the form $\hat{\phi} = \frac{1}{m}\phi$ is necessary. This gives the action

$$\begin{aligned} S''''[h_{\mu\nu}, A_\mu, \phi] = \int d^4x \Big\{ &\mathcal{L}_{m=0}(h_{\mu\nu}, \partial_\lambda h_{\mu\nu}) - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \\ &- 2m(h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) - 2(h_{\mu\nu} \partial^\mu \partial^\nu \phi - h \Box \phi) \Big\} \end{aligned} \tag{3.2.8}$$

and the corresponding gauge transformations

$$h_{\mu\nu} \longmapsto h_{\mu\nu}, \quad A_\mu \longmapsto A_\mu + \partial_\mu \Lambda, \quad \phi \longmapsto \phi - m\Lambda. \tag{3.2.9}$$

In order to see that this action in fact always gives five degrees of freedom, one last substitution needs to be performed:

$$h_{\mu\nu} = h'_{\mu\nu} + \phi g_{\mu\nu}. \tag{3.2.10}$$

This results in the final form of the action, which will be referred to as S_{Stb}.

$$\begin{aligned} S_{Stb}[h'_{\mu\nu}, A_\mu, \phi] = \int d^4x \Big\{ &\mathcal{L}_{m=0}(h'_{\mu\nu}, \partial_\lambda h'_{\mu\nu}) - \frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \\ &+ 3\phi(\Box + 2m^2)\phi - 2m(h'_{\mu\nu}\partial^\mu A^\nu - h'\partial_\mu A^\mu) \\ &+ 3(m^2 h'\phi + 2m\phi\partial_\mu A^\mu)\Big\}. \end{aligned} \tag{3.2.11}$$

It is invariant under the gauge transformations

$$h'_{\mu\nu} \longmapsto h'_{\mu\nu} + \delta^i h'_{\mu\nu}, \quad A_\mu \longmapsto A_\mu + \delta^i A_\mu, \quad \phi \longmapsto \phi + \delta^i \phi \tag{3.2.12}$$

with

$$\delta^1 h'_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta^1 A_\mu = -m\xi_\mu, \quad \delta^1\phi = 0 \quad \text{and} \tag{3.2.13}$$

$$\delta^2 h'_{\mu\nu} = m\Lambda g_{\mu\nu}, \quad \delta^2 A_\mu = \partial_\mu \Lambda, \quad \delta^2\phi = -m\Lambda. \tag{3.2.14}$$

The corresponding Lagrangian, i.e. the integrand of (3.2.11), shall be denoted as $\mathcal{L}_{Stb}$. For $m = 0$ one can see that S_{Stb} simply is the action for a massless spin-2 field, a massless spin-1 field and a massless scalar field, which completely decouple from each other. They carry the desired five degrees of freedom. Also the gauge transformations turn into the ones which are characteristic for those fields.

Furthermore, in the massive case all the field configurations of the Stückelberg fields are equivalent with respect to gauge transformations. By choosing the gauge transformations with $\xi_\mu = \frac{1}{m}A_\mu$ and $\Lambda = \frac{1}{m}\phi$, one regains the Fierz-Pauli action for $h'_{\mu\nu}$. For $m = 0$ the Stückelberg fields become physical and offer the remaining three degrees of freedom, just as described above.

As a last statement of this section note that the gauge transformations (3.2.13) and (3.2.14) do nothing but adding extra terms to the fields, which themselves are invariant under gauge transformations. So they clearly commute with each other.

3.3 Gauge Fixings for Spin-2 Particles

The next step is to formulate gauge fixings that are suitable for a quantization of the fields. But before a gauge fixed action can be derived, one has to ensure that all the gauge transformations are independent from each other, as pointed out in Section 2.4.2. By using the De Witt notation, the gauge transformations can be written as

$$\delta^1 h'_{\mu\nu}(x) = \xi^{\lambda y}\delta_{1\lambda y} h'_{\mu\nu x} \quad \text{with} \quad \delta_{1\lambda y} h'_{\mu\nu x} = -\Big(g_{\lambda\nu}\frac{\partial}{\partial y^\mu} + g_{\lambda\mu}\frac{\partial}{\partial y^\nu}\Big)\delta(x-y), \tag{3.3.1}$$

$$\delta^1 A_\mu(x) = \xi^{\lambda y}\delta_{1\lambda y} A_{\mu x} \quad \text{with} \quad \delta_{1\lambda y} A_{\mu x} = -m g_{\lambda\mu}\delta(x-y), \tag{3.3.2}$$

$$\delta^1 \phi(x) = \xi^{\lambda y}\delta_{1\lambda y}\phi_x \quad \text{with} \quad \delta_{1\lambda y}\phi_x = 0, \tag{3.3.3}$$

$$\delta^2 h'_{\mu\nu}(x) = \Lambda^y \delta_{2y} h'_{\mu\nu x} \quad \text{with} \quad \delta_{2y} h'_{\mu\nu x} = m g_{\mu\nu}\delta(x-y), \tag{3.3.4}$$

$$\delta^2 A_\mu(x) = \Lambda^y \delta_{2y} A_{\mu x} \quad \text{with} \quad \delta_{2y} A_{\mu x} = -\frac{\partial}{\partial y^\mu}\delta(x-y) \quad \text{and} \tag{3.3.5}$$

$$\delta^2 \phi(x) = \Lambda^y \delta_{2y}\phi_x \quad \text{with} \quad \delta_{2y}\phi_x = -m\delta(x-y). \tag{3.3.6}$$

So to check the independence of all gauge transformations, the relation

$$\xi^{\lambda y}\delta_{1\lambda y} + \Lambda^y \delta_{2y} = 0 \quad \Leftrightarrow \quad \xi_\mu = 0 \quad \text{and} \quad \Lambda = 0 \tag{3.3.7}$$

has to be shown. For the massive case this is quite trivial. By applying $\xi^{\lambda y}\delta_{1\lambda y} + \Lambda^y \delta_{2y}$ to ϕ one gets

$$0 = \xi^{\lambda y}\delta_{1\lambda y}\phi_x + \Lambda^y \delta_{2y}\phi_x = -m\Lambda(x) \tag{3.3.8}$$

and therefore $\Lambda = 0$. This result can be used to derive

$$0 = \xi^{\lambda y}\delta_{1\lambda y}A_{\mu x} + \Lambda^y \delta_{2y}A_{\mu x} = -m\xi_\mu(x), \tag{3.3.9}$$

which implies $\xi_\mu = 0$.

The massless case however requires a closer look. Here one gets

$$0 = \xi^{\lambda y}\delta_{1\lambda y}A_{\mu x} + \Lambda^y \delta_{2y}A_{\mu x} = \partial_\mu \Lambda(x), \tag{3.3.10}$$

so Λ has to be constant and in order for it to vanish at infinity, it has to vanish everywhere. ξ_μ can be treated in a similar way. The relation

$$0 = \xi^{\lambda y}\delta_{1\lambda y}h'_{\mu\nu x} + \Lambda^y \delta_{2y}h'_{\mu\nu x} = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \tag{3.3.11}$$

implies

$$\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) = 0. \tag{3.3.12}$$

This is the so-called *Killing equation* for flat spacetime. Its solutions are of the form

$$\xi_\mu(x) = K_{\mu\nu}x^\nu + c_\mu, \tag{3.3.13}$$

where c_μ is a constant vector and $K_{\mu\nu}$ a constant antisymmetric matrix, which is proven in [15]. So again, in order for ξ_μ to vanish at infinity it has to vanish everywhere. Therefore, the gauge transformations are always independent and the gauge fixing procedure can be performed without any problems for any mass m.

Now proper gauge fixings have to be identified. Those should guarantee that the terms in S_{Stb} that mix the fields $h'_{\mu\nu}$, A_μ and ϕ vanish after the gauge fixing. Gauge fixing terms of the form

$$-\int d^4x \Big\{ \frac{1}{2\alpha}\mathcal{G}^1_\mu \mathcal{G}^{1\mu} + \mathcal{G}^2\mathcal{G}^2 \Big\} \tag{3.3.14}$$

with

$$\mathcal{G}^1_\mu = \partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + 2\alpha m A_\mu \tag{3.3.15}$$

and

$$\mathcal{G}^2 = \partial_\mu A^\mu + m\Big(\frac{1}{2}h' + 3\phi\Big) \tag{3.3.16}$$

satisfy this criterion.

Remark When $\mathcal{G}^1_\mu$ is chosen to have the given form (3.3.15), it is easy to verify that the second gauge fixing must be the one given in (3.3.16), in order to let the remaining couplings disappear. It contains no free parameter like α in the $\mathcal{G}^1_\mu$ case. To see this, consider the mixing terms of $S_{Stb} - \int d^4x \frac{1}{2\alpha}\mathcal{G}^1_\mu \mathcal{G}^{1\mu}$. With the method of partial integration, they can be brought into the form

$$\int d^4x \Big\{ mh'\partial_\mu A^\mu + 3(m^2h'\phi + 2m\partial_\mu A^\mu\phi) \Big\}. \tag{3.3.17}$$

The gauge fixing that eliminates them must be scalar, since the theory only comes with five independent gauge transformations from which four already have been exploited to formulate $\mathcal{G}^1_\mu$. So one can make the general ansatz

$$\mathcal{G}^2 = a\partial_\mu A^\mu + mbh' + mc\phi. \tag{3.3.18}$$

It clearly contains all possible scalars that can be constructed out of the fields and contain at most one derivative. Terms with higher order derivatives, such as $\partial_\mu\partial_\nu h'^{\mu\nu}$, can be neglected, since they cause terms in the gauge fixed action that are quadratic in second or higher order derivatives of the fields. Such terms obviously cause difficulties during the construction of a Hamiltonian for the gauge fixed action, which is necessary for an appropriate quantization of the theory.

The coefficients a, b and c now have to be determined. The mixing terms that result from $\mathcal{G}^2\mathcal{G}^2$ are

$$2abmh'\partial_\mu A^\mu + 2bcm^2h'\phi + 2acm\partial_\mu A^\mu\phi. \tag{3.3.19}$$

In order to eliminate the remaining mixing terms of the action one finds

$$2ab = 1, \quad 2bc = 3 \quad \text{and} \quad 2ac = 6. \tag{3.3.20}$$

This implies

$$a = \pm 1, \quad b = \pm\frac{1}{2} \quad \text{and} \quad c = \pm 3, \tag{3.3.21}$$

and therefore gives

$$\mathcal{G}^2 = \pm\Big(\partial_\mu A^\mu + m\Big(\frac{1}{2}h' + 3\phi\Big)\Big). \tag{3.3.22}$$

This is up to a sign just the gauge fixing given in (3.3.16). The existence of a free parameter like α, which can be chosen arbitrarily, would have led to a $\mathcal{G}^2$ that contains such a parameter. So such a free parameter does not exist for $\mathcal{G}^2$. The freedom of the choice of the sign does not really lead to two different gauge fixings. The only difference between the two gauge fixed actions is the sign of the corresponding ghost term. This does not lead to different results since one can always replace one ghost with its negative via substitution.

It is important to ensure that the two gauge fixings $\mathcal{G}^1_\mu$ and $\mathcal{G}^2$ are compatible. A worst case scenario would be that the two gauge fixing conditions $\mathcal{G}^1_\mu = 0$ and $\mathcal{G}^2 = 0$ contradict each other, since their operator versions hold on the physical sector, as pointed out in Section 2.4.4. The easiest way to avoid such effects is to choose $\mathcal{G}^1_\mu$ in such a way that it is invariant under δ^2 and vice versa, i.e. $\mathcal{G}^1_\mu$ is supposed to fix only δ^1 and $\mathcal{G}^2$ should only fix δ^2. $\delta^1\mathcal{G}^2 = 0$ is already given. To ensure that $\delta^2\mathcal{G}^1_\mu = m(2\alpha - 1)\partial_\mu\Lambda$ vanishes as well the parameter α has to be set to $\frac{1}{2}$. The resulting gauge fixings are

$$\mathcal{G}^1_\mu = \partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + mA_\mu \tag{3.3.23}$$

and of course $\mathcal{G}^2$ as given in (3.3.16). They are presented in [5] as well. However, the argumentation from above clarifies that the ansatz (3.3.15) uniquely determines $\mathcal{G}^2$ and also α, if one additionally demands that $\mathcal{G}^1_\mu$ only fixes δ^1 and $\mathcal{G}^2$ only fixes δ^2. Furthermore, note that

$$\delta^1 \mathcal{G}^1_\mu = (\Box - m^2)\xi_\mu \tag{3.3.24}$$

holds. So the μ-th component $\mathcal{G}^1_\mu$ is invariant under the gauge transformations that are offered by ξ_ν for all $\nu \neq \mu$. This means that $\mathcal{G}^1_\mu$ just fixes the part of δ^1 that is generated by the μ-th component ξ_μ. Consequently, one can rest assured that the components $\mathcal{G}^1_\mu$ are gauge fixings that do not contradict each other as well.

Now the only thing left to do, in order to finish the gauge fixing procedure, is to calculate the terms that contain the ghost fields. By exploiting the invariance of $\mathcal{G}^1_\mu$ under δ^2 and the respective invariance of $\mathcal{G}^2$ under δ^1, one finds that the matrix $\delta_N \mathcal{G}_M$ has the block structure

$$\delta_N \mathcal{G}_M = \begin{pmatrix} \delta_{1\nu y}\mathcal{G}^1_{\mu x} & 0 \\ 0 & \delta_{2y}\mathcal{G}^2_x \end{pmatrix}. \tag{3.3.25}$$

So the ghost terms that occur in the gauge fixed action S_{GF} can be written as

$$\bar{\eta}^M \eta^N \delta_N \mathcal{G}_M = \bar{\eta}^{\mu x}\eta^{\nu y}\delta_{1\nu y}\mathcal{G}^1_{\mu x} + \bar{\zeta}^x \zeta^y \delta_{2y}\mathcal{G}^2_x, \tag{3.3.26}$$

with vectorial ghosts $\bar{\eta}_\mu$ and η_μ and scalar ones $\bar{\zeta}$ and ζ. By a direct calculation one gets

$$\begin{aligned} \delta_{1\nu y} h'_{\rho\sigma y'} \frac{\delta_L \mathcal{G}^1_{\mu x}}{\delta h'_{\rho\sigma y'}} =& \int d^4y' \Big\{ \Big(g_{\nu\sigma}\frac{\partial}{\partial y^\rho} + g_{\nu\rho}\frac{\partial}{\partial y^\sigma}\Big)\delta(y-y') \\ &\times \frac{\partial}{\partial y'^\lambda}\delta(y'-x)\Big(g^{\lambda(\rho}g^{\sigma)}{}_\mu - \frac{1}{2}g^{\rho\sigma}g^\lambda{}_\mu\Big)\Big\} \\ =&\Big(-g_{\mu\nu}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y_\rho} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\Big)\delta(x-y), \end{aligned} \tag{3.3.27}$$

$$\delta_{1\nu y} A_{\lambda y'} \frac{\delta_L \mathcal{G}^1_{\mu x}}{\delta A_{\lambda y'}} = -\int d^4y' m^2 g_{\nu\lambda}\delta(y-y')\delta(x-y')g^\lambda{}_\mu = -m^2 g_{\mu\nu}\delta(x-y) \tag{3.3.28}$$

and

$$\delta_{1\nu y}\phi_{y'}\frac{\delta_L \mathcal{G}^1_{\mu x}}{\delta \phi_{y'}} = 0. \tag{3.3.29}$$

This then gives

$$\begin{aligned} \bar{\eta}^{\mu x}\eta^{\nu y}\delta_{1\nu y}\mathcal{G}^1_{\mu x} =& \bar{\eta}^{\mu x}\eta^{\nu y}\Big(\delta_{1\nu y}h'_{\rho\sigma y'}\frac{\delta_L \mathcal{G}^1_{\mu x}}{\delta h'_{\rho\sigma y'}} + \delta_{1\nu y}A_{\lambda y'}\frac{\delta_L \mathcal{G}^1_{\mu x}}{\delta A_{\lambda y'}} + \delta_{1\nu y}\phi_{y'}\frac{\delta_L \mathcal{G}^1_{\mu x}}{\delta \phi_{y'}}\Big) \\ =& \int d^4x \bar{\eta}^\mu (\Box - m^2)\eta_\mu. \end{aligned} \tag{3.3.30}$$

In an analogous way it follows

$$\begin{aligned} \delta_{2y}\mathcal{G}^2_x =& \delta_{2y}h'_{\rho\sigma y'}\frac{\delta_L \mathcal{G}^2_x}{\delta h'_{\rho\sigma y'}} + \delta_{2y}A_{\lambda y'}\frac{\delta_L \mathcal{G}^2_x}{\delta A_{\lambda y'}} + \delta_{2y}\phi_{y'}\frac{\delta_L \mathcal{G}^2_x}{\delta \phi_{y'}} \\ =& 2m^2\delta(x-y) - \frac{\partial}{\partial y^\rho}\frac{\partial}{\partial x_\rho}\delta(x-y) - 3m^2\delta(x-y) \end{aligned} \tag{3.3.31}$$

and therefore

$$\bar{\zeta}^x \zeta^y \delta_{2y} \mathcal{G}_x^2 = \int d^4x \bar{\zeta}(\Box - m^2)\zeta. \tag{3.3.32}$$

So the gauge fixed action S_{GF} has the very nice form

$$\begin{aligned} S_{GF}[h'_{\mu\nu}, A_\mu, \phi, \bar{\eta}_\mu, \eta_\mu, \bar{\zeta}, \zeta] = \int d^4x \Big\{ & \frac{1}{2} h'_{\mu\nu}(\Box - m^2) h'^{\mu\nu} - \frac{1}{4} h'(\Box - m^2) h' \\ & + A_\mu(\Box - m^2)A^\mu + 3\phi(\Box - m^2)\phi \\ & + \bar{\eta}_\mu(\Box - m^2)\eta^\mu + \bar{\zeta}(\Box - m^2)\zeta \Big\}. \end{aligned} \tag{3.3.33}$$

At this point it appears to be a good idea to rescale the Stückelberg fields once more in order to ensure that the usual commutation relations for the creation and annihilation operators in momentum space fit with the canonical commutation relations for the fields and their conjugate momenta. The rescaled fields

$$A'_\mu = \sqrt{2} A_\mu \quad \text{and} \quad \phi' = \sqrt{6}\phi \tag{3.3.34}$$

do the job. The gauge transformations of these new fields are

$$\delta^1 A'_\mu = -m\sqrt{2}\xi_\mu, \quad \delta^1 \phi' = 0 \quad \text{and} \tag{3.3.35}$$

$$\delta^2 A'_\mu = \partial_\mu \Lambda, \quad \delta^2 \phi' = -m\sqrt{6}\Lambda. \tag{3.3.36}$$

The gauge fixings have the form

$$\mathcal{G}^1_\mu = \partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + \frac{m}{\sqrt{2}} A'_\mu \quad \text{and} \tag{3.3.37}$$

$$\mathcal{G}^2 = \frac{1}{\sqrt{2}} \partial_\mu A'^\mu + m\Big(\frac{1}{2} h' + \sqrt{\frac{3}{2}}\phi'\Big). \tag{3.3.38}$$

The action reads

$$\begin{aligned} S_{GF} = \int d^4x \Big\{ & \frac{1}{2} h'_{\mu\nu}(\Box - m^2) h'^{\mu\nu} - \frac{1}{4} h'(\Box - m^2) h' \\ & + \frac{1}{2} A'_\mu(\Box - m^2)A'^\mu + \frac{1}{2}\phi'(\Box - m^2)\phi' \\ & + \bar{\eta}_\mu(\Box - m^2)\eta^\mu + \bar{\zeta}(\Box - m^2)\zeta \Big\}. \end{aligned} \tag{3.3.39}$$

$\mathcal{G}^1_\mu = 0$ turns into the so-called *de Donder gauge* (see [5]) for $m = 0$, which only concerns $h'_{\mu\nu}$, and $\mathcal{G}^2 = 0$ becomes just the Feynman gauge. So at this point it can already be seen that for the massless case the BRST transformations of the Stückelberg fields will have nothing to do with the ones of the spin-2 field. Therefore, the BRST quantization will essentially reduce to the one for massless spin-2 particles without Stückelberg fields and the one for massless spin-1 fields, since the scalar field does not carry any gauge freedom.

3.4 Hamiltonian Formulation of Spin-2 Particles

In order to perform the quantization, the momenta of the corresponding Hamiltonian have to be derived. Just as in the case of the $U(1)$ gauge field, it is not possible to perform a Legendre transformation of $\mathcal{L}_{Stb}$ without fixing the gauge. The corresponding momenta

$$\frac{\delta_L L_{Stb}}{\delta \dot{h}'_{\mu\nu}} = \dot{h}'^{\mu\nu} + 2\partial^{(\mu} h'^{\nu)0} - g^{0(\mu}\partial^{\nu)} h' + \partial^k h'_{k0} g^{\mu\nu} - \dot{h}'^k{}_k g^{\mu\nu}, \tag{3.4.1}$$

which give

$$\frac{\delta_L L_{Stb}}{\delta \dot{h}'_{00}} = \partial_k h'^{k0}, \quad \frac{\delta_L L_{Stb}}{\delta \dot{h}'_{0i}} = \partial^i h'^{00} + \frac{1}{2}\partial^i h' \quad \text{and} \tag{3.4.2}$$

$$\frac{\delta_L L_{Stb}}{\delta \dot{h}'_{ij}} = \dot{h}^{ij} + 2\partial^{(i} h^{j)0} + \partial^k h'_{k0}\delta^{ij} - \dot{h}^k{}_k \delta^{ij}, \tag{3.4.3}$$

can not be inverted such that all $\dot{h}'_{\mu\nu}$ can be expressed in terms of them, since the $\dot{h}'_{0\mu}$ do not appear in any of them. From (3.2.11) it is easy to see that they also do not appear in any of the momenta of the Stückelberg fields. A similar problem occurs for the massive spin-1 field. Furthermore, note that

$$\frac{\delta_L L_{Stb}}{\delta \dot{h}'_{\mu\nu}} = \frac{\delta_L L}{\delta \dot{h}'_{\mu\nu}} = \frac{\delta_L L_{m=0}}{\delta \dot{h}'_{\mu\nu}} \tag{3.4.4}$$

holds. So the fact that one can not perform an appropriate Legendre transformation does not originate from the Stückelberg fields but is already present for the Fierz-Pauli Lagrangian, just as it is the case for the Lagrangian of a massive spin-1 particle, which is discussed in Section 2.6.

As it can be deduced from (3.3.39), the Lagrangian that results from the gauge fixings $\mathcal{G}^1_\mu$ and $\mathcal{G}^2$ reads

$$\begin{aligned}\mathcal{L}_{GF} = &-\frac{1}{2}\partial_\lambda h'_{\mu\nu}\partial^\lambda h'^{\mu\nu} - \frac{1}{2}m^2 h'_{\mu\nu}h'^{\mu\nu} + \frac{1}{4}\partial_\lambda h'\partial^\lambda h' + \frac{1}{4}m^2 h'h' \\ &- \frac{1}{2}\partial_\lambda A'_\mu \partial^\lambda A'^\mu - \frac{1}{2}m^2 A'_\mu A'^\mu - \frac{1}{2}\partial_\lambda\phi'\partial^\lambda\phi' - \frac{1}{2}m^2\phi'\phi' \\ &- \partial_\lambda\bar{\eta}_\mu\partial^\lambda\eta^\mu - m^2\bar{\eta}_\mu\eta^\mu - \partial_\lambda\bar{\zeta}\partial^\lambda\zeta - m^2\bar{\zeta}\zeta.\end{aligned} \tag{3.4.5}$$

It offers the canonical momenta

$$\pi^{\mu\nu} = \frac{\delta_L L_{GF}}{\delta \dot{h}'_{\mu\nu}} = \dot{h}'^{\mu\nu} - \frac{1}{2}\dot{h}' g^{\mu\nu}, \quad \Sigma^\mu = \frac{\delta_L L_{GF}}{\delta \dot{A}'_\mu} = \dot{A}'^\mu, \quad \gamma = \frac{\delta_L L_{GF}}{\delta \dot{\phi}'} = \dot{\phi}', \tag{3.4.6}$$

$$\bar{r}^\mu = \frac{\delta_L L_{GF}}{\delta \dot{\bar{\eta}}_\mu} = \dot{\eta}^\mu, \quad r^\mu = \frac{\delta_L L_{GF}}{\delta \dot{\eta}_\mu} = -\dot{\bar{\eta}}^\mu, \tag{3.4.7}$$

$$\bar{z} = \frac{\delta_L L_{GF}}{\delta \dot{\bar{\zeta}}} = \dot{\zeta}, \quad z = \frac{\delta_L L_{GF}}{\delta \dot{\zeta}} = -\dot{\bar{\zeta}}. \tag{3.4.8}$$

The only nontrivial inversion of these concerns the $\pi^{\mu\nu}$. By contracting with $g_{\mu\nu}$ one gets $\pi = -\dot{h}'$ and therefore

$$\pi^{\mu\nu} = \dot{h}^{\mu\nu} + \frac{1}{2}\pi g^{\mu\nu} \quad \Leftrightarrow \quad \dot{h}_{\mu\nu} = \pi_{\mu\nu} - \frac{1}{2}\pi g_{\mu\nu}. \tag{3.4.9}$$

So in terms of the momenta the Lagrangian reads

$$
\begin{aligned}
\mathcal{L}_{GF} =& \frac{1}{2}\Big(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{2}\pi\pi\Big) - \frac{1}{2}\Big(\partial_i h'_{\mu\nu}\partial^i h'^{\mu\nu} - \frac{1}{2}\partial_i h'\partial^i h'\Big) - \frac{1}{2}m^2\Big(h'_{\mu\nu}h'^{\mu\nu} - \frac{1}{2}h'h'\Big) \\
&+ \frac{1}{2}\Sigma_\mu\Sigma^\mu - \frac{1}{2}\partial_i A'_\mu\partial^i A'^\mu - \frac{1}{2}m^2 A'_\mu A'^\mu \\
&+ \frac{1}{2}\gamma\gamma - \frac{1}{2}\partial_i\phi'\partial^i\phi' - \frac{1}{2}m^2\phi'\phi' \\
&- r_\mu\bar{r}^\mu - \partial_i\bar{\eta}_\mu\partial^i\eta^\mu - m^2\bar{\eta}_\mu\eta^\mu - z\bar{z} - \partial_i\bar{\zeta}\partial^i\zeta - m^2\bar{\zeta}\zeta,
\end{aligned}
\tag{3.4.10}
$$

which leads to the Hamiltonian

$$
\begin{aligned}
\mathcal{H} =& \dot{h}'_{\mu\nu}\pi^{\mu\nu} + \dot{A}'_\mu\Sigma^\mu + \dot{\phi}'\gamma + \dot{\eta}_\mu r^\mu + \dot{\bar{\eta}}_\mu\bar{r}^\mu + \dot{\zeta}z + \dot{\bar{\zeta}}\bar{z} - \mathcal{L}_{GF} \\
=& \frac{1}{2}\Big(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{2}\pi\pi\Big) + \frac{1}{2}\Big(\partial_i h'_{\mu\nu}\partial^i h'^{\mu\nu} - \frac{1}{2}\partial_i h'\partial^i h'\Big) + \frac{1}{2}m^2\Big(h'_{\mu\nu}h'^{\mu\nu} - \frac{1}{2}h'h'\Big) \\
&+ \frac{1}{2}\Sigma_\mu\Sigma^\mu + \frac{1}{2}\partial_i A'_\mu\partial^i A'^\mu + \frac{1}{2}m^2 A'_\mu A'^\mu \\
&+ \frac{1}{2}\gamma\gamma + \frac{1}{2}\partial_i\phi'\partial^i\phi' + \frac{1}{2}m^2\phi'\phi' \\
&+ \bar{r}_\mu r^\mu + \partial_i\bar{\eta}_\mu\partial^i\eta^\mu + m^2\bar{\eta}_\mu\eta^\mu + \bar{z}z + \partial_i\bar{\zeta}\partial^i\zeta + m^2\bar{\zeta}\zeta.
\end{aligned}
\tag{3.4.11}
$$

Chapter 4

The BRST Formalism for Massive Spin-2 Fields

Now all the necessary preparations for the BRST quantization have been made. For the formulation of the quantization, the Nakanishi-Lautrup fields will be integrated out. Consequently, the gauge fixed Lagrangian $\mathcal{L}_{GF}$ and the corresponding equations of motion will be used for the quantization of the fields.

4.1 Canonical Quantization of the Fields

If one assumes $\mathcal{L}_{GF}$ as the Lagrangian that is suitable for a quantization of spin-2 fields, the usual canonical quantization can be performed. This leads to field operators that generate a vector space $\mathcal{V}$ with an inner product that is neither positive nor definite. The BRST formalism then constructs the actual physical space from $\mathcal{V}$.

4.1.1 Quantization of the Spin-2 Field

The Euler-Lagrange equations for $h'_{\mu\nu}$, which originate from (3.3.39), are

$$(\Box - m^2)h'_{\mu\nu} - \frac{1}{2}(\Box - m^2)h' g_{\mu\nu} = 0. \tag{4.1.1}$$

By contracting them with $g^{\mu\nu}$, it follows that they imply $(\Box - m^2)h' = 0$ and therefore the massive Klein-Gordon equation $(\Box - m^2)h'_{\mu\nu} = 0$ for every field component. The Klein-Gordon equations on the other hand also imply $(\Box - m^2)h' = 0$ and therefore (4.1.1). This means

$$(\Box - m^2)h'_{\mu\nu} - \frac{1}{2}(\Box - m^2)h' g_{\mu\nu} = 0 \quad \Leftrightarrow \quad (\Box - m^2)h'_{\mu\nu} = 0. \tag{4.1.2}$$

Since these Klein-Gordon equations describe a particle with mass m, their solutions in momentum space have to satisfy the on-shellness condition $p^2 = -m^2$. Like in the photonic case (see [12]), the classical positive energy solutions of the spin-2 field then have the form $\varepsilon_{\mu\nu}(\mathbf{p})e^{\pm ipx}$ for $p^2 = -m^2$ and $p^0 > 0$, and vanish for all other p. The symmetric tensor $\varepsilon_{\mu\nu}(\mathbf{p})$ appears to be a generalization of the polarization vectors of the photonic case and will be called *polarization tensor* in the following. Since the only restriction for such tensors is that they have to be symmetric, they span a 10-dimensional space. So the quantized spin-2 field can be expressed as

$$\begin{aligned} h'_{\mu\nu}(x) &= \int \widetilde{dp} \sum_{r=0}^{9} \Big(\varepsilon^r_{\mu\nu}(\mathbf{p}) n^r_{\mathbf{p}} e^{ipx} + \varepsilon^{r*}_{\mu\nu}(\mathbf{p}) n^{r\dagger}_{\mathbf{p}} e^{-ipx} \Big) \\ &= \int \widetilde{dp} \Big\{ n_{\mu\nu}(\mathbf{p}) e^{ipx} + n^{\dagger}_{\mu\nu}(\mathbf{p}) e^{-ipx} \Big\}, \end{aligned} \tag{4.1.3}$$

with

$$n_{\mu\nu}(\mathbf{p}) = \sum_{r=0}^{9} \varepsilon^r_{\mu\nu}(\mathbf{p}) n^r_{\mathbf{p}}. \tag{4.1.4}$$

In order to obtain matching completeness and orthogonality conditions for the polarization tensors $\varepsilon^r_{\mu\nu}(\mathbf{p})$ and commutation relations for the corresponding creation and annihilation operators $n^{r\dagger}_{\mathbf{p}}$ and $n^r_{\mathbf{p}}$ the canonical commutation relations, which are motivated from the Poisson brackets of the corresponding classical fields,

$$[h'_{\mu\nu}(x), \pi_{\rho\sigma}(y)]_-\big|_{x^0=y^0} = i g_{\mu(\rho} g_{\sigma)\nu} \delta(\mathbf{x}-\mathbf{y}) \tag{4.1.5}$$

have to be established. By a straightforward calculation and the application of $[n^r_{\mathbf{p}}, n^l_{\mathbf{q}}]_- = 0$, this leads to

$$\begin{aligned} [h'_{\mu\nu}(x), \pi_{\rho\sigma}(y)]_-\big|_{x^0=y^0} = i \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \int \frac{d^3\mathbf{q}}{2(2\pi)^3} \sum_{r,l=0}^{9} \Big(\\ \varepsilon^r_{\mu\nu}(\mathbf{p}) X^{l*}_{\rho\sigma}(-\mathbf{q}) [n^r_{\mathbf{p}}, n^{l\dagger}_{-\mathbf{q}}]_- e^{-iE_{\mathbf{p}}x^0 + iE_{\mathbf{q}}y^0} \\ + \varepsilon^{r*}_{\mu\nu}(-\mathbf{p}) X^l_{\rho\sigma}(\mathbf{q}) [n^l_{\mathbf{q}}, n^{r\dagger}_{-\mathbf{p}}]_- e^{iE_{\mathbf{p}}x^0 - iE_{\mathbf{q}}y^0} \Big) e^{i\mathbf{p}\mathbf{x} + i\mathbf{q}\mathbf{y}}\big|_{x^0=y^0}, \end{aligned} \tag{4.1.6}$$

where

$$X^l_{\rho\sigma}(\mathbf{p}) = \varepsilon^l_{\rho\sigma}(\mathbf{p}) - \frac{1}{2}\varepsilon^l(\mathbf{p}) g_{\rho\sigma}. \tag{4.1.7}$$

In order to interpret the $n^{r\dagger}_{\mathbf{p}}$ and $n^r_{\mathbf{p}}$ as creation and annihilation operators respectively they have to satisfy commutation relations of the form

$$[n^r_{\mathbf{p}}, n^{l\dagger}_{\mathbf{q}}]_- = 2E_{\mathbf{p}} (2\pi)^3 G^{rl} \delta(\mathbf{p}-\mathbf{q}), \tag{4.1.8}$$

where the G^{rl} are constant real parameters, which are not determined yet. A first naive approach would be to choose $G^{rl} = \delta^{rl}$, but this will cause contradictions with the canonical commutation relations, as will be shown later on. However, (4.1.8) implies

$$\begin{aligned} [h'_{\mu\nu}(x), \pi_{\rho\sigma}(y)]_-\big|_{x^0=y^0} = &i \int \frac{d^3\mathbf{p}}{2(2\pi)^3} \sum_{r,l=0}^{9} G^{rl} \Big(\varepsilon^r_{\mu\nu}(\mathbf{p}) X^{l*}_{\rho\sigma}(\mathbf{p}) \\ &+ \varepsilon^{r*}_{\mu\nu}(-\mathbf{p}) X^l_{\rho\sigma}(-\mathbf{p}) \Big) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \stackrel{!}{=} i g_{\mu(\rho} g_{\sigma)\nu} \delta(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{4.1.9}$$

and therefore it follows

$$\sum_{r,l=0}^{9} G^{rl} \Big(\varepsilon^r_{\mu\nu}(\mathbf{p}) X^{l*}_{\rho\sigma}(\mathbf{p}) + \varepsilon^{r*}_{\mu\nu}(-\mathbf{p}) X^l_{\rho\sigma}(-\mathbf{p}) \Big) = 2 g_{\mu(\rho} g_{\sigma)\nu}. \tag{4.1.10}$$

By using the additional assumption that $\sum_{r,l=0}^{9} G^{rl}\varepsilon^{r}_{\mu\nu}(\mathbf{p})X^{l*}_{\rho\sigma}(\mathbf{p})$ is real and independent of $\mathbf{p}$, the relation

$$\sum_{r,l=0}^{9} G^{rl}\varepsilon^{r}_{\mu\nu}(\mathbf{p})\Big(\varepsilon^{l*}_{\rho\sigma}(\mathbf{p}) - \frac{1}{2}\varepsilon^{l*}(\mathbf{p})g_{\rho\sigma}\Big) = g_{\mu(\rho}g_{\sigma)\nu} \tag{4.1.11}$$

can be derived. By contracting with $g^{\rho\sigma}$ one gets $\sum_{r,l=0}^{9} G^{rl}\varepsilon^{r}_{\mu\nu}(\mathbf{p})\varepsilon^{l*}(\mathbf{p}) = -g_{\mu\nu}$, which can be inserted into (4.1.11) to find

$$\sum_{r,l=0}^{9} G^{rl}\varepsilon^{r}_{\mu\nu}(\mathbf{p})\varepsilon^{l*}_{\rho\sigma}(\mathbf{p}) = g_{\mu(\rho}g_{\sigma)\nu} - \frac{1}{2}g_{\mu\nu}g_{\rho\sigma}. \tag{4.1.12}$$

This relation can be used to derive

$$[n_{\mu\nu}(\mathbf{p}), n^{\dagger}_{\rho\sigma}(\mathbf{q})]_{-} = \Big(g_{\mu(\rho}g_{\sigma)\nu} - \frac{1}{2}g_{\mu\nu}g_{\rho\sigma}\Big)2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \tag{4.1.13}$$

by a straightforward calculation. At this point it is quite easy to see that G^{rl} can not be δ^{rl}. By assuming the opposite one can contract (4.1.12) with $g^{\mu\nu}g^{\rho\sigma}$ to find

$$\sum_{r=0}^{9} \varepsilon^{r}(\mathbf{p})\varepsilon^{r*}(\mathbf{p}) = -4. \tag{4.1.14}$$

But the left hand side of this is just a sum of non-negative values and therefore can not be negative. Thus G^{rl} can not be a Kronecker delta. Such problems are not unknown. Something similar occurs for photons in Feynman gauge. This case is treated in [3]. The following idea of how to treat this subtlety is strongly motivated from the appropriate quantization in the photonic case.

First one has to keep in mind that the $n^{r}_{\mathbf{p}}$ shall correspond to different independent polarizations of $h'_{\mu\nu}$. Therefore, $n^{r}_{\mathbf{p}}$ and $n^{l\dagger}_{\mathbf{q}}$ should commute for different r and l. This means that G^{rl} has to be diagonal. Furthermore, not all of the entries can be non-negative. Otherwise, $\sum_{r,l=0}^{9} G^{rl}\varepsilon^{r}(\mathbf{p})\varepsilon^{l*}(\mathbf{p})$ would still be a sum of non-negative parameters and the argument above also holds for this more general case. So the consequence is that some of the entries have to be negative. Also note that all $n^{r}_{\mathbf{p}}$ and $n^{r\dagger}_{\mathbf{p}}$ shall be interpreted as pairs of creation and annihilation operators, so they are not allowed to commute, i.e. no G^{rr} is zero. Therefore, by reparametrizing the $\varepsilon^{r}_{\mu\nu}(\mathbf{p})$, the G^{rl} can be chosen to be $(-1)^{\lambda_r}\delta^{rl}$, where $\lambda_r \in \{0,1\}$ for all r. This corresponds to the commutation relations

$$[n^{r}_{\mathbf{p}}, n^{l\dagger}_{\mathbf{q}}]_{-} = (-1)^{\lambda_r}2E_{\mathbf{p}}(2\pi)^3\delta^{rl}\delta(\mathbf{p}-\mathbf{q}). \tag{4.1.15}$$

Now choose $\varepsilon^{0}_{\mu\nu}(\mathbf{p}) = \frac{1}{2}g_{\mu\nu}$ and $\varepsilon^{r}(\mathbf{p}) = 0$ for all $r > 0$ and $\mathbf{p} \in \mathbb{R}^3$, i.e. only $\varepsilon^{0}_{\mu\nu}$ shall contribute to the trace of $h'_{\mu\nu}$. This immediately shows, by setting $\lambda_0 = 1$,

$$\sum_{r=1}^{9}(-1)^{\lambda_r}\varepsilon^{r}_{\mu\nu}(\mathbf{p})\varepsilon^{r*}_{\rho\sigma}(\mathbf{p}) = g_{\mu(\rho}g_{\sigma)\nu} - \frac{1}{4}g_{\mu\nu}g_{\rho\sigma} = P_{\mu\nu\rho\sigma}. \tag{4.1.16}$$

$P_{\mu\nu\rho\sigma}$ is the projector onto the symmetric traceless tensors, which directly follows from $P_{\mu\nu\rho\sigma} = P_{\nu\mu\rho\sigma}$, $P^{\mu}{}_{\mu\rho\sigma} = 0$, $P_{\mu\nu}{}^{\alpha\beta}P_{\alpha\beta\rho\sigma} = P_{\mu\nu\rho\sigma}$ and $P_{\mu\nu}{}^{\alpha\beta}K_{\alpha\beta} = K_{\mu\nu}$ for any symmetric and traceless tensor $K_{\mu\nu}$. Consequently, the remaining $\varepsilon^r_{\mu\nu}$ can be chosen to be an orthogonal basis of the subspace of the symmetric and traceless tensors. By doing so, it follows

$$\begin{aligned} \varepsilon^l_{\mu\nu}(\mathbf{p}) = P_{\mu\nu}{}^{\rho\sigma}\varepsilon^l_{\rho\sigma}(\mathbf{p}) &= \sum_{r=1}^{9}(-1)^{\lambda_r}\varepsilon^r_{\mu\nu}(\mathbf{p})\varepsilon^{r*\rho\sigma}(\mathbf{p})\varepsilon^l_{\rho\sigma}(\mathbf{p}) \\ &= (-1)^{\lambda_l}\varepsilon^l_{\mu\nu}(\mathbf{p})\varepsilon^{l*\rho\sigma}(\mathbf{p})\varepsilon^l_{\rho\sigma}(\mathbf{p}), \end{aligned} \tag{4.1.17}$$

since $\varepsilon^{r*\rho\sigma}(\mathbf{p})\varepsilon^l_{\rho\sigma}(\mathbf{p})$ vanishes for $r \neq l$. So the orthogonality condition

$$\varepsilon^{r*\rho\sigma}(\mathbf{p})\varepsilon^l_{\rho\sigma}(\mathbf{p}) = (-1)^{\lambda_l}\delta^{rl}, \tag{4.1.18}$$

for $l > 0$, has been derived.

Note that not all λ_r for $r > 0$ can be chosen to be 0. This follows from (4.1.16). By setting $\mu = \rho = i$ and $\nu = \sigma = 0$ one gets

$$\sum_{r=1}^{9}(-1)^{\lambda_r}\varepsilon^r_{i0}(\mathbf{p})\varepsilon^{r*}_{i0}(\mathbf{p}) = g_{i(i}g_{0)0} - \frac{1}{4}g_{i0}g_{i0} = \frac{1}{2}g_{ii}g_{00} = -\frac{1}{2} \tag{4.1.19}$$

and the same argument as in the previous cases can be applied: Would all λ_r be 0 for $r > 0$, the left hand side of this equality would be non-negative, which is a contradiction to the right hand side.

To get a better understanding of the effects that are caused by the polarization tensors with $\lambda_r = 1$, it is useful to take a closer look at $\langle 0|n^r_{\mathbf{p}}n^{r\dagger}_{\mathbf{q}}|0\rangle$. A simple calculation shows

$$\langle 0|n^r_{\mathbf{p}}n^{r\dagger}_{\mathbf{q}}|0\rangle = \langle 0|[n^r_{\mathbf{p}}, n^{r\dagger}_{\mathbf{q}}]_-|0\rangle = (-1)^{\lambda_r}2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}). \tag{4.1.20}$$

Thus, just as in the photonic case (see [3]), the polarizations with $\lambda_r = 1$ cause states with negative norm. So it is important to ensure that states with such polarizations do not contribute to physical phenomena. To do so it is necessary to get a better understanding of the BRST charge, which is subject of Section 4.2.

To conclude the quantization of $h'_{\mu\nu}$, a certain basis $\varepsilon^r_{\mu\nu}(\mathbf{p})$ of polarization tensors shall be presented, that turns out to be quite useful for the analysis of the physical sector. For this, let $\hat{k}$ be the 4-momentum of a particle with mass m in the rest frame, i.e. $\hat{k} = (m, 0, 0, 0)$. By defining the polarization tensors $\varepsilon^r_{\mu\nu}(\hat{\mathbf{k}})$ it is possible to construct $\varepsilon^r_{\mu\nu}(\mathbf{p})$ for any momentum p with $p^2 = -m^2$ and $p^0 > 0$, simply by choosing a Lorentz transformation $L(p)$ that turns $\hat{k}$ into p and setting $\varepsilon^r_{\mu\nu}(\mathbf{p}) = L(p)_\mu{}^\rho L(p)_\nu{}^\sigma\varepsilon^r_{\rho\sigma}(\hat{\mathbf{k}})$ (as it is done in [5]). A suitable choice for a basis is

$$\varepsilon^0_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon^1_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\varepsilon^2_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^3_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\varepsilon^4_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon^5_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.1.21}$$
$$\varepsilon^6_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad \varepsilon^7_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\varepsilon^8_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \varepsilon^9_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Just as described above, $\varepsilon^0_{\mu\nu}(\hat{\mathbf{k}})$ is simply $\frac{1}{2}g_{\mu\nu}$. Furthermore, (4.1.18) can be used to find $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$ and $\lambda_5 = ... = \lambda_9 = 0$, while λ_0 has to be set to 1. It is also easy to check that $\hat{k}^\mu\varepsilon^r_{\mu\nu}(\hat{\mathbf{k}}) \neq 0$ for $0 \leq r \leq 4$ and $\hat{k}^\mu\varepsilon^r_{\mu\nu}(\hat{\mathbf{k}}) = 0$ for $r \geq 5$ and $\varepsilon^r(\hat{\mathbf{k}}) = 0$ for $r > 0$ hold. All these properties are invariant under Lorentz transformations. Therefore, they are true for all p, i.e.

$$\varepsilon^0_{\mu\nu}(\mathbf{p}) = \frac{1}{2}g_{\mu\nu}, \quad p^\mu\varepsilon^r_{\mu\nu}(\mathbf{p}) \neq 0 \quad \text{for} \quad 0 \leq r \leq 4, \tag{4.1.22}$$
$$p^\mu\varepsilon^r_{\mu\nu}(\mathbf{p}) = 0 \quad \text{for} \quad r \geq 5 \quad \text{and} \quad \varepsilon^r(\mathbf{p}) = 0 \quad \text{for} \quad r > 0. \tag{4.1.23}$$

Also the λ_r do not change for different p^μ. So $\varepsilon^5_{\mu\nu}(\mathbf{p})$,..., $\varepsilon^9_{\mu\nu}(\mathbf{p})$ form an orthonormal basis that spans the space of all symmetric traceless tensor fields $K_{\mu\nu}(\mathbf{p})$ such that $p^\mu K_{\mu\nu}(\mathbf{p})$ disappears. Therefore, the projector $\widetilde{P}_{\mu\nu\rho\sigma}$ onto that space can be written as

$$\widetilde{P}_{\mu\nu\rho\sigma}(\mathbf{p}) = \sum_{r=5}^{9} \varepsilon^r_{\mu\nu}(\mathbf{p})\varepsilon^{r*}_{\rho\sigma}(\mathbf{p}). \tag{4.1.24}$$

The space spanned by $\varepsilon^5_{\mu\nu}(\mathbf{p})$,..., $\varepsilon^9_{\mu\nu}(\mathbf{p})$ clearly is just the one that is generated by the classical polarizations of spin-2 particles. Furthermore, all their corresponding creation operators correspond to positive norm states, as discussed in (4.1.20). So one would hope that the polarizations $\varepsilon^0_{\mu\nu}(\mathbf{p})$,..., $\varepsilon^4_{\mu\nu}(\mathbf{p})$ turn out to be the unphysical ones, in order for the formalism to be consistent with the classical formulation.

4.1.2 Quantization of the Stückelberg Fields

The Euler-Lagrange equations for A'_μ and ϕ' are obviously just massive Klein-Gordon equations. For the quantization of the spin-1 field the same strategy as for the spin-2 field can be applied. The field has the classical positive energy solutions $\epsilon_\mu(\mathbf{p})e^{\pm ipx}$. Therefore, the quantized field has the form

$$A'_\mu(x) = \int \widetilde{dp} \sum_{r=1}^{4} \left\{ \epsilon^r_\mu(\mathbf{p}) a^r_{\mathbf{p}} e^{ipx} + \epsilon^{r*}_\mu(\mathbf{p}) a^{r\,\dagger}_{\mathbf{p}} e^{-ipx} \right\} = \int \widetilde{dp} \Big\{ a_\mu(\mathbf{p}) e^{ipx} + a^\dagger_\mu(\mathbf{p}) e^{-ipx} \Big\}. \quad (4.1.25)$$

Here the ϵ^r_μ form a basis of the polarization vectors and $a_\mu(\mathbf{p}) = \sum_{r=1}^4 \epsilon^r_\mu(\mathbf{p}) a^r_{\mathbf{p}}$. To choose an appropriate basis of polarization vectors, an analogous approach as in the spin-2 case can be used. The choice

$$\epsilon^1_\mu(\hat{\mathbf{k}}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^2_\mu(\hat{\mathbf{k}}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^3_\mu(\hat{\mathbf{k}}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^4_\mu(\hat{\mathbf{k}}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (4.1.26)$$

turns out to be quite useful for the analysis of BRST transformations. $\epsilon^r_\mu(\mathbf{p})$ is then defined by applying the same Lorentz transformation $L(p)_\nu{}^\mu$ as for $\varepsilon^r_{\mu\nu}(\mathbf{p})$ on $\epsilon^r_\mu(\hat{\mathbf{k}})$. This leads together with the canonical commutation relations, which are again motivated from the corresponding Poisson brackets,

$$[A'_\mu(x), \Sigma_\nu(y)]_-\big|_{x^0=y^0} = i g_{\mu\nu} \delta(\mathbf{x}-\mathbf{y}), \quad (4.1.27)$$

the relation

$$\sum_{r=1}^{4} (-1)^{\lambda_r+1} \epsilon^r_\mu(\hat{\mathbf{k}}) \epsilon^{r*}_\nu(\hat{\mathbf{k}}) = g_{\mu\nu} = \sum_{r=1}^{4} (-1)^{\lambda_r+1} \epsilon^r_\mu(\mathbf{p}) \epsilon^{r*}_\nu(\mathbf{p}) \quad (4.1.28)$$

and analogous arguments as in the spin-2 case to the commutation relations

$$[a^r_{\mathbf{p}}, a^{l\dagger}_{\mathbf{q}}]_- = (-1)^{\lambda_r+1} 2E_{\mathbf{p}} (2\pi)^3 \delta^{lr} \delta(\mathbf{p}-\mathbf{q}) \quad \text{and} \quad [a^r_{\mathbf{p}}, a^l_{\mathbf{q}}]_- = 0. \quad (4.1.29)$$

Here λ_r is defined just as in the spin-2 case. In particular, this implies

$$[a_\mu(\mathbf{p}), a^\dagger_\nu(\mathbf{q})]_- = g_{\mu\nu} 2E_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p}-\mathbf{q}). \quad (4.1.30)$$

Furthermore, note that the polarization vectors come with the orthogonality condition

$$\epsilon^r_\mu(\mathbf{p}) \epsilon^{l\mu*}(\mathbf{p}) = (-1)^{\lambda_r+1} \delta^{rl}. \quad (4.1.31)$$

By keeping $\epsilon^r_\mu(\mathbf{p})\epsilon^{l\mu*}(\mathbf{p}) = \epsilon^r_\mu(\hat{\mathbf{k}})\epsilon^{l\mu*}(\hat{\mathbf{k}})$ in mind, this is easy to see.

The scalar field can be quantized in the same way. By introducing its creation and annihilation operators $b^\dagger_{\mathbf{p}}$ and $b_{\mathbf{p}}$ in momentum space via

$$\phi'(x) = \int \widetilde{dp} \Big\{ b_{\mathbf{p}} e^{ipx} + b^\dagger_{\mathbf{p}} e^{-ipx} \Big\} \quad (4.1.32)$$

and demanding the canonical commutation relation

$$[\phi'(x), \gamma(y)]_-\big|_{x^0=y^0} = i\delta(\mathbf{x}-\mathbf{y}) \quad (4.1.33)$$

one obtains the commutation relations

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}]_- = 2E_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p}-\mathbf{q}) \quad \text{and} \quad [b_{\mathbf{p}}, b_{\mathbf{q}}]_- = 0. \quad (4.1.34)$$

4.1.3 Quantization of the Ghost Fields

For the ghosts the commutation relations become a little more subtle. By introducing the relation[1]

$$[\eta_\mu(x), r_\nu(y)]_+\big|_{x^0=y^0} = -ig_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}) \tag{4.1.35}$$

together with the momentum space expressions

$$\bar{\eta}_\mu(x) = \int \widetilde{dp}\Big\{\bar{v}_\mu(\mathbf{p})e^{ipx} - \bar{v}^\dagger_\mu(\mathbf{p})e^{-ipx}\Big\}, \quad \eta_\mu(x) = \int \widetilde{dp}\Big\{v_\mu(\mathbf{p})e^{ipx} + v^\dagger_\mu(\mathbf{p})e^{-ipx}\Big\} \tag{4.1.36}$$

for the ghosts[2] and making use of $[\bar{v}_\mu(\mathbf{p}), v_\nu(\mathbf{q})]_+ = 0$, one finds

$$\begin{aligned}[\eta_\mu(x), r_\nu(y)]_+\big|_{x^0=y^0} = \int \widetilde{dp} \int \frac{d^3\mathbf{q}}{2(2\pi)^3}\Big\{&i[v_\mu(\mathbf{p}), \bar{v}^\dagger_\nu(\mathbf{q})]_+ e^{ipx-iqy} \\ &+ i[v^\dagger_\mu(\mathbf{p}), \bar{v}_\nu(\mathbf{q})]_+ e^{-ipx+iqy}\Big\}\big|_{x^0=y^0}.\end{aligned} \tag{4.1.37}$$

This strongly suggests to introduce the commutation relations

$$[v_\mu(\mathbf{p}), \bar{v}^\dagger_\nu(\mathbf{q})]_+ = -g_{\mu\nu}2E_\mathbf{p}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \tag{4.1.38}$$

in order to obtain

$$[\eta_\mu(x), r_\nu(y)]_+\big|_{x^0=y^0} = -i\int \widetilde{dp}2E_\mathbf{p}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}g_{\mu\nu} = -ig_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}). \tag{4.1.39}$$

By using the same polarization vectors as for the spin-1 field to express $v_\mu(\mathbf{p})$ and $\bar{v}_\mu(\mathbf{p})$ in terms of this basis as

$$v_\mu(\mathbf{p}) = \sum_{r=1}^{4}\epsilon^r_\mu(\mathbf{p})v^r_\mathbf{p}, \quad \bar{v}_\mu(\mathbf{p}) = \sum_{r=1}^{4}\epsilon^r_\mu(\mathbf{p})\bar{v}^r_\mathbf{p}, \tag{4.1.40}$$

one finds that this is compatible with

$$[v^r_\mathbf{p}, \bar{v}^{l\dagger}_\mathbf{q}]_+ = (-1)^{\lambda_r}2E_\mathbf{p}(2\pi)^3\delta^{rl}\delta(\mathbf{p}-\mathbf{q}) \quad \text{and} \tag{4.1.41}$$

$$[\bar{v}^r_\mathbf{p}, v^l_\mathbf{q}]_+ = [\bar{v}^r_\mathbf{p}, \bar{v}^l_\mathbf{q}]_+ = [v^r_\mathbf{p}, v^l_\mathbf{q}]_+ = [v^r_\mathbf{p}, v^{l\dagger}_\mathbf{q}]_+ = [\bar{v}^r_\mathbf{p}, \bar{v}^{l\dagger}_\mathbf{q}]_+ = 0 \tag{4.1.42}$$

to obtain (4.1.38). The parameter λ_r again is the same as in the spin-2 case. Furthermore, (4.1.38) also guarantees the remaining canonical commutation relation

$$[\bar{\eta}_\mu(x), \bar{r}_\nu(y)]_+\big|_{x^0=y^0} = -ig_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}). \tag{4.1.43}$$

The quantization of the scalar ghost fields works just the same way. The representation in momentum space

$$\bar{\zeta}(x) = \int \widetilde{dp}\Big\{\bar{c}_\mathbf{p}e^{ipx} - \bar{c}^\dagger_\mathbf{p}e^{-ipx}\Big\}, \quad \zeta(x) = \int \widetilde{dp}\Big\{c_\mathbf{p}e^{ipx} + c^\dagger_\mathbf{p}e^{-ipx}\Big\} \tag{4.1.44}$$

[1] Note that the sign convention in this commutation relation is motivated from the convention that is used in [9]. It is important in order to ensure that the BRST charge Ω, which will be derived in the next section, generates the correct transformations. This also justifies the sign convention for the scalar ghosts.

[2] The minus sign in the expression of $\bar{\eta}_\mu$ is necessary in order to ensure its antihermiticity.

together with the canonical commutation relations

$$[\zeta(x), z(y)]_+\big|_{x^0=y^0} = -i\delta(\mathbf{x}-\mathbf{y}) \quad \text{and} \quad [\bar{\zeta}(x), \bar{z}(y)]_+\big|_{x^0=y^0} = -i\delta(\mathbf{x}-\mathbf{y}) \tag{4.1.45}$$

suggests to use

$$[c_{\mathbf{p}}, \bar{c}^\dagger_{\mathbf{q}}]_+ = -2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \tag{4.1.46}$$

and

$$[\bar{c}_{\mathbf{p}}, c_{\mathbf{q}}]_+ = [\bar{c}_{\mathbf{p}}, \bar{c}_{\mathbf{q}}]_+ = [c_{\mathbf{p}}, c_{\mathbf{q}}]_+ = [c_{\mathbf{p}}, c^\dagger_{\mathbf{q}}]_+ = [\bar{c}_{\mathbf{p}}, \bar{c}^\dagger_{\mathbf{q}}]_+ = 0 \tag{4.1.47}$$

as commutation relations.

Remark Note that the orthogonality conditions (4.1.18) and (4.1.31) imply together with $\varepsilon^r(\mathbf{p}) = 0$ for $r > 0$ and $\varepsilon^0_{\mu\nu}(\mathbf{p}) = \frac{1}{2}g_{\mu\nu}$ that the spatial integral over the Hamiltonian (3.4.11) is diagonal in the chosen set of creation and annihilation operators, i.e.

$$\begin{aligned} H = \int d^3\mathbf{x}\mathcal{H} = \int \widetilde{dp} E_{\mathbf{p}} \Big\{ &\sum_{r=0}^{9} (-1)^{\lambda_r} \Big(n^{r\dagger}_{\mathbf{p}} n^r_{\mathbf{p}} + \frac{1}{2}[n^r_{\mathbf{p}}, n^{r\dagger}_{\mathbf{p}}]_- \Big) \\ &+ \sum_{l=1}^{4} (-1)^{\lambda_l+1} \Big(a^{l\dagger}_{\mathbf{p}} a^l_{\mathbf{p}} - \bar{v}^{l\dagger}_{\mathbf{p}} v^l_{\mathbf{p}} - v^{l\dagger}_{\mathbf{p}} \bar{v}^l_{\mathbf{p}} + \frac{1}{2}[a^l_{\mathbf{p}}, a^{l\dagger}_{\mathbf{p}}]_- + [\bar{v}^l_{\mathbf{p}}, v^{l\dagger}_{\mathbf{p}}]_+ \Big) \\ &+ b^\dagger_{\mathbf{p}} b_{\mathbf{p}} - \bar{c}^\dagger_{\mathbf{p}} c_{\mathbf{p}} - c^\dagger_{\mathbf{p}} \bar{c}_{\mathbf{p}} + \frac{1}{2}[b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}]_- + [\bar{c}_{\mathbf{p}}, c^\dagger_{\mathbf{p}}]_+ \Big\}. \end{aligned} \tag{4.1.48}$$

This can be shown by a straightforward calculation.

4.2 The BRST Transformations and Charge

The operators that have been obtained by the canonical quantization of the fields now can be used to find an explicit expression of the BRST charge operator Ω. To do so, one has to derive it by using the Noether procedure.

The gauge transformations (3.2.13) and (3.2.14) of the spin-2 field and the ones of the Stückelberg fields, (3.3.35) and (3.3.36), give the BRST transformation δ_θ with

$$\delta_\theta h'_{\mu\nu} = \theta(\partial_\mu\eta_\nu + \partial_\nu\eta_\mu + m\zeta g_{\mu\nu}), \quad \delta_\theta A'_\mu = \theta\sqrt{2}(-m\eta_\mu + \partial_\mu\zeta), \quad \delta_\theta\phi' = -\theta\sqrt{6}m\zeta, \tag{4.2.1}$$

$$\delta_\theta\bar{\eta}_\mu = -\theta B_\mu, \quad \delta_\theta\bar{\zeta} = -\theta C, \tag{4.2.2}$$

$$\delta_\theta\eta_\mu = \delta_\theta\zeta = 0, \quad \delta_\theta B_\mu = \delta_\theta C = 0, \tag{4.2.3}$$

where B_μ and C are the Nakanishi-Lautrup fields corresponding to η_μ and ζ, respectively. These transformations come with a Slavnov operator s that allows to write the full BRST-invariant action S_{BRST} in the form

$$\begin{aligned} S_{BRST} &= S_{Stb} - \mathrm{s}\int d^4x \Big\{ \bar{\eta}^\mu \Big(\mathcal{G}^1_\mu + \frac{1}{4}B_\mu \Big) + \bar{\zeta}\Big(\mathcal{G}^2 + \frac{1}{4}C \Big) \Big\} \\ &= S_{Stb} + \int d^4x \Big\{ \mathcal{G}^1_\mu B^\mu + \frac{1}{4}B_\mu B^\mu + \mathcal{G}^2 C + \frac{1}{4}C^2 + \bar{\eta}_\mu(\Box - m^2)\eta^\mu + \bar{\zeta}(\Box - m^2)\zeta \Big\}. \end{aligned} \tag{4.2.4}$$

By integrating over the auxiliary fields B_μ and C, they can be replaced by $-2\mathcal{G}^1_\mu$ and $-2\mathcal{G}^2$ respectively, which leads to the gauge fixed action S_{GF} and

$$\mathrm{s}\,\bar{\eta}_\mu = 2\mathcal{G}^1_\mu, \quad \mathrm{s}\,\bar{\zeta} = 2\mathcal{G}^2. \tag{4.2.5}$$

To explicitly calculate the BRST charge, it is necessary to calculate $\mathrm{s}\,\mathcal{L}_{GF}$ first and identify a fitting $\tilde{J}^\lambda$ such that $\mathrm{s}\,\mathcal{L}_{GF} = \partial_\lambda \tilde{J}^\lambda$ holds. It is easy to verify that $\mathrm{s}\,\mathcal{L}_{GF}$ can be brought into the form

$$\begin{aligned}\mathrm{s}\,\mathcal{L}_{GF} = \partial_\lambda\Big(&- 2\partial_\mu\eta_\nu\partial^\mu h'^{\lambda\nu} - 2m^2\eta_\mu h'^{\lambda\mu} + \partial_\mu\eta^\lambda\partial^\mu h' + m^2\eta^\lambda h' \\ &- \sqrt{2}\partial_\mu\zeta\partial^\mu A'^\lambda - \sqrt{2}m^2\zeta A'^\lambda\Big),\end{aligned} \tag{4.2.6}$$

which offers a $\tilde{J}^\lambda$ with the desired property. The Noether current J^λ that corresponds to δ_θ then has the form

$$\begin{aligned}J^\lambda = \tilde{J}^\lambda - \sum_\Phi \mathrm{s}\,\Phi\frac{\partial_L\mathcal{L}_{GF}}{\partial\partial_\lambda\Phi} =&2\partial_\mu\eta_\nu(\partial^\lambda h'^{\mu\nu} - \partial^\mu h'^{\lambda\nu}) + (m^2\eta^\lambda + m\partial^\lambda\zeta)h' \\ &+ 2\partial^\lambda\eta_\mu\partial_\nu h'^{\mu\nu} - 2m^2\eta_\mu h'^{\lambda\mu} + (\partial_\mu\eta^\lambda - \partial^\lambda\eta_\mu)\partial^\mu h' \\ &- (\partial_\mu\eta^\mu + m\zeta)\partial^\lambda h' + \sqrt{2}\partial_\mu\zeta(\partial^\lambda A'^\mu - \partial^\mu A'^\lambda) \\ &- \sqrt{2}m\eta_\mu\partial^\lambda A'^\mu + \sqrt{2}m\partial^\lambda\eta_\mu A'^\mu - \sqrt{2}m^2\zeta A'^\lambda \\ &+ \partial^\lambda\zeta(\sqrt{6}m\phi' + \sqrt{2}\partial_\mu A'^\mu) - \sqrt{6}m\zeta\partial^\lambda\phi'.\end{aligned} \tag{4.2.7}$$

The sum over Φ shall denote the sum over all fields. By using the momenta from (3.4.6), (3.4.7) and (3.4.8), the BRST charge $\Omega = \int d^3\mathbf{x} J^0$ can be brought into the form

$$\begin{aligned}\Omega = \int d^3\mathbf{x}\Big\{&\partial_k\eta^0\partial^k h' + m^2\eta^0 h' - 2\partial_k\eta_\nu\partial^k h'^{0\nu} - 2m^2\eta_\mu h'^{0\mu} - \sqrt{2}\partial_k\zeta\partial^k A'^0 \\ &- \sqrt{2}m^2\zeta A'^0 - 2\bar{r}_\mu\pi^{0\mu} - 2\partial_k\eta_\mu\pi^{k\mu} - m\zeta\pi + \sqrt{2}m\eta_\mu\Sigma^\mu - \sqrt{2}\bar{z}\Sigma^0 \\ &- \sqrt{2}\partial_k\zeta\Sigma^k + m\sqrt{6}\zeta\gamma - \bar{r}_\mu(2\partial_k h'^{k\mu} + \sqrt{2}mA'^\mu) + \bar{r}_k\partial^k h' \\ &- \bar{z}(\sqrt{2}\partial_k A'^k + mh' + \sqrt{6}m\phi')\Big\}.\end{aligned} \tag{4.2.8}$$

In terms of creation and annihilation operators in momentum space, Ω takes the form

$$\begin{aligned}\Omega = \int \widetilde{dp}\Big\{&2p_\mu v_\nu(\mathbf{p})n^{\mu\nu\dagger}(\mathbf{p}) + 2p_\mu v^\dagger_\nu(\mathbf{p})n^{\mu\nu}(\mathbf{p}) - p_\mu v^\mu(\mathbf{p})n^\dagger(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p})n(\mathbf{p}) \\ &+ imc_\mathbf{p} n^\dagger(\mathbf{p}) - imc^\dagger_\mathbf{p} n(\mathbf{p}) + i\sqrt{2}mv_\mu(\mathbf{p})a^{\mu\dagger}(\mathbf{p}) - i\sqrt{2}mv^\dagger_\mu(\mathbf{p})a^\mu(\mathbf{p}) \\ &+ \sqrt{2}c_\mathbf{p} p_\mu a^{\mu\dagger}(\mathbf{p}) + \sqrt{2}c^\dagger_\mathbf{p} p_\mu a^\mu(\mathbf{p}) + i\sqrt{6}mc_\mathbf{p} b^\dagger_\mathbf{p} - i\sqrt{6}mc^\dagger_\mathbf{p} b_\mathbf{p}\Big\}.\end{aligned} \tag{4.2.9}$$

Remark Note that this form of Ω makes the fact that it commutes with the Hamiltonian operator H quite trivial: Obviously, the field operators only appear in pairs of creation and annihilation operators in Ω, i.e. terms of the form $G^\dagger_\mathbf{p} D_\mathbf{p}$, where $G_\mathbf{p}$ and $D_\mathbf{p}$ shall resemble arbitrary annihilation operators of the fields. By keeping (4.1.48) in mind, it is easy to see that H basically is an integral over the number operator $N_\mathbf{p}$ for particles with momentum $\mathbf{p}$, i.e. the operator that satisfies

$$[N_\mathbf{p}, G^\dagger_\mathbf{q}]_- = G^\dagger_\mathbf{q}\delta(\mathbf{p} - \mathbf{q}). \tag{4.2.10}$$

In terms of $N_{\mathbf{p}}$, H reads

$$H = \int d^3\mathbf{p} E_{\mathbf{p}} N_{\mathbf{p}}. \tag{4.2.11}$$

Consequently, since

$$\begin{aligned}[H, G^\dagger_{\mathbf{q}} D_{\mathbf{q}}]_- &= \int d^3\mathbf{p} E_{\mathbf{p}} [N_{\mathbf{p}}, G^\dagger_{\mathbf{p}} D_{\mathbf{q}}]_- = \int d^3\mathbf{p} E_{\mathbf{p}} \Big\{ [N_{\mathbf{p}}, G^\dagger_{\mathbf{q}}] D_{\mathbf{q}} + G^\dagger_{\mathbf{q}} [N_{\mathbf{p}}, D_{\mathbf{q}}]_- \Big\} \\ &= E_{\mathbf{q}} G^\dagger_{\mathbf{q}} D_{\mathbf{q}} - E_{\mathbf{q}} G^\dagger_{\mathbf{q}} D_{\mathbf{q}} = 0,\end{aligned} \tag{4.2.12}$$

one obtains

$$[H, \Omega]_- = 0. \tag{4.2.13}$$

Furthermore, (4.2.9) obviously implies $|0\rangle \in \ker \Omega$.

The BRST transformations of the creation and annihilation operators of the various fields are therefore given by

$$[\Omega, n_{\mu\nu}(\mathbf{p})]_- = -p_\mu v_\nu(\mathbf{p}) - p_\nu v_\mu(\mathbf{p}) + imc_{\mathbf{p}} g_{\mu\nu}, \tag{4.2.14}$$
$$[\Omega, n^\dagger_{\mu\nu}(\mathbf{p})]_- = p_\mu v^\dagger_\nu(\mathbf{p}) + p_\nu v^\dagger_\mu(\mathbf{p}) + imc^\dagger_{\mathbf{p}} g_{\mu\nu}, \tag{4.2.15}$$
$$[\Omega, a_\mu(\mathbf{p})]_- = -i\sqrt{2} m v_\mu(\mathbf{p}) - \sqrt{2} p_\mu c_{\mathbf{p}}, \quad [\Omega, a^\dagger_\mu(\mathbf{p})]_- = -i\sqrt{2} m v^\dagger_\mu(\mathbf{p}) + \sqrt{2} p_\mu c^\dagger_{\mathbf{p}}, \tag{4.2.16}$$
$$[\Omega, b_{\mathbf{p}}]_- = -i\sqrt{6} m c_{\mathbf{p}}, \quad [\Omega, b^\dagger_{\mathbf{p}}]_- = -i\sqrt{6} m c^\dagger_{\mathbf{p}}, \tag{4.2.17}$$
$$[\Omega, \bar{v}_\mu(\mathbf{p})]_+ = -2p^\nu n_{\mu\nu}(\mathbf{p}) + p_\mu n(\mathbf{p}) + i\sqrt{2} m a_\mu(\mathbf{p}), \tag{4.2.18}$$
$$[\Omega, \bar{v}^\dagger_\mu(\mathbf{p})]_+ = -2p^\nu n^\dagger_{\mu\nu}(\mathbf{p}) + p_\mu n^\dagger(\mathbf{p}) - i\sqrt{2} m a^\dagger_\mu(\mathbf{p}), \tag{4.2.19}$$
$$[\Omega, v_\mu(\mathbf{p})]_+ = [\Omega, v^\dagger_\mu(\mathbf{p})]_+ = 0, \tag{4.2.20}$$
$$[\Omega, \bar{c}_{\mathbf{p}}]_+ = -\sqrt{2} p^\mu a_\mu(\mathbf{p}) + imn(\mathbf{p}) + i\sqrt{6} m b_{\mathbf{p}}, \tag{4.2.21}$$
$$[\Omega, \bar{c}^\dagger_{\mathbf{p}}]_+ = -\sqrt{2} p^\mu a^\dagger_\mu(\mathbf{p}) - imn^\dagger(\mathbf{p}) - i\sqrt{6} m b^\dagger_{\mathbf{p}}, \tag{4.2.22}$$
$$[\Omega, c_{\mathbf{p}}]_+ = [\Omega, c^\dagger_{\mathbf{p}}]_+ = 0. \tag{4.2.23}$$

4.3 Analysis of the Physical Sector

Since the BRST charge now has been derived, it is possible to make quantitative statements about the physical sector $\ker \Omega$. The first step is to examine the subspace of $\ker \Omega$ that only contains spin-2 fields. From the physical point of view this clearly is the most interesting part of the kernel of Ω.

4.3.1 First Results for the Structure of the Physical Sector

Let $\mathcal{V}^{h'} \subset \mathcal{V}$ be the subspace that is generated by the $n^{r\dagger}$ operators, i.e. the space that only contains spin-2 particles. In order to learn more about the subspace $\ker \Omega \cap \mathcal{V}^{h'}$ of $\ker \Omega$ it is sufficient to restrict Ω to the very simple subspace $\mathcal{V}^{h'}$. This reduces Ω to

$$\Omega\big|_{\mathcal{V}^{h'}} = \int \widetilde{dp} \Big\{ 2p_\mu v^\dagger_\nu(\mathbf{p}) n^{\mu\nu}(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p}) n(\mathbf{p}) - imc^\dagger_{\mathbf{p}} n(\mathbf{p}) \Big\}. \tag{4.3.1}$$

Now let $|\psi\rangle$ be an arbitrary element of $\ker\Omega \cap \mathcal{V}^{h'}$. Then

$$\Omega|\psi\rangle = \int \widetilde{dp} \Big\{ \big(2p_\mu v_\nu^\dagger(\mathbf{p}) n^{\mu\nu}(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p}) n(\mathbf{p})\big)|\psi\rangle - imc_{\mathbf{p}}^\dagger n(\mathbf{p})|\psi\rangle \Big\} \tag{4.3.2}$$

is a linear combination of vectors with one $v_\mu^\dagger(\mathbf{p})$ ghost and no $c_{\mathbf{p}}^\dagger$ ghost and vectors with one $c_{\mathbf{p}}^\dagger$ ghost and no $v_\mu^\dagger(\mathbf{p})$ ghosts. Since these two types of ghosts are independent from each other, $(2p_\mu v_\nu^\dagger(\mathbf{p}) n^{\mu\nu}(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p}) n(\mathbf{p}))|\psi\rangle$ and $imc_{\mathbf{p}}^\dagger n(\mathbf{p})|\psi\rangle$ both have to disappear in order to obtain $\Omega|\psi\rangle = 0$. Therefore, since $v_\mu^\dagger(\mathbf{p})$ and $c_{\mathbf{p}}^\dagger$ are both creation operators,

$$n(\mathbf{p})|\psi\rangle = 0 \quad \text{and} \quad p^\mu n_{\mu\nu}(\mathbf{p})|\psi\rangle = 0 \tag{4.3.3}$$

follows. By setting

$$h'^{(+)}_{\mu\nu}(x) = \int \widetilde{dp}\, n_{\mu\nu}(\mathbf{p}) e^{ipx} \tag{4.3.4}$$

this can be equivalently written as

$$h'^{(+)}(x)|\psi\rangle = 0 \quad \text{and} \quad \partial^\mu h'^{(+)}_{\mu\nu}(x)|\psi\rangle = 0, \tag{4.3.5}$$

which is quite similar to the Gupta-Bleuler supplementary condition (2.4.10) for the $U(1)$ gauge field. Together with the Klein-Gordon equation $(\Box - m^2) h'^{(+)}_{\mu\nu}|\psi\rangle = 0$, which of course is a direct consequence of the on-shellness of the particles, these conditions resemble the operator version of the Euler-Lagrange equations (3.1.26) for classical massive spin-2 particles. By using the polarization tensors $\varepsilon^r_{\mu\nu}(\mathbf{p})$, as defined in Section 4.1.1, it follows from $p^\mu \varepsilon^r_{\mu\nu}(\mathbf{p}) = 0$ for all $r \geq 5$ and $\varepsilon^r(\mathbf{p}) = 0$ for all $r > 0$ that (4.3.3) can also be written as

$$\varepsilon^0(\mathbf{p}) n^0_{\mathbf{p}}|\psi\rangle = 0 \quad \text{and} \quad \sum_{r=1}^{4} p^\mu \varepsilon^r_{\mu\nu}(\mathbf{p}) n^r_{\mathbf{p}}|\psi\rangle = 0, \tag{4.3.6}$$

where the first equation has been used to eliminate the $\varepsilon^0_{\mu\nu}(\mathbf{p})$ term in the sum of the second one. A very desirable result would be that $n^r_{\mathbf{p}}|\psi\rangle$ disappears for all $r \leq 4$. This would mean that in $\ker\Omega \cap \mathcal{V}^{h'}$ only the polarization tensors from the classical spin-2 particle show up. In fact this is true, as can be shown in the following way:

$n^0_{\mathbf{p}}|\psi\rangle = 0$ is already given. To show the analogous relation for the remaining four polarizations, it is useful to have a look at $\hat{k}^\mu \varepsilon^r_{\mu\nu}(\hat{\mathbf{k}})$ for $1 \leq r \leq 4$. These are, up to normalization, just the standard basis vectors. In particular, they are linearly independent from each other. Since the $p^\mu \varepsilon^r_{\mu\nu}(\mathbf{p})$ are just Lorentz transformations of their $\hat{k}$ versions, they are also linearly independent. So

$$M = \begin{pmatrix} | & | & | & | \\ p^\mu \varepsilon^1_{\mu\nu}(\mathbf{p}) & p^\mu \varepsilon^2_{\mu\nu}(\mathbf{p}) & p^\mu \varepsilon^3_{\mu\nu}(\mathbf{p}) & p^\mu \varepsilon^4_{\mu\nu}(\mathbf{p}) \\ | & | & | & | \end{pmatrix}, \tag{4.3.7}$$

i.e. the matrix with $p^\mu \varepsilon^r_{\mu\nu}(\mathbf{p})$ as r-th column, is invertible. Using this matrix notation, the second part of (4.3.6) can be written as

$$\begin{pmatrix} | & | & | & | \\ p^\mu \varepsilon^1_{\mu\nu}(\mathbf{p}) & p^\mu \varepsilon^2_{\mu\nu}(\mathbf{p}) & p^\mu \varepsilon^3_{\mu\nu}(\mathbf{p}) & p^\mu \varepsilon^4_{\mu\nu}(\mathbf{p}) \\ | & | & | & | \end{pmatrix} \begin{pmatrix} n^1_{\mathbf{p}}|\psi\rangle \\ n^2_{\mathbf{p}}|\psi\rangle \\ n^3_{\mathbf{p}}|\psi\rangle \\ n^4_{\mathbf{p}}|\psi\rangle \end{pmatrix} = 0. \tag{4.3.8}$$

The invertibility of M then implies $n^r_{\mathbf{p}}|\psi\rangle = 0$ for all $r = 1, ..., 4$. So $\ker\Omega \cap \mathcal{V}^{h'}$ in fact is just what would have been expected from a naive quantization, ignoring the problem that the momenta can not be defined without introducing Stückelberg fields.

An even more intriguing result can be obtained by looking at a $|\Phi\rangle \in \ker\Omega$ of the form

$$|\Phi\rangle = \int \widetilde{dp}\Big\{ f_1^{\mu\nu}(\mathbf{p}) n^\dagger_{\mu\nu}(\mathbf{p}) + f_2^{\mu}(\mathbf{p}) a^\dagger_{\mu}(\mathbf{p}) + f_3(\mathbf{p}) b^\dagger_{\mathbf{p}} \Big\} |\Psi\rangle, \tag{4.3.9}$$

where $|\Psi\rangle$ itself shall also be contained in $\ker\Omega$. Using the commutation relations of Ω with creation operators, it is easy to check that

$$\begin{aligned} a^\dagger_\mu(\mathbf{p}) &= \frac{i}{\sqrt{2}m}[\Omega, \bar{v}^\dagger_\mu(\mathbf{p})]_+ + \frac{i\sqrt{2}}{m} p^\nu n^\dagger_{\mu\nu}(\mathbf{p}) - \frac{i}{\sqrt{2}m} p_\mu n^\dagger(\mathbf{p}) \quad \text{and} \\ b^\dagger_{\mathbf{p}} &= \frac{i}{\sqrt{6}m}[\Omega, \bar{c}^\dagger_{\mathbf{p}} + \frac{i}{m} p^\mu \bar{v}^\dagger_\mu(\mathbf{p})]_+ - \frac{1}{m^2}\sqrt{\frac{2}{3}} p^\mu p^\nu n^\dagger_{\mu\nu}(\mathbf{p}) - \sqrt{\frac{2}{3}} n^\dagger(\mathbf{p}) \end{aligned} \tag{4.3.10}$$

hold. Therefore, together with $\Omega|\Psi\rangle = 0$, $|\Phi\rangle$ can be written as

$$\begin{aligned} |\Phi\rangle = \int \widetilde{dp}\Big\{ \Big(& f_1^{\mu\nu}(\mathbf{p}) + \frac{i\sqrt{2}}{m} p^{(\mu} f_2^{\nu)}(\mathbf{p}) - \frac{i}{\sqrt{2}m} p_\lambda f_2^\lambda(\mathbf{p}) g^{\mu\nu} \\ & - \sqrt{\frac{2}{3}}\Big(\frac{1}{m^2} p^\mu p^\nu + g^{\mu\nu}\Big) f_3(\mathbf{p})\Big) n^\dagger_{\mu\nu}(\mathbf{p})\Big\} |\Psi\rangle \\ & + \Omega \int \widetilde{dp}\Big\{ \frac{i}{\sqrt{2}m} f_2^\mu(\mathbf{p}) \bar{v}^\dagger_\mu(\mathbf{p}) - \frac{1}{\sqrt{6}m^2} f_3(\mathbf{p}) p^\mu \bar{v}^\dagger_\mu(\mathbf{p}) + \frac{i}{\sqrt{6}m} f_3(\mathbf{p}) \bar{c}^\dagger_{\mathbf{p}} \Big\} |\Psi\rangle. \end{aligned} \tag{4.3.11}$$

So, by using $\Omega^2 = 0$, the question if $\Omega|\Phi\rangle = 0$ holds, reduces to the question what properties a $f^{\mu\nu}(\mathbf{p})$ must have, such that $\Omega \int \widetilde{dp} f^{\mu\nu}(\mathbf{p}) n^\dagger_{\mu\nu}(\mathbf{p})|\Psi\rangle = 0$ holds. Here $f^{\mu\nu}$ obviously can be assumed to be symmetric. By using $\Omega|\Psi\rangle = 0$ once more, one gets

$$\begin{aligned} \Omega \int \widetilde{dp} f^{\mu\nu}(\mathbf{p}) n^\dagger_{\mu\nu}(\mathbf{p}) |\Psi\rangle &= \int \widetilde{dp} f^{\mu\nu}(\mathbf{p}) [\Omega, n^\dagger_{\mu\nu}(\mathbf{p})]_- |\Psi\rangle \\ &= \int \widetilde{dp}\Big\{ 2p_\mu f^{\mu\nu}(\mathbf{p}) v^\dagger_\nu(\mathbf{p}) + imf(\mathbf{p}) c^\dagger(\mathbf{p}) \Big\} |\Psi\rangle. \end{aligned} \tag{4.3.12}$$

The vectors $2p_\mu f^{\mu\nu}(\mathbf{p}) v^\dagger_\nu(\mathbf{p})|\Psi\rangle$ and $imf(\mathbf{p})c^\dagger(\mathbf{p})|\Psi\rangle$ clearly are linearly independent from each other. Consequently, one obtains $f(\mathbf{p}) = 0$ and $p_\mu f^{\mu\nu}(\mathbf{p}) = 0$. This on the other hand implies $f^{\mu\nu}(\mathbf{p}) = \widetilde{P}^{\mu\nu\rho\sigma} f_{\rho\sigma}(\mathbf{p})$, with $\widetilde{P}^{\mu\nu\rho\sigma}$ defined as in (4.1.24) and therefore

$$\begin{aligned} f^{\mu\nu}(\mathbf{p}) n^\dagger_{\mu\nu}(\mathbf{p}) &= \widetilde{P}^{\mu\nu\rho\sigma} f_{\rho\sigma}(\mathbf{p}) \sum_{r=0}^{9} \varepsilon^{r*}_{\mu\nu}(\mathbf{p}) n^{r\dagger}_{\mathbf{p}} \\ &= f_{\rho\sigma}(\mathbf{p}) \sum_{r=0}^{9} \widetilde{P}^{\rho\sigma\mu\nu*} \varepsilon^{r*}_{\mu\nu}(\mathbf{p}) n^{r\dagger}_{\mathbf{p}} = f^{\rho\sigma}(\mathbf{p}) \sum_{r=5}^{9} \varepsilon^{r*}_{\rho\sigma}(\mathbf{p}) n^{r\dagger}_{\mathbf{p}}. \end{aligned} \tag{4.3.13}$$

So only the polarizations from $\ker\Omega \cap \mathcal{V}^{h'}$ contribute. This means that a $|\Phi\rangle \in \ker\Omega$ of the form (4.3.9) can always be written as

$$|\Phi\rangle = \int \widetilde{dp} f^{\mu\nu}(\mathbf{p}) n^\dagger_{\mu\nu}(\mathbf{p}) |\Psi\rangle + \Omega|\varsigma\rangle, \tag{4.3.14}$$

with some $|\varsigma\rangle \in \mathcal{V}$ and a $f^{\mu\nu}(\mathbf{p})$ that only generates the polarizations from $\ker\Omega \cap \mathcal{V}^{h'}$.

This result raises the question whether every element of $\ker\Omega$ can be written as a sum of an element of $\ker\Omega \cap \mathcal{V}^{h'}$ and one of $\operatorname{im}\Omega$. In fact this is true. In order to prove it, one has to understand the behavior of Ω on the level of creation and annihilation operators for the different polarizations of the fields. For this, one can use the relations

$$[\Omega, n^0_{\mathbf{p}}]_- = \frac{1}{2}[\Omega, n_{\mu\nu}(\mathbf{p})]_- g^{\mu\nu}, \tag{4.3.15}$$

$$[\Omega, n^r_{\mathbf{p}}]_- = (-1)^{\lambda_r}[\Omega, n_{\mu\nu}(\mathbf{p})]_- \varepsilon^{r\mu\nu *}(\mathbf{p}) \quad \text{for} \quad 1 \le r \le 9, \tag{4.3.16}$$

$$[\Omega, a^r_{\mathbf{p}}]_- = (-1)^{\lambda_r+1}[\Omega, a_\mu(\mathbf{p})]_- \epsilon^{r\mu *}(\mathbf{p}) \quad \text{for} \quad 1 \le r \le 4, \tag{4.3.17}$$

$$[\Omega, \bar{v}^r_{\mathbf{p}}]_+ = (-1)^{\lambda_r+1}[\Omega, \bar{v}_\mu(\mathbf{p})]_+ \epsilon^{r\mu *}(\mathbf{p}) \quad \text{for} \quad 1 \le r \le 4 \quad \text{and} \tag{4.3.18}$$

$$[\Omega, v^r_{\mathbf{p}}]_+ = (-1)^{\lambda_r+1}[\Omega, v_\mu(\mathbf{p})]_+ \epsilon^{r\mu *}(\mathbf{p}) \quad \text{for} \quad 1 \le r \le 4. \tag{4.3.19}$$

They are an immediate implication of (4.1.18) and (4.1.31). Together with (4.2.14)-(4.2.23) and the fact that

$$\begin{aligned} &\epsilon^l_\nu(\mathbf{p}) p_\mu \varepsilon^{r\mu\nu *}(\mathbf{p}) = \epsilon^l_\nu(\hat{\mathbf{k}}) \hat{k}_\mu \varepsilon^{r\mu\nu *}(\hat{\mathbf{k}}), \quad p^\mu \epsilon^l_\mu(\mathbf{p}) = \hat{k}^\mu \epsilon^l_\mu(\hat{\mathbf{k}}) \quad \text{and} \\ &\epsilon^l_\nu(\mathbf{p}) \epsilon^{s\nu *}(\mathbf{p}) = \epsilon^l_\nu(\hat{\mathbf{k}}) \epsilon^{s\nu *}(\hat{\mathbf{k}}) \end{aligned} \tag{4.3.20}$$

hold for all $r = 0, ..., 9$, $l = 1, ..., 4$ and $s = 1, ..., 4$, which is a direct consequence of the construction of the polarization tensors and vectors via Lorentz transformations, one finds

$$[\Omega, n^0_{\mathbf{p}}]_- = -mv^1_{\mathbf{p}} + i2mc_{\mathbf{p}}, \quad [\Omega, n^1_{\mathbf{p}}]_- = \sqrt{3}mv^1_{\mathbf{p}}, \tag{4.3.21}$$

$$[\Omega, n^r_{\mathbf{p}}]_- = \sqrt{2}mv^r_{\mathbf{p}} \quad \text{for} \quad r = 2,3,4, \quad [\Omega, n^l_{\mathbf{p}}]_- = 0 \quad \text{for} \quad l \ge 5, \tag{4.3.22}$$

$$[\Omega, a^r_{\mathbf{p}}]_- = -im\sqrt{2}v^r_{\mathbf{p}} + \sqrt{2}m\delta^{r1}c_{\mathbf{p}} \quad \text{for} \quad r = 1,2,3,4, \quad [\Omega, b_{\mathbf{p}}]_- = -im\sqrt{6}c_{\mathbf{p}}, \tag{4.3.23}$$

$$[\Omega, \bar{v}^1_{\mathbf{p}}]_+ = -mn^0_{\mathbf{p}} - \sqrt{3}mn^1_{\mathbf{p}} + i\sqrt{2}ma^1_{\mathbf{p}}, \tag{4.3.24}$$

$$[\Omega, \bar{v}^r_{\mathbf{p}}]_+ = -\sqrt{2}mn^r_{\mathbf{p}} + i\sqrt{2}ma^r_{\mathbf{p}} \quad \text{for} \quad r = 2,3,4, \tag{4.3.25}$$

$$[\Omega, \bar{c}_{\mathbf{p}}]_+ = i2mn^0_{\mathbf{p}} - \sqrt{2}ma^1_{\mathbf{p}} + i\sqrt{6}mb_{\mathbf{p}} \quad \text{and} \tag{4.3.26}$$

$$[\Omega, v^r_{\mathbf{p}}]_+ = [\Omega, c_{\mathbf{p}}]_+ = 0 \quad \text{for} \quad r = 1,2,3,4. \tag{4.3.27}$$

In particular, the $n^r_{\mathbf{p}}$ commute with Ω for $r \ge 5$ and can not be written as anticommutator of Ω with some linear combination of the ghost fields. This can be seen easily, since these $n^r_{\mathbf{p}}$ do not appear in any of the anticommutators of Ω with ghosts. According to Section 2.4.4, the space that is generated by such fields is a good candidate for a space that is isometric isomorphic to the cohomology $\mathfrak{V}$. The generated space in this case is just $\ker\Omega \cap \mathcal{V}^{h'}$, which directly follows from its analysis above. This is another result that supports the assumption that any element of $\ker\Omega$ can be written as sum of an element of $\ker\Omega \cap \mathcal{V}^{h'}$ and one of $\operatorname{im}\Omega$.

Furthermore, $\ker\Omega \cap \mathcal{V}^{h'}$ already contains five independent polarizations, i.e. degrees of freedom. The classical theory that is described by S_{Stb} also only has five degrees of freedom, since the introduced Stückelberg fields offer no additional ones. So in order for the BRST formalism, as it is presented here, to make sense, all additional independent degrees of freedom in $\ker\Omega$ have to be unphysical.

The proposition that all $n^r_{\mathbf{p}}$ for $r \ge 5$ offer independent physical degrees of freedom is equivalent to the statement $\left(\ker\Omega \cap \mathcal{V}^{h'}\right) \cap \operatorname{im}\Omega = \{0\}$. It is plausible, since none of these

$n^r_{\mathbf{p}}$ can be created by applying $[\Omega, \cdot]_+$ to any ghost configuration. A formal proof is given in the next subsection. However, this justifies to label the polarizations corresponding to $n^r_{\mathbf{p}}$ for $r \geq 5$ and the states in $\ker \Omega \cap \mathcal{V}^{h'}$ *physical* from now on. All other polarizations, including the Stückelberg and ghost fields, shall be denoted as *unphysical*. Furthermore, the notation $\mathcal{V}_{phys} = \ker \Omega \cap \mathcal{V}^{h'}$ shall be introduced. So if it can be proven that any element $|\Psi\rangle \in \ker \Omega$ has the form

$$|\Psi\rangle = |\psi\rangle + \Omega|\varsigma\rangle \tag{4.3.28}$$

with a unique $|\psi\rangle \in \mathcal{V}_{phys}$ and $|\varsigma\rangle \in \mathcal{V}$, according to Theorem 2.4.8, one would have shown that $\mathcal{V}_{phys}$ is isometric isomorphic to $\mathfrak{V} = \ker \Omega / \operatorname{im} \Omega$.

4.3.2 The Quartet Mechanism for the Massive Case

There is a very elegant way to prove this quite desirable statement by exploiting the algebraic structure of the participating operators. This is known as the *quartet mechanism*. The proof presented here is basically an adaptation of the more general formulation given in [8] and [9] to the special case of spin-2 particles.

The first step is to introduce new fields by the following substitutions:

$$X^0_{\mathbf{p}} = -in^0_{\mathbf{p}} + \frac{1}{\sqrt{2}}a^1_{\mathbf{p}} - i\sqrt{\frac{3}{2}}b_{\mathbf{p}}, \quad Y^0_{\mathbf{p}} = -\frac{i}{3}n^0_{\mathbf{p}} + \frac{1}{3\sqrt{2}}a^1_{\mathbf{p}} + \frac{i}{\sqrt{6}}b_{\mathbf{p}}, \tag{4.3.29}$$

$$X^1_{\mathbf{p}} = \frac{1}{2}n^0_{\mathbf{p}} + \frac{\sqrt{3}}{2}n^1_{\mathbf{p}} - \frac{i}{\sqrt{2}}a^1_{\mathbf{p}}, \quad Y^1_{\mathbf{p}} = -\frac{1}{3}n^0_{\mathbf{p}} + \frac{1}{\sqrt{3}}n^1_{\mathbf{p}} + i\frac{\sqrt{2}}{3}a^1_{\mathbf{p}}, \tag{4.3.30}$$

$$X^r_{\mathbf{p}} = \frac{1}{\sqrt{2}}(n^r_{\mathbf{p}} - ia^r_{\mathbf{p}}), \quad Y^r_{\mathbf{p}} = \frac{1}{\sqrt{2}}(n^r_{\mathbf{p}} + ia^r_{\mathbf{p}}) \quad \text{for} \quad r = 2, 3, 4. \tag{4.3.31}$$

It is quite easy to check that these new fields satisfy the commutation relations

$$[X^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_- = (-1)^{\lambda_r} 2E_{\mathbf{p}}(2\pi)^3 \delta^{lr} \delta(\mathbf{p} - \mathbf{q}), \tag{4.3.32}$$

$$[X^r_{\mathbf{p}}, X^{l\dagger}_{\mathbf{q}}]_- = [Y^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_- = [X^r_{\mathbf{p}}, X^l_{\mathbf{q}}]_- = [Y^r_{\mathbf{p}}, Y^l_{\mathbf{q}}]_- = 0 \tag{4.3.33}$$

for all $l, r = 0, ..., 4$. Furthermore, they also commute with all the operators corresponding to physical polarizations and ghosts. So the $X^r_{\mathbf{p}}$ and $Y^{r\dagger}_{\mathbf{p}}$ form pairs of annihilation and creation operators that are independent from each other and all other fields. One can easily verify that

$$n^0_{\mathbf{p}} = \frac{1}{3}X^1_{\mathbf{p}} - \frac{1}{2}Y^1_{\mathbf{p}} + i\Big(\frac{1}{3}X^0_{\mathbf{p}} + Y^0_{\mathbf{p}}\Big), \quad n^1_{\mathbf{p}} = \frac{1}{2\sqrt{3}}(2X^1_{\mathbf{p}} + 3Y^1_{\mathbf{p}}), \tag{4.3.34}$$

$$n^r_{\mathbf{p}} = \frac{1}{\sqrt{2}}(X^r_{\mathbf{p}} + Y^r_{\mathbf{p}}) \quad \text{for} \quad r = 2, 3, 4, \tag{4.3.35}$$

$$a^1_{\mathbf{p}} = \frac{1}{\sqrt{2}}\Big(\frac{1}{3}X^0_{\mathbf{p}} + Y^0_{\mathbf{p}} + i\Big(\frac{2}{3}X^1_{\mathbf{p}} - Y^1_{\mathbf{p}}\Big)\Big), \quad a^r_{\mathbf{p}} = \frac{i}{\sqrt{2}}(X^r_{\mathbf{p}} - Y^r_{\mathbf{p}}) \quad \text{for} \quad r = 2, 3, 4 \tag{4.3.36}$$

$$\text{and} \quad b_{\mathbf{p}} = \frac{i}{\sqrt{6}}(X^0_{\mathbf{p}} - 3Y^0_{\mathbf{p}}) \tag{4.3.37}$$

hold. Therefore, the $X^r_{\mathbf{p}}$ and $Y^r_{\mathbf{p}}$ form just another set of operators that can be used, together with the ghost generators, to generate any configuration of unphysical polarizations. However, this special choice of operators offers the remarkable form of BRST transformations

$$[\Omega, Y^0_{\mathbf{p}}]_- = 2mc_{\mathbf{p}}, \quad [\Omega, \bar{c}_{\mathbf{p}}]_+ = -2mX^0_{\mathbf{p}}, \tag{4.3.38}$$

$$[\Omega, Y^r_{\mathbf{p}}]_- = 2mv^r_{\mathbf{p}}, \quad [\Omega, \bar{v}^r_{\mathbf{p}}]_+ = -2mX^r_{\mathbf{p}} \quad \text{for} \quad r = 1, ..., 4. \tag{4.3.39}$$

That means the fields $X^0_{\mathbf{p}}$, $Y^0_{\mathbf{p}}$, $c_{\mathbf{p}}$ and $\bar{c}_{\mathbf{p}}$ form a so-called *quartet*, just as $X^r_{\mathbf{p}}$, $Y^r_{\mathbf{p}}$, $v^r_{\mathbf{p}}$ and $\bar{v}^r_{\mathbf{p}}$ for $r = 1, ..., 4$. These transformations in particular imply that $X^l_{\mathbf{p}}$ commutes with Ω for all $l = 0, ..., 4$. Thus the $X^r_{\mathbf{p}}$ and $Y^r_{\mathbf{p}}$ offer a way to reduce the more or less complex structure of the unphysical polarizations to the much easier case of independent pairs of creation and annihilation operators that all turn into independent ghost fields under BRST transformations.

The second step is to have a closer look at the projector $P^{(n)}$ onto the subspace of $\mathcal{V}$ that contains all the states with n unphysical fields. $P^{(0)}$ therefore is the projector onto $\mathcal{V}_{phys}$. According to [8] $P^{(n)}$ can be inductively formulated as

$$\begin{aligned} P^{(n)} = \frac{1}{n} \int \widetilde{dp} \Big\{ &- X^{0\dagger}_{\mathbf{p}} P^{(n-1)} Y^0_{\mathbf{p}} - Y^{0\dagger}_{\mathbf{p}} P^{(n-1)} X^0_{\mathbf{p}} - \bar{c}^\dagger_{\mathbf{p}} P^{(n-1)} c_{\mathbf{p}} - c^\dagger_{\mathbf{p}} P^{(n-1)} \bar{c}_{\mathbf{p}} \\ &+ \sum_{r=1}^{4} (-1)^{\lambda_r} \Big(X^{r\dagger}_{\mathbf{p}} P^{(n-1)} Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}} P^{(n-1)} X^r_{\mathbf{p}} + \bar{v}^{r\dagger}_{\mathbf{p}} P^{(n-1)} v^r_{\mathbf{p}} + v^{r\dagger}_{\mathbf{p}} P^{(n-1)} \bar{v}^r_{\mathbf{p}} \Big) \Big\} \end{aligned} \tag{4.3.40}$$

for $n \geq 1$.

Remark It is easy to understand that $P^{(n)}$ in fact is the projector onto the space with n unphysical fields. Consider the operator

$$\int \widetilde{dp} (-1)^{\lambda_r} X^{r\dagger}_{\mathbf{p}} P^{(n-1)} Y^r_{\mathbf{p}} \tag{4.3.41}$$

for some $r = 0, ..., 4$ and let it act onto a vector $|m\rangle$ with m unphysical fields. $Y^r_{\mathbf{p}}$ annihilates one certain type of unphysical polarization. Therefore, $Y^r_{\mathbf{p}}|m\rangle$ is an element of the space that contains only vectors with $m-1$ unphysical polarizations. So $P^{(n-1)} Y^r_{\mathbf{p}}|m\rangle$ clearly vanishes if $m \neq n$.

Now assume that $m = n$. Since $P^{(n-1)}$ is the projector onto the space with $n-1$ unphysical polarizations, it leaves $Y^r_{\mathbf{p}}|n\rangle$ invariant. So the operator (4.3.41) takes the form

$$\int \widetilde{dp} (-1)^{\lambda_r} X^{r\dagger}_{\mathbf{p}} Y^r_{\mathbf{p}} \tag{4.3.42}$$

on the space with n unphysical polarizations and vanishes on any space that corresponds to a different number.

This argumentation now can be applied to all the remaining terms of $P^{(n)}$ to find that it takes the form

$$\begin{aligned} P^{(n)} = \frac{1}{n} \int \widetilde{dp} \Big\{ &- X^{0\dagger}_{\mathbf{p}} Y^0_{\mathbf{p}} - Y^{0\dagger}_{\mathbf{p}} X^0_{\mathbf{p}} - \bar{c}^\dagger_{\mathbf{p}} c_{\mathbf{p}} - c^\dagger_{\mathbf{p}} \bar{c}_{\mathbf{p}} \\ &+ \sum_{r=1}^{4} (-1)^{\lambda_r} \Big(X^{r\dagger}_{\mathbf{p}} Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}} X^r_{\mathbf{p}} + \bar{v}^{r\dagger}_{\mathbf{p}} v^r_{\mathbf{p}} + v^{r\dagger}_{\mathbf{p}} \bar{v}^r_{\mathbf{p}} \Big) \Big\} \end{aligned} \tag{4.3.43}$$

on the space with n unphysical polarizations and is zero on all spaces with a different number of unphysical fields. The integral in (4.3.43) clearly is just the number operator of the unphysical polarizations. The signs of the different terms are chosen in such a way that they fit to the signs that come from the corresponding commutation relations. So on the space with n unphysical particles this integral is just $n\mathbb{1}$ and therefore $P^{(n)}$ is just the unit operator on this space. This shows that $P^{(n)}$ in fact projects onto it.

This form of $P^{(n)}$ offers the possibility to prove the following quite remarkable theorem.

Theorem 4.3.1 *The projectors $P^{(n)}$ commute with Ω for all $n \in \mathbb{N}_0$.*

Proof: This statement can be shown by using the method of complete induction. For $n = 0$ the statement is obviously true, since $P^{(0)}$ can be written as a linear combination of projectors of the form

$$\frac{1}{s!}\int \widetilde{dp}_1...\widetilde{dp}_s n^{r_1\dagger}_{\mathbf{p}_1}...n^{r_s\dagger}_{\mathbf{p}_s}|0\rangle\langle 0|n^{r_s}_{\mathbf{p}_s}...n^{r_1}_{\mathbf{p}_1}, \tag{4.3.44}$$

where $r_1, ..., r_s \geq 5$ and $s \in \mathbb{N}_0$. Since Ω commutes with all these creation and annihilation operators and $\Omega|0\rangle = 0$ holds, Ω commutes with these projectors and consequently with $P^{(0)}$ as well.

Now assume that $[\Omega, P^{(n)}]_- = 0$ holds for one particular $n \in \mathbb{N}_0$, then it follows, by using that Ω also (anti)commutes with all the $X^r_{\mathbf{p}}$, $v^r_{\mathbf{p}}$ and $c_{\mathbf{p}}$,

$$\begin{aligned}
[\Omega, P^{(n+1)}]_- =&\frac{1}{n+1}\int \widetilde{dp}\Big\{ - X^{0\dagger}_{\mathbf{p}} P^{(n)}[\Omega, Y^0_{\mathbf{p}}]_- - [\Omega, Y^{0\dagger}_{\mathbf{p}}]_- P^{(n)} X^0_{\mathbf{p}} - [\Omega, \bar{c}^\dagger_{\mathbf{p}}]_+ P^{(n)} c_{\mathbf{p}} \\
&+ c^\dagger_{\mathbf{p}} P^{(n)}[\Omega, \bar{c}_{\mathbf{p}}]_+ + \sum_{r=1}^{4}(-1)^{\lambda_r}\Big(X^{r\dagger}_{\mathbf{p}} P^{(n)}[\Omega, Y^r_{\mathbf{p}}]_- + [\Omega, Y^{r\dagger}_{\mathbf{p}}]_- P^{(n)} X^r_{\mathbf{p}} \\
&+ [\Omega, \bar{v}^{r\dagger}_{\mathbf{p}}]_+ P^{(n)} v^r_{\mathbf{p}} - v^{r\dagger}_{\mathbf{p}} P^{(n)}[\Omega, \bar{v}^r_{\mathbf{p}}]_+\Big)\Big\} \\
=&\frac{2m}{n+1}\int \widetilde{dp}\Big\{ - X^{0\dagger}_{\mathbf{p}} P^{(n)} c_{\mathbf{p}} + c^\dagger_{\mathbf{p}} P^{(n)} X^0_{\mathbf{p}} + X^{0\dagger}_{\mathbf{p}} P^{(n)} c_{\mathbf{p}} \\
&- c^\dagger_{\mathbf{p}} P^{(n)} X^0_{\mathbf{p}} + \sum_{r=1}^{4}(-1)^{\lambda_r}\Big(X^{r\dagger}_{\mathbf{p}} P^{(n)} v^r_{\mathbf{p}} - v^{r\dagger}_{\mathbf{p}} P^{(n)} X^r_{\mathbf{p}} \\
&- X^{r\dagger}_{\mathbf{p}} P^{(n)} v^r_{\mathbf{p}} + v^{r\dagger}_{\mathbf{p}} P^{(n)} X^r_{\mathbf{p}}\Big)\Big\} = 0.
\end{aligned} \tag{4.3.45}$$

This completes the induction and the statement is proven. □

Now the proof of the isometric isomorphy can be performed more or less easily. It is subject of the following two theorems.

Theorem 4.3.2 *The intersection of the space of physical polarizations and the image of the BRST charge operator is trivial, i.e. $\mathcal{V}_{phys} \cap \operatorname{im}\Omega = \{0\}$.*

Proof: First consider an arbitrary vector $|\psi_n\rangle$ with exactly $n \geq 1$ unphysical polarizations, i.e. $|\psi_n\rangle \in P^{(n)}\mathcal{V}$. Then Theorem 4.3.1, together with the projector properties of $P^{(n)}$, implies

$$\Omega|\psi_n\rangle = \Omega P^{(n)}|\psi_n\rangle = P^{(n)}\Omega|\psi_n\rangle. \tag{4.3.46}$$

So $\Omega|\psi_n\rangle$ is also contained in $P^{(n)}\mathcal{V}$. This also shows that the image of an arbitrary element

$$|\psi'\rangle = \sum_{n=1}^{\infty} c_n|\psi_n\rangle, \quad c_n \in \mathbb{C} \tag{4.3.47}$$

of the vector space $\mathcal{V}_{unphys}$, spanned by all the vectors with at least one unphysical polarization, is still contained in $\mathcal{V}_{unphys}$. Now let $|\psi_0\rangle$ be an arbitrary element of $\mathcal{V}_{phys}$, i.e. the space that contains no unphysical polarizations. Since this space is created by the generators $n^{r\dagger}_{\mathbf{p}}$ for $r \geq 5$, the property $\Omega|\psi_0\rangle = 0$ always holds. Therefore, the image of an arbitrary element $|\psi\rangle \in \mathcal{V}$, which can of course be written as a linear combination of the form $|\psi\rangle = \sum_{n=0}^{\infty} c_n|\psi_n\rangle$ with $c_n \in \mathbb{C}$, always is contained in $\mathcal{V}_{unphys}$:

$$\Omega|\psi\rangle = c_0\Omega|\psi_0\rangle + \sum_{n=1}^{\infty} c_n\Omega|\psi_n\rangle = \sum_{n=1}^{\infty} c_n\Omega|\psi_n\rangle \in \mathcal{V}_{unphys}. \tag{4.3.48}$$

Thus any element of $\mathcal{V}_{phys} \cap \operatorname{im}\Omega$ is also an element of $\mathcal{V}_{phys} \cap \mathcal{V}_{unphys}$ and therefore has to be 0, since $\mathcal{V}_{phys} \cap \mathcal{V}_{unphys} = \{0\}$. □

Theorem 4.3.3 *Let $|\Psi\rangle \in \ker\Omega$. Then there is a unique $|\psi\rangle \in \mathcal{V}_{phys}$ and a $|\varsigma\rangle \in \mathcal{V}$ such that*

$$|\Psi\rangle = |\psi\rangle + \Omega|\varsigma\rangle \tag{4.3.49}$$

holds.

Proof: Theorem 4.3.1 can be applied to show that

$$P^{(n)} = [\Omega, R^{(n)}]_+ \tag{4.3.50}$$

holds for $n \geq 1$, where $R^{(n)}$ is defined as

$$\begin{aligned} R^{(n)} = \frac{1}{2mn} \int \widetilde{dp} \Big\{ & \bar{c}^{\dagger}_{\mathbf{p}} P^{(n-1)} Y^0_{\mathbf{p}} + Y^{0\dagger}_{\mathbf{p}} P^{(n-1)} \bar{c}_{\mathbf{p}} \\ & - \sum_{r=1}^{4} (-1)^{\lambda_r} \Big(\bar{v}^{r\dagger}_{\mathbf{p}} P^{(n-1)} Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}} P^{(n-1)} \bar{v}^r_{\mathbf{p}} \Big) \Big\}. \end{aligned} \tag{4.3.51}$$

Therefore, by keeping $\sum_{n=0}^{\infty} P^{(n)} = \mathbb{1}$ in mind, $|\Psi\rangle$ can be written as

$$|\Psi\rangle = P^{(0)}|\Psi\rangle + \sum_{n=1}^{\infty} P^{(n)}|\Psi\rangle = P^{(0)}|\Psi\rangle + \sum_{n=1}^{\infty} [\Omega, R^{(n)}]_+|\Psi\rangle = P^{(0)}|\Psi\rangle + \Omega \sum_{n=1}^{\infty} R^{(n)}|\Psi\rangle, \tag{4.3.52}$$

since $\Omega|\Psi\rangle = 0$. By setting $|\psi\rangle = P^{(0)}|\Psi\rangle$, which per construction is contained in $\mathcal{V}_{phys}$, and $|\varsigma\rangle = \sum_{n=1}^{\infty} R^{(n)}|\Psi\rangle$, the existence is proven. Now assume that there is another $|\psi'\rangle \in \mathcal{V}_{phys}$ for that a $|\varsigma'\rangle \in \mathcal{V}$ exists, such that

$$|\Psi\rangle = |\psi'\rangle + \Omega|\varsigma'\rangle \tag{4.3.53}$$

holds. Then it follows

$$|\psi\rangle - |\psi'\rangle + \Omega|\varsigma\rangle - \Omega|\varsigma'\rangle = 0 \quad \Leftrightarrow \quad |\psi\rangle - |\psi'\rangle = \Omega\Big(|\varsigma'\rangle - |\varsigma\rangle\Big). \tag{4.3.54}$$

But this implies $|\psi\rangle - |\psi'\rangle \in \mathcal{V}_{phys} \cap \operatorname{im}\Omega$. Theorem 4.3.2 states $\mathcal{V}_{phys} \cap \operatorname{im}\Omega = \{0\}$, which implies $|\psi\rangle = |\psi'\rangle$ and therefore the uniqueness of $|\psi\rangle$. □

This last theorem finally shows the desired isometric isomorphy

$$\mathfrak{V} \sim \mathcal{V}_{phys}. \tag{4.3.55}$$

So the physical space that follows from the BRST formalism basically is the space one would have expected from the classical point of view. This shows that the introduction of Stückelberg fields does not change the physical structure of the theory at all. They are just auxiliary fields that enable the introduction of canonical momenta for all components of the tensor field $h'_{\mu\nu}$ and therefore allow a canonical quantization of the field.

Furthermore, note that the inner product of $\mathcal{V}$ in fact is positive definite on $\mathcal{V}_{phys}$. This is a direct consequence of the commutation relations of $n^r_{\mathbf{p}}$ and $n^{l\dagger}_{\mathbf{p}}$ for $r, l \geq 5$ (see (4.1.20)). So the completion $\mathfrak{H}$ of $\mathfrak{V}$ in fact is a Hilbert space.

Remark The explicit form of the fields $X^r_{\mathbf{p}}$ and $Y^r_{\mathbf{p}}$ can be obtained simply by guessing. If one demands the $X^r_{\mathbf{p}}$ to correlate to the BRST transformations of the ghosts in the given way (see (4.3.38) and (4.3.39)), they are basically given by (4.3.24), (4.3.25) and (4.3.26). The only thing left to do is to find $Y^r_{\mathbf{p}}$ such that the conditions (4.3.32), (4.3.33), (4.3.38) and (4.3.39) hold. However, for the massless case, which is discussed in Chapter 5, a systematic approach for the construction of the $Y^r_{\mathbf{p}}$ is presented.

Chapter 5

The BRST Formalism for Massless Spin-2 Fields

In addition to the BRST quantization of massive spin-2 particles, the same shall be done for the massless case. The methods that are used to quantize the massless fields are basically the same as for the massive ones. It is interesting, how these two cases are related to each other and whether there is a way to interpret the massless case as high energy limit of the massive one on the ground of the algebraic structure that is given by the BRST transformations of the different polarization generators.

5.1 Canonical Quantization and BRST Transformations of the Massless Fields

The quantization of the spin-2, Stückelberg and ghost fields can be performed in a similar way as in the massive case. The form of the canonical momenta (3.4.6), (3.4.7) and (3.4.8) is the same as for a finite mass. Consequently, the condition (4.1.11) for the polarization tensors and its analog for the polarization vectors stay the same. The momenta p now only have to satisfy the on-shellness condition $p^2 = 0$, since the massive Klein-Gordon equations for all the fields become massless Klein-Gordon equations. The first significant difference to the massive case occurs at the choice of $\hat{k}$. Since all massless particles move with the speed of light, there is no such thing as a rest frame. Instead of this the usual choice is a momentum that describes a particle moving in the x^3-direction, i.e. $\hat{k} = (k^0, 0, 0, k^0)$. Since the commutation relations for the corresponding creation and annihilation operators are already known, the explicit values of the polarization tensors and vectors given in (4.1.21) and (4.1.26) shall not be changed. One only has to keep in mind that $\hat{k}$ now is no longer $(m, 0, 0, 0)$ but $(k^0, 0, 0, k^0)$. The form of the polarizations for different momenta then are created once more by performing suitable Lorentz transformations on the $\varepsilon^r_{\mu\nu}(\hat{\mathbf{k}})$ and $\epsilon^r_\mu(\hat{\mathbf{k}})$. This leads once more to the commutation relations (4.1.15), (4.1.29), (4.1.34),(4.1.41), (4.1.42), (4.1.46) and (4.1.47), where $E_{\mathbf{p}}$ now is $|\mathbf{p}|$.

Also some results according to BRST transformations can be adapted from the massive case. The transformations of the fields in position space are just the ones given in (4.2.1) to (4.2.3) for $m = 0$, i.e.

$$\delta_\theta h'_{\mu\nu} = \theta(\partial_\mu \eta_\nu + \partial_\nu \eta_\mu), \quad \delta_\theta A'_\mu = \theta\sqrt{2}\partial_\mu \zeta, \quad \delta_\theta \phi' = 0, \tag{5.1.1}$$
$$\delta_\theta \bar{\eta}_\mu = -\theta B_\mu, \quad \delta_\theta \bar{\zeta} = -\theta C, \tag{5.1.2}$$
$$\delta_\theta \eta_\mu = \delta_\theta \zeta = 0, \quad \delta_\theta B_\mu = \delta_\theta C = 0. \tag{5.1.3}$$

The auxiliary fields B_μ and C again can be replaced by $-2\mathcal{G}^1_\mu\big|_{m=0} = -2\partial^\nu h'_{\mu\nu} + \partial_\mu h'$ and $-2\mathcal{G}^2\big|_{m=0} = -\sqrt{2}\partial^\mu A'_\mu$ respectively, by integrating over them. At this point it should be clear that the spin-2 field completely decouples from the Stückelberg fields and the scalar ghost fields on the level of BRST transformations, as mentioned before. Also the vectorial Stückelberg field no longer couples to the vectorial ghost or the scalar field. This means that the three sets of fields $\{h'_{\mu\nu}, \bar{\eta}_\mu, \eta_\mu\}$, $\{A'_\mu, \bar{c}, c\}$ and $\{\phi\}$ can be treated completely independently.

Furthermore, the BRST charge Ω is just the one given in (4.2.8) and (4.2.9) for $m = 0$, i.e.

$$\begin{aligned}
\Omega = &\int d^3\mathbf{x}\Big\{\partial_k \eta^0 \partial^k h' - 2\partial_k \eta_\nu \partial^k h'^{0\nu} - \sqrt{2}\partial_k \zeta \partial^k A'^0 - 2\bar{r}_\mu \pi^{0\mu} - 2\partial_k \eta_\mu \pi^{k\mu} - \sqrt{2}\bar{z}\Sigma^0 \\
&- \sqrt{2}\partial_k \zeta \Sigma^k - 2\bar{r}_\mu \partial_k h'^{k\mu} + \bar{r}_k \partial^k h' - \sqrt{2}\bar{z}\partial_k A'^k\Big\} \\
= &\int \widetilde{dp}\Big\{2p_\mu v_\nu(\mathbf{p}) n^{\mu\nu\dagger}(\mathbf{p}) + 2p_\mu v^\dagger_\nu(\mathbf{p}) n^{\mu\nu}(\mathbf{p}) - p_\mu v^\mu(\mathbf{p}) n^\dagger(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p}) n(\mathbf{p}) \\
&+ \sqrt{2}c_\mathbf{p} p_\mu a^{\mu\dagger}(\mathbf{p}) + \sqrt{2}c^\dagger_\mathbf{p} p_\mu a^\mu(\mathbf{p})\Big\},
\end{aligned} \tag{5.1.4}$$

which then of course leads to the corresponding transformation relations for the annihilation operators:

$$[\Omega, n_{\mu\nu}(\mathbf{p})]_- = -p_\mu v_\nu(\mathbf{p}) - p_\nu v_\mu(\mathbf{p}), \quad [\Omega, a_\mu(\mathbf{p})]_- = -\sqrt{2}p_\mu c_\mathbf{p}, \tag{5.1.5}$$
$$[\Omega, b_\mathbf{p}]_- = 0, \quad [\Omega, \bar{v}_\mu(\mathbf{p})]_+ = -2p^\nu n_{\mu\nu}(\mathbf{p}) + p_\mu n(\mathbf{p}), \tag{5.1.6}$$
$$[\Omega, v_\mu(\mathbf{p})]_+ = 0, \quad [\Omega, \bar{c}_\mathbf{p}]_+ = -\sqrt{2}p^\mu a_\mu(\mathbf{p}) \quad [\Omega, c_\mathbf{p}]_+ = 0. \tag{5.1.7}$$

The transformations of the creation operators can be obtained by conjugating the ones of the annihilation operators and multiplying with -1 in the case of commutators.

5.2 Analysis of the Physical Sector

As mentioned before, the BRST procedure shall be performed for the spin-2 field and the Stückelberg fields separately. The reason for that is the fact that if one is only interested in the massless case of spin-2 particles, no Stückelberg fields have to be introduced, since the corresponding action already carries a gauge invariance that can be used for the BRST formalism. However, to understand the relation between the massless and the massive case, it is necessary to treat the Stückelberg fields for $m = 0$ as well.

5.2.1 The Physical Space of the Massless Spin-2 Field

The examination of $\ker \Omega \cap \mathcal{V}^{h'}$ is a little more subtle than in the massive case. The BRST charge, reduced to the space $\mathcal{V}^{h'}$, takes the form

$$\Omega\big|_{\mathcal{V}^{h'}} = \int \widetilde{dp}\Big\{2p_\mu v^\dagger_\nu(\mathbf{p}) n^{\mu\nu}(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p}) n(\mathbf{p})\Big\}. \tag{5.2.1}$$

Therefore, any $|\psi\rangle \in \ker\Omega \cap \mathcal{V}^{h'}$ has to satisfy

$$\Omega|\psi\rangle = \int \widetilde{dp} \Big\{ 2p_\mu v_\nu^\dagger(\mathbf{p}) n^{\mu\nu}(\mathbf{p}) - p_\mu v^{\mu\dagger}(\mathbf{p}) n(\mathbf{p}) \Big\} |\psi\rangle = 0, \tag{5.2.2}$$

which implies, since $|\psi\rangle$ contains no ghosts,

$$\Big(p_\mu n^{\mu\nu}(\mathbf{p}) - \frac{1}{2} p^\nu n(\mathbf{p}) \Big) |\psi\rangle = 0. \tag{5.2.3}$$

This can once more be expressed in a Gupta-Bleuler like manner:

$$\mathcal{G}_\nu^{1(+)}(x)|\psi\rangle = \Big(\partial^\mu h'^{(+)}_{\mu\nu}(x) - \frac{1}{2} \partial_\nu h'^{(+)}(x) \Big) |\psi\rangle = 0. \tag{5.2.4}$$

Unlike as in the massive case, the intersection of $\ker\Omega \cap \mathcal{V}^{h'}$ and $\operatorname{im}\Omega$ is not trivial. A proof of this will be given later on. However, it is easy to see that the opposite result would be unfavorable. A straight forward calculation shows

$$\hat{k}^\mu \varepsilon^0_{\mu\nu}(\hat{\mathbf{k}}) - \frac{1}{2}\hat{k}_\nu \varepsilon^0(\hat{\mathbf{k}}) = \hat{k}^\mu \varepsilon^5_{\mu\nu}(\hat{\mathbf{k}}) - \frac{1}{2}\hat{k}_\nu \varepsilon^5(\hat{\mathbf{k}}) = \hat{k}^\mu \varepsilon^7_{\mu\nu}(\hat{\mathbf{k}}) - \frac{1}{2}\hat{k}_\nu \varepsilon^7(\hat{\mathbf{k}}) = 0 \tag{5.2.5}$$

and therefore

$$p^\mu \varepsilon^0_{\mu\nu}(\mathbf{p}) - \frac{1}{2} p_\nu \varepsilon^0(\mathbf{p}) = p^\mu \varepsilon^5_{\mu\nu}(\mathbf{p}) - \frac{1}{2} p_\nu \varepsilon^5(\mathbf{p}) = p^\mu \varepsilon^7_{\mu\nu}(\mathbf{p}) - \frac{1}{2} p_\nu \varepsilon^7(\mathbf{p}) = 0. \tag{5.2.6}$$

In particular, this implies

$$p^\mu n_{\mu\nu}(\mathbf{p}) - \frac{1}{2} p_\nu n(\mathbf{p}) = \sum_{\substack{r=0 \\ r\neq 0,5,7}}^{9} \Big(p^\mu \varepsilon^r_{\mu\nu}(\mathbf{p}) - \frac{1}{2} p_\nu \varepsilon^r(\mathbf{p}) \Big) n^r_{\mathbf{p}}. \tag{5.2.7}$$

Therefore, (5.2.3) ensures that the space that is generated by the $n^{0\dagger}_{\mathbf{p}}$, $n^{5\dagger}_{\mathbf{p}}$ and $n^{7\dagger}_{\mathbf{p}}$ is a subspace of $\ker\Omega \cap \mathcal{V}^{h'}$. So if $\operatorname{im}\Omega \cap \ker\Omega \cap \mathcal{V}^{h'} = \{0\}$, the corresponding three polarizations all would be physical and independent from each other. This contradicts the classical picture, in which a massless spin-2 particle only has two physical degrees of freedom.

In order to identify the physical polarizations $\ker\Omega \cap \mathcal{V}^{h'\bar{\eta}\eta}$ has to be examined. Here $\mathcal{V}^{h'\bar{\eta}\eta}$ is the space that is generated by the spin-2 field and the two vectorial ghosts. It is again possible to construct a set of quartets that allow to identify

$$\mathfrak{V}^{h'} = \big(\ker\Omega \cap \mathcal{V}^{h'\bar{\eta}\eta} \big) / \operatorname{im}\Omega \tag{5.2.8}$$

with a subspace of $\ker\Omega \cap \mathcal{V}^{h'\bar{\eta}\eta}$ via isomorphy. To do so, it is necessary to have a closer look at the BRST transformations of the creation and annihilation operators of the different polarizations. The strategy to obtain them is just the same as in the massive case, i.e. (4.3.15), (4.3.16), (4.3.18) and (4.3.19) can be applied together with (4.3.20) to obtain

$$[\Omega, n^0_{\mathbf{p}}]_- = -k^0(v^1_{\mathbf{p}} + v^4_{\mathbf{p}}), \quad [\Omega, n^1_{\mathbf{p}}]_- = k^0\Big(\sqrt{3}v^1_{\mathbf{p}} - \frac{1}{\sqrt{3}}v^4_{\mathbf{p}}\Big), \quad [\Omega, n^2_{\mathbf{p}}]_- = \sqrt{2}k^0 v^2_{\mathbf{p}}, \tag{5.2.9}$$

$$[\Omega, n^3_{\mathbf{p}}]_- = \sqrt{2}k^0 v^3_{\mathbf{p}}, \quad [\Omega, n^4_{\mathbf{p}}]_- = \sqrt{2}k^0(-v^1_{\mathbf{p}} + v^4_{\mathbf{p}}), \quad [\Omega, n^5_{\mathbf{p}}]_- = 0, \tag{5.2.10}$$

$$[\Omega, n^6_{\mathbf{p}}]_- = 2\sqrt{\frac{2}{3}}k^0 v^4_{\mathbf{p}}, \quad [\Omega, n^7_{\mathbf{p}}]_- = 0, \quad [\Omega, n^8_{\mathbf{p}}]_- = -\sqrt{2}k^0 v^2_{\mathbf{p}}, \tag{5.2.11}$$

$$[\Omega, n^9_{\mathbf{p}}]_- = -\sqrt{2}k^0 v^3_{\mathbf{p}} \tag{5.2.12}$$

and

$$[\Omega, \bar{v}^1_{\mathbf{p}}]_+ = -k^0(n^0_{\mathbf{p}} + \sqrt{3}n^1_{\mathbf{p}} + \sqrt{2}n^4_{\mathbf{p}}), \tag{5.2.13}$$

$$[\Omega, \bar{v}^2_{\mathbf{p}}]_+ = -k^0(\sqrt{2}n^2_{\mathbf{p}} + \sqrt{2}n^8_{\mathbf{p}}), \tag{5.2.14}$$

$$[\Omega, \bar{v}^3_{\mathbf{p}}]_+ = -k^0(\sqrt{2}n^3_{\mathbf{p}} + \sqrt{2}n^9_{\mathbf{p}}), \tag{5.2.15}$$

$$[\Omega, \bar{v}^4_{\mathbf{p}}]_+ = -k^0\Big(- n^0_{\mathbf{p}} + \frac{1}{\sqrt{3}}n^1_{\mathbf{p}} + \sqrt{2}n^4_{\mathbf{p}} - 2\sqrt{\frac{2}{3}}n^6_{\mathbf{p}}\Big). \tag{5.2.16}$$

Since the operators $n^5_{\mathbf{p}}$ and $n^7_{\mathbf{p}}$ commute with Ω, their creation operators are good candidates for the generators of the space that is isomorphic to $\mathfrak{V}^{h'}$. In fact, this assumption turns out to be true. Therefore, the space generated by the $n^{5\dagger}_{\mathbf{p}}$ and $n^{7\dagger}_{\mathbf{p}}$ shall be denoted as $\mathcal{V}^{h'}_{phys}$ for the rest of this chapter. The corresponding polarizations shall be called *physical*. All other polarizations of the spin-2 field and the ghost fields shall be denoted as *unphysical*. Another argument that supports the latest assumption is the form of the corresponding polarization tensors $\varepsilon^5_{\mu\nu}$ and $\varepsilon^7_{\mu\nu}$. At $\hat{k}$ they have the form

$$\varepsilon^5_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varepsilon^7_{\mu\nu}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.2.17}$$

So they are the polarizations that are transverse to $\hat{k}$. This is a Lorentz invariant condition and therefore true for all p with $p^2 = 0$. So, like in the photonic case (see [3]), they are the two polarizations for spin-2 particles one would naively expect.

To construct the quartets, a more systematic approach than for the massive case is necessary. Since there are four ghost-anti-ghost pairs involved, one would expect to find four quartets. The operators $X^r_{\mathbf{p}}$, which are supposed to satisfy $[\Omega, \bar{v}^r_{\mathbf{p}}]_+ \propto X^r_{\mathbf{p}}$, can be identified quite easily. By using the relations (5.2.13) to (5.2.16), one finds that the choice

$$X^1_{\mathbf{p}} = n^0_{\mathbf{p}} + \sqrt{3}n^1_{\mathbf{p}} + \sqrt{2}n^4_{\mathbf{p}}, \quad X^2_{\mathbf{p}} = \sqrt{2}(n^2_{\mathbf{p}} + n^8_{\mathbf{p}}), \tag{5.2.18}$$

$$X^3_{\mathbf{p}} = \sqrt{2}(n^3_{\mathbf{p}} + n^9_{\mathbf{p}}) \quad \text{and} \quad X^4_{\mathbf{p}} = -n^0_{\mathbf{p}} + \frac{1}{\sqrt{3}}n^1_{\mathbf{p}} + \sqrt{2}n^4_{\mathbf{p}} - 2\sqrt{\frac{2}{3}}n^6_{\mathbf{p}} \tag{5.2.19}$$

lead to the relations

$$[\Omega, \bar{v}^r_{\mathbf{p}}]_+ = -k^0 X^r_{\mathbf{p}} \quad \text{and} \quad [X^r_{\mathbf{p}}, X^{l\dagger}_{\mathbf{q}}] = [X^r_{\mathbf{p}}, X^l_{\mathbf{q}}] = 0 \tag{5.2.20}$$

for all $r, l = 1, ..., 4$. So the only thing left to do is to find the corresponding $Y^r_{\mathbf{p}}$ such that the following commutation relations hold for all $l, r = 1, ..., 4$:

$$[X^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_- = (-1)^{\lambda_r} 2E_{\mathbf{p}}(2\pi)^3 \delta^{lr}\delta(\mathbf{p} - \mathbf{q}) \quad \text{and} \tag{5.2.21}$$

$$[Y^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_- = [Y^r_{\mathbf{p}}, Y^l_{\mathbf{q}}]_- = 0. \tag{5.2.22}$$

Remark The special choice that $(-1)^{\lambda_r}$ shall determine the sign of $[X^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_-$ is not necessary. One could replace $Y^r_{\mathbf{p}}$ by $-Y^r_{\mathbf{p}}$ for any r and all the following arguments would hold in a quite analogous way. However, the signs that are chosen here allow to write the following formulas in a more compact way.

Furthermore, the $Y^r_{\mathbf{p}}$ have to satisfy

$$[\Omega, Y^r_{\mathbf{p}}]_- \propto v^r_{\mathbf{p}} \tag{5.2.23}$$

in order to turn the sets $\{X^r_{\mathbf{p}}, Y^r_{\mathbf{p}}, \bar{v}^r_{\mathbf{p}}, v^r_{\mathbf{p}}\}$ into quartets. All these commutation relations offer equations that allow to determine the $Y^r_{\mathbf{p}}$. The ansatz $Y^1_{\mathbf{p}} = \alpha_0 n^0_{\mathbf{p}} + \sqrt{3}\alpha_1 n^1_{\mathbf{p}} + \sqrt{2}\alpha_4 n^4_{\mathbf{p}}$ with $\alpha_0, \alpha_1 \alpha_4 \in \mathbb{R}$ can be made to obtain

$$[X^1_{\mathbf{p}}, Y^{1\dagger}_{\mathbf{q}}]_- = (-\alpha_0 + 3\alpha_1 - 2\alpha_4)2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \stackrel{!}{=} 2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}), \tag{5.2.24}$$

$$[Y^1_{\mathbf{p}}, Y^{1\dagger}_{\mathbf{q}}]_- = (-\alpha_0^2 + 3\alpha_1^2 - 2\alpha_4^2)2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \stackrel{!}{=} 0 \quad \text{and} \tag{5.2.25}$$

$$[\Omega, Y^1_{\mathbf{p}}]_- = k^0(-\alpha_0 + 3\alpha_1 - 2\alpha_4)v^1_{\mathbf{p}} + k^0(-\alpha_0 - \alpha_1 + 2\alpha_4)v^4_{\mathbf{p}} \stackrel{!}{\propto} v^1_{\mathbf{p}} \tag{5.2.26}$$

and therefore

$$-\alpha_0 + 3\alpha_1 - 2\alpha_4 = 1, \quad -\alpha_0^2 + 3\alpha_1^2 - 2\alpha_4^2 = 0 \quad \text{and} \quad -\alpha_0 - \alpha_1 + 2\alpha_4 = 0. \tag{5.2.27}$$

Note that this set of equations already implies $[\Omega, Y^1_{\mathbf{p}}]_- = k^0 v^1_{\mathbf{p}}$. Furthermore, it has the unique solution

$$\alpha_0 = -\frac{5}{16}, \quad \alpha_1 = \frac{3}{16} \quad \text{and} \quad \alpha_4 = -\frac{1}{16}. \tag{5.2.28}$$

So $Y^1_{\mathbf{p}}$ has the form

$$Y^1_{\mathbf{p}} = \frac{1}{16}(-5n^0_{\mathbf{p}} + 3\sqrt{3}n^1_{\mathbf{p}} - \sqrt{2}n^4_{\mathbf{p}}). \tag{5.2.29}$$

The remaining commutation relations $[X^r_{\mathbf{p}}, Y^{1\dagger}_{\mathbf{q}}]_- = 0$ for $r \neq 1$ are also fulfilled. The commutation relations for $r = 2$ and $r = 3$ are simple enough to guess the correct $Y^r_{\mathbf{p}}$. One finds

$$Y^2_{\mathbf{p}} = \frac{1}{2\sqrt{2}}(n^2_{\mathbf{p}} - n^8_{\mathbf{p}}) \quad \text{and} \quad Y^3_{\mathbf{p}} = \frac{1}{2\sqrt{2}}(n^3_{\mathbf{p}} - n^9_{\mathbf{p}}). \tag{5.2.30}$$

$Y^4_{\mathbf{p}}$ is again a more challenging case. By making the ansatz

$$Y^4_{\mathbf{p}} = -\beta_0 n^0_{\mathbf{p}} + \frac{1}{\sqrt{3}}\beta_1 n^1_{\mathbf{p}} + \sqrt{2}\beta_4 n^4_{\mathbf{p}} - 2\sqrt{\frac{2}{3}}\beta_6 n^6_{\mathbf{p}}, \tag{5.2.31}$$

with the real coefficients β_0, β_1, β_4 and β_6, one finds

$$[X^4_{\mathbf{p}}, Y^{4\dagger}_{\mathbf{q}}]_- = \Big(-\beta_0 + \frac{1}{3}\beta_1 - 2\beta_4 + \frac{8}{3}\beta_6\Big)2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \stackrel{!}{=} -2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}), \tag{5.2.32}$$

$$[Y^1_{\mathbf{p}}, Y^{4\dagger}_{\mathbf{q}}]_- = \frac{1}{16}(-5\beta_0 + 3\beta_1 + 2\beta_4)2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \stackrel{!}{=} 0, \tag{5.2.33}$$

$$[Y^4_{\mathbf{p}}, Y^{4\dagger}_{\mathbf{q}}]_- = \Big(-\beta_0^2 + \frac{1}{3}\beta_1^2 - 2\beta_4^2 + \frac{8}{3}\beta_6^2\Big)2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}) \stackrel{!}{=} 0 \quad \text{and} \tag{5.2.34}$$

$$[\Omega, Y^4_{\mathbf{p}}]_- = k^0(\beta_0 + \beta_1 - 2\beta_4)v^1_{\mathbf{p}} + k^0\Big(\beta_0 - \frac{1}{3}\beta_1 + 2\beta_4 - \frac{8}{3}\beta_6\Big)v^4_{\mathbf{p}} \overset{!}{\propto} v^4_{\mathbf{p}} \tag{5.2.35}$$

and therefore the following set of equations for the coefficients:

$$-\beta_0 + \frac{1}{3}\beta_1 - 2\beta_4 + \frac{8}{3}\beta_6 = -1, \quad -5\beta_0 + 3\beta_1 + 2\beta_4 = 0 \tag{5.2.36}$$

$$-\beta_0^2 + \frac{1}{3}\beta_1^2 - 2\beta_4^2 + \frac{8}{3}\beta_6^2 = 0 \quad \text{and} \quad \beta_0 + \beta_1 - 2\beta_4 = 0. \tag{5.2.37}$$

Similar to the case of $Y^1_{\mathbf{p}}$, this set of equations already implies $[\Omega, Y^4_{\mathbf{p}}]_- = k^0 v^4_{\mathbf{p}}$. The unique solution of this set reads

$$\beta_0 = \beta_1 = \beta_4 = -\beta_6 = \frac{3}{16}. \tag{5.2.38}$$

So $Y^4_{\mathbf{p}}$ has the form

$$Y^4_{\mathbf{p}} = \frac{3}{16}\Big(- n^0_{\mathbf{p}} + \frac{1}{\sqrt{3}}n^1_{\mathbf{p}} + \sqrt{2}n^4_{\mathbf{p}} + 2\sqrt{\frac{2}{3}}n^6_{\mathbf{p}}\Big). \tag{5.2.39}$$

It is easy to check that all remaining commutation relations are fulfilled, too. These constructions of the $Y^r_{\mathbf{p}}$ give the following relations:

$$[X^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_- = (-1)^{\lambda_r} 2E_{\mathbf{p}}(2\pi)^3\delta^{lr}\delta(\mathbf{p}-\mathbf{q}), \tag{5.2.40}$$

$$[X^r_{\mathbf{p}}, X^{l\dagger}_{\mathbf{q}}]_- = [Y^r_{\mathbf{p}}, Y^{l\dagger}_{\mathbf{q}}]_- = [X^r_{\mathbf{p}}, X^l_{\mathbf{q}}]_- = [Y^r_{\mathbf{p}}, Y^l_{\mathbf{q}}]_- = 0, \tag{5.2.41}$$

$$[\Omega, Y^r_{\mathbf{p}}]_- = k^0 v^r_{\mathbf{p}}, \quad \text{and} \quad [\Omega, \bar{v}^r_{\mathbf{p}}]_+ = -k^0 X^r_{\mathbf{p}} \tag{5.2.42}$$

for $r = 1, ..., 4$. So the four quartets have been found. Since

$$n^0_{\mathbf{p}} = \frac{5}{16}X^1_{\mathbf{p}} - Y^1_{\mathbf{p}} - \frac{3}{16}X^4_{\mathbf{p}} - Y^4_{\mathbf{p}}, \quad n^1_{\mathbf{p}} = \frac{3\sqrt{3}}{16}X^1_{\mathbf{p}} + \sqrt{3}Y^1_{\mathbf{p}} - \frac{\sqrt{3}}{16}X^4_{\mathbf{p}} - \frac{1}{\sqrt{3}}Y^4_{\mathbf{p}}, \tag{5.2.43}$$

$$n^2_{\mathbf{p}} = \frac{1}{2\sqrt{2}}X^2_{\mathbf{p}} + \sqrt{2}Y^2_{\mathbf{p}}, \quad n^3_{\mathbf{p}} = \frac{1}{2\sqrt{2}}X^3_{\mathbf{p}} + \sqrt{2}Y^3_{\mathbf{p}}, \tag{5.2.44}$$

$$n^4_{\mathbf{p}} = \frac{1}{8\sqrt{2}}(X^1_{\mathbf{p}} - 16Y^1_{\mathbf{p}} + 3X^4_{\mathbf{p}} + 16Y^4_{\mathbf{p}}), \quad n^6_{\mathbf{p}} = 2\sqrt{\frac{2}{3}}Y^4_{\mathbf{p}} - \frac{\sqrt{3}}{4\sqrt{2}}X^4_{\mathbf{p}}, \tag{5.2.45}$$

$$n^8_{\mathbf{p}} = \frac{1}{2\sqrt{2}}X^2_{\mathbf{p}} - \sqrt{2}Y^2_{\mathbf{p}} \quad \text{and} \quad n^9_{\mathbf{p}} = \frac{1}{2\sqrt{2}}X^3_{\mathbf{p}} - \sqrt{2}Y^3_{\mathbf{p}}, \tag{5.2.46}$$

the quartets generate all the configurations of unphysical polarizations, just as in the massive case.

Now in order to prove that $\mathcal{V}^{h'}_{phys}$ is isometric isomorphic to $\mathfrak{V}^{h'}$ one has to show that for all $|\Psi\rangle \in \ker\Omega \cap \mathcal{V}^{h'\bar{\eta}\eta}$ there is a unique $|\psi\rangle \in \mathcal{V}^{h'}_{phys}$ and a $|\varsigma\rangle \in \mathcal{V}$ such that $|\Psi\rangle = |\psi\rangle + \Omega|\varsigma\rangle$ holds. The proof of this can be performed in a quite analogous way as for the massive case.

Let $P^{(0)}$ be the projector onto $\mathcal{V}^{h'}_{phys}$. The projector $P^{(n)}$ onto the subspace of $\mathcal{V}^{h'\bar{\eta}\eta}$ with $n > 0$ unphysical polarizations once more can be expressed inductively as

$$P^{(n)} = \frac{1}{n}\int \widetilde{dp} \sum_{r=1}^{4}(-1)^{\lambda_r}\Big(X^{r\dagger}_{\mathbf{p}}P^{(n-1)}Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}}P^{(n-1)}X^r_{\mathbf{p}} + \bar{v}^{r\dagger}_{\mathbf{p}}P^{(n-1)}v^r_{\mathbf{p}} + v^{r\dagger}_{\mathbf{p}}P^{(n-1)}\bar{v}^r_{\mathbf{p}}\Big). \tag{5.2.47}$$

A complete induction, which can be performed in the same way as for the massive case, shows

$$[\Omega, P^{(n)}]_- = 0 \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{5.2.48}$$

This property and the fact that all generators of the space $\mathcal{V}^{h'}_{phys}$ commute with Ω are the only ingredients necessary to prove Theorem 4.3.2. So $\mathcal{V}^{h'}_{phys} \cap \operatorname{im}\Omega = \{0\}$ also holds in the massless case. Furthermore, it is easy to verify that

$$P^{(n)} = [\Omega, R^{(n)}]_+, \tag{5.2.49}$$

where $R^{(n)}$ is defined as

$$R^{(n)} = -\frac{1}{k^0 n} \int \widetilde{dp} \sum_{r=1}^{4} (-1)^{\lambda_r} \left(\bar{v}^{r\dagger}_{\mathbf{p}} P^{(n-1)} Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}} P^{(n-1)} \bar{v}^r_{\mathbf{p}} \right), \tag{5.2.50}$$

holds for all $n \geq 1$. Therefore, the massless version of Theorem 4.3.3 can be proven in just the same way as in the massive case. So once more the isometric isomorphy

$$\mathfrak{V}^{h'} \sim \mathcal{V}^{h'}_{phys} \tag{5.2.51}$$

holds. And since $\mathcal{V}^{h'}_{phys}$ is generated by operators with commutation relations that correspond to a positive definite inner product (see (4.1.20)), the completion $\mathfrak{H}^{h'}$ of $\mathfrak{V}^{h'}$ is a Hilbert space.

5.2.2 The Physical Space of the Massless Stückelberg Fields

The BRST formalism for the Stückelberg fields is basically just the one for a massless spin-1 field and a massless scalar field that completely decouples from the former. It works in quite the same way as in the spin-2 case.

Before it is performed explicitly, consider the space $\mathcal{V}^{A'\phi}$ that is generated by the polarizations of A'_μ and ϕ'. Restricted to that subspace, Ω takes the form

$$\Omega\big|_{\mathcal{V}^{A'\phi}} = \sqrt{2} \int \widetilde{dp} c^\dagger_{\mathbf{p}} p_\mu a^\mu(\mathbf{p}). \tag{5.2.52}$$

So one finds that a state $|\psi\rangle \in \ker\Omega \cap \mathcal{V}^{A'\phi}$ has to satisfy

$$p_\mu a^\mu(\mathbf{p})|\psi\rangle = 0 \tag{5.2.53}$$

or equivalently

$$\partial^\mu A'^{(+)}_\mu |\psi\rangle = 0. \tag{5.2.54}$$

This is just the Gupta-Bleuler condition, which shows the consistency of the BRST formalism with the manual quantization of the spin-1 field, that was presented in Section 2.4.1.

Now let $\mathcal{V}^{A'\phi'\bar{c}c}$ be the subspace of $\mathcal{V}$ that is generated by the Stückelberg fields and the scalar ghosts. Similar to the previous cases, a subspace $\mathcal{V}^{A'\phi'}_{phys} \subset \mathcal{V}^{A'\phi'\bar{c}c}$ has to be identified, that is isometric isomorphic to

$$\mathfrak{V}^{A'\phi'} = \left(\ker\Omega \cap \mathcal{V}^{A'\phi'\bar{c}c}\right) / \operatorname{im}\Omega. \tag{5.2.55}$$

First one has to compute the commutators of the BRST charge with all participating polarization generators. By using (4.3.17) and (4.3.20), one finds

$$[\Omega, a^1_{\mathbf{p}}]_- = \sqrt{2}k^0 c_{\mathbf{p}}, \quad [\Omega, a^2_{\mathbf{p}}]_- = 0, \quad [\Omega, a^3_{\mathbf{p}}]_- = 0, \quad [\Omega, a^4_{\mathbf{p}}]_- = -\sqrt{2}k^0 c_{\mathbf{p}}, \tag{5.2.56}$$

$$[\Omega, b_{\mathbf{p}}]_- = 0 \quad \text{and} \quad [\Omega, \bar{c}_{\mathbf{p}}]_+ = -\sqrt{2}k^0(a^1_{\mathbf{p}} + a^4_{\mathbf{p}}). \tag{5.2.57}$$

So, by taking the previous cases as motivation, the space that is generated by the $a^{2\dagger}_{\mathbf{p}}$, $a^{3\dagger}_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ is a good candidate for a space that is isometric isomorphic to $\mathfrak{V}^{A'\phi'}$ and will be denoted as $\mathcal{V}^{A'\phi'}_{phys}$ in the following.

This time there is only one quartet. Inspired by $[\Omega, \bar{c}_{\mathbf{p}}]_+$, one can choose

$$X^0_{\mathbf{p}} = \sqrt{2}(a^1_{\mathbf{p}} + a^4_{\mathbf{p}}) \quad \text{and} \quad Y^0_{\mathbf{p}} = \frac{1}{2\sqrt{2}}(a^1_{\mathbf{p}} - a^4_{\mathbf{p}}) \tag{5.2.58}$$

to find

$$[X^0_{\mathbf{p}}, Y^{0\dagger}_{\mathbf{q}}]_- = -2E_{\mathbf{p}}(2\pi)^3\delta(\mathbf{p} - \mathbf{q}), \tag{5.2.59}$$

$$[X^0_{\mathbf{p}}, X^{0\dagger}_{\mathbf{q}}]_- = [Y^0_{\mathbf{p}}, Y^{0\dagger}_{\mathbf{q}}]_- = [X^0_{\mathbf{p}}, X^0_{\mathbf{q}}]_- = [Y^0_{\mathbf{p}}, Y^0_{\mathbf{q}}]_- = 0, \tag{5.2.60}$$

$$[\Omega, Y^0_{\mathbf{p}}]_- = k^0 c_{\mathbf{p}} \quad \text{and} \quad [\Omega, \bar{c}_{\mathbf{p}}]_+ = -k^0 X^0_{\mathbf{p}}. \tag{5.2.61}$$

Therefore, $\{X^0_{\mathbf{p}}, Y^0_{\mathbf{p}}, \bar{c}_{\mathbf{p}}, c_{\mathbf{p}}\}$ is a quartet. Since

$$a^1_{\mathbf{p}} = \frac{1}{2\sqrt{2}}X^0_{\mathbf{p}} + \sqrt{2}Y^0_{\mathbf{p}} \quad \text{and} \quad a^4_{\mathbf{p}} = \frac{1}{2\sqrt{2}}X^0_{\mathbf{p}} - \sqrt{2}Y^0_{\mathbf{p}}, \tag{5.2.62}$$

the space that is generated by this quartet is just the one generated by the $a^1_{\mathbf{p}}$, $a^4_{\mathbf{p}}$, $\bar{c}_{\mathbf{p}}$ and $c_{\mathbf{p}}$. The corresponding polarizations will be denoted as *unphysical*. From this point forward, the proof that $\mathcal{V}^{A'\phi'}_{phys}$ is isometric isomorphic to $\mathfrak{V}^{A'\phi'}$ works in the exact same way as before. The projector onto the space with $n > 0$ unphysical polarizations is

$$P^{(n)} = -\frac{1}{n}\int \widetilde{dp}\Big\{X^{0\dagger}_{\mathbf{p}}P^{(n-1)}Y^0_{\mathbf{p}} + Y^{0\dagger}_{\mathbf{p}}P^{(n-1)}X^0_{\mathbf{p}} + \bar{c}^\dagger_{\mathbf{p}}P^{(n-1)}c_{\mathbf{p}} + c^\dagger_{\mathbf{p}}P^{(n-1)}\bar{c}_{\mathbf{p}}\Big\}, \tag{5.2.63}$$

where $P^{(0)}$ now denotes the projector onto $\mathcal{V}^{A'\phi'}_{phys}$. It commutes with Ω and can be written as

$$P^{(n)} = [\Omega, R^{(n)}]_+ \tag{5.2.64}$$

with

$$R^{(n)} = \frac{1}{k^0 n}\int \widetilde{dp}\Big\{\bar{c}^\dagger_{\mathbf{p}}P^{(n-1)}Y^0_{\mathbf{p}} + Y^{0\dagger}_{\mathbf{p}}P^{(n-1)}\bar{c}_{\mathbf{p}}\Big\}. \tag{5.2.65}$$

Furthermore, the generators of the space $\mathcal{V}^{A'\phi'}_{phys}$ of physical polarizations, i.e. $a^{2\dagger}_{\mathbf{p}}$, $a^{3\dagger}_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ all commute with Ω and therefore for any $|\Psi\rangle \in \big(\ker\Omega \cap \mathcal{V}^{A'\phi'\bar{c}c}\big)$ again a unique $|\psi\rangle \in \mathcal{V}^{A'\phi'}_{phys}$ and a $|\varsigma\rangle \in \mathcal{V}^{A'\phi'\bar{c}c}$ exists, such that

$$|\Psi\rangle = |\psi\rangle + \Omega|\varsigma\rangle \tag{5.2.66}$$

holds, which proves the isometric isomorphy

$$\mathfrak{V}^{A'\phi'} \sim \mathcal{V}^{A'\phi'}_{phys}. \tag{5.2.67}$$

And again all generators of $\mathcal{V}^{A'\phi'}_{phys}$ have the correct commutation relations to turn the completion $\mathfrak{H}^{A'\phi'}$ of $\mathfrak{V}^{A'\phi'}$ into a Hilbert space.

5.3 The Physical Space for the Full Theory

Now after the structure of the physical states has been examined separately for the spin-2 and the Stückelberg fields, the identification of the physical states in all of $\mathcal{V}$ is easy to perform. As mentioned before, the space of the spin-2 and the Stückelberg fields have nothing to do with each other with regard to the BRST formalism. So one would expect $\mathcal{V}_{phys} = \mathcal{V}^{h'}_{phys} \otimes \mathcal{V}^{A'\phi'}_{phys}$ to be isometric isomorphic to $\mathfrak{V} = \ker\Omega/\operatorname{im}\Omega$. In fact this is what turns out when the quartet mechanism is applied to all of $\mathcal{V}$. One can identify the five quartets $\{X^0_{\mathbf{p}}, Y^0_{\mathbf{p}}, \bar{c}_{\mathbf{p}}, c_{\mathbf{p}}\}$ and $\{X^r_{\mathbf{p}}, Y^r_{\mathbf{p}}, \bar{v}^r_{\mathbf{p}}, v^r_{\mathbf{p}}\}$, which correspond to creation operators that generate the same space as $a^{1\dagger}_{\mathbf{p}}$, $a^{4\dagger}_{\mathbf{p}}$, $n^{0\dagger}_{\mathbf{p}}$, $n^{1\dagger}_{\mathbf{p}}$,..., $n^{4\dagger}_{\mathbf{p}}$, $n^{6\dagger}_{\mathbf{p}}$, $n^{8\dagger}_{\mathbf{p}}$, $n^{9\dagger}_{\mathbf{p}}$, $c^{\dagger}_{\mathbf{p}}$, $\bar{c}^{\dagger}_{\mathbf{p}}$ $v^{r\dagger}_{\mathbf{p}}$ and $\bar{v}^{r\dagger}_{\mathbf{p}}$, i.e. the space of all the unphysical polarizations. The projector onto the space of states with exactly n unphysical polarizations then has the form

$$P^{(n)} = \frac{1}{n}\int \widetilde{dp}\Big\{ - X^{0\dagger}_{\mathbf{p}} P^{(n-1)} Y^0_{\mathbf{p}} - Y^{0\dagger}_{\mathbf{p}} P^{(n-1)} X^0_{\mathbf{p}} - \bar{c}^{\dagger}_{\mathbf{p}} P^{(n-1)} c_{\mathbf{p}} - c^{\dagger}_{\mathbf{p}} P^{(n-1)} \bar{c}_{\mathbf{p}} \\ + \sum_{r=1}^{4}(-1)^{\lambda_r}\left(X^{r\dagger}_{\mathbf{p}} P^{(n-1)} Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}} P^{(n-1)} X^r_{\mathbf{p}} + \bar{v}^{r\dagger}_{\mathbf{p}} P^{(n-1)} v^r_{\mathbf{p}} + v^{r\dagger}_{\mathbf{p}} P^{(n-1)} \bar{v}^r_{\mathbf{p}} \right)\Big\}, \tag{5.3.1}$$

where $P^{(0)}$ now is the projector onto $\mathcal{V}_{phys}$. The rest of the proof works in the same way as before. One finds $[\Omega, P^{(n)}]_- = 0$, which can be applied to find $P^{(n)} = [\Omega, R^{(n)}]_+$ with

$$R^{(n)} = \frac{1}{k^0 n}\int \widetilde{dp}\Big\{ \bar{c}^{\dagger}_{\mathbf{p}} P^{(n-1)} Y^0_{\mathbf{p}} + Y^{0\dagger}_{\mathbf{p}} P^{(n-1)} \bar{c}_{\mathbf{p}} \\ - \sum_{r=1}^{4}(-1)^{\lambda_r}\left(\bar{v}^{r\dagger}_{\mathbf{p}} P^{(n-1)} Y^r_{\mathbf{p}} + Y^{r\dagger}_{\mathbf{p}} P^{(n-1)} \bar{v}^r_{\mathbf{p}} \right)\Big\} \tag{5.3.2}$$

and $\mathcal{V}_{phys} \cap \ker\Omega = \{0\}$. These results then lead to the isometric isomorphy

$$\mathfrak{V} \sim \mathcal{V}_{phys}. \tag{5.3.3}$$

The space $\mathcal{V}_{phys}$ clearly has a positive definite inner product and therefore, once more the completion $\mathfrak{H}$ of $\mathfrak{V}$ is a Hilbert space.

Both the massive and the massless case deliver a physical space with a total number of five degrees of freedom. For a finite mass they correspond to $n^5_{\mathbf{p}}$,..., $n^9_{\mathbf{p}}$ while for a vanishing mass $n^6_{\mathbf{p}}$, $n^8_{\mathbf{p}}$ and $n^9_{\mathbf{p}}$ are replaced by $a^2_{\mathbf{p}}$, $a^3_{\mathbf{p}}$ and $b_{\mathbf{p}}$. This can be interpreted as the correction of the vDVZ discontinuity for the Quantum theory of spin-2 particles.

5.4 The Relation Between the Massless and the Massive Case

This final result raises the question whether it is possible to find a connection between the algebraic structures of the massive and the massless case, i.e. is there a way to continuously transform all the (anti)commutators of the BRST charge and the different fields for $m > 0$ into their analogs for $m = 0$? In fact this is possible. The trick is to choose $\hat{k}$ not to represent a resting particle in the massive case, but a particle that moves with an arbitrary speed in the

x^3-direction, i.e. $\hat{k} = (k^0, 0, 0, k^3)$ with $\hat{k}^2 = -m^2$. By applying the exact same steps as in the previous cases, one gets

$$[\Omega, n^0_{\mathbf{p}}]_- = -(k^0 v^1_{\mathbf{p}} + k^3 v^4_{\mathbf{p}}) + i2mc_{\mathbf{p}}, \quad [\Omega, n^1_{\mathbf{p}}]_- = \sqrt{3}k^0 v^1_{\mathbf{p}} - \frac{k^3}{\sqrt{3}} v^4_{\mathbf{p}}, \tag{5.4.1}$$

$$[\Omega, n^2_{\mathbf{p}}]_- = \sqrt{2}k^0 v^2_{\mathbf{p}}, \quad [\Omega, n^3_{\mathbf{p}}]_- = \sqrt{2}k^0 v^3_{\mathbf{p}}, \quad [\Omega, n^4_{\mathbf{p}}]_- = \sqrt{2}(-k^3 v^1_{\mathbf{p}} + k^0 v^4_{\mathbf{p}}), \tag{5.4.2}$$

$$[\Omega, n^5_{\mathbf{p}}]_- = 0, \quad [\Omega, n^6_{\mathbf{p}}]_- = 2\sqrt{\frac{2}{3}} k^3 v^4_{\mathbf{p}}, \quad [\Omega, n^7_{\mathbf{p}}]_- = 0 \quad [\Omega, n^8_{\mathbf{p}}]_- = -\sqrt{2}k^3 v^2_{\mathbf{p}}, \tag{5.4.3}$$

$$[\Omega, n^9_{\mathbf{p}}]_- = -\sqrt{2}k^3 v^3_{\mathbf{p}}, \tag{5.4.4}$$

$$[\Omega, a^1_{\mathbf{p}}]_- = -\sqrt{2}(imv^1_{\mathbf{p}} - k^0 c_{\mathbf{p}}), \quad [\Omega, a^2_{\mathbf{p}}]_- = -im\sqrt{2}v^2_{\mathbf{p}}, \tag{5.4.5}$$

$$[\Omega, a^3_{\mathbf{p}}]_- = -im\sqrt{2}v^3_{\mathbf{p}}, \quad [\Omega, a^4_{\mathbf{p}}]_- = -\sqrt{2}(imv^4_{\mathbf{p}} + k^3 c_{\mathbf{p}}), \tag{5.4.6}$$

$$[\Omega, b_{\mathbf{p}}]_- = -im\sqrt{6}c_{\mathbf{p}}, \tag{5.4.7}$$

$$[\Omega, \bar{v}^1_{\mathbf{p}}]_+ = -(k^0 n^0_{\mathbf{p}} + \sqrt{3}k^0 n^1_{\mathbf{p}} + \sqrt{2}k^3 n^4_{\mathbf{p}} - i\sqrt{2}ma^1_{\mathbf{p}}), \tag{5.4.8}$$

$$[\Omega, \bar{v}^2_{\mathbf{p}}]_+ = -\sqrt{2}(k^0 n^2_{\mathbf{p}} + k^3 n^8_{\mathbf{p}} - ima^2_{\mathbf{p}}), \quad [\Omega, \bar{v}^3_{\mathbf{p}}]_+ = -\sqrt{2}(k^0 n^3_{\mathbf{p}} + k^3 n^9_{\mathbf{p}} - ima^3_{\mathbf{p}}), \tag{5.4.9}$$

$$[\Omega, \bar{v}^4_{\mathbf{p}}]_+ = k^3 n^0_{\mathbf{p}} - \frac{1}{\sqrt{3}} k^3 n^1_{\mathbf{p}} - \sqrt{2}k^0 n^4_{\mathbf{p}} + 2\sqrt{\frac{2}{3}} k^3 n^6_{\mathbf{p}} + i\sqrt{2}ma^4_{\mathbf{p}} \quad \text{and} \tag{5.4.10}$$

$$[\Omega, \bar{c}_{\mathbf{p}}]_+ = -\sqrt{2}(k^0 a^1_{\mathbf{p}} + k^3 a^4_{\mathbf{p}} - i\sqrt{2}mn^0_{\mathbf{p}} - i\sqrt{3}mb_{\mathbf{p}}). \tag{5.4.11}$$

For $m = 0$, which implies $k^3 = k^0$, these are just the transformations for the massless case, while for $k^3 = 0$ they become the ones presented in Section 4.3.1. So this method offers a way to continuously turn the one set of transformations into the other. This allows to interpret the massless case as the high energy limit of the massive case. For high energies m becomes insignificantly small compared to k^0 and k^3. So one finds $k^3 \approx k^0$ and $m \approx 0$, which means that the transformations above become the ones for a massless particle at the high energy limit.

Another property worth to mention is that the only polarizations of the used basis that remain physical for all $\hat{k} = (k^0, 0, 0, k^3)$ are $\varepsilon^5_{\mu\nu}(\mathbf{p})$ and $\varepsilon^7_{\mu\nu}(\mathbf{p})$. But this is not surprising, since they are the only ones that are transverse to p for all $\hat{k}$. The other physical polarizations are linear combinations of the remaining ones for $m \neq 0$ and $k^3 \neq 0$. The reason for that is the fact that the chosen basis of polarization tensors is constructed in such a way that the physical polarizations for a resting particle can easily be identified. For a moving particle they would need to be adapted by applying suitable Lorentz transformations. But this is just the way how the polarizations for arbitrary momenta are constructed in Section 4.1. So one would just get the (anti)commutators with Ω that are presented in the corresponding chapter.

From the case of massive spin-1 particles it is known that for a fast moving particle two physical polarizations remain unchanged while the third begins to align itself with the 4-momentum of the particle and disappears in the high energy limit. For the spin-2 case a similar effect can be observed. In the high energy limit, i.e. the massless case, three of the five physical polarizations of the spin-2 field vanish and are replaced by physical polarizations of the Stückelberg fields, since they are constructed in such a way that they conserve the total number of physical polarizations.

Chapter 6

Coupling to External Sources

The physical vector space that describes spin-2 particles of arbitrary mass is now very well understood. The next step is to couple the spin-2 fields to external sources. At this point the spin-2 fields do not interact with any other field and therefore just form an isolated system that has nothing to do with any other particle, which might be actually observed in nature.

A first approach to change this is to introduce sources that can actually generate spin-2 fields. While doing so one has to ensure that such a source does not produce particles that carry unphysical polarizations, i.e. states that are not contained in the kernel of Ω. The easiest way to guarantee this is to demand that the new action, which contains also the external sources, is still BRST-invariant.

6.1 The General Structure of External Sources

The gauge fixed action S_{GF} can be extended by additional terms that represent couplings of the spin-2 and Stückelberg fields to external sources, i.e.

$$S_{GF} \longrightarrow S_{GF} + S_S, \tag{6.1.1}$$

with

$$S_S[h'_{\mu\nu}, A'_\mu, \phi'] = \int d^4x \Big\{ h'_{\mu\nu} T^{\mu\nu} + A'_\mu j^\mu + \phi' f \Big\}. \tag{6.1.2}$$

Here $T^{\mu\nu}$, j^μ and f is a tensorial, a vectorial and a scalar external source, respectively. Since $h'_{\mu\nu}$ is symmetric, $T^{\mu\nu}$ can be assumed to be symmetric without loss of generality.

By demanding that $S_{GF} + S_S$ is still BRST-invariant, it is possible to derive certain properties the external sources have to fulfill. Since S_{GF} is already BRST-invariant, the only thing that has to be ensured is

$$\begin{aligned} \mathrm{s}\, S_S = \int d^4x \Big\{ &(2\partial_\mu \eta_\nu + m\zeta g_{\mu\nu}) T^{\mu\nu} + \sqrt{2}(-m\eta_\mu + \partial_\mu \zeta) j^\mu - \sqrt{6} m\zeta f \\ &+ h'_{\mu\nu}\, \mathrm{s}\, T^{\mu\nu} + A'_\mu\, \mathrm{s}\, j^\mu + \phi'\, \mathrm{s}\, f \Big\} = 0. \end{aligned} \tag{6.1.3}$$

The fields $h'_{\mu\nu}$, A'_μ and ϕ' only appear in the terms $h'_{\mu\nu}\, \mathrm{s}\, T^{\mu\nu}$, $A'_\mu\, \mathrm{s}\, j^\mu$ and $\phi'\, \mathrm{s}\, f$, respectively. So, in order to ensure that $\mathrm{s}\, S_S$ vanishes for all possible field configurations, one finds

$$\mathrm{s}\, T^{\mu\nu} = 0, \quad \mathrm{s}\, j^\mu = 0 \quad \text{and} \quad \mathrm{s}\, f = 0. \tag{6.1.4}$$

Therefore and by using the method of partial integration once more, it follows

$$\mathrm{s}\, S_S = \int d^4x \Big\{ -(2\partial_\mu T^{\mu\nu} + \sqrt{2}mj^\nu)\eta_\nu + (mT - \sqrt{2}\partial_\mu j^\mu - \sqrt{6}mf)\zeta \Big\} = 0. \tag{6.1.5}$$

So the same argument as above implies

$$2\partial_\mu T^{\mu\nu} + \sqrt{2}mj^\nu = 0 \quad \text{and} \quad mT - \sqrt{2}\partial_\mu j^\mu - \sqrt{6}mf = 0. \tag{6.1.6}$$

For $m = 0$ this gives

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu j^\mu = 0. \tag{6.1.7}$$

So both $T^{\mu\nu}$ and j^μ have to be conserved, while f can be chosen arbitrarily, as long as $\mathrm{s}\, f = 0$ holds. In the massive case one finds

$$j^\mu = -\frac{\sqrt{2}}{m}\partial_\nu T^{\mu\nu} \quad \text{and} \quad f = \frac{1}{\sqrt{6}}T + \frac{\sqrt{2}}{\sqrt{3}m^2}\partial_\mu\partial_\nu T^{\mu\nu}, \tag{6.1.8}$$

while $T^{\mu\nu}$ can be any source for that $\mathrm{s}\, T^{\mu\nu}$ vanishes. So for any such $T^{\mu\nu}$ there are unique j^μ and f, such that S_S is invariant under BRST transformations.

The form of j^μ and f also could have been obtained in a different way. For this consider the Fierz-Pauli action S and extend it with a term that couples $h_{\mu\nu}$ to $T^{\mu\nu}$, i.e.

$$S[h_{\mu\nu}] \longrightarrow S[h_{\mu\nu}] + \int d^4x h_{\mu\nu} T^{\mu\nu}. \tag{6.1.9}$$

Then apply the Stückelberg trick and write $h_{\mu\nu}$ in terms of $h'_{\mu\nu}$. This effectively means to perform the replacement

$$h_{\mu\nu} \longrightarrow h'_{\mu\nu} + \frac{1}{\sqrt{6}}\phi' g_{\mu\nu} + \frac{1}{\sqrt{2}m}(\partial_\mu A'_\nu + \partial_\nu A'_\mu) + \frac{\sqrt{2}}{\sqrt{3}m^2}\partial_\mu\partial_\nu\phi' \tag{6.1.10}$$

in $S[h_{\mu\nu}] + \int d^4x h_{\mu\nu} T^{\mu\nu}$. By shifting all the partial derivatives from the Stückelberg fields to $T^{\mu\nu}$ via partial integration, one finds

$$\int d^4x h_{\mu\nu} T^{\mu\nu} \longrightarrow \int d^4x \Big\{ h'_{\mu\nu} T^{\mu\nu} + A'_\mu \Big(-\frac{\sqrt{2}}{m}\partial_\nu T^{\mu\nu}\Big) + \phi' \Big(\frac{1}{\sqrt{6}}T + \frac{\sqrt{2}}{\sqrt{3}m^2}\partial_\mu\partial_\nu T^{\mu\nu}\Big)\Big\}, \tag{6.1.11}$$

which is just what one gets when the j^μ and f from (6.1.8) are inserted into S_S. This is a very satisfying result. From the physical point of view it is at first not clear, why one should couple the Stückelberg fields to external sources, since they are known to be unphysical. However, the argumentation from above shows that the only way to do this appropriately is to couple the spin-2 field $h_{\mu\nu}$ to an external source and then perform the Stückelberg trick, as it is done for example in [5].

Now it is an interesting question to ask what external sources can be formulated in terms of other fields that are supposed to couple to massive spin-2 particles. In the following subsections this is done for scalar, vector and spinor fields. In order to ensure that the limit $m \longrightarrow 0$ can be performed without any problems, $\partial_\mu T^{\mu\nu} = 0$ will be used as an additional restriction. So the resulting sources will be just the ones that also can be coupled to massless spin-2 particles. Furthermore, since the sources are supposed to be external, the corresponding fields will be assumed to be on-shell, i.e. to solve the corresponding Euler-Lagrange equations. Interactions of the fields with themselves will not be considered. Therefore, the sources will be assumed to be a sum of terms that only consist of products of two fields. In addition, each term will be assumed to contain only up to two derivatives.

Furthermore, it shall be investigated what sources can be constructed that only couple to $h'_{\mu\nu}$ and not to the Stückelberg fields. They clearly have to fulfill the additional condition $T = 0$ (see (6.1.8)).

In order to ensure that the sources are invariant under BRST transformations, one would need to know how they behave under the gauge transformations δ^1 and δ^2. This will not be discussed. The sources will simply be assumed to be BRST-invariant.

6.2 Coupling to Scalar Fields

Consider a scalar field χ with mass $M \neq 0$. It is described by the action

$$S_\chi[\chi] = \int d^4x \Big\{ \frac{1}{2}\chi(\Box - M^2)\chi \Big\}. \tag{6.2.1}$$

So the corresponding Euler-Lagrange equation is just the Klein-Gordon equation

$$(\Box - M^2)\chi = 0. \tag{6.2.2}$$

By applying this to eliminate redundancies such as $\Box(\chi\chi) = 2M^2\chi\chi + 2\partial_\lambda\chi\partial^\lambda\chi$, it turns out that a symmetric source $T_\chi^{\mu\nu}$, with the restrictions from above, always has the form

$$T_\chi^{\mu\nu} = \alpha\partial^\mu\chi\partial^\nu\chi + \beta\partial^\mu\partial^\nu\chi\chi + \gamma\partial_\lambda\chi\partial^\lambda\chi g^{\mu\nu} + \delta M^2\chi\chi g^{\mu\nu}, \tag{6.2.3}$$

with real parameters α, β, γ and δ. By making use of the Euler-Lagrange equation, one finds

$$\partial_\mu T_\chi^{\mu\nu} = (\alpha + \beta + 2\delta)M^2\chi\partial^\nu\chi + (\alpha + \beta + 2\gamma)\partial_\lambda\chi\partial^\nu\partial^\lambda\chi. \tag{6.2.4}$$

So by demanding $\partial_\mu T_\chi^{\mu\nu} = 0$, it follows

$$(\alpha + \beta + 2\delta) = (\alpha + \beta + 2\gamma) = 0, \tag{6.2.5}$$

which is equivalent to

$$\gamma = \delta = -\frac{1}{2}(\alpha + \beta). \tag{6.2.6}$$

By applying this to (6.2.3) and then performing the substitution $\alpha = \alpha' - \beta$, one gets

$$T_\chi^{\mu\nu} = \alpha'\Big(\partial^\mu\chi\partial^\nu\chi - \frac{1}{2}(\partial_\lambda\chi\partial^\lambda\chi + M^2\chi\chi)g^{\mu\nu}\Big) + \beta(\partial^\mu\partial^\nu\chi\chi - \partial^\mu\chi\partial^\nu\chi). \tag{6.2.7}$$

It turns out that the remaining parameters α' and β can not be chosen in such a way that $T_\chi^{\mu\nu}$ is unequal to zero and only couples to $h'_{\mu\nu}$. As mentioned before, such a source would require to have a vanishing trace. So one has to demand

$$0 = T_\chi = -(\alpha' + \beta)\partial_\lambda\chi\partial^\lambda\chi - (2\alpha' - \beta)M^2\chi\chi. \tag{6.2.8}$$

This implies $\alpha' + \beta = 0$ and $2\alpha' - \beta = 0$ and therefore one gets $\alpha' = \beta = 0$.

This changes for the massless case. Consider χ to be massless. Then it satisfies the Euler-Lagrange equation $\Box\chi = 0$. This can again be used to formulate a general symmetric external source:

$$T^{\mu\nu}_{\chi} = \alpha\partial^\mu\chi\partial^\nu\chi + \beta\partial^\mu\partial^\nu\chi\chi + \gamma\partial_\lambda\chi\partial^\lambda\chi g^{\mu\nu} + \delta\mu^2\chi\chi g^{\mu\nu}. \tag{6.2.9}$$

Here μ is a parameter with the dimension of a mass. It is introduced to ensure that all the other appearing parameters have the same dimension. The appearance of an additional mass parameter, which is not the mass of the corresponding field, seems somehow unnatural. It is used here only for the sake of completeness. The corresponding term turns out to vanish when $\partial_\mu T^{\mu\nu} = 0$ is demanded. Together with $\Box\chi = 0$, this condition leads to

$$0 = \partial_\mu T^{\mu\nu}_{\chi} = (\alpha + \beta + 2\gamma)\partial_\lambda\chi\partial^\nu\partial^\lambda\chi + 2\delta\mu^2\chi\partial^\nu\chi \tag{6.2.10}$$

and therefore implies

$$\gamma = -\frac{1}{2}(\alpha + \beta) \quad \text{and} \quad \delta = 0. \tag{6.2.11}$$

So after the substitution $\alpha = \alpha' - \beta$ the resulting source takes the form

$$T^{\mu\nu}_{\chi} = \alpha'\Big(\partial^\mu\chi\partial^\nu\chi - \frac{1}{2}\partial_\lambda\chi\partial^\lambda\chi g^{\mu\nu}\Big) + \beta(\partial^\mu\partial^\nu\chi\chi - \partial^\mu\chi\partial^\nu\chi). \tag{6.2.12}$$

Now the condition $T_\chi = 0$ implies

$$0 = T_\chi = -(\alpha' + \beta)\partial_\lambda\chi\partial^\lambda\chi \tag{6.2.13}$$

and therefore $\beta = -\alpha'$. So the source that couples only to the spin-2 field is, up to a multiplicative constant, uniquely determined and has the form

$$T^{\mu\nu}_{\chi} = \alpha'\Big(2\partial^\mu\chi\partial^\nu\chi - \partial^\mu\partial^\nu\chi\chi - \frac{1}{2}\partial_\lambda\chi\partial^\lambda\chi g^{\mu\nu}\Big). \tag{6.2.14}$$

6.3 Coupling to Vector Fields

For vector fields the same procedure as for scalar fields can be applied. First consider V_μ to be a vector field with mass $M \neq 0$, i.e. a massive spin-1 field. According to Section 2.6, the corresponding Euler-Lagrange equations can be brought into the form

$$(\Box - M^2)V_\mu = 0, \quad \partial_\lambda V^\lambda = 0. \tag{6.3.1}$$

They can again be used to eliminate redundant terms in a general ansatz for a symmetric source $T^{\mu\nu}_V$. Hence it can be written in the form

$$\begin{aligned} T^{\mu\nu}_V =& \alpha\partial^\mu V^\lambda\partial^\nu V_\lambda + \beta\partial^\mu\partial^\nu V_\lambda V^\lambda + \gamma\partial_\lambda\partial^{(\mu}V^{\nu)}V^\lambda + \delta\partial_\lambda V^{(\mu}\partial^{\nu)}V^\lambda + \varepsilon\partial_\lambda V^\mu\partial^\lambda V^\nu \\ &+ \vartheta\partial_\rho V_\lambda\partial^\rho V^\lambda g^{\mu\nu} + \kappa\partial_\rho V_\lambda\partial^\lambda V^\rho g^{\mu\nu} + \sigma M^2 V_\lambda V^\lambda g^{\mu\nu} + \varphi M^2 V^\mu V^\nu, \end{aligned} \tag{6.3.2}$$

with real parameters α, β, γ, δ, ε, ϑ, κ, σ and φ. Furthermore, the Euler-Lagrange equations can be used to find

$$\begin{aligned} \partial_\mu T^{\mu\nu}_V =& (\alpha + \beta + 2\vartheta)\partial_\rho V_\lambda\partial^\nu\partial^\rho V^\lambda + \frac{1}{2}(\gamma + \delta + 2\varepsilon)\partial_\rho\partial_\lambda V^\nu\partial^\rho V^\lambda \\ &+ \frac{1}{2}(\gamma + \delta + 4\kappa)\partial_\rho V_\lambda\partial^\nu\partial^\lambda V^\rho + (\alpha + \beta + 2\sigma)M^2 V^\lambda\partial^\nu V_\lambda \\ &+ \frac{1}{2}(\gamma + \delta + 2\varphi)M^2\partial_\lambda V^\nu V^\lambda. \end{aligned} \tag{6.3.3}$$

So by demanding $\partial_\mu T_V^{\mu\nu} = 0$ once more it follows

$$\alpha + \beta + 2\vartheta = \gamma + \delta + 2\varepsilon = \gamma + \delta + 4\kappa = \alpha + \beta + 2\sigma = \gamma + \delta + 2\varphi = 0, \tag{6.3.4}$$

which is equivalent to

$$\vartheta = \sigma = -\frac{1}{2}(\alpha + \beta), \quad \varepsilon = 2\kappa = \varphi = -\frac{1}{2}(\gamma + \delta). \tag{6.3.5}$$

Therefore, by additionally substituting $\alpha = \alpha' - \beta$ and $\delta = \delta' - \gamma$, $T_V^{\mu\nu}$ can be brought into the form

$$\begin{aligned} T_V^{\mu\nu} =& \alpha' \Big(\partial^\mu V^\lambda \partial^\nu V_\lambda - \frac{1}{2}(\partial_\rho V_\lambda \partial^\rho V^\lambda + M^2 V_\lambda V^\lambda) g^{\mu\nu} \Big) \\ &+ \beta(\partial^\mu \partial^\nu V_\lambda V^\lambda - \partial^\mu V_\lambda \partial^\nu V^\lambda) + \gamma(\partial_\lambda \partial^{(\mu} V^{\nu)} V^\lambda - \partial_\lambda V^{(\mu} \partial^{\nu)} V^\lambda) \\ &+ \delta' \Big(\partial_\lambda V^{(\mu} \partial^{\nu)} V^\lambda - \frac{1}{2} \partial_\lambda V^\mu \partial^\lambda V^\nu - \frac{1}{4} \partial_\rho V_\lambda \partial^\lambda V^\rho g^{\mu\nu} - \frac{1}{2} M^2 V^\mu V^\nu \Big). \end{aligned} \tag{6.3.6}$$

To identify the part of $T_V^{\mu\nu}$ that only couples to the spin-2 field, one has to demand $T_V = 0$. This implies, together with the Euler-Lagrange equations,

$$0 = T_V = -\frac{1}{2}(2\alpha' + 2\beta + \delta')\partial_\rho V_\lambda \partial^\rho V^\lambda - \gamma \partial_\rho V_\lambda \partial^\lambda V^\rho + \frac{1}{2}(-4\alpha' + 2\beta - \delta') M^2 V_\lambda V^\lambda. \tag{6.3.7}$$

So one gets

$$2\alpha' + 2\beta + \delta' = \gamma = -4\alpha' + 2\beta - \delta' = 0, \tag{6.3.8}$$

which is equivalent to

$$\beta = \frac{1}{2}\alpha', \quad \gamma = 0, \quad \delta' = -3\alpha'. \tag{6.3.9}$$

Therefore, the $T_V^{\mu\nu}$ that only couples to $h'_{\mu\nu}$ is, up to a multiplicative constant, once more uniquely determined and has the form

$$\begin{aligned} T_V^{\mu\nu} = \alpha' \Big(& \frac{1}{2} \partial^\mu V_\lambda \partial^\nu V^\lambda + \frac{1}{2} \partial^\mu \partial^\nu V_\lambda V^\lambda - 3 \partial_\lambda V^{(\mu} \partial^{\nu)} V^\lambda + \frac{3}{2} \partial_\lambda V^\mu \partial^\lambda V^\nu \\ &+ \frac{3}{2} M^2 V^\mu V^\nu + \frac{1}{4} \big(3 \partial_\rho V_\lambda \partial^\lambda V^\rho - 2 \partial_\rho V_\lambda \partial^\rho V^\lambda - 2 M^2 V_\lambda V^\lambda \big) g^{\mu\nu} \Big). \end{aligned} \tag{6.3.10}$$

The procedure for the massless case works basically in the same way. Since a massless vector field is described by the action

$$S_V[V_\mu] = \int d^4x \Big\{ -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) \Big\}, \tag{6.3.11}$$

the corresponding Euler-Lagrange equations are

$$\Box V^\mu = \partial^\mu \partial_\lambda V^\lambda. \tag{6.3.12}$$

They can again be used to eliminate redundancies in order to get the general ansatz

$$\begin{aligned} T_V^{\mu\nu} =& \alpha \partial^\mu V^\lambda \partial^\nu V_\lambda + \beta \partial^\mu \partial^\nu V_\lambda V^\lambda + \gamma \partial_\lambda \partial^{(\mu} V^{\nu)} V^\lambda + \delta \partial^{(\mu} V^{\nu)} \partial_\lambda V^\lambda + \varepsilon \partial_\lambda V^{(\mu} \partial^{\nu)} V^\lambda \\ &+ \vartheta \partial_\lambda V^\mu \partial^\lambda V^\nu + \kappa \Box V^{(\mu} V^{\nu)} + \sigma \partial_\rho V_\lambda \partial^\rho V^\lambda g^{\mu\nu} + \varphi \partial_\rho V_\lambda \partial^\lambda V^\rho g^{\mu\nu} \\ &+ \psi \partial_\rho V^\rho \partial_\lambda V^\lambda g^{\mu\nu} + \eta \Box V_\lambda V^\lambda g^{\mu\nu} + \xi \mu^2 V^\mu V^\nu + \varrho \mu^2 V_\lambda V^\lambda g^{\mu\nu} \end{aligned} \tag{6.3.13}$$

for a symmetric source. The parameters α, β, γ, δ, ε, ϑ, κ, σ, φ, ψ, η, ξ and ϱ are again assumed to be real and μ is supposed to have the dimension of a mass. It once more is only included for the sake of completeness. The corresponding terms vanish when $\partial_\mu T_V^{\mu\nu} = 0$ is demanded. This leads, together with the Euler-Lagrange equations, to

$$\begin{aligned} 0 = \partial_\mu T_V^{\mu\nu} =& \frac{1}{2}(2\alpha + \delta + \varepsilon + 2\eta)\Box V_\lambda \partial^\nu V^\lambda + (\alpha + \beta + 2\sigma)\partial_\rho V_\lambda \partial^\nu \partial^\rho V^\lambda \\ &+ \frac{1}{2}(2\beta + 2\gamma + \kappa + 2\eta)V_\lambda \partial^\nu \Box V^\lambda + \frac{1}{2}(2\delta + \kappa + 4\psi)\Box V^\nu \partial_\lambda V^\lambda \\ &+ \frac{1}{2}(\delta + \varepsilon + 2\vartheta + \kappa)\Box V^\lambda \partial_\lambda V^\nu + \frac{1}{2}(\gamma + \varepsilon + 4\varphi)\partial_\rho V_\lambda \partial^\nu \partial^\lambda V^\rho \\ &+ \frac{1}{2}(\gamma + \varepsilon + 2\vartheta)\partial_\rho \partial_\lambda V^\nu \partial^\rho V^\lambda + \frac{1}{2}\kappa \Box \partial_\lambda V^\lambda V^\nu + \xi\mu^2 \partial_\lambda V^\lambda V^\nu \\ &+ \xi\mu^2 V^\lambda \partial_\lambda V^\nu + 2\varrho\mu^2 V_\lambda \partial^\nu V^\lambda \end{aligned} \tag{6.3.14}$$

and therefore implies

$$\begin{aligned} 0 &= 2\alpha + \delta + \varepsilon + 2\eta = \alpha + \beta + 2\sigma = 2\beta + 2\gamma + \kappa + 2\eta = 2\delta + \kappa + 4\psi \\ &= \delta + \varepsilon + 2\vartheta + \kappa = \gamma + \varepsilon + 4\varphi = \gamma + \varepsilon + 2\vartheta = \kappa = \xi = \varrho. \end{aligned} \tag{6.3.15}$$

This is equivalent to

$$\begin{aligned} &\alpha = \beta + \frac{1}{2}(\gamma - \varepsilon), \quad \sigma = -\beta + \frac{1}{4}(\varepsilon - \gamma), \quad \vartheta = 2\varphi = -\frac{1}{2}(\varepsilon + \gamma), \\ &\delta = -2\psi = \gamma, \quad \eta = -(\beta + \gamma), \quad \kappa = \xi = \varrho = 0. \end{aligned} \tag{6.3.16}$$

So $T_V^{\mu\nu}$ has the form

$$\begin{aligned} T_V^{\mu\nu} =& \beta\big(\partial^\mu V^\lambda \partial^\nu V_\lambda + \partial^\mu \partial^\nu V_\lambda V^\lambda - (\partial_\rho V_\lambda \partial^\rho V^\lambda + \Box V_\lambda V^\lambda)g^{\mu\nu}\big) \\ &+ \gamma\Big(\frac{1}{2}\partial^\mu V^\lambda \partial^\nu V_\lambda + \partial_\lambda \partial^{(\mu} V^{\nu)} V^\lambda + \partial^{(\mu} V^{\nu)} \partial_\lambda V^\lambda - \frac{1}{2}\partial_\lambda V^\mu \partial^\lambda V^\nu \\ &- \frac{1}{4}(\partial_\rho V_\lambda \partial^\rho V^\lambda + \partial_\rho V_\lambda \partial^\lambda V^\rho + 2\partial_\rho V^\rho \partial_\lambda V^\lambda + 4\Box V_\lambda V^\lambda)g^{\mu\nu}\Big) \\ &\varepsilon\Big(-\frac{1}{2}\partial^\mu V^\lambda \partial^\nu V_\lambda + \partial_\lambda V^{(\mu} \partial^{\nu)} V^\lambda - \frac{1}{2}\partial_\lambda V^\mu \partial^\lambda V^\nu \\ &+ \frac{1}{4}(\partial_\rho V_\lambda \partial^\rho V^\lambda - \partial_\rho V_\lambda \partial^\lambda V^\rho)g^{\mu\nu}\Big). \end{aligned} \tag{6.3.17}$$

Again, $T_V = 0$ enables to identify the $T_V^{\mu\nu}$ that only couple to the spin-2 fields. One finds

$$0 = T_V = -(3\beta + \gamma)\partial_\rho V_\lambda \partial^\rho V^\lambda - 3(\beta + \gamma)\Box V_\lambda V^\lambda - \gamma \partial_\rho V^\rho \partial_\lambda V^\lambda - \gamma \partial_\rho V_\lambda \partial^\lambda V^\rho \tag{6.3.18}$$

and therefore

$$\beta = \gamma = 0. \tag{6.3.19}$$

So the corresponding source again is, up to a multiplicative constant, uniquely determined:

$$\begin{aligned} T_V^{\mu\nu} = \varepsilon\Big(&-\frac{1}{2}\partial^\mu V^\lambda \partial^\nu V_\lambda + \partial_\lambda V^{(\mu} \partial^{\nu)} V^\lambda - \frac{1}{2}\partial_\lambda V^\mu \partial^\lambda V^\nu \\ &+ \frac{1}{4}(\partial_\rho V_\lambda \partial^\rho V^\lambda - \partial_\rho V_\lambda \partial^\lambda V^\rho)g^{\mu\nu}\Big). \end{aligned} \tag{6.3.20}$$

Note that this $T_V^{\mu\nu}$ is invariant under gauge transformations of the form

$$V_\mu \longmapsto V_\mu + \partial_\mu a, \tag{6.3.21}$$

with a function a that falls off sufficiently fast at infinity. This is a quite nice feature. The action (6.3.11), which is implicitly used by applying the corresponding Euler-Lagrange equations, of course has the same gauge invariance. This means that a theory that also treats V_μ as a dynamic field most likely comes with a corresponding BRST transformation of V_μ. Therefore, the BRST transformation for all the fields, including spin-2 and Stückelberg fields, also takes the transformation of V_μ into account. The gauge invariance of $T_V^{\mu\nu}$ now ensures that the coupling terms of the fields are invariant under this expanded BRST transformation, which is important to guarantee that the over all action is invariant under BRST transformations.

6.4 Coupling to Spinor Fields

The construction of the source that consists of a spinor field works basically just as the previous cases. A spinor field with mass M is described by the action

$$S_\psi[\bar{\psi}, \psi] = \int d^4x \Big\{ -i\bar{\psi}\partial\!\!\!/\psi - M\bar{\psi}\psi \Big\}, \tag{6.4.1}$$

with $\partial\!\!\!/ = \gamma^\lambda \partial_\lambda$ and $\bar{\psi} = \psi^\dagger \gamma^0$. The γ^μ are the *Dirac matrices*. The corresponding Euler-Lagrange equations

$$(-i\partial\!\!\!/ - M)\psi = 0 \quad \text{and} \quad i\partial_\lambda \bar{\psi}\gamma^\lambda - M\bar{\psi} = 0 \tag{6.4.2}$$

in particular imply

$$(\Box - M^2)\psi = -i\partial\!\!\!/(-i\partial\!\!\!/ - M)\psi = 0 \quad \text{and} \quad (\Box - M^2)\bar{\psi} = i\partial_\rho(i\partial_\lambda\bar{\psi}\gamma^\lambda - M\bar{\psi})\gamma^\rho = 0, \tag{6.4.3}$$

which is a direct consequence of $[\gamma^\mu, \gamma^\nu]_+ = -2g^{\mu\nu}\mathbb{1}$.

Now consider $M \neq 0$. For the source $T_\psi^{\mu\nu}$ the ansatz

$$T_\psi^{\mu\nu} = \alpha\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi + \beta\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi + \delta M\bar{\psi}\psi g^{\mu\nu} \tag{6.4.4}$$

is made. When the redundancies that follow from the Euler-Lagrange equations are taken into account, it follows that this form contains all possible symmetric terms that consist of Dirac matrices, spinors and up to one partial derivative. The terms with two partial derivatives are neglected here, since they would require to introduce a new factor with the inverse dimension of a mass in order to ensure that the corresponding parameters have the same dimension as α, β and δ. M^{-1} would be a natural candidate, but this would mean that the source strongly couples to the spin-2 field in the limit $M \longrightarrow 0$, which does not appear to be reasonable.

Furthermore, note that the term $\bar{\psi}\gamma^{(\mu}\gamma^{\nu)}\psi$ is also indirectly taken into account, since $[\gamma^\mu, \gamma^\nu]_+ = -2g^{\mu\nu}$ implies $\bar{\psi}\gamma^{(\mu}\gamma^{\nu)}\psi = -\bar{\psi}\psi g^{\mu\nu}$. It is also important to mention that the parameters α, β and δ now have to be assumed to be complex in order to ensure that $T_\psi^{\mu\nu}$ is real.

From $\partial_\mu T_\psi^{\mu\nu} = 0$ and the Euler-Lagrange equations one gets

$$0 = \partial_\mu T_\psi^{\mu\nu} = \frac{1}{2}(\alpha + \beta)\partial_\lambda\bar{\psi}\gamma^\nu\partial^\lambda\psi + \frac{1}{2}(\alpha + \beta)M^2\bar{\psi}\gamma^\nu\psi + \delta M\partial^\nu\bar{\psi}\psi + \delta M\bar{\psi}\partial^\nu\psi. \tag{6.4.5}$$

This implies

$$\beta = -\alpha \quad \text{and} \quad \delta = 0 \tag{6.4.6}$$

and therefore

$$T_\psi^{\mu\nu} = \alpha(\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi - \bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi). \tag{6.4.7}$$

Furthermore, by demanding $T_\psi^{\mu\nu *} = T_\psi^{\mu\nu}$ and applying $\bar{\psi} = \psi^\dagger\gamma^0$, $\gamma^0\gamma^0 = \mathbb{1}$ and $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$, one gets

$$T_\psi^{\mu\nu *} = -\alpha^*(\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi - \bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi) \overset{!}{=} \alpha(\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi - \bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi) = T_\psi^{\mu\nu} \tag{6.4.8}$$

and therefore $\alpha^* = -\alpha$. So there is a $\varepsilon \in \mathbb{R}$ such that $\alpha = i\varepsilon$ holds, which leads to

$$T_\psi^{\mu\nu} = \varepsilon(i\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi - i\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi). \tag{6.4.9}$$

So the condition $\partial_\mu T_\psi^{\mu\nu} = 0$ already uniquely determines $T_\psi^{\mu\nu}$. Since

$$T_\psi = 2M\bar{\psi}\psi \neq 0, \tag{6.4.10}$$

it follows that there is no $T_\psi^{\mu\nu}$ that only couples to $h'_{\mu\nu}$.

This changes for the massless case. Here the Euler-Lagrange equations deliver a very similar ansatz for $T_\psi^{\mu\nu}$ as in the massive case:

$$T_\psi^{\mu\nu} = \alpha\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi + \beta\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi + \delta\mu\bar{\psi}\psi g^{\mu\nu}. \tag{6.4.11}$$

Again the terms with two derivatives are neglected in order to avoid the introduction of a new parameter with the inverse dimension of a mass. μ again is a parameter with the dimension of a mass. Once more it is only introduced to show that the corresponding parameter δ has to vanish in order to get $\partial_\mu T_\psi^{\mu\nu} = 0$:

$$0 = \partial_\mu T_\psi^{\mu\nu} = \frac{1}{2}(\alpha + \beta)\partial_\lambda\bar{\psi}\gamma^\nu\partial^\lambda\psi + \delta\mu\partial^\nu\bar{\psi}\psi + \delta\mu\bar{\psi}\partial^\nu\psi. \tag{6.4.12}$$

This implies $\beta = -\alpha$ and $\delta = 0$. So $T_\psi^{\mu\nu}$ has just the same form as for the massive case. The additional fact that it has to be real again implies that there is a real parameter ε such that $\alpha = i\varepsilon$ holds. The resulting source

$$T_\psi^{\mu\nu} = \varepsilon(i\partial^{(\mu}\bar{\psi}\gamma^{\nu)}\psi - i\bar{\psi}\gamma^{(\mu}\partial^{\nu)}\psi) \tag{6.4.13}$$

is uniquely determined (up to a multiplicative constant) and has a vanishing trace. This is a direct consequence of the Euler-Lagrange equations. Thus it only couples to the spin-2 field.

Chapter 7

Further Aspects of Spin-2 Particles

To finish the discussion of spin-2 particles and their quantization some additional properties of them shall be mentioned. It is an interesting question to ask in what way the action S_{Stb}, which has been derived by introducing Stückelberg fields, is unique or how spin-2 particles relate to gravity. Furthermore, it is interesting to analyze the propagators of the theory and their relations to each other.

7.1 Propagators and Slavnov-Taylor Identities

The quantized spin-2, Stückelberg and ghost fields can be used to derive the corresponding Feynman propagators. This procedure is a very well-known calculation. Therefore, it is only sketched in this section. For an arbitrary field Φ the Feynman propagator has the form

$$\begin{aligned}\langle 0|T\Phi(x)\Phi(y)|0\rangle =&\Theta(x^0-y^0)\langle 0|\Phi^{(+)}(x)\Phi^{(-)}(y)|0\rangle\\ &\pm\Theta(y^0-x^0)\langle 0|\Phi^{(+)}(y)\Phi^{(-)}(x)|0\rangle\\ =&\Theta(x^0-y^0)\langle 0|[\Phi^{(+)}(x),\Phi^{(-)}(y)]_{\mp}|0\rangle\\ &\pm\Theta(y^0-x^0)\langle 0|[\Phi^{(+)}(y),\Phi^{(-)}(x)]_{\mp}|0\rangle.\end{aligned} \tag{7.1.1}$$

The T denotes the time ordering operator. As usual, the commutator version is used for the bosonic fields, while the anticommutator refers to the ghosts. Furthermore, in the case of ghosts one has to consider a time ordered product of a ghost and its corresponding anti-ghost in (7.1.1), not a time ordered product of two (anti)ghosts. The integral expression of the Heavyside function

$$\Theta(x^0-y^0)=\lim_{\epsilon\to 0}\frac{i}{2\pi}\int dp\frac{1}{p+i\epsilon}e^{-ip(x^0-y^0)} \tag{7.1.2}$$

can, together with the commutation relations (4.1.13), (4.1.30), (4.1.34), (4.1.38) and (4.1.46) for the creation and annihilation operators in momentum space, be used to get

$$\langle 0|Th'_{\mu\nu}(x)h'_{\rho\sigma}(y)|0\rangle=\lim_{\epsilon\to 0}\int\frac{d^4p}{(2\pi)^4}\frac{i\big(g_{\mu(\rho}g_{\sigma)\nu}-\frac{1}{2}g_{\mu\nu}g_{\rho\sigma}\big)}{-p^2-m^2+i\epsilon}e^{ip(x-y)}, \tag{7.1.3}$$

$$\langle 0|TA'_{\mu}(x)A'_{\nu}(y)|0\rangle=\lim_{\epsilon\to 0}\int\frac{d^4p}{(2\pi)^4}\frac{ig_{\mu\nu}}{-p^2-m^2+i\epsilon}e^{ip(x-y)}, \tag{7.1.4}$$

$$\langle 0|T\phi(x)\phi(y)|0\rangle=\lim_{\epsilon\to 0}\int\frac{d^4p}{(2\pi)^4}\frac{i}{-p^2-m^2+i\epsilon}e^{ip(x-y)}, \tag{7.1.5}$$

$$\langle 0|T\bar{\eta}_\mu(x)\eta_\nu(y)|0\rangle = \lim_{\epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{-ig_{\mu\nu}}{-p^2-m^2+i\epsilon}e^{ip(x-y)} \quad \text{and} \tag{7.1.6}$$

$$\langle 0|T\bar{\zeta}(x)\zeta(y)|0\rangle = \lim_{\epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{-i}{-p^2-m^2+i\epsilon}e^{ip(x-y)}. \tag{7.1.7}$$

These expressions are true for an arbitrary mass $m \geq 0$.

The BRST formalism offers a way to identify relations among Green's functions. The idea behind this is to exploit

$$\langle 0|[\Omega, \Phi]_\pm|0\rangle = 0 \tag{7.1.8}$$

for some arbitrary field operator Φ. Relations that originate from such a condition are called *Slavnov-Taylor identities* (see [9]).

Consider for example $\langle 0|[\Omega, T\bar{\eta}_\rho(x)h'_{\mu\nu}(y)]_+|0\rangle = 0$. Since Ω is time independent, it commutes with the time ordering operator. This leads to

$$\begin{aligned}
0 =&\langle 0|[\Omega, T\bar{\eta}_\rho(x)h'_{\mu\nu}(y)]_+|0\rangle = \langle 0|T[\Omega, \bar{\eta}_\rho(x)h'_{\mu\nu}(y)]_+|0\rangle \\
=&\langle 0|T[\Omega, \bar{\eta}_\rho(x)]_+h'_{\mu\nu}(y)|0\rangle - \langle 0|T\bar{\eta}_\rho(x)[\Omega, h'_{\mu\nu}(y)]_-|0\rangle \\
=&i\langle 0|T\big(2\partial^\lambda h'_{\lambda\rho}(x) - \partial_\rho h'(x) + \sqrt{2}mA'_\rho(x)\big)h'_{\mu\nu}(y)|0\rangle \\
&- i\langle 0|T\bar{\eta}_\rho(x)\big(\partial_\mu\eta_\nu(y) + \partial_\nu\eta_\mu(y) + m\zeta(y)g_{\mu\nu}\big)|0\rangle \\
=&i\langle 0|T\big(2\partial^\lambda h'_{\lambda\rho}(x) - \partial_\rho h'(x)\big)h'_{\mu\nu}(y)|0\rangle - i\langle 0|T\bar{\eta}_\rho(x)\big(\partial_\mu\eta_\nu(y) + \partial_\nu\eta_\mu(y)\big)|0\rangle.
\end{aligned} \tag{7.1.9}$$

So the relation

$$\langle 0|T\big(2\partial^\lambda h'_{\lambda\rho}(x) - \partial_\rho h'(x)\big)h'_{\mu\nu}(y)|0\rangle = \langle 0|T\bar{\eta}_\rho(x)\big(\partial_\mu\eta_\nu(y) + \partial_\nu\eta_\mu(y)\big)|0\rangle \tag{7.1.10}$$

can be derived. In an analogous way one can use $\langle 0|[\Omega, T\bar{\zeta}(x)h'_{\mu\nu}(y)]_+|0\rangle = 0$ to get

$$m\langle 0|Th'(x)h'_{\mu\nu}(y)|0\rangle = m\langle 0|\bar{\zeta}(x)\zeta(y)|0\rangle g_{\mu\nu}. \tag{7.1.11}$$

In the massive case this results in the relation

$$\langle 0|Th'(x)h'_{\mu\nu}(y)|0\rangle = \langle 0|T\bar{\zeta}(x)\zeta(y)|0\rangle g_{\mu\nu}. \tag{7.1.12}$$

The explicit expressions of the propagators, which are given above, can be used to verify that this relation is also true for $m = 0$.

Furthermore, $\langle 0|[\Omega, T\bar{\eta}_\mu(x)A'_\nu(y)]_+|0\rangle = 0$ leads to

$$m\langle 0|TA'_\mu(x)A'_\nu(y)|0\rangle = -m\langle 0|T\bar{\eta}_\mu(x)\eta_\nu(y)|0\rangle. \tag{7.1.13}$$

For $m \neq 0$ this implies

$$\langle 0|TA'_\mu(x)A'_\nu(y)|0\rangle = -\langle 0|T\bar{\eta}_\mu(x)\eta_\nu(y)|0\rangle. \tag{7.1.14}$$

This again is also true in the massless case, just as

$$\langle 0|T\phi'(x)\phi'(y)|0\rangle = -\langle 0|T\bar{\zeta}(x)\zeta(y)|0\rangle, \tag{7.1.15}$$

which follows from $\langle 0|[\Omega, T\bar{\zeta}(x)\phi'(y)]_+|0\rangle = 0$ in the massive case.

The last relation that shall be mentioned here is

$$2\langle 0|T\partial^\lambda h'_{\lambda\rho}(x)h'_{\rho\sigma}(y)|0\rangle = \langle 0|T\big(\bar{\eta}_\rho(x)(\partial_\mu\eta_\nu(y) + \partial_\nu\eta_\mu(y)) + \partial_\rho\bar{\zeta}(x)\zeta(y)g_{\mu\nu}\big)|0\rangle. \qquad (7.1.16)$$

It follows from $\langle 0|[\Omega, T\big(m\bar{\eta}_\rho(x) + \partial_\rho\bar{\zeta}(x)\big)h'_{\mu\nu}(y)]_+|0\rangle = 0$ in the massive case.

Note that all these relations can also be derived from the explicit expressions of the propagators. However, the BRST formalism allows to find them only by using the algebraic structure of the theory. No explicit quantization of the fields is necessary for this. One only has to know how the different fields are related to each other via BRST transformations. So they offer a very effective way to check whether the derivation of the propagators has been done correctly.

In non-abelian gauge theories, there are Slavnov-Taylor identities that are less trivial. In [10] some examples can be found.

7.2 Gauge Theoretic Approach

The action S_{Stb} for massive spin-2 fields has been constructed from the Fierz-Pauli action S in such a way that it is invariant under the gauge transformations δ^1 and δ^2. This raises the question whether it is uniquely determined by those invariances, i.e. is any action that is invariant under δ^1 and δ^2 proportional to S_{Stb}? The discussion of the Stückelberg formalism in Section 2.6 clearly shows that this can not be true. One can take an arbitrary action of $h'_{\mu\nu}$ and introduce A_μ and ϕ in just the same way as for S_{Stb}. This strategy clearly always leads to an action that is invariant under the corresponding gauge transformations. So in order to find some kind of uniqueness for S_{Stb} further restrictions have to be considered.

Two very intuitive properties one could demand are first that the gauge invariant action $\widetilde{S}$ should come with a Lagrangian $\widetilde{\mathcal{L}}$ that only depends on the fields and their first derivatives, i.e. no derivatives of higher order should show up in $\mathcal{L}$, and second that $\widetilde{\mathcal{L}}$ is Lorentz invariant. Furthermore, only terms that are products of two fields shall be considered for the Lagrangian, i.e. the corresponding action is supposed to describe non-interacting fields with no sources. In fact even those restrictions are not enough to uniquely determine $\widetilde{S}$. To see this, the most general action $\widetilde{S}$ that is gauge invariant and offers such a Lagrangian shall be derived.

Before the derivation is performed, one can reduce the liberties for the different terms in $\widetilde{S}$ by some careful considerations. The following notation is quite useful for them: For two arbitrary tensor fields Φ_1 and Φ_2, $(\Phi_1\Phi_2)$ shall denote the integral over an arbitrary contraction of their indices. For example, $(\partial A\partial A)$ shall represent any of the contractions

$$\int d^4x\partial_\mu A_\nu\partial^\mu A^\nu, \quad \int d^4x\partial_\mu A_\nu\partial^\nu A^\mu = \int d^4x\partial_\mu A^\mu\partial_\nu A^\nu. \qquad (7.2.1)$$

Clearly, this notation only makes sense when the total number of indices of the fields is even, such that one can contract the fields to a scalar. It is quite obvious that the terms that are represented by $(\partial\Phi_1\Phi_2)$ are the same as for $(\Phi_1\partial\Phi_2)$, since they can be turned into each other via partial integration. Therefore, one does not have to distinguish between them. So the only important things about such terms that show up in $\widetilde{S}$ are their total number of derivatives and the type of the involved fields.

This is the most important conclusion for the following considerations. The gauge transformations of $\widetilde{S}$ in particular contain terms that include the ξ_μ and Λ together with the other fields and their derivatives. To ensure that $\widetilde{S}$ is gauge invariant, all the terms with the same types of fields and number of derivatives have to cancel each other.

As an example, consider the $(\partial A \partial A)$ terms. Under δ^2 they deliver $(\partial\partial\Lambda\partial A)$ terms. The only other terms that include Λ and A under δ^2 are (AA), $(h'\partial A)$ and $(\phi\partial A)$. But they give at most two derivatives not three, as $(\partial\partial\Lambda\partial A)$. Therefore, the $(\partial A\partial A)$ terms in $\widetilde{S}$ must have a structure that guarantees that the resulting $(\partial\partial\Lambda\partial A)$ terms cancel each other. But this implies that the $(\partial A\partial A)$ part of $\widetilde{S}$ must be proportional to

$$\int d^4x F_{\mu\nu}F^{\mu\nu}, \tag{7.2.2}$$

as is well-known from electrodynamics.

It is also easy to see that no $\int d^4x A_\mu A^\mu$ term, which is the only existing (AA) term, appears in $\widetilde{S}$. Under δ^1 it produces a term that is proportional to $\int d^4x \xi_\mu A^\mu$, which no other term does.

Furthermore, the $(\partial\phi\partial h')$ terms lead under δ^1, in particular, to terms that include ϕ, ξ_μ and three derivatives. They are the only terms that show that pattern and therefore the $(\partial\phi\partial\delta^1 h')$ terms also cancel each other. By making the ansatz

$$\widetilde{S}_{(\partial\phi\partial h')} = \int d^4x \Big\{\alpha\partial_\mu\phi\partial_\nu h'^{\mu\nu} + \beta\partial_\mu\phi\partial^\mu h'\Big\} \tag{7.2.3}$$

for the corresponding part of the action, one gets

$$0 \overset{!}{=} \delta^1\widetilde{S}_{(\partial\phi\partial h')} = 2(\alpha+\beta)\int d^4x\partial_\mu\phi\Box\xi^\mu \tag{7.2.4}$$

and therefore $\alpha = -\beta$.

The liberties for the $(\partial h'\partial h')$ part $\widetilde{S}_{(\partial h'\partial h')}$ of $\widetilde{S}$ also can be reduced by a closer look. It leads under δ^1 to $(\partial\partial\xi\partial h')$ expressions, in particular. They include one ξ_μ and one $h'_{\mu\nu}$. The only other terms that do that are the $(h'\partial\delta^1 A)$ terms. But they include only one derivative, not three as $(\partial\partial\xi\partial h')$. Therefore, the $(\partial\partial\xi\partial h')$ terms that originate from $\widetilde{S}_{(\partial h'\partial h')}$ have to vanish independently from all the other terms as well. The ansatz

$$\widetilde{S}_{(\partial h'\partial h')}[h'_{\mu\nu}] = \int d^4x\Big\{a_1\partial_\lambda h'_{\mu\nu}\partial^\lambda h'^{\mu\nu} + a_2\partial_\lambda h'_{\mu\nu}\partial^\mu h'^{\nu\lambda} + a_3\partial^\lambda h'_{\lambda\nu}\partial^\nu h' + a_4\partial_\lambda h'\partial^\lambda h'\Big\} \tag{7.2.5}$$

contains all possible $(\partial h'\partial h')$ terms that are not equivalent in the context of partial integration. Now relations between the coefficients $a_1, ..., a_4$ can be determined. By performing the gauge transformation and exploiting the symmetry of $h'_{\mu\nu}$, one gets

$$\begin{aligned}\delta^1\widetilde{S}_{(\partial h'\partial h')} = \int d^4x\Big\{&\big((4a_1+2a_2)\partial_\mu\partial_\lambda\xi_\nu + 2a_2\partial_\mu\partial_\nu\xi_\lambda\big)\partial^\lambda h'^{\mu\nu}\\ &+\big(a_3\Box\xi_\nu + (a_3+4a_4)\partial_\nu\partial^\mu\xi_\mu\big)\partial^\nu h'\\ &+2a_3\partial_\nu\partial^\mu\xi_\mu\partial_\lambda h'^{\lambda\nu}\Big\} + \widetilde{S}_{(\partial h'\partial h')}[\partial_\mu\xi_\nu + \partial_\nu\xi_\mu].\end{aligned} \tag{7.2.6}$$

Note that $2\widetilde{S}_{(\partial h'\partial h')}[\partial_\mu\xi_\nu+\partial_\nu\xi_\mu]$ is just the part of $\delta^1\widetilde{S}_{(\partial h'\partial h')}$ with all the $(\partial\partial\xi\partial h')$ expressions for the special choice $h'_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$. Therefore, if the $(\partial\partial\xi\partial h')$ terms cancel each other, $\widetilde{S}_{(\partial h'\partial h')}[\partial_\mu\xi_\nu+\partial_\nu\xi_\mu]=0$ and consequently $\delta^1\widetilde{S}_{(\partial h'\partial h')}=0$ is an immediate consequence.

A naive conclusion would be to demand that all the coefficients in front of the different products of $h'_{\mu\nu}$ and ξ_μ have to vanish. But this is a too strong restriction. It is possible that the different terms can be combined to form total divergences, which then would only vanish when the integration is performed explicitly. This subtlety is best dealt with by shifting all the derivatives in $\delta^1 S_{(\partial h'\partial h')}$ to ξ_μ via partial integration. Since in this form no derivatives of $h'_{\mu\nu}$ show up, there can be no more hidden total divergence. The result is

$$\begin{aligned}\delta^1\widetilde{S}_{(\partial h'\partial h')}=\int d^4x\Big\{&-2(2a_1+a_2)\Box\partial_\mu\xi_\nu h'^{\mu\nu}\\&-2(a_2+a_3)\partial_\mu\partial_\nu\partial^\lambda\xi_\lambda h'^{\mu\nu}\\&-2(a_3+2a_4)\Box\partial^\lambda\xi_\lambda h'\Big\}+\widetilde{S}_{(\partial h'\partial h')}[\partial_\mu\xi_\nu+\partial_\nu\xi_\mu].\end{aligned}\tag{7.2.7}$$

Since every line in the upper expression corresponds to a different type of contraction between third order derivatives of ξ_ν and $h'_{\mu\nu}$, they all have to disappear independently. Consequently, one gets

$$2a_1+a_2=a_2+a_3=a_3+2a_4=0.\tag{7.2.8}$$

This is equivalent to

$$a_2=-2a_1,\quad a_3=2a_1\quad\text{and}\quad a_4=-a_1.\tag{7.2.9}$$

By substituting $a_1=-\frac{1}{2}b_1$, one finds

$$\widetilde{S}_{(\partial h'\partial h')}=b_1\int d^4x\Big\{-\frac{1}{2}\partial_\lambda h'_{\mu\nu}\partial^\lambda h'^{\mu\nu}+\partial_\lambda h'_{\mu\nu}\partial^\mu h'^{\nu\lambda}-\partial^\lambda h'_{\lambda\nu}\partial^\nu h'+\frac{1}{2}\partial_\lambda h'\partial^\lambda h'\Big\}.\tag{7.2.10}$$

So $\widetilde{S}_{(\partial h'\partial h')}$ is, up to a multiplicative constant, uniquely determined as the massless Fierz-Pauli action $S_{m=0}$.

These preliminary thoughts lead to the following ansatz for an action $\widetilde{S}$:

$$\begin{aligned}\widetilde{S}[h'_{\mu\nu},A_\mu,\phi]=&b_1S_{m=0}+\int d^4x\Big\{b_2m^2h'_{\mu\nu}h'^{\mu\nu}+b_3m^2h'h'+b_4F_{\mu\nu}F^{\mu\nu}+b_5\partial_\lambda\phi\partial^\lambda\phi\\&+b_6m^2\phi\phi+b_7mh'_{\mu\nu}\partial^\mu A^\nu+b_8mh'\partial_\mu A^\mu+b_9(\partial_\mu\phi\partial_\nu h'^{\mu\nu}-\partial_\mu\phi\partial^\mu h')\\&+b_{10}m^2\phi h'+b_{11}m\phi\partial_\mu A^\mu\Big\}.\end{aligned}\tag{7.2.11}$$

It contains all the considerable terms with at most two derivatives, up to equivalent expressions via partial integrations. At this point m should not be interpreted as the mass of $h'_{\mu\nu}$ yet, but as a given constant with the dimension of a mass that ensures that all the parameters b_r do not carry a physical dimension. The same strategy as in the $\widetilde{S}_{(\partial h'\partial h')}$ case leads to

$$\begin{aligned}0\overset{!}{=}\delta^1\widetilde{S}=\int d^4x\Big\{&(4b_2-b_7)m^2\partial_\mu\xi_\nu h'^{\mu\nu}+(4b_3-b_8)m^2\partial_\mu\xi^\mu h'\\&+(4b_4-b_7)m\Box\xi_\nu A^\nu-(4b_4+b_7+2b_8)m\partial_\mu\partial_\nu\xi^\mu A^\nu\\&+(2b_{10}-b_{11})m^2\partial_\mu\xi^\mu\phi\Big\}+\widetilde{S}[\partial_\mu\xi_\nu+\partial_\nu\xi_\mu,-m\xi_\mu,0].\end{aligned}\tag{7.2.12}$$

Again $\widetilde{S}[\partial_\mu\xi_\nu + \partial_\nu\xi_\mu, -m\xi_\mu, 0]$ will vanish automatically when the rest of $\delta^1\widetilde{S}$ does. Once more, all the arguments of the different contractions have to vanish independently. By some straightforward calculations, this leads to the relations

$$b_2 = -b_3 = b_4 = \frac{1}{4}b_7 = -\frac{1}{4}b_8 \quad \text{and} \quad b_{10} = \frac{1}{2}b_{11}. \tag{7.2.13}$$

From this argumentation the form

$$\begin{aligned} \widetilde{S} = b_1 S_{m=0} + \int d^4x \Big\{ b_2\Big(& m^2(h'_{\mu\nu}h'^{\mu\nu} - h'h') + F_{\mu\nu}F^{\mu\nu} + 4mh'_{\mu\nu}\partial^\mu A^\nu \\ & - 4mh'\partial_\mu A^\mu\Big) + b_5\partial_\lambda\phi\partial^\lambda\phi + b_6 m^2\phi\phi \\ & + b_9(\partial_\mu\phi\partial_\nu h'^{\mu\nu} - \partial_\mu\phi\partial^\mu h') + b_{10}(m^2\phi h' + 2m\phi\partial_\mu A^\mu)\Big\} \end{aligned} \tag{7.2.14}$$

of the desired action follows. Now by performing just the same procedure with δ^2, one gets

$$\begin{aligned} \delta^2\widetilde{S} = \int d^4x \Big\{ & (2b_1 + 4b_2 + b_9)m(\partial_\mu\partial_\nu\Lambda h'^{\mu\nu} - \Box\Lambda h') \\ & + (6b_2 + b_{10})m^2(2\partial_\mu\Lambda A^\mu - m\Lambda h') \\ & + (2b_5 + 3b_9 + 2b_{10})m\Box\Lambda\phi \\ & + (4b_{10} - 2b_6)m^3\Lambda\phi\Big\} + \widetilde{S}[m\Lambda g_{\mu\nu}, \partial_\mu\Lambda, -m\Lambda]. \end{aligned} \tag{7.2.15}$$

Therefore, in order for $\delta^2\widetilde{S}$ to vanish, the remaining parameters have to satisfy

$$\begin{pmatrix} 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_5 \\ b_6 \\ b_9 \\ b_{10} \end{pmatrix} = 0. \tag{7.2.16}$$

This again will ensure that $\widetilde{S}[m\Lambda g_{\mu\nu}, \partial_\mu\Lambda, -m\Lambda]$ vanishes as well. All lines of the matrix M in the expression above are obviously linear independent. Therefore, M has full rank, which implies that its kernel is two dimensional. In fact

$$\ker(M) = \text{span}\,\big\{(1,0,3,0,-2,0), (1,-1/2,-3,6,0,3)\big\}. \tag{7.2.17}$$

The first of the two vectors that span $\ker(M)$ corresponds to an action

$$\widetilde{S}_1 = S_{m=0} + \int d^4x \Big\{3\partial_\lambda\phi\partial^\lambda\phi + 2(\partial_\mu\phi\partial^\mu h' - \partial_\mu\phi\partial_\nu h'^{\mu\nu})\Big\}, \tag{7.2.18}$$

i.e. a massless 2-tensor field and a scalar field that carry the gauge invariances δ^1 and δ^2. The second vector gives

$$\begin{aligned} \widetilde{S}_2 = S_{m=0} + \int d^4x \Big\{ & -\frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'h') - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2m(h'_{\mu\nu}\partial^\mu A^\nu - h'\partial_\mu A^\mu) \\ & - 3\partial_\lambda\phi\partial^\lambda\phi + 6m^2\phi\phi + 3(m^2\phi h' + 2m\phi\partial_\mu A^\mu)\Big\}, \end{aligned} \tag{7.2.19}$$

which is just S_{Stb}. So any linear combination of $\widetilde{S}_1$ and $\widetilde{S}_2$ gives an action with the desired gauge invariances. To determine a unique action, some additional conditions have to be established. A quite natural one would be to claim that the $h'_{\mu\nu}$ field should satisfy the massive Klein-Gordon equation $(\Box - m^2)h'_{\mu\nu} = 0$ when A_μ and ϕ are set to zero via gauge fixing, i.e. $h'_{\mu\nu}$ is supposed to be a particle of mass $m > 0$. The Euler-Lagrange equations in this case are

$$\begin{aligned}\frac{\delta_L \widetilde{S}[h'_{\mu\nu}, 0, 0]}{\delta h'_{\mu\nu}} =& b_1\big(\Box h'_{\mu\nu} - \partial^\lambda\partial_\mu h'_{\lambda\nu} - \partial^\lambda\partial_\nu h'_{\lambda\mu} + \partial_\mu\partial_\nu h' + (\partial_\lambda\partial_\rho h'^{\lambda\rho} - \Box h')g_{\mu\nu}\big) \\ &+ 2b_2 m^2(h'_{\mu\nu} - h' g_{\mu\nu}) = 0.\end{aligned} \tag{7.2.20}$$

Similar as in Section 3.1, the contraction with ∂^μ, together with the assumption that b_2 is not zero, leads to

$$\partial^\mu h'_{\mu\nu} - \partial_\nu h' = 0. \tag{7.2.21}$$

By applying this to the Euler-Lagrange equations, one gets

$$b_1(\Box h'_{\mu\nu} - \partial_\mu\partial_\nu h') + 2b_2 m^2(h'_{\mu\nu} - h' g_{\mu\nu}) = 0. \tag{7.2.22}$$

Contracting with $g_{\mu\nu}$ gives $h' = 0$, which leads together with (7.2.22) to

$$(b_1\Box + 2b_2 m^2)h'_{\mu\nu} = 0. \tag{7.2.23}$$

So by demanding that $h'_{\mu\nu}$ has to satisfy the Klein-Gordon equation for a particle of mass m, one enforces the relation $b_2 = -\frac{1}{2}b_1$ and therefore makes $(b_1, b_2, b_5, b_6, b_9, b_{10})$ linear dependent to the second vector in (7.2.17) and by this $\widetilde{S}$ proportional to $\widetilde{S}_{Stb}$.

Remark The fact that $\widetilde{S}$ is not uniquely determined by the conditions that have been stated at the beginning of this section can also be obtained by making use of the properties of the Stückelberg formalism. The gauge transformations δ^1 and δ^2 commute with each other. Furthermore, ϕ is invariant under δ^1 and $h'_{\mu\nu} + \phi g_{\mu\nu}$ clearly transforms just like $h'_{\mu\nu}$ under δ^1. So if one applies the Stückelberg trick to $S_{m=0}$ in order to make it invariant under δ^2, by replacing $h'_{\mu\nu}$ with $h'_{\mu\nu} + \phi g_{\mu\nu}$, the result is an an action that is invariant under δ^1 and δ^2, according to the results of Section 2.6. This new action does not contain A_μ and therefore is not proportional to S_{Stb}. In fact it turns out that it is just $\widetilde{S}_1$, as can be shown by a straightforward calculation. This argument already shows that one can expect at least two independent actions that are invariant under δ^1 and δ^2. The explicit calculation from above shows that there are in fact only two such actions.

Remark During the derivation of $\widetilde{S}$ it has been shown that the $(\partial h'\partial h')$ terms in $\widetilde{S}$ are (up to a multiplicative constant) uniquely determined by δ^1 to form $S_{m=0}$. This result can also be used to show that every action that is invariant under δ^1 and only contains $h'_{\mu\nu}$ and its first derivatives is proportional to the Fierz-Pauli action. An ansatz for such an action contains the $(\partial h'\partial h')$ terms and possibly $(h'h')$ terms. The structure of the former is already uniquely determined. Since they form a gauge invariant action, the $(h'h')$ terms must do the same. By making the general ansatz

$$\widetilde{S}_{(h'h')} = m^2 \int d^4x \Big\{ \alpha h'_{\mu\nu} h'^{\mu\nu} + \beta h' h' \Big\} \tag{7.2.24}$$

for the corresponding part of the action, one finds

$$0 \stackrel{!}{=} \delta^1 \widetilde{S}_{(h'h')} = 4m^2 \int d^4x \Big\{ \alpha \partial_\mu \xi_\nu h'^{\mu\nu} + \beta \partial_\mu \xi^\mu h' \Big\} + \widetilde{S}_{(h'h')}[\partial_\mu \xi_\nu + \partial_\nu \xi_\mu]. \tag{7.2.25}$$

Consequently, it follows $\alpha = \beta = 0$ and therefore $\widetilde{S}_{(h'h')} = 0$. This proves the statement.

7.3 Connection to General Relativity

Spin-2 particles also play an important role in the context of quantizing gravity. Usually the exchange particles that are assumed to cause gravitational effects, the so-called *gravitons* (see [15]), are modeled to be massless spin-2 particles. To understand this picture consider the *Einstein-Hilbert action*

$$S_{EH} = \frac{1}{16\pi} \int d^4x \sqrt{-\det(g_{\mu\nu})} R, \tag{7.3.1}$$

which is used to describe gravity in the vacuum. Here $g_{\mu\nu}$ now is an arbitrary spacetime metric with inverse $g^{\mu\nu}$ and $R = g^{\mu\nu} R^\rho{}_{\mu\rho\nu}$ the *Ricci scalar*. $R^\rho{}_{\mu\sigma\nu}$ is the *curvature tensor*, which is defined as

$$R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\sigma\lambda} \Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\sigma\mu}. \tag{7.3.2}$$

The *Christoffel symbols* $\Gamma^\rho{}_{\mu\nu}$ are given by

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \Big(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \Big). \tag{7.3.3}$$

The variation principle $\frac{\delta_L}{\delta g_{\mu\nu}} S_{EH} = 0$ leads to the Einstein equations for the vacuum

$$R^\lambda{}_{\mu\lambda\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{7.3.4}$$

The solutions of them usually are referred to as *gravitational waves*. The mathematical models and concepts that are used to formulate general relativity will not be discussed in this thesis. An introduction can be found in [2] and [15]. For this section one should just take S_{EH} as the action that describes the correct dynamics for the metric tensor and therefore models gravity.

In order to find some kind of particle that can be identified with gravity, consider the ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{7.3.5}$$

i.e. one examines cases in which the metric tensor only differs by a small variation $h_{\mu\nu}$ from the one for a flat spacetime. This variation is used as graviton (see [15]). To be more precise, gravitons are modeled as small excitations of the metric tensor from its ground state $\eta_{\mu\nu}$. Expressing S_{EH} in terms of the $h_{\mu\nu}$ leads to extremely complicated self interaction terms of arbitrary order. This causes great difficulties for renormalization approaches. Nevertheless, the expansion of S_{EH} up to second order gives the non-interacting part of the gravitons. So if this part of the action turns out to be the massless Fierz-Pauli action, one would have justified the assumption that gravitons are spin-2 particles.

In fact this is what turns out to be the case. In order to perform this expansion of S_{EH} appropriately, several steps are necessary. Note that all the following contractions are done according to the Minkowski metric $\eta_{\mu\nu}$. Furthermore, since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, its first order approximation in terms of $h_{\mu\nu}$ has the form

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}h_{\alpha\beta}\eta^{\beta\nu} + \mathcal{O}(h^2) = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2). \tag{7.3.6}$$

The term $\mathcal{O}(h^2)$ shall denote the remaining terms of order 2 or higher in $h_{\mu\nu}$.

A good point to start with is to derive the second order approximation of the Christoffel symbols. It has the form

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda})\Big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\Big) + \mathcal{O}(h^3). \tag{7.3.7}$$

Clearly, it starts with the terms that are first order in $h_{\mu\nu}$. Since the Ricci scalar only consists of Christoffel symbols, its approximation also starts with the first order. This is quite convenient, because therefore $\sqrt{-\det(g_{\mu\nu})}$ only needs to be expanded up to first order for the second order approximation of $\sqrt{-\det{(g_{\mu\nu})}}R$. To do so, the well-known formula

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\det(X+\epsilon H) = \det{(X)}\sum_{i,j}(X^{-1})_{ij}H_{ji}, \tag{7.3.8}$$

for the derivative of the determinant at an invertible X in direction H, can be used to get

$$\sqrt{-\det(\eta_{\mu\nu}+h_{\mu\nu})} = \sqrt{-\det(\eta_{\mu\nu})} - \frac{\det(\eta_{\mu\nu})}{2\sqrt{-\det(\eta_{\mu\nu})}}\eta^{\rho\sigma}h_{\rho\sigma} + \mathcal{O}(h^2) = 1 + \frac{1}{2}h + \mathcal{O}(h^2). \tag{7.3.9}$$

This leads to

$$\begin{aligned} S_{EH} = \frac{1}{16\pi}\int d^4x\Big\{ &\partial_\mu\partial_\nu h^{\mu\nu} - \Box h + \frac{3}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + h_{\mu\nu}\Box h^{\mu\nu} \\ &- \partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} - 2h_{\lambda\mu}\partial_\nu\partial^\lambda h^{\mu\nu} \\ &+ \partial_\mu h^{\mu\nu}\partial_\nu h + h_{\mu\nu}\partial^\mu\partial^\nu h + \frac{1}{2}h\partial_\mu\partial_\nu h^{\mu\nu} \\ &- \frac{1}{4}\partial_\lambda h\partial^\lambda h - \frac{1}{2}h\Box h + \mathcal{O}(h^3)\Big\} \end{aligned} \tag{7.3.10}$$

and therefore, by assuming that $h_{\mu\nu}$ falls off sufficiently fast for $x \longrightarrow \infty$, gives

$$\begin{aligned} S_{EH} = \frac{1}{32\pi}\int d^4x\Big\{ &-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} \\ &- \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h + \mathcal{O}(h^3)\Big\}. \end{aligned} \tag{7.3.11}$$

So the non-interacting part is basically just the Fierz-Pauli action for $m = 0$. Therefore, gravitons have to be massless spin-2 particles.

Alternatively to this calculation one can also apply the ansatz $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to the Einstein equations (7.3.4) and expand them up to first order in $h_{\mu\nu}$. This leads to the equations of motion that correspond to the massless Fierz-Pauli action (see for example [2] or [15]). Since one already knows that they are equivalent to the defining equations of massless spin-2 particles, this leads to the desired result as well. Such approximations of $g_{\mu\nu}$ are sometimes referred to as linearized gravity (see for example [2]).

This relation between massless spin-2 particles and general relativity also offers a nice way to interpret the invariance of the Fierz-Pauli action under the gauge transformations

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{7.3.12}$$

In fact it is a relict of the invariance of the Einstein-Hilbert action under coordinate transformations. To see this[1], consider a coordinate transformation of the form

$$x^\mu \longmapsto x'^\mu = x^\mu - \xi^\mu. \tag{7.3.13}$$

The ξ^μ shall be chosen such that the variation $h'_{\mu\nu}$ from $\eta_{\mu\nu}$ of the metric tensor $g'_{\mu\nu}$ in the new coordinates is still small. To be more precise, one assumes that $\frac{\partial}{\partial x^\mu}\xi^\nu$ has an order of magnitude that does not exceed the one of $h_{\mu\nu}$. This assumption justifies to consider only the terms that do not contain any products of $\frac{\partial}{\partial x^\mu}\xi^\nu$ and $h_{\mu\nu}$ for the calculation of $g'_{\mu\nu}$ and therefore leads to the following approximation[2]:

$$\begin{aligned} g_{\mu\nu} = g'_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} &= (\eta_{\rho\sigma} + h'_{\rho\sigma})(\delta^\rho_\mu - \frac{\partial}{\partial x^\mu}\xi^\rho)(\delta^\sigma_\nu - \frac{\partial}{\partial x^\nu}\xi^\sigma) \\ &\approx \eta_{\mu\nu} + h'_{\mu\nu} - \frac{\partial}{\partial x^\mu}\xi_\nu - \frac{\partial}{\partial x^\nu}\xi_\mu. \end{aligned} \tag{7.3.14}$$

Consequently, one finds

$$h'_{\mu\nu} \approx h_{\mu\nu} + \frac{\partial}{\partial x^\mu}\xi^\nu + \frac{\partial}{\partial x^\nu}\xi^\mu \tag{7.3.15}$$

for this weak field approximation. This clearly is just the gauge transformation for the spin-2 field. So one can interpret the gauge transformations as infinitesimal coordinate transformations. And consequently the gauge invariance of the Fierz-Pauli action resembles the invariance under infinitesimal coordinate transformations.

[1]The following discussion is taken from [15].

[2]The approximation sign $\approx$ shall indicate that all products of $\frac{\partial}{\partial x^\mu}\xi^\nu$ and $h'_{\mu\nu}$ are neglected.

Chapter 8

Conclusion

The BRST formalism has been successfully applied to quantize spin-2 particles of arbitrary mass. In both cases, the massive and the massless, the correct candidate for a subspace of $\ker \Omega$ that is isometric isomorphic to $\mathfrak{V}$ can already be guessed from the form of the BRST transformations of the annihilation operators. This subspace is generated by the classical polarizations. So one can identify the physical space that originates from the BRST formalism with the one generated by the classical polarizations, without any problems. Especially in the massive case this is a neat result, since it shows that the introduction of Stückelberg fields does not change the physical system, but is a legitimate trick to introduce a gauge invariance that allows for a BRST quantization of the spin-2 fields.

A further remarkable result is that the presented couplings to external sources are uniquely determined by the corresponding restrictions, if they even exist. An interesting subject for further studies is the treatment of the corresponding external fields as interacting particles by adding dynamical terms for them to the action. In particular, it is a fascinating question to ask whether it is possible to introduce the gauge transformations δ^1 and δ^2 for those fields in such a way that one can regain the coupling terms simply by demanding that their action is gauge invariant. In other words, is it possible to treat the spin-2 and Stückelberg fields as gauge fields that have to be introduced (just as the photon field has to be introduced in electromagnetism) in order to ensure the gauge invariance of a certain action?

Another possible extension of the results of this thesis would be the introduction of interaction terms for the spin-2 fields. A natural first approach would be to consider the lowest order interaction terms that result from a power series expansion of the Einstein-Hilbert action and analyze whether it is possible to find similar gauge transformations as δ^1 and δ^2 for the resulting action.

However, for both those considerations a more careful BRST quantization will be necessary. Since these expansions include interactions, not all appearing fields can be assumed to be external. Therefore, one needs to use the Nakanishi-Lautrup fields in order to formulate the BRST transformations appropriately.

So the results of this thesis are just a small part of all the quite interesting properties of quantized spin-2 particles. They offer a good starting point for further investigations, which will hopefully be performed.

Bibliography

[1] Bekaert, X. and Boulanger, N.: *The Unitary Representations of the Poincaré Group in any Spacetime Dimension.* Expanded lecture notes for the Modave summer school in mathematical physics in Belgium, August 2006. arXiv:hep-th/0611263v1, November 2006. Mentioned on page 12, 13, 14, 15, 16

[2] Carroll, S.: *Spacetime and Geometry. An Introduction to General Relativity.* Addison Wesley, San Francisco – Boston – New York – Capetown, 2004. Mentioned on page 100, 101, 102

[3] Das, A.: *Lectures on Quantum Field Theory.* World Scientific Publishing Co. Pte. Ltd., Singapore – Hackensack – London, 2008. Mentioned on page 1, 2, 3, 16, 17, 18, 22, 59, 60, 78

[4] Henneaux, M. and Teitelboim, C.: *Quantization of Gauge Systems.* Princeton University Press, Princeton, 1992. Mentioned on page 43

[5] Hinterbichler, K.: *Theoretical Aspects of Massive Gravity.* Reviews of Modern Physics, Volume 84, 671-710, 2012. Mentioned on page 2, 39, 40, 43, 47, 48, 53, 54, 60, 86

[6] Kilian, W., Ohl, T., Reuter, J. and Sekulla, M.: *Resonances at the LHC Beyond the Higgs Boson: The Scalar/Tensor Case.* Physical Review D, Volume 93, 036004, 2016. Mentioned on page 2

[7] Kögerler, R. and Sladkowski, J.: *The BRST Analysis of the Stückelberg Formalism.* Acta Physica Polonica B, Volume 26, 903-912, 1995. Mentioned on page 2, 39

[8] Kugo, T. and Ojima, I.: *Local Covariant Operator Formalism of Non-Abelian Gauge Theories and Quark Confinement Problem.* Progress of Theoretical Physics Supplements, Volume 66, 1-130, 1979. Mentioned on page 2, 26, 29, 30, 70, 71

[9] Nakanishi, N. and Ojima, I.: *World Scientific Lecture Notes in Physics Vol. 27. Covariant Operator Formalism of Gauge Theories and Quantum Gravity.* World Scientific Publishing Co. Pte. Ltd., Singapore – New Jersey – London – Hong Kong, 1990. Mentioned on page 2, 11, 17, 19, 24, 25, 26, 28, 29, 30, 31, 35, 63, 70, 94

[10] Ohl, T.: *Quantum Field Theory 2.0.* Lecture notes for Quantenfeldtheorie II, Summer term 2016, Julius-Maximilians-Universität Würzburg. July, 2016. Mentioned on page 2, 11, 19, 20, 95

[11] Ohl, T.: *Theoretical (Elementary) Particle Physics.* Lecture notes for Theoretische Teilchenphysik, Summer term 2015, Julius-Maximilians-Universität Würzburg. July, 2015. Mentioned on page 2, 39

[12] Peskin, M. E. and Schroeder D. V.: *An Introduction to Quantum Field Theory.* Westview Press, Colorado, 1995. Mentioned on page 2, 3, 11, 17, 23, 36, 57

[13] Ruegg, H. and Ruiz-Altaba, M.: *The Stueckelberg Field.* International Journal of Modern Physics A, Volume 19, 3265-3347, 2004, arXiv:hep-th/0304245v2. Mentioned on page 2, 39, 40

[14] Salam, A. and Strathdee J.: *Equivalent Formulations of Massive Vector Field Theories.* Physical Review D, Volume 2, 2869-2876, 1970. Mentioned on page 39

[15] Weinberg, S.: *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley & Sons, New York – Chichester – Brisbane – Toronto – Singapore, 1972. Mentioned on page 51, 100, 101, 102

[16] Weinberg, S.: *The Quantum Theory of Fields. Volume I: Foundations.* Cambridge University Press, Cambridge – New York – Melbourne, 1995. Mentioned on page 3, 14, 36, 37

[17] Weinberg, S.: *The Quantum Theory of Fields. Volume II: Modern Applications.* Cambridge University Press, Cambridge – New York – Melbourne, 1996. Mentioned on page 2, 17, 19, 20, 22, 23, 25, 27

[18] Zeidler, E.: *Quantum Field Theory. Volume I: Basics in Mathematics and Physics.* Springer-Verlag, Berlin – Heidelberg, 2006. Mentioned on page 36, 37

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen verwendet und die Arbeit keiner anderen Prüfungsbehörde unter Erlangung eines akademischen Grades vorgelegt habe.

Würzburg, den

(Raimond Abt)