Master Thesis

The Complex Mass Scheme, Gauge Dependence and Unitarity

in Perturbative Quantum Field Theory

submitted by

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1. Introduction

As a matter of fact, most particles, either fundamental or composed, are unstable, but in everyday life unstable particles play a minor role, for instance, directly we do only observe photons, and in chemistry or biology the electron and proton play major roles where the proton is believed to be stable. The situation is completely different at todays colliders such as the Large Hadron Collider. Even though stable particles are brought to reaction and in the final states one recovers stable particles, intermediate unstable particles are produced and give in many cases dominant contributions to reactions.
The Standard Model (SM) is one of the most successful theories nowadays because of its precise predictions which are confirmed by experiments. It is based on the principles of Quantum Field Theory (QFT). Four fundamental forces are known whereas the SM does incorporate three of them - the electroweak and strong force. The interaction is mediated by so-called gauge bosons which is characteristic for gauge theories.
Often, calculations are carried out in the framework of perturbative QFT which is assumed to be a valid perturbative expansion.
In view of the SM, even most fundamental particles are unstable and in contrast to stable ones, unstable particles do not fit in a pure perturbative QFT. In certain cases it turns out that a perturbative treatment is possible, but not always recommended, in particular not near thresholds where observables take on unphysical values. As for Quantum Electrodynamics (QED) one does not encounter such problems - photons are stable and so are electrons though they have mass. But also when intermediate particles are present perturbative QFT might work out, for instance, the muon decay is well-approximated by first-order perturbation theory even though there is an unstable intermediate particle, the W boson. As soon as unstable particles are produced 'on-shell' finite-width effects play a crucial role. It is known how to account for finite-widths in perturbative QFT - one needs a mechanism transforming the Feynman propagator into a propagator with a finite-width, but this cannot be done arbitrarily and should be done such that none of the defining symmetries of the theory are violated. For instance, simply adding by hand a finite-width would violate gauge independence.
In this work we consider the Complex Mass Scheme (CMS) which is one of the methods available dealing with unstable particles in a perturbative QFT while guaranteeing exact gauge invariance. The manipulations associated to the CMS introduce a finite-width to

the propagator of unstable particles such as complex couplings and the question arises how unitarity is implemented. Unitarity is not expected to be violated because for non-perturbative QFT it has been shown that unitarity is fulfilled given that unstable particles are excluded from asymptotic states. Since perturbation theory is a perturbative expansion around the full theory, if done correctly, the CMS should thus not violate unitarity. Introducing a finite-width and complex couplings, the Cutkosky cutting rules are no longer valid. These rules express perturbative unitarity in the case of stable particles only and it is desirable to have corresponding rules for unstable particles. In the first part of this work we derive such rules, similar to the cutting rules, but based on Veltman's Largest Time Equation (LTE) and we discuss unitarity using several examples.

In the second part of this work we go deeper into the foundation of the CMS. For non-Abelian gauge theories the local gauge invariance has to be replaced by Becchi, Rouet, Stora and Tyutin (BRST) invariance. There is an invariance known as the extended BRST invariance which allows to study with ease gauge (in-)dependence. We discuss whether such a symmetry can be implemented for a quantum theory. From an algebraic point of view it has been proven that such an invariance can always be enforced given that the former invariance, i.e. the usual BRST invariance, does not exhibit a quantum anomaly. Using the example of an SU(2) Higgs model we study the case when Spontaneous Symmetry Breaking (SSB) is present and we revise certain arguments which were given by Piguet for the pure Yang-Mills case.

As a result of our discussion we demonstrate the gauge independence of the definition of the physical mass and the gauge independence of the S matrix. In view of the CMS, the manipulations do neither violate the BRST invariance nor gauge dependence is introduced by the renormalization condition which can be proved via the extended BRST invariance, thus the CMS yields a gauge-independent S matrix.

2. Principles of Quantum Field Theory

The language of modern particle physics is QFT, a quantum theory with infinite degrees of freedom. Interactions occur between fields and fields consist of field quanta which can be interpreted as particles. Classical field theory serves as a starting point where the field content and interactions are given by the Lagrangian $\mathcal{L}$. The equations of motion (EOM) follow from the action principle

$$\delta S[\varphi] = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\mathcal{L}}{\partial \partial_\mu \varphi} + \text{higher derivatives} \stackrel{!}{=} 0.$$

From Quantum Mechanics (QM) it is known that position and momentum cannot be measured simultaneously which is most naturally expressed by the commutator relation

$$[q, p] = \mathrm{i}\hbar$$

leading to the Heisenberg uncertainty inequality. Alike to QM there is a canonical construction of QFT. Fields are promoted to operator-valued distributions living on a suited Hilbert space $\mathcal{H}$.

In relativistic QFT free particles are characterized according to Wigner's classification, i.e. free particles are represented as states of $\mathcal{H}$ which transform as irreducible unitary representations of the Poincaré group. The Poincaré group unifies the group of spacetime translations and Lorentz transformations and can be seen as the relativistic pendant to the Galilean group. States are uniquely characterized by the two Casimir invariants, which, roughly speaking, define the mass[1] and the spin or helicity of the particle, depending whether the particle is massive. Besides the space-time symmetries there are also internal symmetries. One-particle states are represented as

$$|p, \alpha\rangle = \hat{a}^\dagger(p, \alpha)\, |0\rangle \,,$$

where p is the four momentum, $p^2 = m^2$, m is the mass, α represents further quantum numbers such as spin et cetera and $\hat{a}^\dagger(p, \alpha)$ is the creation operator which creates out of the vacuum state $|0\rangle$ that one-particle state. The creation and annihilation operators

[1] One has to impose the condition that the mass is greater or equal to zero.

$\hat{a}^\dagger(p,\alpha), \hat{a}(p,\alpha)$ satisfy commutator or anti-commutator relations. Characteristic for a QFT is the fact that interactions are not restricted to finite number of particles. From the creation and annihilation operators one constructs n-particle Hilbert spaces

$$\hat{a}^\dagger : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n+1}$$
$$\hat{a} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}$$

where $\mathcal{H}^{\otimes n} := \overbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}^{n}$. Allowing arbitrary number of particles, the operators act on the Fock space $\mathcal{F}$

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}.$$

Observables which can be measured by todays collider experiments are of special interest. For instance, the momentum and energy of particles can be determined precisely by detectors, and this suggests to favour the momentum basis over the position space. Having characterized particles, particle physicists are interested in the probability for a configuration of particles going over to another configuration via decays, scattering or recombination. This probability can be expressed with the S matrix. Given initial and final (n-particle) states $|\Phi_i(t)\rangle, \langle\Phi_f(t)|$ at $t = -\infty, +\infty$, respectively, we define the S matrix S_{fi} as

$$S_{fi} := \langle \Phi_f(+\infty) \, | \Phi_i(+\infty) \rangle := \langle \Phi_f(+\infty) | \, S \, | \Phi_i(-\infty) \rangle \,,$$

where the operator S transforms the initial state into a final state. Hence, S holds all information about the time evolution and the interaction. Likewise to usual QM, $|S_{fi}|^2$ is interpreted as transition probability $i \to f$.

2.1. Perturbative Quantum Field Theory

Any quantum mechanical problem which is defined by a Hamiltonian can be formally transformed to a functional integral known as the path integral. The interpretation of the path integral is that the system takes any possible configuration in traveling from an initial to a final state. These configurations are most naturally expressed by Feynman diagrams which are a graphical representation of the perturbative expansion of path integrals.

In the following we define Feynman diagrams and we introduce our notation for their visualization. Then, we sketch the concept of Green's functions, one-particle irreducible functions and we establish their connection to the S matrix which is given by the Lehmann Symanzik Zimmermann (LSZ) reduction formula.

Definition 1 (Feynman diagram)**.** A Feynman diagram $\mathcal{F}$ is characterized by n vertices and m links. Each link represents a Feynman propagator $\Delta_{\mathrm{F}ij}$ connecting to vertices i, j and for every vertex i there is an associated coupling constant g_i. Both, links and vertices depend on the theory and are given by the Feynman rules. We denote vertices by spacetime points x_i, $i = 1, \ldots, n$. Generically, the diagrams are written as

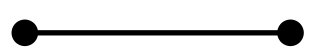
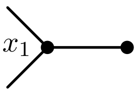
$$\mathcal{F}(x_1, \ldots, x_n) = \left(\prod_{i<j} c_{ij} \, \mathrm{i}\Delta_{\mathrm{F}ij}(x_i - x_j) \right) \times cc, \tag{2.1}$$

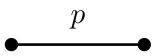
where the coefficients c_{ij} are given by

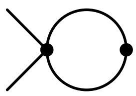
$$c_{ij} = \begin{cases} 1: & i \text{ connected to } j \\ 0: & i \text{ not connected to } j \end{cases}. \tag{2.2}$$

The coupling constants (cc) can differ from vertex to vertex. Generically, they are given by $cc = \prod_{i=1}^{n} \mathrm{i}g_i$ where g_i is a coupling specified by the vertex i.

In a scalar theory, the Feynman propagator is denoted by a straight line going from one spacetime point to another. Spacetime points or vertices are denoted by dots. Integrating out the inner spacetime points and going to the momentum space, we leave out the spacetime point indication. Instead, a momentum is associated to each propagator. When connecting propagators a coupling insertion is implied. Asymptotic states are represented by straight lines without any dots indicating their end. Connecting asymptotic states to any propagator a coupling insertion is implied.

$\mathrm{i}\Delta_{\mathrm{F}}(x_1, x_2) = x_1$  x_2, $\quad \mathrm{i}g \; \mathrm{i}\Delta_{\mathrm{F}}(x_1, x_2) =$ x_1  x_2. (2.3)

$\mathrm{i}\Delta_{\mathrm{F}}(p) =$ 

, $\quad \mathrm{i}g \; \mathrm{i}\Delta_{\mathrm{F}}(p) \; \mathrm{i}g \; \mathrm{i}\Delta_{\mathrm{F}}(q-p) =$ 

. (2.4)

2.2. Green's Function and Functional Calculus

The main ingredient for the S matrix are vacuum expectation values (vev) of products of fields. The n-point Green's function in QFT is defined as the time-ordered vev

$$\begin{aligned} G_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n) &:= \langle 0|\,\mathcal{T}\hat{\psi}_{\alpha_1}(x_1)\ldots\hat{\psi}_{\alpha_n}(x_n)\,|0\rangle \\ &= \frac{\left\langle 0^0\right|\mathcal{T}\hat{\psi}^0_{\alpha_1}(x_1)\ldots\hat{\psi}^0_{\alpha_n}(x_n)\exp\left(-\mathrm{i}S_{\text{Int}}\left(\hat{\psi}^0\right)\right)\left|0^0\right\rangle}{\left\langle 0^0\right|\mathcal{T}\exp\left(-\mathrm{i}S_{\text{Int}}\left(\hat{\psi}^0\right)\right)\left|0^0\right\rangle}, \end{aligned} \tag{2.5}$$

where the second line is the famous Gell-Man Low formula. $\mathcal{T}$ stands for the time ordering operator and quantities with an upper 0 are in the interaction picture. General solutions are nearly impossible to obtain and one falls back to a perturbative calculation. This is where the representation (2.5) is advantageous, but nowadays such calculations are performed with path integrals and an equivalent representation of (2.5) is given by the path integral

$$G_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n) = \frac{\int \mathrm{D}\left[\psi\right]\psi_{\alpha_1}(x_1)\ldots\psi_{\alpha_n}(x_n)\exp\left(\mathrm{i}S[\psi]\right)}{\int \mathrm{D}\left[\psi\right]\exp\left(\mathrm{i}S[\psi]\right)}, \tag{2.6}$$

where $\mathrm{D}\left[\psi\right]$ is the path integral measure and $S[\psi]$ is the action functional. The measure is not well-defined, but can formally be written as $\mathrm{D}\left[\psi\right] = \prod_{x\in\mathbb{R}^4}\mathrm{d}\psi(x)$. In the path integral formulation the fields ψ are no longer distribution-valued operators, but numbers or Grassmann-valued numbers in the case of fermionic fields.

The form above motivates the definition of a generating functional of n-point Green's functions.

$$Z[j] := \int \mathrm{D}\left[\psi\right]\exp\left(\mathrm{i}S[\psi] + \mathrm{i}\int \mathrm{d}^4x\; j_\alpha(x)\psi_\alpha(x)\right)$$

It follows from the rules of functional derivation that

$$G_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n) = \frac{1}{Z[0]}\prod_{i=1}^{n}\frac{\delta}{\mathrm{i}\delta j_{\alpha_i}(x_i)}Z[j]\bigg|_{j=0}. \tag{2.7}$$

The perturbative expansion is done in two steps. One separates the action in two parts, namely into the free part which can be solved exactly and into the interaction part

$$S[\psi] = S_{\text{free}}[\psi] + S_{\text{int}}[\psi],$$

then the argument of interaction is replaced by functional derivatives

$$Z[j] = \exp\left(\mathrm{i}S_{\text{int}}\left[\frac{\delta}{\mathrm{i}\delta j}\right]\right)\int \mathrm{D}\left[\psi\right]\exp\left(\mathrm{i}S_{\text{free}}[\psi] + \mathrm{i}\int \mathrm{d}^4x\; j_\alpha(x)\psi_\alpha(x)\right).$$

The free part $S_{\text{free}}[\psi]$ is by definition quadratic in the fields ψ, and in this case the path integral can be solved exactly by Gaussian integrals yielding

$$Z[j] = \exp\left(\mathrm{i}S_{\text{int}}\left[\frac{\delta}{\mathrm{i}\delta j}\right]\right)\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; j_\alpha(x)G_{\alpha\beta}(x-y)j_\beta(y)\right), \tag{2.8}$$

up to unimportant normalizations, where the free Green's function is defined via

$$\mathcal{L}_{\text{free}}(x) = \frac{1}{2}\psi_\alpha(x)G^{-1}_{\alpha\beta}\psi_\beta(x).$$

Example 2.1. Consider the Klein-Gordon field defined by $\mathcal{L}_{\text{free}}(x) = \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right)$. Partial integration yields $G^{-1} = -\Box - m^2$, thus

$$G = \frac{1}{-\Box - m^2} \to \tilde{G}(p) = \frac{1}{p^2 - m^2}$$

2.3. The LSZ Reduction Formula

The time-ordered correlation function encodes the full physics, but more than necessary. Indeed, the external legs corresponding to space-time points in (2.7) are not necessarily physical, i.e. neither must the external momenta fulfill the 4-momentum conservation nor must they be on-shell. The S matrix, however, is a physical observable and there should be a way to project on the physical part of the correlation functions which should be somehow related to S and this is exactly what the LSZ reduction does. Moreover, the LSZ reduction, being at the very heart of scattering theory, gives, in contrast to other approaches, a precise definition of asymptotically free states and the connection to interacting fields.

For the LSZ reduction to hold the fields must fulfill several properties, among these is the adiabatic hypothesis

$$'' \lim_{t\to=\pm\infty} \hat{\phi}(x) = \sqrt{R}\,\hat{\phi}_{\text{out/in}}(x)'',$$

which must be understood in the weak sense. The asymptotic fields do not coincide with the interacting ones, even not in the limit of infinite time, but only being proportional to each other. The reason for this is that interacting fields do not create one-particle states with probability one, as it is the case for free fields, but are able of creating multi-particle states.

The LSZ reduction depends on certain normalizations and one of them being that the vevs of the one-point functions vanishes, i.e. $\langle 0|\,\hat{\phi}(x)\,|0\rangle = 0$. The other one is related to the adiabatic hypothesis, namely the one-particle weight $\left|\langle 0|\,\hat{\phi}(x)\,|p,a\rangle\right|^2 = R$. One

can show that the latter is equivalent to the residue of the two-point function (2.10) where the multi-particle contributions vanish because of strong oscillations (Riemann-Lebesgue lemma).
So far the results can be compared to the full interacting theory and from a spectral point of view the full propagator of the interacting field ϕ has the structure

$$\langle 0| \mathcal{T}\phi(x)\phi(0) |0\rangle = \int_0^\infty \mathrm{d}s \, \rho\,(s) \, \langle 0| \mathcal{T}\phi^0(x)\phi^0(0) |0\rangle \tag{2.9}$$

$$= \int \mathrm{d}^4p \, \mathrm{e}^{-\mathrm{i}px} \left(\frac{R}{p^2 - m^2 + \mathrm{i}\epsilon} + \int_{\tilde{m}^2}^\infty \mathrm{d}s \, \sigma\,(s) \, \frac{1}{p^2 - s + \mathrm{i}\epsilon} \right), \tag{2.10}$$

which is known as the Källén-Lehmann representation. The spectral density ρ has been decomposed $\rho(s) = R\,\delta(s - m^2) + \sigma(s)$ into its one-particle contribution (δ) and multi-particle continuum (σ). ϕ^0 represents a free scalar field satisfying the Klein-Gordon equation. This result together with the Canonical Commutation Relations (CCR) leads to the requirement $0 \leq R < 1$.
At this point we must be very careful since we are still talking of bare quantities which are not renormalized. It turns out that perturbation theory is inconsistent and these inconsistencies manifest themselves in divergences, but renormalization, which takes place at the level of the Lagrangian and which enters the theory in form of renormalization constants, helps out and is explained later. Nevertheless, we shall now investigate the field-renormalization. We can define renormalized fields $\sqrt{Z_R}\phi_R = \phi$ such that

$$|\langle 0| \phi_R(x) |p, a\rangle|^2 \stackrel{!}{=} 1 \quad \Rightarrow Z = R,$$

which is a choice we can make and which does have no physical consequences. Clearly, since the canonical fields ϕ fulfilled the CCR, the renormalized ones do not. Further, we define a renormalized Green's function by replacing fields with the renormalized ones. Such a choice does of course affect n-point functions and especially the residue of the two-point function and it seems that the connection to the full theory (2.10) is lost, but this is not the case. One expects that the perturbative result for Z approximately matches with the one of the full theory, but naive perturbative computation demonstrates the opposite and Z is actually divergent and not between zero and one. The solution to this problem is that one should not expect the correction to Z to be small and indeed they are not. One way to deal with this[2] is via the renormalization group where resummations are performed yielding a finite and well-approximated result for Z.
Coming back to the scattering matrix S, given that all fields have vanishing vev and that the one-particle weight is normalized to R_{ω_i}, where ω_i specifies the field, the LSZ

[2] See discussion in [Col84]

reduction formula reads

$$S = \langle k_1 \ldots, \mathrm{out}| \ldots k_n, \mathrm{in}\rangle_c = \left(\prod_{i=1}^{n} \lim_{k_i^2 \to m_i^2} \frac{\mathrm{i}}{\sqrt{R_i}} (k_i^2 - m_i^2) \epsilon^{\omega_i} \right) G_{\omega_1,\ldots,\omega_n}, \tag{2.11}$$

where the product goes over all external particles ω_i and, further, the ϵ^{ω_i} are contracted to G and the momenta k_i fulfill 4-momentum conservation, i.e. $\sum_i k_i = 0$. The derivation of the formula is standard and is found in the literature, e.g. [IZ80]. Later, after having discussed extended BRST transformations, we come back to this formula showing that the $\sqrt{R_i}$ are very important when gauge fields are involved. We explicitly demonstrate in section 7.6.2 the gauge independence of the S matrix and that it is unitary.

2.4. Connected Green's function, the Linked Cluster Theorem and the Vertex Functional

The definition of the S matrix is yet not complete and naive computation of Green's function via the generating functional Z could lead to too high cross sections. The problem with Z is that it generates disconnected results and these are uncorrelated. Of physical interest are only those contribution to the S matrix which are connected to external vertices as well as among each other. A nice discussion about this can be found in [Wei96],[Ede+66]. There it is worked out that disconnected processes factorize into connected subprocesses which, accordingly, have nothing to do with each other. Physically spoken, processes taking place at sufficiently distant places do not affect the outcome of each other and only connected S matrix elements should be compared with the experiment.

In practice, the generating functional is replaced by the generating functional for connected Green's function. Let $Z[j]$ be the generating functional for Green's function, then the linked cluster theorem states that the generating functional for connected Green's function $Z_c[j]$ is related to $Z[j]$ by

$$Z[j] = \mathrm{e}^{\mathrm{i}Z_c[j]}. \tag{2.12}$$

According to the formula (2.7) Green's function are normalized with $Z[0]$. This accounts to, speaking in terms of Feynman diagrams, eliminating vacuum contributions which after the previous discussion do not contribute anyway to physical processes[3].

[3] The statement follows from the linked-cluster theorem. Vacuum contributions are not connected to the rest of the process and therefore one can exponentiate the result to connected vacuum bubbles. Dividing by $Z[0]$, where only vacuum bubbles are left, cancels all of them.

The last, but very important generating functional is the vertex functional Γ also known as the generating functional for one-particle irreducible (1PI) Green's functions. Γ is defined as the Legendre transform of Z_c with respect to the source j

$$\Gamma[\phi] = Z_c[j(\phi)] - \int \mathrm{d}^4x \; j(x)\phi(x).$$

To lowest order which is tree level, the vertices are given by the action S where the Feynman rules for a vertex can be computed by multiple functional derivatives of S with respect to the fields participating at the vertex, i.e. ${}^{\delta^n S}\!/\!{}_{\delta\phi_1 \ldots \delta\phi_n}$. Higher contribution involve loops and the generating functional Z_c can also be thought of as the generating functional of tree diagrams where the vertices are computed not by S, but by taking Γ instead. In this sense Γ represents the quantum action whereas S is the classical action [Pok03]. All generating functionals are equivalent and one is free to choose the most suited one. With regard to proofs of renormalizability Γ plays the major role.

2.5. The Optical Theorem

The history of the optical theorem can be traced back to the nineteenth century. When light penetrates a medium refraction is observed. The process can be fully described by the knowledge of the refractive index in the case of plane waves and it was Rayleigh who discovered a connection between the absorptive (imaginary) part of the refractive index and forward scattering of photons. This connections is nothing but the optical theorem.

Nowadays, the theorem is mostly referred to the full cross-section of a scattering process rather than refractive indices:

$$\sigma_{\text{tot}} = \text{flux factor} \times \text{Im}\left[\mathcal{T}_{ii}\right], \tag{2.13}$$

where σ_{tot} is the total cross-section and $\text{Im}\left[\mathcal{T}_{ii}\right]$ is the imaginary part of the forward scattering amplitude. The proof is trivial and follows directly from unitarity which is equivalent to conservation of probability.

The situation is nearly the same for QFT, but with a crucial difference: One cannot infer from classical field theory that the built-in symmetries and especially unitarity hold for the quantized version. The phenomenon of field theories with symmetries which cannot be promoted to a quantum theory is called an anomaly.

In the language of QFT unitarity implies that the S matrix fulfills

$$\mathcal{S}^\dagger \mathcal{S} = \mathbb{1}. \tag{2.14}$$

Separating the non-scattered contributions from $\mathcal{S}$ via $\mathcal{S} =: \mathbb{1} + \mathrm{i}\mathcal{T}$, we can deduce from the unitarity condition (2.14)

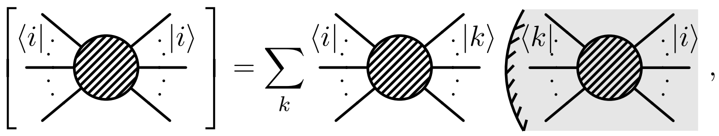
$$\begin{aligned}\left(\mathbb{1} - \mathrm{i}\mathcal{T}^\dagger\right)(\mathbb{1} + \mathrm{i}\mathcal{T}) = \mathbb{1} - \mathrm{i}\left(\mathcal{T}^\dagger - \mathcal{T}\right) + \mathcal{T}^\dagger\mathcal{T} \overset{!}{=} \mathbb{1} \\ \Rightarrow \quad \mathcal{T}^\dagger\mathcal{T} = \mathrm{i}\left(\mathcal{T}^\dagger - \mathcal{T}\right)\end{aligned} \tag{2.15}$$

To establish the connection to the optical theorem we contract the transition operator with initial and final states

$$\begin{aligned}\mathrm{i}\langle f|\,\mathcal{T}^\dagger - \mathcal{T}\,|i\rangle = \mathrm{i}\left(\underbrace{\langle f|\,\mathcal{T}^\dagger\,|i\rangle}_{(\langle i|\mathcal{T}|f\rangle)^*} - \underbrace{\langle f|\,\mathcal{T}\,|i\rangle}_{=:\mathcal{T}_{fi}}\right) \\ \Rightarrow \mathrm{i}\left(\mathcal{T}^*_{if} - \mathcal{T}_{fi}\right) = \sum_k \mathcal{T}^*_{kf}\mathcal{T}_{ki},\end{aligned} \tag{2.16}$$

where the sum runs over all possible intermediate states and total 4-momentum conservation is implied. Forward scattering is when initial and final states coincide, then we have

$$2\mathrm{Im}\left[\mathcal{T}_{ii}\right] = -2\mathrm{Re}$$



where the shadowed region on the right-hand side (rhs) of the equation refers to the complex conjugated transition operator $\mathcal{T}^*$. The cross section is proportional to the square of the absolute value of the transition amplitude and working out the kinematics one recovers the optical theorem (2.13).

From the rules of perturbative QFT we obtain for a given process the S matrix elements or transition amplitude $\mathcal{T}$. With these objects we can calculate the left-hand side (lhs) of the unitarity equation (2.16) and we denote the result as left-hand side of the unitarity equation (LHSUE). Perturbative unitarity is fulfilled if the lhs equals right-hand side of the unitarity equation (RHSUE), i.e. (2.16), or at least up to the given perturbative order. For a scalar theory the expression (2.16) further simplifies. The Feynman propagator is invariant under inversion $x^\mu \to -x^\mu$ wherewith we conclude that $\mathcal{T}_{fi} = \mathcal{T}_{if}$ and consequently

$$\Rightarrow \overbrace{-2\mathrm{Re}\left[\mathrm{i}(\mathcal{T}_{fi})\right]}^{\text{LHSUE}} = \overbrace{\sum_k \mathcal{T}^*_{kf}\mathcal{T}_{ki}}^{\text{RHSUE}}\,. \tag{2.17}$$

This will be the formula we will work with in chapter 3. In section 3.1 we demonstrate how to systematically evaluate the LHSUE with the help of cutting rules for stable particles and in section 3.2 we extend the rules to unstable particles.

2.6. Renormalization

In the previous section 2.5 we mentioned that canonical QFT has some inconsistencies and that divergences occur beyond tree level. These divergences make any theory useless, but there is a way out. Renormalization renders divergent vertex functions finite giving the formal expression (2.5) a physical meaning in a unique way, but not all theories are renormalizable. There are basically two things one must distinguish - the defining symmetries of the field theory can or cannot be implemented in the corresponding quantum theory or the interaction operators are of too high dimension making the theory non-renormalizable. The former is disastrous and will be discussed in more detail in 2.6.3, the latter is not that problematic. Let us first clarify the notion of renormalizability. Renormalizability is defined as follows - one distinguishes between non-renormalizable, renormalizable and super-renormalizable theories. A theory is called super-renormalizable if there is only a finite number of divergent diagrams. A theory is renormalizable if a finite set of counter-terms is sufficient to cancel all divergences. And finally, a theory is non-renormalizable if an infinite set of counter-terms is necessary. To determine whether a diagram is divergent or not one usually does a power-counting analysis leading to the so-called superficial degree of divergence. It is often possible to eliminate the dependence of vertices and to find a closed expression which then allows to investigate whole classes of diagrams where the degree of divergence does only depend on the space-time dimension d and on how many external particles participate. Let γ denote a generic diagram, then one finds in general for the superficial degree of divergence ω

$$\omega(\gamma) = d - \sum_{N_\phi \in \gamma} d_\phi + \sum_{N_{\text{vert}} \in \gamma} d_{\text{vert}},$$

where the first sum runs over the number of external fields having the canonical dimension d_ϕ. The second sum runs over special vertices which may raise the power-counting dimension and which are taken into account in the superficial degree of divergence by assigning a dimension for these vertices. It is noted that a power-counting divergent graph is not necessarily divergent. Power-counting is rather a necessary condition for a diagram being divergent, but symmetries in theories may be responsible for high momentum cancellations which is not captured by naive power-counting.

In the old days renormalizability was often assumed because of convenience. For instance, the SM was postulated to be a renormalizable theory, but actually it could have been otherwise. Todays results from high-energy colliders confirm that the relevant and marginal part of the field operators, that is the renormalizable operators, are in agreement with the observations and if the SM is an effective theory, which it certainly is, then the cutoff must be very high, where the scale is most often referred to the Planck scale. Non-renormalizable theories are not bad and the modern way of thinking of QFT is in terms of effective theories. These theories must not be renormalizable. Anyway, as fundamental theories renormalizable QFT are desirable since they are more predictive in the sense that they are valid up to arbitrary high scales.

2.6.1. Procedure of Renormalization

So far any known QFT has bad large momentum behavior, and beyond tree level diagrams are most likely divergent. The parameters in the Lagrangian are said to be non observable, these are often called bare parameters and the idea of renormalization is to distinguish bare parameters from renormalized ones which are by definition finite. Formally, bare fields, e.g. ϕ_0, ψ_0, and bare parameters, e.g. m_0, g_0 are related to renormalized ones via

$$\phi_0 = Z_\phi^{\frac{1}{2}} \phi_R, \quad m_0 = Z_m m_R, \quad g_0 = Z_g g_R, \tag{2.18}$$

In view of gauge theories it turns out that another convention concerning gauge couplings is more suited. Given a gauge field A^μ, the corresponding gauge coupling g_0 shall renormalize as follows

$$g_0 = \frac{Z_g}{Z_A^{\frac{3}{2}}} g_R. \tag{2.19}$$

In perturbative QFT the (divergent) renormalization factors Z are expanded in loop orders and because at tree level no renormalization is necessary the expansion reads[4]

$$Z = 1 + \hbar \delta Z^1 + \hbar^2 \delta Z^2 + \mathcal{O}(\hbar^3).$$

The renormalized parameter can be identified as physical parameter, for instance, the renormalized mass may be equal to the physical one, i.e. $m_R = m$, where m is defined as the pole position of the 2-point function. This case is known as the mass scheme and it is important to note that this is a special case and renormalized quantities must not be observable.

The difference in $m_R - m$ does depend on the finite part of the counter terms which is convention. In this sense there is an arbitrariness in the finite part of counter-terms

[4]There is a correspondence of loop order to orders in $\hbar$. The correspondence follows from Euler's famous formula for planar graphs.

and renormalized quantities which does not have any physical meaning. This invariance is naturally understood by the renormalization group which expresses the invariance of the theory under rescaling, i.e. it is irrelevant at which scale the theory has been renormalized. In practice, one does not simply calculate observables in the full theory and perturbation theory is a common choice. The prediction of the full theory is unaffected by the scale, but choosing a scale does have an impact on the perturbative results which may lead to quite wrong results and the renormalization point can be crucial for the validity of the perturbative result.

2.6.2. Unstable Particles and the Complex Mass Scheme

From the Källén-Lehmann representation (2.10) follows that the two-point function has poles on the real axis ($p^2 \in \mathbb{R}$) starting with the one-particle pole followed by branch cuts originating from multi-particle states. The thresholds are fixed by the explicit value of masses and it is possible to move the one-particle pole into the branch cut. In this case the particle has no longer an asymptotic limit according to the LSZ and the particle is unstable. It is well-known that the phenomenon of unstable particles is of non-perturbative nature and it is impossible to change the pole structure of an analytic function without resummation, i.e. without a geometric series. Veltman has shown [Vel63] that for a super renormalizable theory the S matrix in non-perturbative QFT was unitary on the Hilbert space spanned by only stable particles. He assumed the existence of a Källén-Lehmann representation for unstable particles which lacks a one-particle pole on the real axis and he showed via a LTE unitarity for dressed propagators.

There is yet no fully established treatment of unstable particles within perturbation theory. The problem comes, as already mentioned, from the need of resummed self-energies. However, this resummation, if done wrong, leads to violation of gauge independence, e.g. the naive manipulation of the propagator as it is done in the fixed-width scheme introduces gauge dependence[5]. There were several attempts on including unstable particles in perturbative QFT while respecting gauge invariance. The most straightforward way is found in [BVZ92]. They do not perform any resummation nor do they change the propagator of unstable particles, but instead they multiply the complete matrix element with

$$\frac{p^2 - M^2}{p^2 - M^2 + \mathrm{i}\Gamma M},$$

[5] Gauge cancellations take place between Feynman diagrams. When changing the pole structure via a Breit-Wigner propagator these cancellations do no longer occur order by order as has been pointed out in [Stu91].

thus keeping gauge invariance while treating an intermediate particle with momentum p as an unstable one. This procedure has become popular under the name of factorization scheme. In the special case where the unstable particle is resonant the narrow-width approximation is a valid method. Besides these two methods, the very first rigorous treatment was given by R.G. Stuart [Stu91]. He proposed a Laurent expansion of the complete matrix element around the complex poles.
There are further methods and most, especially the mentioned ones, lack in larger phase space validity and are only valid in the region of resonance, but not below the thresholds, in contrast to the CMS [DD06] which is valid in the full phase space. The basic idea is to perform an analytic continuation of masses to the complex plane. The Ward identities, being algebraic relations, are thus not violated by such a modification[6]. Additionally, the renormalization conditions must be gauge independent to guarantee gauge independence of the QFT, which is motivated by the result that the complex pole is gauge independent. A proof of this statement is given in chapter 7.6.1. In practice, the Feynman rules are modified by the rule that any appearing mass corresponding to an unstable particle is replaced by the complex one in such a way that the bare Lagrangian is not modified. We sketch the procedure: In the first step renormalized parameters are introduced. Let m_0 denote the bare mass of an unstable particle, then introduce

$$m_0^2 = Z_\mu \mu^2 = \mu^2 + (Z_\mu - 1)\,\mu^2 =: \mu^2 + \delta\mu^2 \tag{2.20}$$

where μ^2 is resummed in the propagator while $(Z_\mu - 1)\,\mu^2$ is not. This brings us to the CMS propagator

$$\Delta_{\mathrm{F}}(x-y,\mu) := \frac{1}{(2\pi)^4}\int \mathrm{d}^4p\,\frac{\mathrm{e}^{-\mathrm{i}p(x-y)}}{p^2-\mu^2}, \tag{2.21}$$

where the usual causality prescription (see (3.7)) is not necessary due to the finite imaginary part of $\mu^2 = M^2 - \mathrm{i}\Gamma M$. The procedure implies that the mass counter terms are complex. Since the bare mass is real, we derive the following consistency equation

$$\mathrm{Im}\left[\mu^2\right] = -\mathrm{Im}\left[\delta\mu^2\right]. \tag{2.22}$$

Further, it happens that couplings become complex in favor of gauge invariance which is, for instance, the case in the Glashow Salam Weinberg (GSW) theory where the weak mixing angle becomes complex [DD06].
The last missing piece to complete the discussion of renormalization is the renormalization condition which fixes the finite part of counter terms at a given scale. A prominent

[6] Actually, there is another way of thinking of the CMS. In reference [BBC00] they provide an effective approach for the description of finite-width effects. By the addition of gauge-invariant non-local terms they are free to choose the propagator structure. As a special case they obtain the CMS.

example is the on-shell scheme which differs from others in the way that the renormalized parameters are equal to physical observables. The direct connection to physical quantities comes from the fact that the scale is nothing but the physical mass. More concretely, one demands that the renormalized two-point function of a stable particle at its mass $p^2 = m^2$ is given by the Feynman propagator (3.7). This condition does fix both, the mass counter term and the field renormalization (see [Den93]). The on-shell scheme can be extended to the case of unstable particles and the renormalization conditions read

$$\Sigma(p^2)\big|_{p^2=\mu^2} = 0, \;\; \Sigma'(p^2)\big|_{p^2=\mu^2} = 0. \tag{2.23}$$

Σ denotes the self-energy of the unstable particle and Σ' is that self energy, but differentiate with respect to p^2. The renormalization condition together with the consistency equation (2.22) represent a gauge independent definition of the complex pole. Even though the renormalzation is similar to the on-shell scheme, there is a difference and this can be measured in the experiment. For a discussion we refer to [Vel94].

2.6.3. The Quantum Action Principle

Besides the renormalization scheme which fixes the scale there are regularization schemes which are methods to extract divergences in practice. The physics should not depend on the regularization scheme and one must guarantee independence. Schemes such as dimensional regularization or the Bogoliubov, Parasiuk, Hepp, Zimmermann and Lowenstein (BPHZL) method may or may not yield the same result, i.e. a regularized result may depend on the regularization. As we are going to show, QFTs beyond tree level are defined by symmetries and one distinguishes between symmetries which cannot be maintained in the quantum extension from symmetries which can be maintained, but which might be broken by a particular regularization scheme. The hope of renormalization is to obtain a well-defined QFT with the same symmetries as the classical field theory and which is independent of the specific regularization scheme. Therefore, it is desirable to investigate symmetries of QFT on another level independent of regularization schemes and that is where algebraic renormalization and the Quantum Action Principle (QAP) comes to our help. For the understanding of the QAP the notion of insertions is important. Green's function are computed as the vevs of local field operators. When a new operator is added to a specific Green's function, the operation is called an insertion. Concerning the definition of these operators one must be careful in the case of an interacting theory. As in the case of elementary field operators, composed operators are defined via Green's functions obtained formally by the Gell-Mann Low formula or by path integrals, thus, a proper renormalization is necessary. The so-obtained operator is then called a normal product. According to Zimmermann's definition, insertions are thus normal products that are the generalization of normal products of free fields in the sense of Wick's definition. Consider for simplicity only one field $\phi(x)$, the insertion of the normal product $N_d[B](x)$ where B is composed of elementary fields, e.g. $B(x) = (\phi(x))^m$ for some m, is defined by

$$N_d[B\{\phi\}](x) \circ G_n(x_1 \ldots, x_n) := \langle 0 | \mathcal{T} B(x) \phi(x_1) \ldots |0\rangle \,,$$

where the rhs must be understood in the renormalized sense. An insertion of B is therefore nothing but the replacement of vertices by new vertices defined by the tree approximation of B and a proper treatment of quantum corrections, i.e. subtractions are needed. The index d stands for the dimension of B which enters the superficial divergence. For more details we refer to the original literature [Zim73].

2.6.4. The Dyson-Schwinger Equation

To get a feeling what the QAP is useful for we give an example of a QAP, namely the Dyson-Schwinger (DS) equation which can be derived by a heuristic calculation within path integral formulation [PS95]

$$\left.\frac{\delta S[\phi]}{\delta \phi}\right|_{\phi \to \frac{\hbar}{\mathrm{i}} \frac{\delta}{\delta J}} Z[J] + J Z[J] = 0.$$

To establish what is known as a QAP we go over to 1PI functions. In a first step we need to cancel out non-connected diagrams and the trick is the same as usual, i.e. connect the result for $\left.\frac{\delta S[\phi]}{\delta \phi}\right|_{\phi \to \frac{\hbar}{\mathrm{i}} \frac{\delta}{\delta J}}$ to a new source, say $j_{\delta S}$, and divide by Z. Performing the Legendre transform one arrives at

$$\frac{\delta S}{\delta \phi} \cdot \Gamma := \frac{\delta \Gamma[\phi, j_{\delta S}]}{\mathrm{i} \delta j_{\delta S}} = \frac{\delta \Gamma[\phi]}{\delta \phi}. \tag{2.24}$$

The DS equation captures the response of the theory under the change of fields (rhs) which is an insertion (lhs). Actually, the result (2.24) is not always true and depends on the regularization, but in general, the QAP states that whether we change fields or parameter of the theory, within a power-counting renormalizable theory the result is an insertion, i.e.

$$\frac{\delta \Gamma[\phi]}{\delta \phi} = \Delta_\phi \cdot \Gamma, \quad \frac{\partial \Gamma}{\partial \lambda} = \Delta_\lambda \cdot \Gamma.$$

or put in other words: A change of fields or couplings within a vertex function can be represented as a vertex function where several vertices are replaced by new ones. We do not give a proof and the difficult part is to show that regularization preserves this structure. Actually, all known regularization schemes fulfill the QAP, for instance, for the proof within BPHZL scheme we refer to [Low71] and within dimensional regularization the proof has been given in [BM77]. This result makes the QAP very powerful, a tool to investigate renormalization in a regularization independent way. In chapter 6 we investigate gauge theories, spontaneous symmetry breaking and the gauge dependence of such theories. There, we also discuss renormalizability and counter-terms will play a major role. The QAP will help us out and there we make use of the result that counter-terms fall under the certain class, namely so-called symmetric insertions which we will introduce later. In the next section, we briefly discuss the power of the QAP and what we are going to discuss in the forthcoming sections.

2.6.5. From the Tree Approximation to Quantum Corrections

The necessity of renormalization is indisputable and the question is if the procedure of renormalization, which is not simply removing divergences, does violate the defining symmetries at any stage of perturbative expansion. Renormalization can only be a valid way of ”saving” perturbative QFT if it is possible to ensure that the defining symmetries hold up to arbitrary high perturbative orders.

Having discussed the advantage of using the QAP when it comes down to renormalization, we now discuss renormalization within an algebraic approach which is the approach when investigating symmetries.

In our first steps we need to prepare the defining symmetries, i.e. the symmetry transformations are being expressed in a functional way allowing us to study their effect on the generating functional Z and therefore, in general, for all n-point functions. The result is then exponentiated and Legendre transformed, yielding a relation in terms of the vertex functional, this is done in section 7.4.2. We are going to consider a gauge theory (chapter 6) coupled to a scalar SU(2) Higgs which is spontaneously broken (section 7.2). Among those symmetries are the Lorentz invariance which results from the underlying Poincaré symmetry and the local gauge invariance which is the symmetry coupling both kinds of fields, i.e. gauge and matter fields. The local gauge invariance is, as discussed in 7.1, replaced by the BRST invariance.

For the moment we assume that our theory is renormalizable and we briefly discuss the standard proof of whether a given symmetry can be promoted to a quantum symmetry in the algebraic approach via the QAP. To this end, let $\mathcal{S}(\Gamma) = 0$ denote the invariance of the vertex functional under BRST transformation. Assume we have renormalized the theory up to $n-1$ loops. At loop order n after regularization the symmetries might be broken and we have that

$$\mathcal{S}(\Gamma^n) = \Delta^n,$$

where the QAP tells us that to lowest order, here $\mathcal{O}(\hbar^n)$, Δ^n is composed of integrated local field polynomials. Later on we proof one of two useful identities concerning the so-called Slavnov-Taylor (ST) operator $\mathcal{S}_\Gamma$, namely

$$\mathcal{S}_F \mathcal{S}(F) = 0, \ \ \forall F \quad \text{and} \quad \mathcal{S}_F^2 = 0, \ \text{if} \ \ \mathcal{S}(F) = 0, \tag{2.25}$$

but for the moment we take them as granted. From equation (2.25) follows that Δ^n fulfills $\mathcal{S}_\Gamma \Delta^n = 0$. Since the violating terms are of order n, the non-linear ST can be expanded to this order which results in simply replacing Γ by the classical action S in $\mathcal{S}_\Gamma$, i.e. $\mathcal{S}_\Gamma \to \mathcal{S}_S$. Again making use of (7.30) the most general solution Δ^n has the structure

$$\mathcal{S}_S \Delta^n = 0 \quad \Leftrightarrow \quad \Delta^n = \phi^+ + \mathcal{S}_S \phi^0,$$

with $\mathcal{S}_S \phi^+ = 0$ and $\phi^+ \neq \mathcal{S}_S \Phi$. A theory is defined to be non-anomalous if $\phi^+ = 0$, then it is possible to restore the symmetry. Consider the corrected vertex functional $\Gamma^n \to \Gamma^n - \phi^0$ then we have that $\mathcal{S}(\Gamma^n) = \mathcal{O}(\hbar^{n+1})$ if and only if (iff) $\phi^+ = 0$. This has practical applications because one often wants to use dimensional regularization which is, for certain symmetries such as supersymmetry, a non-invariant regularization method. For an introduction to this topic we refer to [GHS01]

The condition $\mathcal{S}_S \Delta^n = 0$ is known as the Wess-Zumino consistency condition and is solved via the so-called descent equation (see literature [PS95]).

Actually, solving the consistency equation does in general not give an answer to the question whether the theory can be extended to a quantum theory or not, but one can determine the full functional dependence up to a factor. The result for ϕ^+ for simple Lie groups reads

$$\phi^+ = r \varepsilon_{\mu\nu\alpha\beta} \operatorname{Tr} \int \mathrm{d}x \; u_a \partial^\mu \left(d^{abc} \partial^\nu A_b^\alpha A_c^\beta + \frac{\mathcal{D}^{abcd}}{12} A_b^\nu A_c^\alpha A_d^\beta \right)$$

where $\mathcal{D}^{abcd} = d^{eab} f^{ecd} + d^{eac} f^{edb} + d^{ead} f^{ebc}$. $\mathbf{A}_\mu$ is the gauge field, $\mathbf{u}$ the ghost field and r an undetermined constant. A one-loop calculation is enough to determine if $r = 0$ which is known as the Adler-Bardeen Theorem [AB69].

Since we consider a SU(2) theory an anomaly is excluded because $d^{abc} = 0 \ \forall \ a, b, c$. Furthermore, it is well known that the usual gauge theories are renormalizable because of BRST invariance, i.e. the divergences can be absorbed with a finite set of counter terms while maintaining the symmetries. Consequently, an SU(2) model should make

perfectly sense for a quantum theory.
We are especially interested in the question of gauge-dependence and we investigate the same theory, but the BRST invariance is replaced by the extended BRST invariance. As we are going to show, within spontaneously broken theories, we need at least two so-called BRST doublets to implement the extended BRST invariance to be able to describe the full gauge-dependence of the theory. In reference [Qua02] (original proof [BDK90]) it has been shown that BRST doublets are not involved in the cohomology of $\mathcal{S}_S$. Put it in other words, this means that we do not induce anomalies when we extend the symmetry and the ϕ^+ coincide in both theories. In section 7.5 we start investigating symmetric insertions which are, roughly speaking, candidates for regularization independent counter terms and we show that the theory is renormalizable for at least the physical sector.

3. Unitarity and the Cutting Equation

3.1. Largest Time Equation

3.1.1. The Landau Equations and the Cutkosky Cutting Rules

In this section we introduce the Cutkosky cutting rules, their basis and why they are no longer valid when CMS propagators are involved.
Long ago, Cutkosky showed [Cut60] that the discontinuity of an arbitrary Feynman graph is given by replacing in all possible ways propagators by δ functions. Keeping the notation of [Cut60] any Feynman graph $\mathcal{F}$ has the structure

$$\mathcal{F} = \int \prod_i \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \frac{B}{A_1 \dots A_N}, \tag{3.1}$$

where B is a polynomial, $A_i = q_i^2 - m_i^2 + \mathrm{i}\epsilon$, q_i being linear combinations of loop momenta k_j and external momenta. Cutkosky states that according to any solution to the Landaus equations, which we introduce below, there is a contribution to the discontinuity of $\mathcal{F}$ and taking along all contributions yields the optical theorem (2.16). Let us first clarify the connection between the optical theorem and discontinuities by taking the example of dispersion relations. Let $\mathcal{M}(z)$ denote an amplitude analytically continued. It follows from the representation (3.1) that the amplitude is analytic everywhere except for possible branch cuts on the real axis (see figure 3.1). Then, from Cauchy's integral formula follows

$$\mathcal{M}(z) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{d}z' \frac{\mathcal{M}(z')}{z'-z}, \tag{3.2}$$

where γ is a closed contour around z not crossing or hitting any branch cuts. Let Γ denote the contour along the branch cut and assume that the amplitude falls off at least as $1/z$, then, in general, we have that

$$\oint_\gamma \mathrm{d}z' \frac{\mathcal{M}(z')}{z'-z} + 2\pi\mathrm{i} \sum_i \frac{\mathrm{Res}\mathcal{M}(z_i)}{z_i - z} + \int_\Gamma \mathrm{d}z' \frac{\mathcal{M}(z')}{z'-z} = 0., \tag{3.3}$$

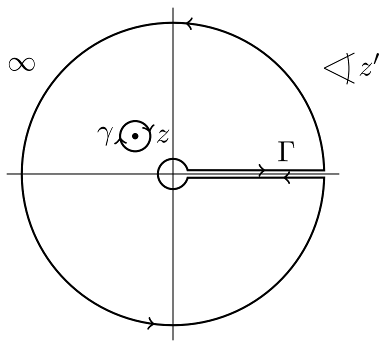


Figure 3.1.: The picture illustrates the contours appearing in (3.3). Shown is the complex plane z', the branch cut goes from zero to $+\infty$ and there is a single pole located at $z' = z$.

the middle part drops out since $\mathcal{M}$ is holomorphic in the upper and lower complex plane, but the last part which is the discontinuity does contribute. For the special case of the S matrix the discontinuity starts at some real threshold value and the cut is placed to $+\infty$, thus the discontinuity can be rewritten as an integral in the upper complex plane over the imaginary part of $\mathcal{M}$. Using (3.2) we can represent the full amplitude as a dispersion relation which is nothing but an integral along the branch cut.
The origin of these branch points are singularities of the integral representation (3.1) which are captured by the Landau equation. Actually, the validity of the Landau equations is not restricted to Feynman graphs and the analysis can be generalized. In the notation of [Ede+66] the most general situation is given by an integral representation of a function $f(z)$ defined via

$$f(z) = \int_H \prod_i \mathrm{d}w_i \, g(z, w), \tag{3.4}$$

where H is a hypercontour in the complex w space. The special case of (3.1) is included in (3.4). The Landau equations have their origin in the study of singularities of the integral representation (3.4). Consider (3.4) for $i = 1$ and assume the integrand $g(z, w)$ has singularities $w_r(z)$ ($r = 1, 2, \dots$), then the representation (3.4) is well-defined as long as the contour can be deformed to avoid singularities. This is prevented when

- there are end-point singularities, i.e. $g(z, w) \in \partial H$ is singular. Actually, exactly this happens for non-renormalized Feynman graphs, but the UV-divergence is cured by regularization.
- there are pinch-singularities that cause the hypercontour to get caught in between singularities, thus it is not possible to deform the contour avoiding the singularities.
- a singularity drags the contour to infinity, thus, an infinite deformation is inevitable

making the integral potentially divergent.

In practice the difficulty lies in determining whether one of these cases occurs and this becomes more and more difficult as the dimensionality of the problem grows. For our purposes it is enough to know that the Cutkosky rules are derived from the Landau equations which deal with the occurrence of pinch singularities and endpoint singularities. Without going too deep into detail, it is possible to formulate with the help of singularity surfaces conditions which can then be applied to the case of Feynman graphs. For instance, for the Feynman parametrization of (3.1)

$$(3.1) = \left(\int \prod_i \mathrm{d}\alpha_i\right) \int \prod_i \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \frac{B}{\left(\sum_{i=1}^N \alpha_i A_i\right)^N} \delta\left(1 - \sum \alpha_i\right),$$

one obtains the following Landau equations

$$\alpha_i A_i = 0, \qquad \frac{\partial}{\partial k_i} \sum_k \alpha_k A_k = 0 \qquad \forall i \tag{3.5}$$

where in this case the singularity surfaces are simply given by the A_i and the k_i are loop momenta. The Feynman parametrization makes the singularity condition obvious, either A_i is zero or α_i. The second condition is more interesting and represents the condition that singularity surfaces/ singularities actually pinch the hypercontour.
In the case of (3.1) where the singularities are simple poles, Cutkosky has shown that the discontinuity of $\mathcal{F}$ is connected to the solution $A_i = 0$, more concretely, the discontinuity is given by residues at position according to the solutions to the Landau equations, thus they can be represented as $\delta(A_i)$.

$$\mathrm{disc}\mathcal{F} = (-2\pi \mathrm{i})^r \int \prod_i \frac{\mathrm{d}^4 k_i}{(2\pi)^4} \frac{\delta^+(A_1)\ldots\delta^+(A_r)}{A_{r+1}\ldots A_n} \tag{3.6}$$

Though, in practice it is not clear how to evaluate the discontinuity (3.6), especially not when CMS propagators are involved where some δ functions would take on complex arguments. We do not define the meaning of the $+$ and r as it will become clear much easier by another method. In this work we focus on another approach by Veltman [Vel63]. His approach has the advantage that it is straightforward, easily derived, does not need any topological arguments and the equation can be taken literally, i.e. one knows exactly the result for the discontinuity.
In this section 3.1 we introduce the LTE which can be seen as the analogue to Cutkoskys cutting rules. We start with a derivation based on a decomposition of the Feynman propagator into advanced and retarded propagators. In section 3.2 we extend the arguments to CMS propagators.

3.1.2. Decomposition of Feynman Propagators in Spacetime

The Feynman propagator in space-time representation reads for stable particles

$$\Delta_{\mathrm{F}}(x-y)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^4}\int \mathrm{d}^4p\,\frac{e^{-\mathrm{i}p(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}. \tag{3.7}$$

It is possible to perform a decomposition of Δ_{F} in positive and negative time parts in such a way that positive (negative) time is connected to positive (negative) energy flow and vice versa[1].

Theorem 3.1 (Decomposition theorem)**.** *There exist functions $\Delta^{\pm}$ with the following properties:*

$$\Delta_{\mathrm{F}}(x_i-x_j)=\theta\left(x_i^0-x_j^0\right)\Delta^{+}(x_i-x_j)+\theta\left(x_j^0-x_i^0\right)\Delta^{-}(x_i-x_j), \tag{3.8}$$

$$\Delta^{\pm}(x_i-x_j)=-\left(\Delta^{\mp}(x_i-x_j)\right)^*=\Delta^{\mp}(x_j-x_i), \tag{3.9}$$

$$\Delta_{\mathrm{F}}{}^*(x_i-x_j)=-\theta\left(x_i^0-x_j^0\right)\Delta^{-}(x_i-x_j)-\theta\left(x_j^0-x_i^0\right)\Delta^{+}(x_i-x_j). \tag{3.10}$$

For the moment it is enough to know that such a decomposition exists, for both stable and CMS propagators (2.21). The proof and the explicit formulae will be given later.

Definition 2 (The underline operation)**.** Given a Feynman diagram in the sense of Definition 1, we define the following operations:

- A space-time point x_i can be underlined: $\underline{x_i}$.

This operation shall have the following consequences for connected vertices:

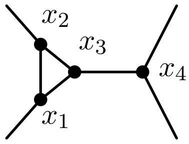
- Δ_{ki} is unchanged if x_k, x_i are unchanged
- Transform $\Delta_{ki}\to\Delta_{ki}^{+}$ if and only if (iff) $x_k\to\underline{x_k}$, but x_i remains unchanged
- Transform $\Delta_{ki}\to\Delta_{ki}^{-}$ iff $x_i\to\underline{x_i}$, but x_k remains unchanged
- If two connected space-time points x_k, x_i are underlined, then $\mathrm{i}\Delta_{ki}\to-\mathrm{i}\Delta_{ki}^{*}$.
- Any underlined space-time point implies a complex conjugation of the coupling constant at this point, i.e. if $x_k\to\underline{x_k}$, then $\mathrm{i}g_k\to-\mathrm{i}g_k$

At the level of Feynman diagrams the underline operation is illustrated as a circle ◯ at the corresponding underlined space-time points.

[1] The energy flow direction is related to the sign of p^0 where p^0 is the zeroth component of the four-momentum. The momentum representation is connected to the space-time representation via Fourier transform.

Example 3.1. Consider the following amplitude

$$\mathcal{F}(x_1, x_2, x_3, x_4) = \tag{3.11}$$



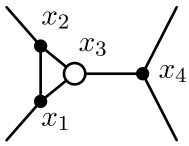
,

where $\mathcal{F}$ is given by

$$\mathcal{F}(x_1, x_2, x_3, x_4) = \mathrm{i}g \; \mathrm{i}\Delta_{12} \; \mathrm{i}g \; \mathrm{i}\Delta_{23} \; \mathrm{i}g \; \mathrm{i}\Delta_{31} \; \mathrm{i}g \; \mathrm{i}\Delta_{34}. \tag{3.12}$$

If we underline the space-time point x_3, we obtain the diagram

$$\mathcal{F}(x_1, x_2, \underline{x_3}, x_4) = \tag{3.13}$$



,

which corresponds to the formula

$$\mathcal{F}(x_1, x_2, \underline{x_3}, x_4) = \mathrm{i}g \; \mathrm{i}\Delta_{12} \; \mathrm{i}g \; \mathrm{i}\Delta^{-}_{23} \; (-\mathrm{i}g^{*}) \; \mathrm{i}\Delta^{+}_{31} \; \mathrm{i}g \; \mathrm{i}\Delta^{+}_{34}. \tag{3.14}$$

Let us assume that in example above the space-time points x_4, x_3 obey the condition $x_4^0 > x_3^0$. According to the decomposition rule (3.8), we can replace Δ_{34} by Δ^{-}_{34}. Performing the underline operation on x_4, i.e. $\underline{x_4}$, we replace Δ_{34} by Δ^{-}_{34} and we get an additional minus sign. Therefore, we obtain the following relation

$$\mathcal{F}(x_1, x_2, x_3, x_4) + \mathcal{F}(x_1, x_2, x_3, \underline{x_4}) = 0 \quad \text{for} \quad x_4^0 > x_3^0. \tag{3.15}$$

Due to the fact that the space-time points x_1 and x_2 are not connected to x_4, we can extend the above relation to

$$\mathcal{F}(x_1, x_2, x_3, x_4) + \mathcal{F}(x_1, x_2, x_3, \underline{x_4}) = 0, \tag{3.16}$$

$$\mathcal{F}(\underline{x_1}, x_2, x_3, x_4) + \mathcal{F}(\underline{x_1}, x_2, x_3, \underline{x_4}) = 0, \tag{3.17}$$

$$\mathcal{F}(x_1, \underline{x_2}, x_3, x_4) + \mathcal{F}(x_1, \underline{x_2}, x_3, \underline{x_4}) = 0, \tag{3.18}$$

$$\mathcal{F}(\underline{x_1}, \underline{x_2}, x_3, x_4) + \mathcal{F}(\underline{x_1}, \underline{x_2}, x_3, \underline{x_4}) = 0. \tag{3.19}$$

Of course, we can proceed in this way for all space-time points which are not connected to x_4.

Let x_3 be underlined, i.e. $\mathcal{F}(x_1, x_2, \underline{x_3}, x_4)$, meaning that Δ_{34} is replaced by Δ^{+}_{34}. Applying the underline operation on x_4 and using the decomposition property, it is easy to see that

$$\mathcal{F}(x_1, x_2, \underline{x_3}, x_4) + \mathcal{F}(x_1, x_2, \underline{x_3}, \underline{x_4}) = 0. \tag{3.20}$$

Consequently, if we have found one and only one space-time point x_k which is the largest time, then we observe that always pairs of LTE components cancel each other. Therefore, given such a x_k, we can sum over all underlinings and obtain a vanishing result.

3.1.3. Largest Time Equation

Theorem 3.2 (Largest Time Equation)**.** *Given a Feynman diagram $\mathcal{F}$ in the sense of Definition 1. If all couplings are real and all propagators fulfill the decomposition theorem 3.1, then the following equation holds*

$$\left(\prod_i \sum_{t_i=0,1}\right) \mathcal{F}(t_1 x_1 + (1-t_1)\underline{x_1}, \dots, t_j x_j + (1-t_j)\underline{x_j}, \dots) = 0,$$

where the sum runs over all possibilities of underlining elements. In total there are 2^N contributions where N is the number of vertices.

Proof. Without loss of generality (wlog), assume x_i^0 is the largest time, i.e.

$$x_i^0 > x_k^0,\ \forall k \neq i, \tag{3.21}$$

then for an arbitrary set A of underlinings $A \in \text{Underlinings}\ \{x\}/\{x_i\}$

$$\mathcal{F}_A(\dots, x_i, \dots) + \mathcal{F}_A(\dots, \underline{x_i}, \dots) = 0. \tag{3.22}$$

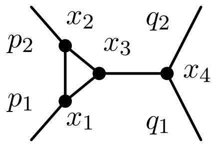
Since A is arbitrary we can sum over all distinct sets of underlinings. Merging all distinct A with the underlinings of x_i we obtain all possible underlinings. $\square$

The proof does not consider equal time cases, but it is easily implemented and works out (by chance!) for scalar particles. We meet a problem when dealing with higher spin particles where the cancellation at equal time is no longer that simple, but the case of equal time is irrelevant when going to Fourier space because isolated points have zero measure and this argument holds as long as these LTE violating contributions do not behave like distributions since then they do contribute[2]. Matrix elements can be

[2] In the case of vector bosons one has to deal with distribution-valued contributions at equal time which violate the LTE. These terms do not affect the outcome in Fourier space. They are so-called contact

expressed in momentum space and fulfill physical constraints such as purely physical ingoing and outgoing particles. To establish the connection between relations from LTE and matrix elements we switch to momentum space and demand that all external particles are purely physical in the sense that they lie on the mass-shell with positive mass. Our first step is to perform a Fourier transform. Let us go back to our first example, i.e

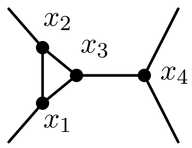
$$\mathcal{F}(x_1(p_1), x_2(p_2), x_3, x_4(q_1, q_2)) = \quad . \tag{3.23}$$



The apparent dependence of space-time points on momenta shall not be understood as a dependence, but as an assignment. Ingoing particles get a factor $e^{\mathrm{i}px}$, while outgoing particles get a factor $e^{-\mathrm{i}px}$. Integrating over all space-time points, we transform $\mathcal{F}$ from space-time to momentum space. Since all propagators in $\mathcal{F}$ are defined by Fourier components (3.7), (2.21), integrating over space-time yields the well-known matrix element expressed by Fourier components

$$(2\pi)^4 \delta(p_1 + p_2 - q_1 - q_2)\ \mathrm{i}\mathcal{M}(p_1, p_2, q_1, q_2) =$$

$$\prod_{i=1,2,3,4} \int \mathrm{d}^4 x_i\ e^{\mathrm{i}(p_1 x_1 + p_2 x_2 - q_1 x_4 - q_2 x_4)} \quad , \tag{3.24}$$



where momentum conservation is fulfilled at every vertex. Physical or asymptotic states are those states with positive energy, mass greater or equal zero, and physical polarization. The cutting rules for stable particles express transformations from off-shell particles to on-shell particles, i.e. to asymptotic states, but for the individual case a non-vanishing contribution is only possible if for a given cut the on-shell region is covered by the off-shell region such that the off-shell particle can be put on-shell. Putting on-shell means to replace the Feynman propagator by $\Theta(\pm p^0)\delta(p^2 - m^2)$ which will, as we shall demonstrate, not always lead to a contribution due to the kinematical situation.

In this section we derive cutting rules for stable and unstable particles and we start by deriving the Fourier components of the decomposition (3.8). Motivated by the result for stable particles, we extend the LTE relations for unstable particles described within the CMS and we start with a proposition which we prove afterwards.

terms which are related to the ill-defined expressions at equal times or equally at large momentum. Renormalization takes care of these divergences [Hoo05].

Proposition 3.1. The Fourier components of $\Delta^{\pm}(x)$ for stable particles read

$$\begin{aligned}\Delta^{+}(k) &= 2\mathrm{i}\theta(k^0)\mathrm{Im}\left[\Delta_{\mathrm{F}}(k)\right], \\ \Delta^{-}(k) &= 2\mathrm{i}\theta(-k^0)\mathrm{Im}\left[\Delta_{\mathrm{F}}(k)\right],\end{aligned} \tag{3.25}$$

where $\Delta_{\mathrm{F}}(k)$ is the Fourier component of the Feynman propagator. In the case of unstable particles the decomposition reads

$$\begin{aligned}\Delta^{+}(k) &= \mathrm{i}\mathrm{Im}\left[\frac{1}{k_1^0(k^0 - k_1^0)}\right], \\ \Delta^{-}(k) &= -\mathrm{i}\mathrm{Im}\left[\frac{1}{k_1^0(k^0 + k_1^0)}\right],\end{aligned} \tag{3.26}$$

where k_1^0 is given by $k_1^0 = \sqrt{\mathbf{k}^2 + M^2 - \mathrm{i}\Gamma M}$.

3.2. Construction of a LTE for CMS propagators

3.2.1. Recapitulation for Stable Particles

The Feynman propagator $\Delta_{\mathrm{F}}(x-y)$ for stable particles with mass m and causality prescription $\epsilon \to 0$ is given by (3.7)

$$\Delta_{\mathrm{F}}(x-y,m) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{-\mathrm{i}k(x-y)}\frac{1}{k^2 - m^2 + \mathrm{i}\epsilon}.$$

For unstable particles with mass M and width Γ the CMS provides the following Feynman propagator (2.21)

$$\Delta_{\mathrm{F}}(x-y,\mu) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{-\mathrm{i}k(x-y)}\frac{1}{k^2 - \mu^2}, \quad \mu^2 = m^2 - \mathrm{i}\Gamma m.$$

The pole structures are similar in both cases, for instance, in the case of unstable particles the poles in the complex plane of the variable k^0 can be read off from figure 3.2. In the following we prove the formulae (3.25) and motivate how the result is generalized for CMS (3.26). The properties (3.9), (3.10) are a direct consequence of equation (3.25). We define the Fourier transform as

$$\mathrm{FT}\left[\circ\right](\mathrm{x}) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4\mathrm{k}\ \mathrm{e}^{-\mathrm{ikx}} \circ (\mathrm{k}).$$

Proof. Further we define advanced and retarded propagators corresponding to the Feyn-

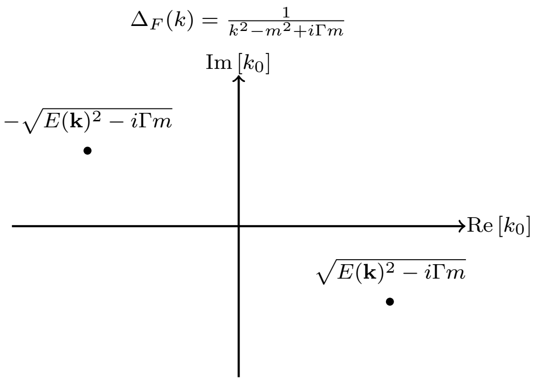


Figure 3.2.: Poles of the Feynman propagator

man propagator for stable particles[3]

$$\Delta_{\mathrm{A}}(k) := \frac{1}{k^2 - m^2 - \mathrm{i}\epsilon\,\mathrm{sgn}(k^0)}, \tag{3.27}$$

$$\Delta_{\mathrm{R}}(k) := \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon\,\mathrm{sgn}(k^0)}. \tag{3.28}$$

We can perform the following decomposition

$$\begin{aligned}\Delta_{\mathrm{F}}(x) &= \theta\left(x^0\right)\Delta_{\mathrm{F}}(x) + \theta\left(-x^0\right)\Delta_{\mathrm{F}}(x)\\ &= \theta\left(x^0\right)\left(\Delta_{\mathrm{F}}(x) - \mathrm{FT}\left[\Delta_{\mathrm{A}}\right](x)\right) + \theta\left(-x^0\right)\left(\Delta_{\mathrm{F}}(x) - \mathrm{FT}\left[\Delta_{\mathrm{R}}\right](x)\right)\end{aligned} \tag{3.29}$$

This equation holds because $\theta\left(x^0\right)\mathrm{FT}\left[\Delta_{\mathrm{A}}\right](x)$ and $\theta\left(-x^0\right)\mathrm{FT}\left[\Delta_{\mathrm{R}}\right](x)$ vanish. For instance, consider the integral $\theta\left(x^0\right)\mathrm{FT}\left[\Delta_{\mathrm{A}}\right](x)$ illustrated in figure 3.3. We can close the contour[4] $\mathcal{C}$ of the integration path k^0 in the lower half plane, but the advanced propagator has no poles there. Therefore the integral vanishes. Proceeding this way, we encounter the following subtraction of Fourier components

$$\begin{aligned}\Delta_{\mathrm{F}}(k) - \Delta_{\mathrm{A}}(k) &= \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon} - \frac{1}{k^2 - m^2 - \mathrm{i}\epsilon\,\mathrm{sgn}\left(k^0\right)}\\ &= \frac{-\mathrm{i}\epsilon\left(\mathrm{sgn}\left(k^0\right) + 1\right)}{\left(k^2 - m^2 + \mathrm{i}\epsilon\right)\left(k^2 - m^2 - \mathrm{i}\epsilon\,\mathrm{sgn}\left(k^0\right)\right)}\\ &= \frac{-\mathrm{i}2\epsilon\theta\left(k^0\right)}{\left(k^2 - m^2 + \mathrm{i}\epsilon\right)\left(k^2 - m^2 - \mathrm{i}\epsilon\right)}\\ &= \theta\left(k^0\right)\left(\Delta_{\mathrm{F}}(k) - \Delta_{\mathrm{F}}^*(k)\right) = 2\mathrm{i}\theta\left(k^0\right)\mathrm{Im}\left[\left(\Delta_{\mathrm{F}}(k)\right)\right].\end{aligned}$$

[3]The following calculation cannot be adopted for CMS because we use analytic continuation. The $\mathrm{sgn}(z)$ function cannot be continued analytically since it is a function of z and z^*.
[4]This is only allowed because of the limit $\epsilon \to 0$.

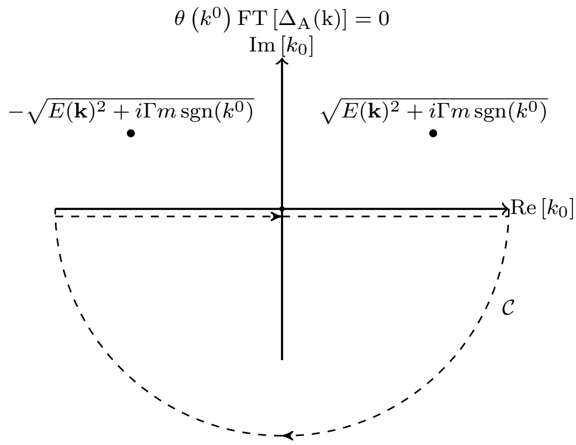


Figure 3.3.: The Fourier transform of the advanced propagator Δ_A. The figure only shows the integration contour of k^0.

Analogously we obtain

$$\Delta_{\mathrm{F}}(k)-\Delta_{\mathrm{R}}(k)=2\mathrm{i}\theta\left(-k^0\right)\mathrm{Im}\left[\left(\Delta_{\mathrm{F}}(k)\right)\right].$$

Using equation (3.29) and the results above, we have proved the identities (3.25). $\square$

3.2.2. Extended LTE for Unstable Particles

The idea is the same for the CMS and we demonstrate how to perform a decomposition similar to (3.29) in the case of CMS-propagator. Such a decomposition in addition to the properties of $\Delta^{\pm}$, which we discuss in more detail below, is the basis for LTE relations. Our procedure is not unique, but the result is easily interpreted. Our approach consists of defining meromorphic functions $\Delta_{\mathrm{A/R}}$ with similar pole structure as in the case of stable particles, and we demand that

$$\int \mathrm{d}p^0\;\Delta_{\mathrm{A/R}}(p^0,\mathbf{p},M,\Gamma)e^{\pm\mathrm{i}p^0|x^0|}=0, \tag{3.30}$$

$$\Delta_{\mathrm{F}}(p,\mu)-\Delta_{\mathrm{A}}(p^0,\mathbf{p},M,\Gamma)\overset{!}{=}\Delta^{+}(p^0,\mathbf{p},M,\Gamma), \tag{3.31}$$

$$\Delta_{\mathrm{F}}(p,\mu)-\Delta_{\mathrm{R}}(p^0,\mathbf{p},M,\Gamma)\overset{!}{=}\Delta^{-}(p^0,\mathbf{p},M,\Gamma). \tag{3.32}$$

The equations above shall be understood in such a way that $\Delta_{\mathrm{A/R}}$ must be chosen so that $\Delta^{\pm}$ fulfills the decomposition theorem 3.1. The first equation (3.30) is the condition that the advanced/retarded propagator has only poles in the upper/lower complex plane. Consequently, we have the same situation as in the stable case, namely $\theta(\pm x^0)\mathrm{FT}\left[\Delta_{\mathrm{A/R}}\right](x)=0$. Formulating the decomposition theorem 3.1 in Fourier space, we obtain the following constraint

$$\Delta^-(p^0, \mathbf{p}, M, \Gamma) = -\left(\Delta^+(-p^0, \mathbf{p}, M, \Gamma)\right)^* . \tag{3.33}$$

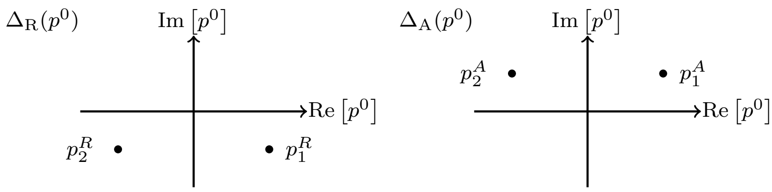
Furthermore, we demand that similar as in the case of stable particles (3.28) the retarded propagator changes over to the advanced propagator by complex conjugations and vice versa

$$\Delta_\mathrm{A}(p^0, \mathbf{p}, M, \Gamma) = \left(\Delta_\mathrm{R}(p^0, \mathbf{p}, M, \Gamma)\right)^* . \tag{3.34}$$

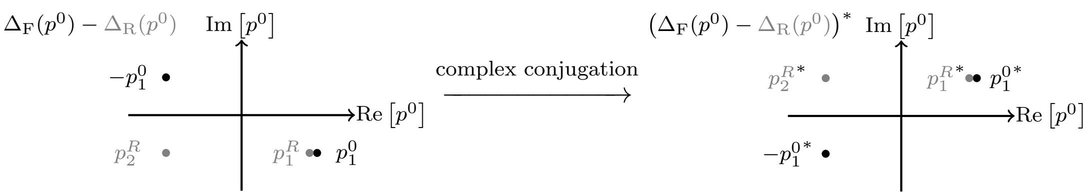
Due to the assumption (3.34) and the fact that the poles of the Feynman propagator are point symmetric to the origin, we show that with (3.33) the poles of $\Delta_{\mathrm{A/R}}$ must be symmetric with respect to the imaginary axis. For the following discussion it proves to be useful to represent the Feynman propagator as a function of p^0 and to reveal its pole structure

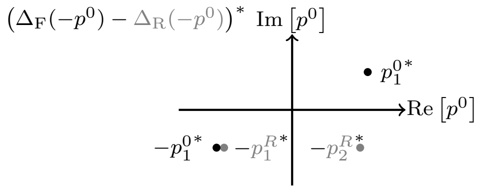
$$\begin{aligned} \Delta_\mathrm{F}(p,\mu) &= \frac{1}{p^0 - p_1^0}\frac{1}{p^0 + p_1^0}, \quad \text{with} \quad p_1^0 = \sqrt{\mathbf{p}^2 + M^2 - \mathrm{i}\Gamma M} \\ \Delta_\mathrm{F}^*(p,\mu) &= \frac{1}{p^0 + p_2^0}\frac{1}{p^0 - p_2^0}, \quad \text{with} \quad p_2^0 = -\sqrt{\mathbf{p}^2 + M^2 + \mathrm{i}\Gamma M} = -\left(p_1^0\right)^* \end{aligned} \tag{3.35}$$

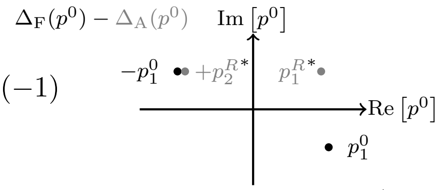
The retarded propagator has poles in the lower complex plane which we denote with p_1^R and p_2^R. The advanced propagator, on the contrary, has poles only in in the upper complex plane



The location of the residues are related via condition (3.34) and we obtain the pole positions of Δ_A: $p_1^{R*} = p_1^A$ and $p_2^{R*} = p_2^A$. With these preliminary words we solve condition (3.33) graphically



$\xrightarrow{p^0 \to -p^0}$ 

$\overset{!}{\underset{(3.33)}{=}} (-1)$ 

(3.36)

In order to fulfill (3.36) the following is necessary

$$p_1^R = p_1^0,\ p_2^R = p_2^0,\quad \mathrm{Res}_{p_1^0}\,\Delta_\mathrm{F} = \mathrm{Res}_{p_1^0}\,\Delta_\mathrm{R},\quad \mathrm{Res}_{p_2^{0*}}\,\Delta_\mathrm{F} = \mathrm{Res}_{p_2^{0*}}\,\Delta_\mathrm{A}$$

We can now give the proof of the decomposition rule for unstable particles (3.26).

Proof. Considering the pole structure allows us to derive necessary conditions, but (3.33) must be valid not only for the poles, but for the whole function. A closer look at the Feynman propagator leads to the partial fraction decomposition

$$\Delta_\mathrm{F}(p^0) = \frac{1}{2p_1^0}\left(\frac{1}{p^0 - p_1^0} - \frac{1}{p^0 + p_1^0}\right). \tag{3.37}$$

As a function of p^0, the Feynman propagator is a superposition of poles of first order with constant coefficients and this motivates the following ansatz:

$$\Delta_\mathrm{R}(p^0) = \frac{f^1}{p^0 - p_1^0} + \frac{f^2}{p^0 - p_2^0} \tag{3.38}$$

Of course, we have to check whether the required conditions are fulfilled or not. The necessary conditions yield

$$\begin{aligned} \frac{1}{p_1^0 + p_1^0} &\overset{(3.37)}{=} \mathrm{Res}_{p_1^0}\,\Delta_\mathrm{F} \overset{!}{=} \mathrm{Res}_{p_1^0}\,\Delta_\mathrm{R} \overset{(3.38)}{=} f^1, \\ \frac{1}{-p_1^0 - p_1^0} \overset{(3.37)}{=} \mathrm{Res}_{p_2^{0*}}\,\Delta_\mathrm{F} &\overset{!}{=} \mathrm{Res}_{p_2^{0*}}\,\Delta_\mathrm{A} \overset{(3.38)}{=} f^{2*}, \quad \Rightarrow f^2 = \frac{1}{2p_2^0}. \end{aligned}$$

And therefore we obtain

$$\Delta_\mathrm{R}(p^0) = \frac{1}{2p_1^0}\frac{1}{p^0 - p_1^0} + \frac{1}{2p_2^0}\frac{1}{p^0 - p_2^0},\ \Delta_\mathrm{A}(p^0) = -\frac{1}{2p_2^0}\frac{1}{p^0 + p_2^0} - \frac{1}{2p_1^0}\frac{1}{p^0 + p_1^0}.$$

Having determined the advanced and retarded propagator, we calculate the Fourier components of $\Delta^\pm$ with the help of (3.31), (3.32) and we obtain

$$\Delta_\mathrm{F} - \Delta_\mathrm{R} = \frac{1}{p^0 - p_1^0}\frac{1}{p^0 + p_1^0} - \frac{1}{2p_1^0}\frac{1}{p^0 - p_1^0} - \frac{1}{2p_2^0}\frac{1}{p^0 - p_2^0}$$

$$
\begin{aligned}
&= \frac{1}{p^0 - p_1^0}\left(\frac{p_1^0 - p^0}{2p_1^0(p^0+p_1^0)}\right) - \frac{1}{2p_2^0}\frac{1}{p^0 - p_2^0} \\
&= -\frac{1}{2p_1^0(p^0+p_1^0)} - \frac{1}{2p_2^0(p^0-p_2^0)} = -\mathrm{iIm}\left[\frac{1}{p_1^0(p^0+p_1^0)}\right] = \Delta^-
\end{aligned}
$$

$$
\Delta_{\mathrm{F}} - \Delta_{\mathrm{A}} = \mathrm{iIm}\left[\frac{1}{p_1^0(p^0-p_1^0)}\right] = \Delta^+ \tag{3.39}
$$

Our solutions fulfill all requirements (3.30), (3.31), (3.32), (3.33), (3.34) and (3.26) is proved. □

Corollary 3.1. *Properties of $\Delta_{\mathrm{F}}, \Delta_{\mathrm{A/R}}, \Delta^{\pm}$:*
From the decomposition theorem (3.29) *and representation of $\Delta^{\pm}$* (3.25), (3.26)*, we can derive the following properties*

$$
\mathrm{Re}\left[\Delta_{\mathrm{A/R}}(p, m/\mu)\right] = \mathrm{Re}\left[\Delta_{\mathrm{F}}(p, m/\mu)\right], \; \textit{valid for stable/unstable} \tag{3.40}
$$

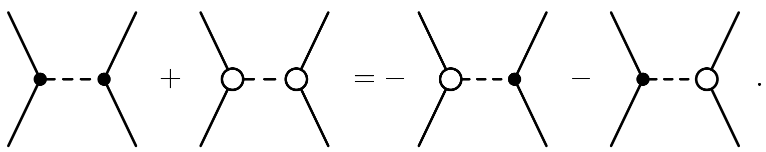
$$
\mathrm{Im}\left[\Delta_{\mathrm{A/R}}(p, m)\right] = \mp\,\mathrm{sgn}(p^0)\mathrm{Im}\left[\Delta_{\mathrm{F}}(p, m)\right], \; \textit{valid for stable} \tag{3.41}
$$

$$
\Delta^+ + \Delta^- = \Delta_{\mathrm{F}} - \Delta_{\mathrm{F}}^* = 2\mathrm{iIm}\left[\Delta_{\mathrm{F}}\right], \; \textit{valid for stable/unstable} \tag{3.42}
$$

Proof. The first relation (3.40) follows from (3.31), (3.32) and the fact that $\mathrm{Re}\left[\Delta^{\pm}\right] = 0$. Relation (3.41) follows from the well-known Sokhotski–Plemelj theorem which can be derived via Cauchy's integral formula

$$
\begin{aligned}
\Delta_{\mathrm{F}}(p, m) &= \mathcal{P}\Delta_{\mathrm{F}} - \mathrm{i}\pi\delta(p^2 - m^2), \\
\Delta_{\mathrm{A/R}}(p, m) &= \mathcal{P}\Delta_{\mathrm{F}} \pm \mathrm{i}\,\mathrm{sgn}\,(p^0)\pi\delta(p^2 - m^2),
\end{aligned} \tag{3.43}
$$

where $\mathcal{P}$ denotes the Cauchy principal value. Relation (3.42) can be proved by direct calculation, but it is more instructive to use the LTE. The LTE states



Dividing this equation by $(ig)^2$, we have proved (3.42). □

Corollary 3.2. *In the limit $\Gamma \to 0$ we recover the Cutkosky cutting rules [Cut60] (see equation* (3.6)*).*

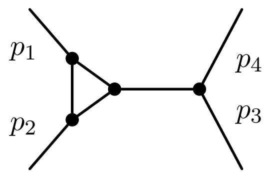
Proof. We have to show that $\Delta^{\pm}(p, \mu) \to \Delta^{\pm}(p, m)$ for $\Gamma \to 0^+$. Consider the singular part of Δ^+

$$
\begin{aligned}
\lim_{\Gamma\to 0^+}\frac{1}{p^0-p_1^0} &= \lim_{\Gamma\to 0^+}\frac{1}{p^0-\sqrt{\mathbf{p}^2+M^2-\mathrm{i}\Gamma M}}\\
&= \frac{1}{p^0-E_p+\mathrm{i}\epsilon}\\
\to \lim_{\Gamma\to 0^+}\mathrm{Im}\left[\frac{1}{p^0-p_1^0}\right] &= -\pi\delta(p^0-E_p)\\
\to \lim_{\Gamma\to 0^+}\Delta^+(p,\mu) = \lim_{\Gamma\to 0^+}\mathrm{iIm}\left[\frac{1}{p_1^0}\frac{1}{p^0-p_1^0}\right] &= -\mathrm{i}\pi\frac{1}{E_p}\delta(p^0-E_p) = -2\mathrm{i}\pi\delta(p^2-m^2)\theta(p^0)\\
&= 2\mathrm{i}\theta(p^0)\mathrm{Im}\left[\Delta_{\mathrm{F}}(p,m)\right] = \Delta^\pm(p,m)
\end{aligned}
$$

□

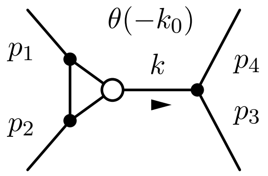
3.2.3. Discussion of Regions and Physical Constraints

The cutting rules are a special subset of the LTE relations and many contributions to the LTE do not contribute because the S-matrix underlies physical restrictions. As we shall show in a moment, the vanishing of LTE contributions is due to kinematical constraints which are enforced by the δs and θs. Since the situation is similar when dealing with unstable particles we consider stable particles first. Therefore, we return to our previous example

$\mathcal{F}(p_1,p_2,p_3,p_4) =$ 

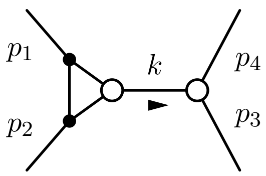
. (3.44)

If we sum over all possible underlinings, the result equals zero due to the LTE, but there are some underlined diagrams which vanish for themselves. The following diagram does vanish

$\mathcal{F}(p_1,p_2,p_3,p_4) =$ 

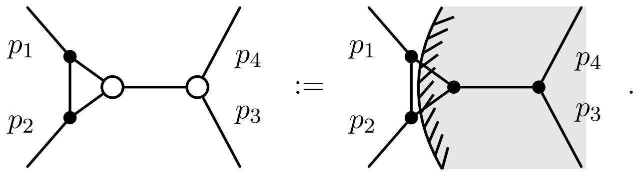
$= 0,$ (3.45)

but

$\mathcal{F}(p_1,p_2,p_3,p_4) =$ 

, (3.46)

does not. In (3.45) a θ-function arises which enforces a backward energy flow. It is therefore no more energy available for producing the outgoing particles p_3, p_4. For the second example, there is no reason why it should vanish.
The example also shows that, at least in this special case, the solutions of the LTE for stable particles tend to split into two separate regions where the normal part (non-shadowed) is given by the black dots, while the complex conjugated part (shadowed) is given by the white circles. In between we have to use cut propagators, i.e. $\Delta^{\pm}$. We visualize cuttings by



(3.47)

In the case of the physical constraints and stable particles there is always a well-defined shadowed- and non-shadowed region which is a consequence of unitarity.

Corollary 3.3. *It is well-known that unitarity follows from the Cutkosky cutting rules [Cut60] which are valid for stable particles and we conclude that the unitarity equation is fulfilled, i.e. we have that* (2.17)

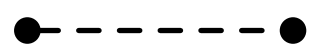
$$-2\mathrm{Re}\left[\mathrm{i}(\mathcal{T}_{fi})\right] = \sum_k \mathcal{T}^*_{kf}\mathcal{T}_{ki},$$

which shows that there is a well-defined shadowed ($\mathcal{T}^*$) and non-shadowed region ($\mathcal{T}$).

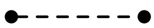
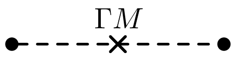
3.2.4. Relevant and Irrelevant Contributions to the LTE

As in the case of stable particles, one would like to have the same structure of the cutting rules for unstable particles, i.e. a split into shadowed and non-shadowed region which is a priori not given. We motivate that we actually obtain the same behavior for unstable particles in a perturbative sense, meaning that those LTE contributions that violate the "split" structure are always of next order in the coupling constant. These violating contributions come from the fact that for the CMS $\Delta^{\pm}$ there is neither a $\theta\left(\pm p^0\right)$ nor a $\delta\left(p^2 - m^2\right)$, but smoothed functions instead. The smoothing does no longer enforce the same strict kinematical constraints as one has with stable particles which is necessary for well-defined shadowed and non-shadowed regions.
Before going further into detail, we discuss how to simplify LTE relations. We show that irrelevant contribution stay irrelevant even if the calculation is extended to higher orders. This mimics the fact that contributions which are kinematically forbidden in

the case of stable particles and correspondingly would be suppressed in the case of unstable particles, do not appear in a higher order calculation. The statement is trivial for stable particles because kinematically forbidden contributions are zero, but it is not immediately clear for unstable particles since suppressed contributions might be of the same order as higher order contributions. The mechanism doing the trick is the resummation and it is closely linked to the imaginary mass counterterm $\mathrm{i}\Gamma M$ appearing in the CMS (see (2.22)). By now we have implicitly accepted that there are no complex couplings and in our current framework we cannot derive a LTE when $\mathrm{i}\Gamma M$ is involved. On the other hand, we do not have to adapt the LTE for complex couplings[5] because $\mathrm{i}\Gamma M$ is related to the imaginary part of complex pole (2.22). The idea is the following: Diagrams containing $\mathrm{i}\Gamma M$ can be related to diagrams where $\mathrm{i}\Gamma M$ is missing and where the LTE relations can be applied. Afterwards one can undo the mapping (which is a differential operation) and obtain the result for the LHSUE.
We start with an oversimplified approach where we take along multiple insertions of $\mathrm{i}\Gamma M$. Then, we perform the resummation to show that if the contribution is irrelevant, then it goes to zero as one performs the limit of infinite insertions. Finally we give the argument for perturbation theory. For all propagators of unstable particles

 , (3.48)

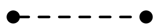
there are corresponding higher order contributions[6]

 + 

$+\ldots,$ (3.49)

and so on. This sum can also be written in the following form

$$\left(\mathrm{e}^{-\xi\frac{\partial}{\partial \Gamma M}}\right)_n \left(\right)\Bigg|_{\xi=\Gamma M}, \tag{3.50}$$



with

$$\left(\mathrm{e}^{-\xi\frac{\partial}{\partial \Gamma M}}\right)_n := \sum_{k=0}^{n} \frac{1}{k!}\left(-\xi\frac{\partial}{\partial \Gamma M}\right)^k. \tag{3.51}$$

[5]In principle, complex couplings are allowed, e.g. angle in CKM, and it is possible to derive a LTE as long as the Lagrangian is real, but the argument fails for $\mathrm{i}\Gamma M$ which has its counterpart in propagator.

[6]These higher orders emerge from more insertions of renormalized self-energies in the 2-point function. Extracting the imaginary part of each mass counter term which is associated with a self-energy, one arrives at the given statement. This will be discussed in more detail later.

Taking the real part commutes with differentiation with respect to a real parameter and we can compute the LHSUE. For a more general amplitude one encounters several unstable propagators and differentiation with respect to ΓM does not make sense anymore. One defines the unstable propagator as $\Delta(p,\mu) \to \Delta(p,\mu)|_{\Gamma M \to \Gamma M + \alpha}$. Then instead of performing the differentiation with respect to ΓM, one can differentiate with respect to α and at the end put α to zero. This procedure can be taken over for all unstable propagators and each differentiation can be uniquely assigned to a given unstable propagator.
Taking the limit $n \to \infty$ the differential operator (3.51) is nothing but the translation operator and in the sense above it acts in the following way on a function P

$$\mathrm{e}^{-\epsilon_0 \frac{\partial}{\partial \epsilon}} P(\epsilon)\Big|_{\epsilon_0=\epsilon} = P(\epsilon - \epsilon_0)|_{\epsilon_0=\epsilon} = P(0)$$

Consider contributions involving $\Delta^{\pm}$ of unstable particles.

$$\Delta^+ = \mathrm{iIm}\left[\frac{1}{p_1^0}\frac{1}{p^0 - p_1^0}\right], \quad \Delta^- = -\mathrm{iIm}\left[\frac{1}{p_1^0}\frac{1}{p^0 + p_1^0}\right]$$

For the whole phase-space the function $\Delta^{\pm}$ is suppressed because of the small imaginary part of p_1^0, except in the region of resonance where the small imaginary part gives rise to a non-negligible contribution. $\Delta^{\pm}$ is a nascent delta function in the limit $\Gamma \to 0$ and on the resonance ($p^0 = \pm E_{\mathbf{p}}$) we have

$$\begin{aligned}
\pm E_{\mathbf{p}} \mp p_1^0 &= \pm E_{\mathbf{p}} \mp \sqrt{E_{\mathbf{p}}^2 - \mathrm{i}\Gamma M} \\
&= \pm E_{\mathbf{p}} \mp E_{\mathbf{p}}\left(1 - \frac{\mathrm{i}\Gamma M}{2E_{\mathbf{p}}^2} + \mathcal{O}\left(\left(\frac{\mathrm{i}\Gamma M}{E_{\mathbf{p}}^2}\right)^2\right)\right) \\
&= \pm E_{\mathbf{p}}\left(\frac{\mathrm{i}\Gamma M}{2E_{\mathbf{p}}^2} + \mathcal{O}\left(\left(\frac{\mathrm{i}\Gamma M}{E_{\mathbf{p}}^2}\right)^2\right)\right) \\
\Rightarrow \Delta^{\pm} &\propto \mathrm{Im}\left[\frac{1}{\mathrm{i}\Gamma M}\right]
\end{aligned}$$

Applying the translation operator on $\Delta^{\pm}$, we obtain a vanishing result if and only if we are not on the resonance since then $\mathrm{Im}\,[\mathbb{R}] = 0$. For $p^2 = M^2$ the operation is undefined. This can also be seen from the geometric series $\mathrm{i}\Delta_{\mathrm{F}} \sum_{k=0}^{n} q^k$ with $q = \mathrm{i}(-\mathrm{i}\Gamma M)\mathrm{i}\Delta_{\mathrm{F}}$. The series converges only for $|q|^2 < 1$ and we have

$$|q|^2 = \frac{\Gamma^2 M^2}{(p^2 - M^2)^2 + \Gamma^2 M^2}, \quad |q|^2 \begin{cases} < 1 & p^2 \neq M^2 \\ = 1 & p^2 = M^2 \end{cases}. \tag{3.52}$$

All contributions with non-resonant $\Delta^{\pm}$ of unstable particles are non-relevant meaning that with the above argumentation they do not contribute at all. As against, resonant unstable $\Delta^{\pm}$ have to be treated separately.
We translate the statement for perturbation theory. Assume we have done a calculation of a process up to order g^n. We evaluate the LHSUE and assume there is somewhere a contribution involving $\Delta^+(p,\mu)$. Furthermore, assume $\Delta^+(p,\mu)$ cannot become resonant due to the kinematical situation. Since it is non-resonant the order is at least $\mathcal{O}\left(g^2\right)$. Formally, this follows from

$$\lim_{\Gamma\to 0, p^0\neq E(\mathbf{p})} \Delta^+ = 0, \qquad \lim_{\Gamma\to 0, p^0\neq E(\mathbf{p})} \left|\frac{\Delta^+}{\Gamma}\right| < \infty, \tag{3.53}$$

or from Taylor expansion. Now assume that the contribution involving that Δ^+ is negligible, then the question is whether this contribution remains of higher order when going to higher perturbative orders $g^{n+1}, g^{n+2}, \ldots$. The argument is the same as before. Since we extend our calculation to higher orders, we have insertions of ΓM and we need to include these contributions and make use of the inverse resummation to show that the contribution is always of higher order. The Δ^+ arises from one specific unstable propagator $\Delta(p,\mu)$ and for the same process up to order g^{n+2m} the same diagram with up to m insertions of ΓM. On the other hand, we can obtain those diagrams if we replace $\Delta(p,\mu)$, in the amplitude we were starting with, with[7]

$$\Delta(p,\mu) \to \left(\mathrm{e}^{-\xi\frac{\partial}{\partial\Gamma M}}\right)_m \Delta(p,\mu)\Big|_{\xi=\Gamma M}, \tag{3.54}$$

where the differentiation takes care of the multiple insertions of ΓM (3.49). Again, we evaluate the LHSUE making use of the LTE. Among the LTE contributions we retrieve the previous one with the Δ^+, and at the end we have to act with the differentiation on what became of $\Delta(p,\mu)$. Since $\Delta \to \Delta^+$, we have the following replacement

$$\Delta^+ \to \left(\mathrm{e}^{-\xi\frac{\partial}{\partial\Gamma M}}\right)_m \Delta^+\Big|_{\xi=\Gamma M}. \tag{3.55}$$

Making use of our order argument and for simplicity assume Δ^+ behaves like $\Gamma M f(p,\mu)$ where $f(p,\mu)$ is a suited non-singular function. For $m=1$ we find

$$\left(1-\xi\frac{\partial}{\partial\Gamma M}\right)\Gamma M f(p,\mu)\Big|_{\xi=\Gamma M} = -\,\xi\Gamma M\frac{\partial}{\partial\Gamma M}f(p,\mu)\Big|_{\xi=\Gamma M} = \mathcal{O}(\alpha^2)\frac{\partial}{\partial\Gamma M}f(p,\mu). \tag{3.56}$$

[7] The width actually depends on the perturbative order and is obtained via the renormalization condition. In the following discussion it is irrelevant what the width looks like as long as the lowest order is $\Gamma M = \mathcal{O}(g^2)$ which is always the case.

Assuming f behaves well which is the case away from the resonance, we conclude that this contribution is negligible compared to $\mathcal{O}(\alpha)$. For the general case we make use of the Hausdorff formula

$$\mathrm{e}^{X} Y \mathrm{e}^{-X} = e^{\mathrm{ad}_X} Y. \tag{3.57}$$

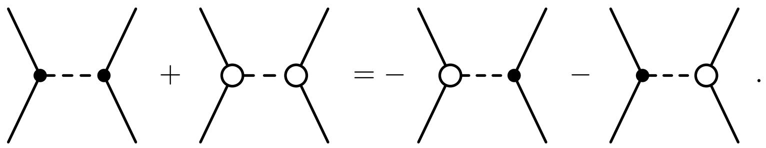
Plugging in $X = -\xi\frac{\partial}{\partial \Gamma M}, Y = \Gamma M, [X, Y] = -\xi \Rightarrow e^{\mathrm{ad}_X} Y = Y - \xi$, we find that

$$\begin{aligned}\left(\mathrm{e}^{-\xi\frac{\partial}{\partial \Gamma M}}\right)_m \Gamma M f(p,\mu)\Big|_{\xi=\Gamma M} &= (\Gamma M - \xi)\left(\mathrm{e}^{-\xi\frac{\partial}{\partial \Gamma M}}\right)_m f(p,\mu)\Big|_{\xi=\Gamma M} \\ &\quad + \mathcal{O}(\xi)\frac{1}{m!}\left(-\xi\frac{\partial}{\partial \Gamma M}\right)^m f(p,\mu)\Big|_{\xi=\Gamma M} \\ &= \mathcal{O}(\Gamma^{m+1}) = \mathcal{O}(g^{2m+2}).\end{aligned} \tag{3.58}$$

where the second line on the rhs contains contributions of higher order. They are present because we cut off the exponential at order $\mathcal{O}(\xi^{m+1})$. We started with a diagram of the order of $\mathcal{O}\left(g^n\right)$ and (3.55) yields a term proportional to $\mathcal{O}\left(g^{2m+2}\right)$ (3.58) and the whole contribution is therefore non-relevant because it is of the order of $\mathcal{O}\left(g^{n+2m+2}\right)$ and our current accuracy is $\mathcal{O}\left(g^{n+2m}\right)$. Therefore, we conclude that LTE contributions involving non-resonant $\Delta^{\pm}(p,\mu)$ of unstable particles of an n-th order diagram do not contribute even if the calculation is extended up to order g^{n+2m}. Because m is arbitrary, non-resonant $\Delta^{\pm}(p,\mu)$ do not contribute at all.

From these results it is clear that a diagram solely involving non-resonant unstable propagators obeys the normal Cutkosky cutting rules without cuttings through unstable propagators because there are only $\Delta^{\pm}$ of stable particles which correspond to real cuts, but we can say more. In the case of stable particles physical external parameters lead to a special subset of LTE relation where a shadowed and non-shadowed region is well-defined. As already discussed, this is no longer the case for the CMS and there are contributions to the LTE that violate this structure, but these violating contributions are exactly the non-relevant contributions. Relevant contributions cannot violate the structure because one could replace the unstable propagator that is responsible for the violation by a stable propagator (LTE still valid) and since relevant contributions are by definition those contributions which can become resonant, the substitution would lead to an on-shell state. Since for stable particles we know that the shadowed and non-shadowed region are well-defined, we conclude that the structure violating contributions must be irrelevant. Finally, resummation of $\mathrm{i}\Gamma M$ makes sure that these contributions are always of higher order and we can neglect them once and for all.

Example 3.2. Consider the s-channel tree level amplitude



On the right hand side we have two contributions, but only the second one is relevant because it can become resonant. If one replaces the intermediate unstable particle by a stable propagator, the first contribution on the right hand side would be exactly zero, but the second one could still become resonant. The example is oversimplified because the resonance condition for the intermediate particle cannot be achieved by a stable particle. However, the example demonstrates that the irrelevant contribution, i.e. the first one on the right hand side, violates our structure argument since the incoming particles are in the shadowed region (white circle).

Corollary 3.4. *Connection to Cutkosky cutting rules.*
The observation above leads to the following consequence: The cutting rules for unstable particles have the same structure as the Cutkosky cutting rules meaning that contributions to the LTE in the case of stable particles that are physically not realized, are, in the case of unstable particles, contributions of higher order.

One can take over the usual Cutkosky cutting rules, but modified by the replacement $\Delta^{\pm}(p, M) \to \Delta^{\pm}(p, \mu)$ for unstable particles. Obviously, relevant $\Delta^{\pm}(p, \mu)$ may not be negligible in the region of the resonance. We cannot tell what happens with regard to unitarity for unstable particles without further investigations and we still have to clarify the meaning of $\Delta^{\pm}(p, \mu)$ in the region of resonance.

Corollary 3.5 (Cutting Equation/Extended Largest Time Equation)**.** *Given a Feynman diagram $\mathcal{F}$ in the sense of Definition 1. If $\mathcal{F}$ fulfills the theorem 3.2 (LTE), then the cutting rules read*

- *We refer cut to $\Delta^{\pm}(p, m)$ in the case of stable particles and pseudo cut to $\Delta^{\pm}(p, \mu)$ in the case of unstable particles.*
- *Write down all possible contributions where propagators are replaced by cut-propagators and pseudo cut-propagators respectively.*
- *Discard all kinematically forbidden contributions and neglect all non-relevant contributions. The results split individually in a shadowed and non-shadowed region (corollary (3.4)).*
- *The non-shadowed region is obtained by applying the Feynman rules while for the shadowed region the complex conjugated Feynman rules must be applied.*

- *Appearing* $\mathrm{i}\Gamma M$ *are expressed as derivatives as explained above.*
- *The result is up to irrelevant contributions equivalent to* $-2 \times \mathrm{Re}\,[\mathcal{F}]$.

In the next part we clarify the meaning of resonant $\Delta^{\pm}(p, \mu)$. Assume we can assign to it a reasonable meaning such on both sides of the unitarity equation we find matching expressions, then we can conclude that unitarity is fulfilled because

- Away from resonances we encounter the usual cuts from the Cutkosky cutting rules and if we can show that the same holds for the region of resonance then we have covered the whole phase-space.
- The structure of the LTE relations within the CMS is the same as for stable particles.

The last missing piece of information is Γ and the LTE does not know anything about it and especially not the relations following from the renormalization procedure. In the next sections we show how the renormalization condition comes in and how to treat resonances. We start with some examples and then we generalize the statements.

4. Unitarity at Tree Level and One-Loop Order

In this section we go further into the question how perturbative unitarity is actually implemented in the CMS. One may imagine that the unitarity equation is fulfilled only for the sum over all diagrams such that only the sums on both sides of the unitarity equation coincide, though, in the case of stable particles, diagrams can be separated according to their topology and loop order. Perturbative unitarity then follows from the fact that the coupling can be chosen arbitrarily meaning that we can, in principle, distinguish between orders by varying the coupling. This argument fails, as we show, when the theory is renormalized according to the CMS. The distinction of loop orders does no longer work because of resummation and we actually have to consider sums of diagrams, but the occurrence of non-trivial dependencies between topologically different Feynman diagrams can be excluded at least in scalar theories.

In the following sections we study two different topologies of $2 \to 2$ processes with regard to unitarity. We apply the LTE to the LHSUE and extract the relevant contributions which follow from the extended cutting rules. The result is then checked against the right hand side of the unitarity equation (2.17). We start by demonstrating how things can be interpreted at tree level. Then we work out the example at NLO and we show how the results can be extended to N^{2}LO and beyond.

Unstable particles are characterized by a complex pole of the propagator which is related to a finite width, but the width or rather the complex pole is, in perturbative quantum field theory, determined from higher perturbative orders, hence even at tree level we have to deal with loops when unstable particles are involved. In practice one simply replaces the width by the experimental value. This approach makes sense even though we explicitly mix up loop orders, but actually, it does make more sense than preserving the loop order. As long as we stay away from the resonance, tree level is assumed to give the leading contribution and as we come close to the resonance we take into account finite-width effects via $\mathrm{i}\Gamma M$ which justifies the approach. The point is that preserving loop order does not preserve the order of accuracy, as we shall see in more detail later.

- Consider the phase-space p^2 for a given unstable propagator with mass M and width Γ. We subdivide the space into two sets $D^\omega_>, D^\omega_<$

$$D^\omega_> := \left\{ p^2 \middle| \left|p^2 - M^2\right| > \omega\, \Gamma M, \omega > 1, \omega = \mathcal{O}(\alpha^0) \right\},$$
$$D^\omega_< := \left\{ p^2 \middle| \left|p^2 - M^2\right| \leq \omega\, \Gamma M, \omega > 1, \omega = \mathcal{O}(\alpha^0) \right\}.$$

$D^\omega_<$ describes the region of resonance while $D^\omega_>$ is the non-resonant region. We demand that for any given order

$$\left.\frac{\mathrm{i}\Sigma_{R,\phi}}{\Gamma M}\left(p^2\right)\right|_{p^2 \in D^\omega_<} = \mathcal{O}(\alpha) \tag{4.1}$$

where $\Sigma_{R,\phi}$ is the renormalized self-energy of the unstable particle ϕ. This statement follows from the renormalization condition

$$\mathrm{i}\Sigma_{\mathrm{R},\phi}\big|_{p^2=\mu^2} = 0.$$

We evaluate the renormalized self-energy $\mathrm{i}\Sigma_{R,\phi}$ for $p^2 \in D^\omega_<$ on the real axis. Therefore and because the unrenormalized self-energy is at least of order $\mathcal{O}(\alpha)$, we expect $\mathrm{i}\Sigma_{R,\phi}$ to behave even of higher order in α. Explicitly:

$$\Sigma\big|_{p^2\in D^\omega_<} = \underbrace{\Sigma\big|_{p^2=\mu^2}}_{=0} - \underbrace{\left(\Sigma_{p^2=\mu^2} - \Sigma_{p^2\in D^\omega_<}\right)}_{\mathcal{O}(\alpha^2)} \tag{4.2}$$

In the last step one could perform a Taylor expansion to formally obtain the statement (4.1), but it may happen that the self-energy cannot be Taylor expanded as it is the case for charged fields (see appendix A).

But nevertheless we assume the contribution (4.1) to be small compared to one.

Remark 4.1. We will often allude to order and we distinguish between two types of orders:

- There is the order in α which we are going to elaborate in more detail below.
- There is the loop order (Feynman rules), i.e. tree level, one-loop, et cetera, which is uniquely defined by the diagrammatical representation.

Within the CMS the Feynman rules do no longer yield a strict perturbative expansion in α, but the orders are mixed and one must determine the order of accuracy of the amplitude which is crucial in view of perturbative unitarity. For instance, a one-loop calculation far from resonances is of next-to-leading order while near any (non-integrated) resonance the accuracy drops and the calculation is only accurate to leading order.

- Tree level with Γ from one-loop is accurate up to LO for the whole phase space.

- One-loop with Γ from two-loop is accurate up to NLO for the whole phase space.

- etc.

Before proceeding we mention two problems we have faced:

- When speaking of a perturbative order, one usually means the Taylor expansion in the coupling constant. For the CMS there is no perturbative order because there is no well-defined order in α due to resummation. The CMS propagator contributes to the perturbative order and, roughly speaking, one has to deal with functions like $\frac{1}{1+x^2}$ which are not Taylor expandable around $x = \pm 1$.

- Beyond that, topologically different diagrams, but of the same loop order may behave differently to the given external parameters with regard to their order in α.

Strictly speaking, we cannot investigate the CMS for perturbative unitarity, however, it turns out that for certain phase-space regions one can extract a well-defined perturbative order. Our strategy is therefore the following:

- Perturbative unitarity is fulfilled if the LHSUE and the RHSUE coincide up to a given accuracy. The idea is to show that for a given accuracy of the RHSUE, we can always find corresponding contributions on the LHSUE such that unitarity is fulfilled. To get that result we perform manipulations that involve higher-order contributions. Finally, one can count the order (if possible) and dismiss higher-order contributions.

- For some phase-space regions the order is ill-defined, but we do not have to worry since unitarity will be fulfilled automatically. One may define perturbative unitarity for the whole phase-space by an interpolation between the cases where a power-counting is possible.

In the examples below we count the order as follows:

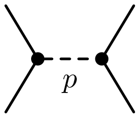
- Count any coupling constant g with $\alpha = g^2$ as one would could the usual perturbative order.

- Count any resonant s-channel propagator with $\frac{1}{\alpha}$ $(s \in D_<)$

We do not take into account any integrated unstable propagator when counting the order. This last point is the critical, especially if the unstable propagator is resonant and when massless particles are involved which we do not consider here (see appendix A) .

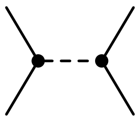
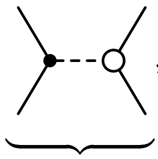
4.1. Unitarity at Tree Level

We start with the most simple case, namely a tree level s-channel example. There are three contributions at tree level for the $2 \to 2$ process and the most interesting one is the s-channel where the unstable particle can be produced on the resonance.

$\mathrm{i}\mathcal{M} =$  (4.3)

Non-resonant region: $p^2 \in D_>$

We apply the extended cutting rules to $\mathrm{i}\mathcal{M}$ and we obtain the LHSUE

$-2\mathrm{Re}\Big[$  $\Big] =$  ,

$\mathcal{O}(\alpha^2)$

where we used arguments of section 3.2.4 concerning the order of non-resonant $\Delta^{\pm}$. The RHSUE has no contribution at LO, i.e. $\mathcal{O}(\alpha)$, and for $p^2 \in D_>$ the LHSUE is of the order of $\mathcal{O}(\alpha^2)$, i.e. NLO which is in agreement with the unitarity equation.

Resonant region: $p^2 \in D_<$

Our cutting rules are useful for the discussion of unitarity as long as no unstable particle is resonant, but they are unpractical in the region of the resonance since information in Γ is missing. In the following we do not directly make use of the LTE and instead we evaluate the LHSUE in the region of resonance simply by taking the real part. This approach has the advantage that one can easily plug in the missing piece of information from Γ and it can be generalized for the n-loop 2-point function, as we shall see later. Evaluating the LHSUE of the amplitude $\mathrm{i}\mathcal{M}$, we obtain

$$\begin{aligned} -2\mathrm{Re}\left[\mathrm{i}g\frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\Gamma M}\mathrm{i}g\right] &= -2(\mathrm{i}g)^2\left|\frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\Gamma M}\right|^2 \mathrm{Re}\left[\frac{p^2 - M^2 - \mathrm{i}\Gamma M}{-\mathrm{i}}\right] \\ &= \mathrm{i}g\frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\Gamma M}2\Gamma M\frac{-\mathrm{i}}{p^2 - M^2 - \mathrm{i}\Gamma M}(-\mathrm{i}g), \end{aligned} \quad (4.4)$$

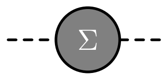
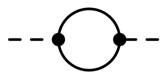
where ΓM is determined by the (one-loop) renormalization equation, i.e.

$$
\begin{aligned}
\mathrm{i}\Sigma^1_{\mathrm{R},\phi}\big|_{p^2=\mu^2} &= 0 = \mathrm{i}\Sigma^1_\phi(p^2)\big|_{p^2=\mu^2} - \mathrm{i}\delta\mu^2 \\
\Rightarrow \mathrm{i}\delta\mu^2 &= \mathrm{i}\Sigma^1_\phi(p^2)\big|_{p^2=\mu^2} \\
\mathrm{Im}\left[\delta\mu^2\right] &= -\mathrm{Im}\left[\mu^2\right] =: \Gamma M \\
\Rightarrow \Gamma M &= -\mathrm{Re}\left[\mathrm{i}\Sigma^1_\phi(p^2)\big|_{p^2=\mu^2}\right]
\end{aligned}
$$

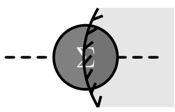
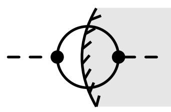
In the next step we make use of equation (4.1) to find a senseful meaning for ΓM

$$
\begin{aligned}
&\mathcal{O}\left(\alpha^2\right) = \mathrm{Re}\left[\mathrm{i}\Sigma_{R,\varphi}|_{p^2\in D_<}\right] = \mathrm{Re}\left[\mathrm{i}\Sigma_\varphi|_{p^2\in D_<}\right] + \Gamma M \\
&\Rightarrow \Gamma M = -\mathrm{Re}\left[\mathrm{i}\Sigma_\varphi|_{p^2\in D_<}\right] + \mathcal{O}\left(\alpha^2\right) = -\mathrm{Re}\left[\mathrm{i}\Sigma_\varphi|_{p^2\in D_<}\right](1+\mathcal{O}(\alpha))
\end{aligned}
$$

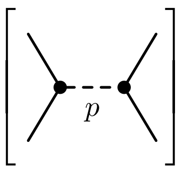
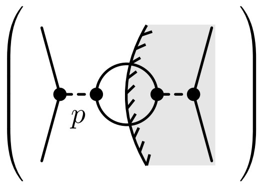
This equation expresses what is known from the usual on-shell scheme, i.e. the width is the cut through loops and can be interpreted as the decay width. At one-loop order the widths of both schemes coincide, but this is no longer true for higher loops and we are not able to argue this way in the general case. Nevertheless, let us make use of this result to demonstrate unitarity. At one-loop order the unrenormalized self-energy is given by

$\mathrm{i}\Sigma^1_\phi =$  $=$  ,

and we can directly apply the Cutkosky cutting rules to it since Σ^1_ϕ does not have any intermediate unstable particles

$-2\mathrm{Re}\left[\mathrm{i}\Sigma^1_\phi\right] =$  $=$  .

Plugging this result into (4.4), we obtain

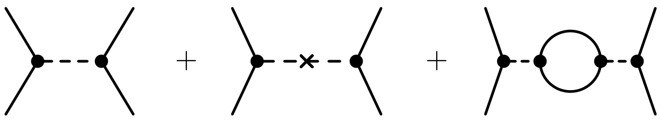
$-\,2\mathrm{Re}$  $=$  $(1+\mathcal{O}(\alpha))\,, \quad p^2 \in D_<. \qquad (4.5)$

The one-loop cut (4.5) does also appear on the RHSUE and is the lowest order contribution for the s-channel topology.

Remark 4.2. It seems that for $p^2 = \inf_{p^2\in D_>}\left|p^2 - M^2\right|$ there is a jump from no cut contribution to a cut contribution (4.5). Since the formula (4.4) holds for any $p^2 \in D_<$, we automatically fulfill the unitarity condition and we do not have to worry about the changeover of orders. On the other hand, this means that the tree level amplitude in the region $D_<$ can produce one-loop cuts. That was to be expected since Γ contains loop informations and as we approach the resonance, the Γ becomes more important which explains loop contributions. This does not mean that unitarity is violated, on the contrary. The calculation is LO for the full phase-space, and unitarity is fulfilled at LO, in contrast to what we show in the next section. If we simply extend the s-channel to one-loop without determining ΓM from higher loop orders, unitarity stays accurate only for LO. Thus, the s-channel resonance reduces the order of accuracy by α.

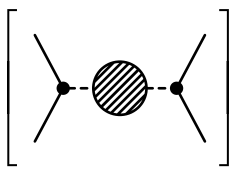
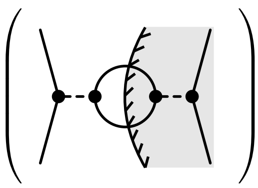
4.2. Unitarity at One-Loop Order: s-channel

In this section we extend the tree level amplitude of the last section to one-loop order while keeping the topological structure.

$i\mathcal{M} =$  (4.6)

① ② ③

If all particles were stable we could separate these diagrams by varying the coupling constant g. As already mentioned before, for the CMS this is not possible because the diagrams become of the same order in g which in turn depends on incoming momentum squared s. The phenomenon becomes significant as we approach the resonance of an unstable propagator, i.e. for $s = M^2$ and as they reach the same order in α cancellations take place due to (4.1). We show at one-loop order that the LHSUE is given by

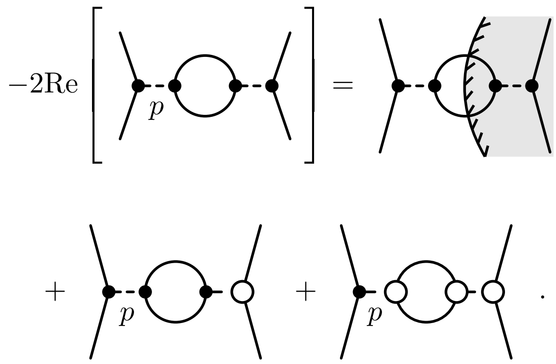
$-2\mathrm{Re}$  $= $  $(1 + \mathcal{O}(\Gamma))\,,$ (4.7)

which agrees with the RHSUE. If this was not the case then perturbative unitarity would be violated since there are no other diagrams with the given topological structure which could solve the problem.

Non-resonant region

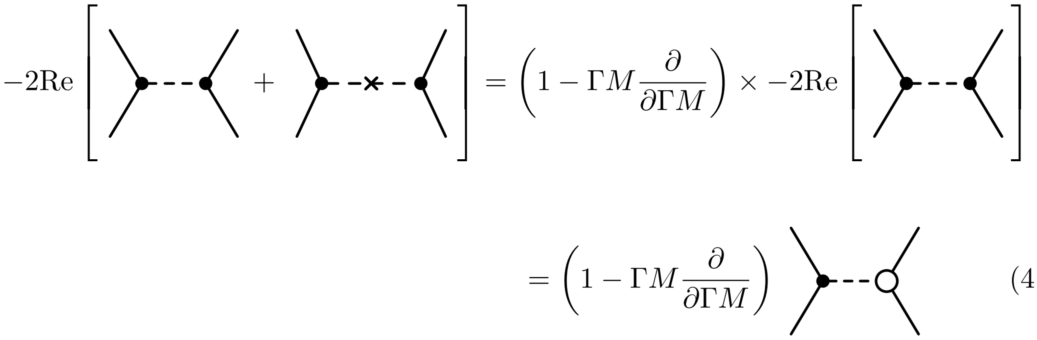
Since there are no resonant pseudo-cuts, we can make use of the usual Cutkosky cutting rules and unitarity is guaranteed. Anyway, we work out this example and show explicitly

how unitarity is implemented.
Taking the real part of the loop contribution ③, we obtain according to the extended cutting rules three relevant contributions



(4.8)

For the diagrams ① + ② we make use of the Dyson resummation trick. Taking the real part, we obtain



(4.9)

For $p^2 \in D_>$ we neglect all pseudo-cuts from one-loop diagrams (4.8). Furthermore, with the argumentation given in corollary 3.4 the contribution from ① + ② (4.9) is also negligible and, as expected, we have a match with unitarity.

Region of the resonance

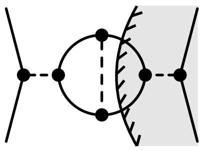
In the region of resonance we can adopt the result from the tree level example and we conclude that in LO the LHSUE agrees with the RHSUE. The leading order is given by the s-channel tree level amplitude which is now of the order of α^0 while the order of the s-channel one-loop contribution ② +③ is α due to (4.1)

$$\mathrm{i}g\ \mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\ \mathrm{i}\Sigma^{1}_{R,\phi}\ \mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\ \mathrm{i}g\big|_{p^2\in D_<} = \mathcal{O}(\alpha).$$

Because of the resonance the amplitude (4.6) can no longer be accurate up to NLO[1], i.e. $\mathcal{O}(\alpha)$ and we would need two-loop contributions in order to fulfill both sides of the

[1]This reflects the necessity of higher-loop contributions for describing resonances.

unitarity equation. For instance,



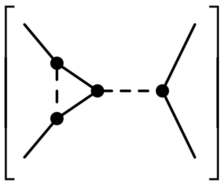
$$\Big|_{p^2 \in D_<} = \mathcal{O}(\alpha)$$

is of the same order as the one-loop contribution of (4.6), and clearly this contribution cannot be obtained from the LHSUE since Γ is determined from one-loop self-energies.

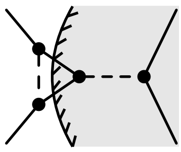
4.3. Unitarity at One-Loop Order: Vertex-function

In this section we closely monitor how the situation changes for another topology. We choose as one-loop contribution for the $2 \to 2$ process a vertex function. The usual Cutkosky rules in the case of stable particles envisage

$$-2\mathrm{Re}\,[\mathcal{F}] := -2\mathrm{Re} \quad \text{(4.10)}$$



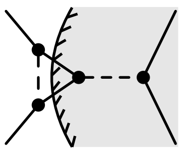
$\overset{?}{=}$



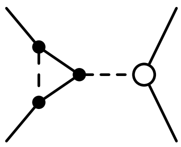
When dealing with stable particles one does only encounter resonant propagators whose momenta are integrated out, meaning that these propagators appear inside an integral. In contrast, unstable particles can be produced on the resonance, and we observe that the singularity is regulated by the width. Describing the process on the resonance, we observe a decrease in the order of α. Since the unstable s-channel propagator can become resonant, we must be careful with our first guess and we should expect other contributions. The following proposition gives more insights.

Proposition 4.1. The relevant contributions to the cutting rules read

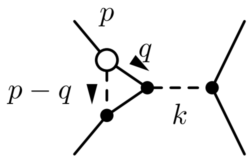
$$-2\mathrm{Re}\,[\mathcal{F}] := \quad \text{(4.11)}$$



$+$



Proof. We have to show that all other contributions do not have a resonant unstable propagator. Consider the following pseudo-cut



$$. \tag{4.12}$$

From the cutting rules we obtain the kinematically allowed phase-space

$$\theta(-q^0), \quad \theta(k^0-2m), \quad \theta(p^0-m) \quad \to q^0-p^0<0. \tag{4.13}$$

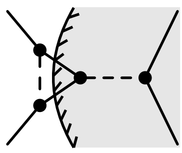
The question is whether $\Delta^+(q-p,\mu)$ can become resonant or not. In the latter case we conclude that it is a non-relevant contribution. The resonance condition for $\Delta^+(q-p,\mu)$ is given by

$$q^0-p^0-\sqrt{(\mathbf{q}-\mathbf{p})^2+M^2-\mathrm{i}\Gamma M}\Big|_{\Gamma=0} \stackrel{!}{=} 0,$$

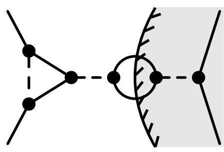
and can due to (4.13) never be fulfilled. The same argument holds for all other non-mentioned contributions to the LTE and we have proved (4.11). □

We might be confronted with a problem. Far from the resonance ($k^2 \in D_>$) equation (4.11) is in accordance with the unitarity condition since the pseudo-cut is suppressed in that phase-space region, but on top of the resonance we have an additional contribution which cannot be swept under the rug. This fact suggests the following equation in the region of resonance

$$-2\mathrm{Re}\,[\mathcal{F}] =$$



$$+$$



$$, \tag{4.14}$$

① ②

and it looks like we are getting into trouble because of the appearance of two-loop contributions. The solution to this problem is very simple, but let us first have a look at the orders. Both amplitudes are of the same order in α in the region of the resonance. For instance, take the first diagram ① and count the couplings. We obtain $g^4 = \mathcal{O}(\alpha^2)$, but near the resonance we have an additional factor of $\frac{1}{\Gamma M}$ from the unstable propagator and therefore the order is $\mathcal{O}(\alpha^2/\alpha) = \mathcal{O}(\alpha)$. For the second diagram ② we can do the same analysis and we obtain

$$① = \begin{cases} \mathcal{O}(\alpha^2), & k^2 \in D_> \\ \mathcal{O}(\alpha), & k^2 \in D_< \end{cases}, \quad ② = \begin{cases} \mathcal{O}(\alpha^3), & k^2 \in D_> \\ \mathcal{O}(\alpha), & k^2 \in D_< \end{cases}. \tag{4.15}$$

This means that if we expand up to a definite perturbative order there is a natural mixing of diagrams on the right-hand side of the unitarity equation. On the resonance, our first guess (4.10) is definitely wrong because there is another contributions of the same order. Returning to the question if (4.14) contributes or not actually depends on our current accuracy. We conclude that on the resonance the contribution is not accurate since LO is α^0 (compare with purely s-channel discussion). This in turn means that on the resonance we discard any of these cut diagrams of (4.14), even though there is a cut one-loop diagram. At first sight it seems that the contributions (4.14) become more significant as $s \to M^2$, but actually the vertex topology becomes irrelevant compared to purely s-channel one-loop cut contribution (4.7). Finally, unitarity is fulfilled because away from the resonance the usual Cutkosky rules are valid and on the resonance there is no cut contribution from this topology at LO.
The Feynman rules do not yield a strict perturbative order and we have seen how different topologies behave differently on the resonance even though they may be of the same loop order. Considering perturbative unitarity, the important question is not which diagram contributes up to a given accuracy, but the question whether all necessary contributions are there such that unitarity is fulfilled. Of course, we must assume that perturbation theory makes sense and that our amplitude is well approximated by loop orders.

5. Perturbative Unitarity in the CMS

5.1. Extending the Examples to N^{2}LO / NLO

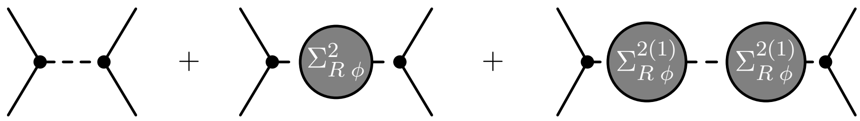
We verified the unitarity equation for selected diagrams of a $2 \to 2$ process at leading order for the whole phase space and at next-to-leading order far from any resonances of unstable particles. On the resonance we proposed an interpretation for Δ^+ which is compatible with the unitarity equation, namely, we made use of the fact that ΓM equals, at leading order, the usual decay width from the on-shell scheme which itself is proportional to cuts through the self-energy. For higher orders ΓM does no longer represent self-energy cuts and we have to find another mechanism which explains cut contributions. In this first part we demonstrate how our previous results are extended to NLO on resonances and to N^{2}LO far from resonances. The mechanism we propose is fairly the same mechanism which takes place when using dressed propagators [Vel63] instead of the CMS propagator. Summing up, we expect perturbative unitarity because of the following arguments:

- Away from resonances the imaginary part of the denominator of the propagator is irrelevant and consequently corresponding pseudo-cuts are negligible. Furthermore, due to resummation of $\mathrm{i}\Gamma M$ pseudo-cuts are irrelevant (corollary (3.4))
- In the region of resonance we can formally use dressed propagators which is compatible with the renormalization condition (4.1) and reexpanding in $\mathrm{i}\Sigma_{R,\varphi}/\Gamma M$ yields the correct perturbative result in the CMS.

Going to higher orders, the renormalization procedure must be repeated and the self energy is evaluated at two-loop order. It becomes more and more intricate and we do not go into detail and we assume the renormalization succeeded, i.e. the CMS conditions are fulfilled.

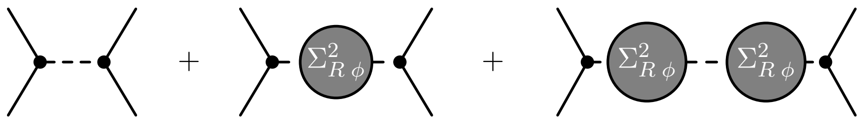
5.2. The Two-Point Function at Two-Loop Order

For a detailed analysis we again face the problem of power-counting and the question which diagrams must be considered for a given accuracy and again we simplify the discussion the same way as we did for the one-loop examples. In contrast to the one-loop example, we now have several two-loop contributions to consider and we hide our ignorance of these contributions simply in the self-energy Σ. For the s-channel, we consider the two-point function with external legs

$\mathrm{i}\mathcal{M} =$  (5.1)

where $\Sigma^2_{R,\phi} = \Sigma^{2(1)}_{R,\phi} + \Sigma^{2(2)}_{R,\phi}$ is the renormalized (according to the CMS) two-loop self-energy of the unstable particle ϕ, and $\Sigma^{2(1)}_{R,\phi}, \Sigma^{2(2)}_{R,\phi}$ denotes the one-loop,two-loop contribution (also renormalized) of the two-loop self-energy, respectively. Clearly, this approach can only make sense if the mass counter-term has been split in two parts in such a way that the one-loop and two-loop contributions of the self-energy can be regularized independently.

To analyze unitarity for arbitrary amplitudes we need to find a systematic way to plug in the information coming from the renormalization procedure. Remembering that the renormalization condition makes use of the self-energy up to a given loop order, the easiest way to achieve our goal is to extend any appearing self-energy to the loop-order at which the renormalization has been carried out. This allows us to analyze the LTE relations on the resonance and finally we can reexpand to obtain the perturbative result. We demand that the amplitude (5.1) is well approximated by

$\mathrm{i}\mathcal{M} =$  (5.2)

for the whole phase space. Away from any s-channel resonance we are accurate up to $\mathrm{N}^2\mathrm{LO}(\alpha, \alpha^2, \alpha^3)$ and on the resonance we have $\mathrm{NLO}(\alpha^0, \alpha^1)$. Clearly, extending the matrix element to (5.2) is valid if perturbation theory within the CMS makes sense. Formally, this means that

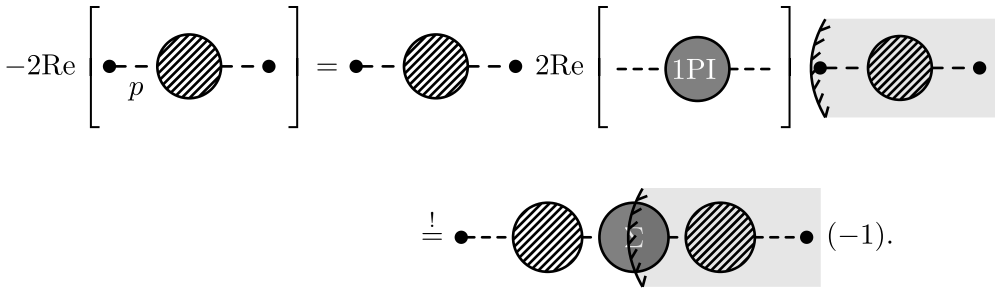
- $\mathcal{O}\left(\alpha^2\right) = \mathcal{O}\left(\Sigma^{2(2)}_{R,\varphi}\right) > \mathcal{O}\left(\Sigma^{2(1)}_{R,\varphi}\right) = \mathcal{O}\left(\alpha\right), \quad p^2 \in D_>.$

- $\mathcal{O}\left(\Sigma^{2(2)}_{R,\varphi}\right) \geq \mathcal{O}\left(\Sigma^{2(1)}_{R,\varphi}\right) = \mathcal{O}\left(\alpha^2\right), \quad p^2 \in D_<.$

The first restriction says that pure two-loop contribution should be $\mathrm{N}^2\mathrm{LO}$ away from resonances. The second one follows immediately from the renormalization condition. As pointed out, ΓM does not represent well-defined cut-contributions beyond one-loop which was to be expected since we can express the resummed two-point function as

$$\frac{\mathrm{i}}{p^2-\mu^2+\Sigma_{R,\varphi}} \;=\; \frac{\mathrm{i}}{p^2-\mu^2+\Sigma_{R,\varphi}}\left(\frac{p^2-\mu^2+\Sigma_{R,\varphi}}{\mathrm{i}}\right)^*\left(\frac{\mathrm{i}}{p^2-\mu^2+\Sigma_{R,\varphi}}\right)^*, \quad (5.3)$$

and taking the real part of this expression, we obtain

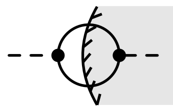


(5.4)

Note that when taking the real part of (5.3) one must only compute the real part of $\left(\mathrm{i}\left(p^2-M^2\right)+\Gamma M-(\mathrm{i}\Sigma_{R,\varphi})^*\right)$ because other factors form a real number and, consequently, one expects that

$$-2\mathrm{Re}\left[\Gamma M-\mathrm{i}\Sigma_{R,\varphi}\right] =: 2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right] \quad (5.5)$$

yields well-defined cuts. On the resonance, the leading-order contribution of (5.5) is exactly ΓM and this case is gone into (5.5), i.e. we have to all orders

$\Gamma M =$  + higher orders. (5.6)

and in the following section we motivate that the result holds in a perturbative sense.

5.2.1. Resonance region

Further difficulties arise, namely the occurrence of unstable resonances in loops. In this section we demonstrate how to systematically eliminate pseudo-cuts in favor of cuts of self-energies and we go on with the example at two-loop level. We reproduce the NLO(α^0,α^1) cut on the RHSUE and since we are only accurate up to NLO, it is sufficient to consider one insertion of the self energy for $p^2\in D_<$. Denoting the renormalized self-energy as $\Sigma^2_{R,\varphi}=\Sigma$, we represent the two-point function (5.2) generically as

$$\begin{aligned}&\frac{\mathrm{i}}{p^2-M^2+\mathrm{i}\Gamma M}\left(1+\mathrm{i}\Sigma\frac{\mathrm{i}}{p^2-M^2+\mathrm{i}\Gamma M}\right)\Bigg|_{p^2\in D_<}\\ &\qquad=\frac{\mathrm{i}}{p^2-M^2+\mathrm{i}\Gamma M+\Sigma}+\frac{\mathrm{i}}{p^2-M^2+\mathrm{i}\Gamma M}\times\mathcal{O}\left(\left(\frac{\mathrm{i}\Sigma}{\Gamma M}\right)^2\right),\ p^2\in D_<,\end{aligned} \quad (5.7)$$

where we used that

$$1 \gg \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\Gamma M}\mathrm{i}\Sigma\bigg|_{p^2 \in D_<} = \mathcal{O}\left(\frac{\mathrm{i}\Sigma}{\Gamma M}\right). \tag{5.8}$$

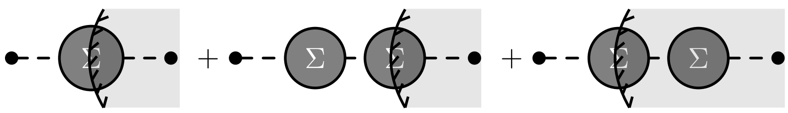
The resummed propagator on the rhs of (5.7) is manipulated as in (5.3). Therefore, we have that (up to higher orders)

$$\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\left(1 + \mathrm{i}\Sigma\,\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right) = \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\Gamma M}\left(1 + \mathrm{i}\Sigma\frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\Gamma M}\right) = (5.3),$$

where the self-energy in (5.3) has to be replaced by the two-loop approximation. We note that the resummation step is not necessary, but it makes the computation of the real part easy and reveals the general structure. From here on we obtain a useful representation for the LHSUE of the amplitude (5.7) by taking the real part of the leading contributions of (5.3).

$$\begin{aligned}
&2\mathrm{Re}\left[\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\left(1 + \mathrm{i}\Sigma\,\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)\right] \\
&\overset{(5.3)}{=} \mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\left(1 + \mathrm{i}\Sigma\,\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)\left(-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]\right)\left(\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\left(1 + \mathrm{i}\Sigma\,\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)\right)^* \\
&= \mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\left(-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]\right)\left(\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^* \text{①} \\
&+ \left[\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\,\mathrm{i}\Sigma\,\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right]\left(-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]\right)\left(\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^* \text{②} \\
&+ \mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\left(-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]\right)\left[\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\,\mathrm{i}\Sigma\,\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right]^* \text{③} \\
&+ \mathcal{O}(\alpha),\ p^2 \in D_<,
\end{aligned} \tag{5.9}$$

where we used the definition $\Gamma M - \mathrm{i}\Sigma =: -\mathrm{i}\tilde{\Sigma}$. As we motivate in section 5.2.3, $-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]$ yields well-defined cuts through stable particles only. With this knowledge the result (5.9) can be visualized as follows



, (5.10)

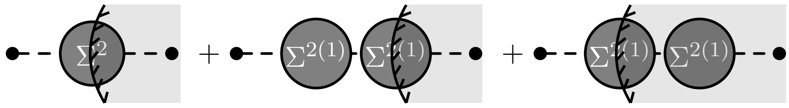
① ② ③

which is nothing but the perturbative expansion of (5.4).

5.2.2. Extension to the Whole Phase Space

Away from the s-channel resonance we make use of the extended cutting rules and we dismiss all non-relevant contributions, especially the s-channel pseudo-cuts and we

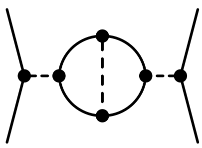
can directly apply the cutting rules to the self-energy parts. The result is a set of cut amplitudes and there are some relevant pseudo-cuts appearing in the two-loop self-energy, as we shall demonstrate.
Both regions can be merged to one final result. We dismiss all contributions for $s \in D_<$ which are of higher order in (5.10) and we obtain

2Re [(5.1)] = 

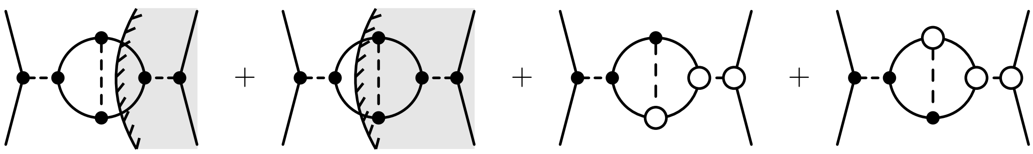
$$+ \begin{cases} \mathcal{O}(\alpha^3) & s \in D_> \\ \mathcal{O}(\alpha) & s \in D_< \end{cases}$$

Even though the results for the different regions have the same structure they are obtained by different arguments. In the region $s \in D_<$ one can study the amplitude with the extended cuttings rules and one must make sense out of the resonant pseudo-cuts. This has been achieved above and it is important to mention that one can only deduce the above equation if the sum of all loop contributions is considered in the region of resonance. The LTE we deduced does not know about that piece of information and we have to put it in by hand.
Coming back to $\mathrm{i}\tilde{\Sigma}$, we argue why the cut-visualization is actually justified. $\mathrm{i}\tilde{\Sigma}$ is given by the self-energy up to two-loop order, but is missing the $\mathrm{i}\Gamma M$ contribution. When taking the real part there is therefore no ΓM contribution and there are only diagrams so that we can apply the extended cutting rules. We find that the one-loop contribution yields exactly the one-loop cut, but for the two-loop contributions we do have resonant pseudo-cuts. The idea is to replace these pseudo-cuts by self-energy cuts.

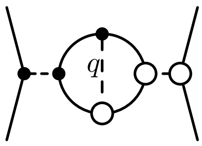
Example 5.1. Consider the two-loop contribution

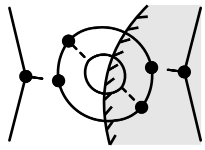
 ,

and apply the extended cutting rules to the two-loop self-energy. We obtain four relevant contributions and plugging in into 1, we obtain



The first two contributions are real cut contributions, while for the other two contributions pseudo-cuts show up. How do we argue about unitarity it this case? We show that the relevant contributions are there, on both sides of the unitarity equation. To that end, we fix the loop-momenta which flow through the pseudo-cut and we consider separately the region of resonance and the non-resonant region of that pseudo-cut. Clearly, we do not have a cut contribution away from the resonance and on the resonance we can just recursively plug in our results. We expect the result to be

LHSUE : $\int \frac{\mathrm{d}^4 q}{(2\pi)^4}$ 

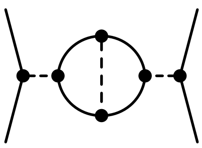
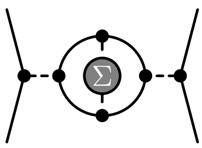
$\stackrel{?}{=} \int \frac{\mathrm{d}^4 q}{(2\pi)^4}$  $(1 + \mathcal{O}(\alpha))$

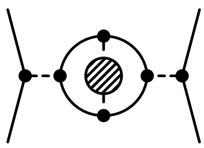
and we are going to use that ΓM represents to lowest order the cut of the one-loop self-energy and this is also true even if ΓM is determined from higher-loop self-energy (5.6). Since the accuracy of our calculation is bound by the two-loop contribution the leading order contribution for the pseudo-cut is enough. Note that the leading contributions to the pseudo-cut forces the unstable particle with momentum q to be produced on-shell and the delta-function is the leading contribution to the pseudo-cut. With the narrow-width approximation we derive the relation

$$\pi\delta(p^2 - M^2) = \pi\delta(p^2 - M^2)\frac{\Gamma M}{\Gamma M} \sim \Delta_{\mathrm{F}}(p,\mu)\Gamma M \left(\Delta_{\mathrm{F}}(p,\mu)\right)^* (1 + \mathcal{O}(\alpha)),$$

which makes only sense under an integral. Plugging in the diagrammatical solution for ΓM (5.6) yields the desired result and matches the RHSUE. In this way one takes care of any two-loop contribution and one can show that $-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]$ yields self-energy cuts up to two loop order.

For a more accurate calculation the narrow-width approximation is not enough, but in that case we find the same diagram with more insertions of self-energies, e.g.

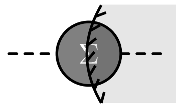
 $+$ 

$=$  ,

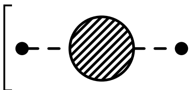
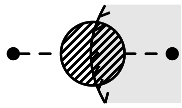
and we can repeat the same calculation. This time, we do not simply replace the relevant pseudo-cut from the previous example by a real cut contributions, but we replace the whole one-loop two-point function. This goes on and on and we replace cut/pseudo-cut two-point functions by real cut contributions until the recursive step stops and it stops either when all pseudo-cuts are replaced by cuts or when contributions become negligible due to the accuracy bound. At this point we already assumed that we can play the game up to arbitrary high orders, but yet we have not shown this. In the next section we show that we can make use of the cutting equation to demonstrate perturbative unitarity for any topology.

5.2.3. Generalization

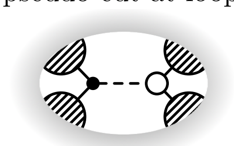
In this section we generalize the ideas from the last sections and we show by induction that $-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]$ as well as of the 2-point function are well-defined perturbative cuts. In section 5.2.2 we discussed and motivated unitarity at two-loop order that we now use as a basis. For the inductive step, we assume that the following is true up to n loops

$$-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right] =$$  $$(1+\mathcal{O}(\alpha)) \tag{5.11}$$

and

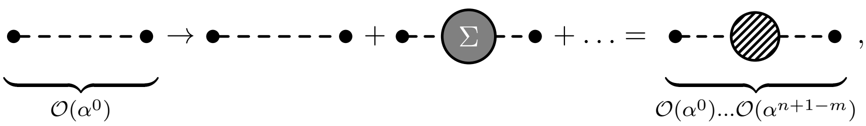
$$-2\mathrm{Re}\Big[$$  $$\Big] =$$  $$(1+\mathcal{O}(\alpha))\,. \tag{5.12}$$

Note that in the last sections we chose the loop order of the renormalization condition to match the loop order of the amplitude we were investigating, but in the following we allow Γ being determined from higher loop order. As long as the loop order of the renormalization condition exceeds the loop order of the given amplitude, the procedure is correct. After all, higher loop orders are supposed to give higher perturbative corrections which we can reject at the end of our calculation. For the inductive step we assume (5.12) to be valid up to n-loop order and we start with the self-energy at $n+1$ loops. The idea is the following: Applying the extended cutting rules to the self-energy $\mathrm{i}\tilde{\Sigma}$, we reduce the problem to two-point functions. Starting from the highest-loop contribution, we look for pseudo-cuts and replace them with cut 2-point functions and this works as follows. Assume we have found a pseudo-cut at loop order m

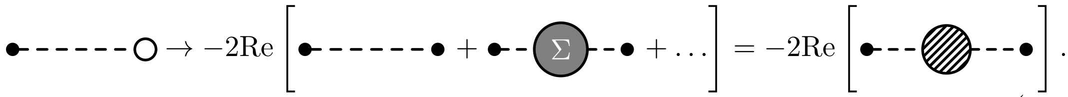


(5.13)

then there are higher order contributions and by these we mean all contributions which emerge when replacing the CMS propagator, which is responsible for the pseudo-cut, with

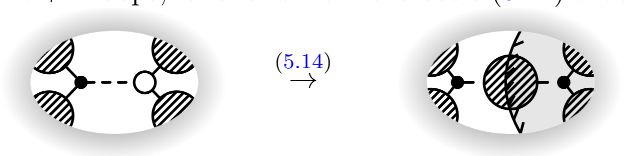


where the above 2-point function has up to $n+1-m$ loops. From the structure of the LTE follows that we do not only have (5.13), but we have the same contribution with the same "background" where pseudo-cut is replaced by



(5.14)

We give an example of this property in the appendix 8. Since the two-loop function has less than $n+1$ loops, it follows from the basis (5.12) that



$$(1+\mathcal{O}(\alpha))\,. \tag{5.15}$$

Again it is important to consider the sum of cut and pseudo-cut contributions which are part of a two-point function because cancellations take place within this structure and we cannot just plug in the leading order contribution to Δ^+. In this way we deal with further pseudo-cuts and we have shown that (5.11) is valid up to $n+1$ loops.
Next we use this result to show that the same holds for the 2-point function at $n+1$ loop-order and we make use of resummation which is allowed if perturbation theory makes sense

$$\frac{\mathrm{i}}{p^2-\mu^2+\Sigma} = \mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\sum_{k=0}^{\infty}\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^k\,, \tag{5.16}$$

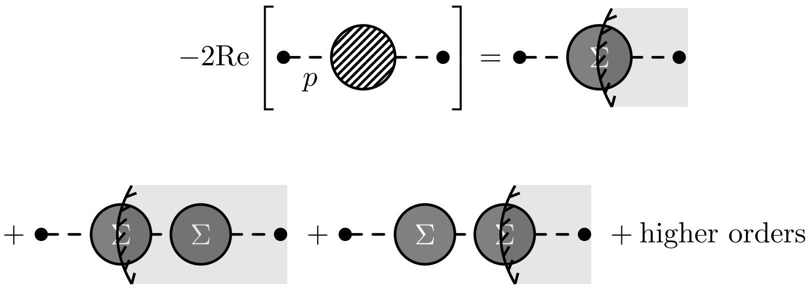
where $\Sigma = \Sigma^n_{R,\varphi}$. Recall the calculation at two-loop order (5.3) where we performed some manipulation to get a more suited form. We proceed the same way and we encounter the following product

$$\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\sum_{k=0}^{\infty}\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^k \times \left(\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\sum_{m=0}^{\infty}\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^m\right)^*\,. \tag{5.17}$$

which is simply the perturbative expansion of both 2-point functions in (5.4). Then we keep $n+1 = m+k$ constant to obtain a perturbative expansion

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^k\left(\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^m\right)^*\\ &\qquad\to\sum_{k=0}^{n+1}\sum_{m=0}^{k}\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^m\left(\left(\mathrm{i}\Sigma\;\mathrm{i}\Delta_{\mathrm{F}}(p,\mu)\right)^*\right)^{k-m} + \text{higher orders}\end{aligned} \tag{5.18}$$

We multiply this result with $-2\mathrm{Re}\left[\mathrm{i}\tilde{\Sigma}\right]$ and obtain the desired result



(5.19)

which represents all possible cuts through the two-point function up to $n+1$ loops. Therefore, the resummation trick tells us again that we can directly apply the extended cutting rules to the self-energy parts and this is what was to be proved.

6. Gauge Theories

6.1. Motivation

It is believed that our world is fully described by four fundamental forces acting between fundamental particles, leptons and quarks. These forces have in common that they are governed by an invariance. Limiting ourselves to electromagnetic phenomena, the Maxwell equations describe our macroscopic world with great success. The corresponding field theory is given by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is called the field strength tensor and j^μ is an electromagnetic conserved current ($\partial_\mu j^\mu = 0$). The underlying invariance is the so-called gauge invariance, i.e. the EOM stay the same under gauge transformations

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi.$$

This invariance expresses the presence of unphysical degrees of freedom which do not enter the EOM. Fixing the gauge explains that the photon field has only two degrees of freedom, namely the transversal modes. The quantum version of the Maxwell equations is known as QED, and one of the most famous predictions of QED is the g factor of the electron. Todays precision measurements reach far beyond QED, and corrections from weak and strong force are essential. The principle of gauge invariance also applies to the weak and strong force which differ from QED, in particular, in their non-Abelian gauge group. For instance, the non-Abelian character plays a crucial role in Quantum Chromodynamics (QCD) and asymptotic freedom. The GSW theory combines QED and the weak force to the electroweak interaction by the gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$. The representations of particles in the in the GSW theory make it difficult to introduce masses without violating gauge invariance. The Higgs mechanism was proposed, solving this problem and giving not only individual masses to all fermions, but also predicting mass relations for the gauge bosons. In fact, there are many reasons to study gauge theories, and the next big step is the unification of all these theories with general relativity where, roughly speaking, the diffeomorphism invariance takes on the role of gauge invariance.

6.2. Construction of Gauge Invariant Lagrangians

In this section we give a short introduction to non-Abelian gauge theories. Non-Abelian gauge theories are characterized by symmetry groups where the fields are in a given representation transforming according to the rules of that specific group. A representation of a group is a homomorphism from the group to the general linear group ($\mathrm{GL}(V)$) on a vector space V. More precisely, a representation is a map [Geo99]

$$\rho\colon G \to \mathrm{GL}(V) \quad \text{such that} \quad \rho(g_1 g_2) = \rho(g_1)\rho(g_2), \qquad \text{for all } g_1, g_2 \in G.$$

On the level of a field theory the Lagrangians are constructed to be invariant under group actions which is then called an internal symmetry. Given a field multiplet ϕ_i, $i = 1, \ldots, N$, we define the group action of an element $U \in G$ as

$$\phi_i \to \phi_i' = \sum_j R_{ij}(U)\phi_j,$$

where R a suited matrix representation matching to the dimension of ϕ. Many groups are possible candidates for model building. For instance, of special interest are the special unitary groups which in the defining representation are given by unitary $N \times N$ matrices with determinant one. As example take the SU(2) in the defining representation which is given by all 2×2 unitary matrices

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \quad \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

An important concept is the Lie algebra which represents the algebraic relations of infinitesimal group transformations. Any element $U \in \mathrm{SU}(2)$ can be written as exponential

$$R(U) = \mathrm{e}^{\mathrm{i}R(t^a)\alpha^a}, a = 1, 2, 3$$

where $R(t^a)$ is the fundamental representation of the generators. These generators satisfy a Lie Algebra $\mathfrak{g}$, i.e.

- bilinearity: $[\alpha\, a + \beta\, b, c] = \alpha\, [a, c] + \beta\, [b, c]$,
- antisymmetry: $[a, b] = -[b, a]$,
- Jacobi identity: $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$.

where $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the Lie bracket. A Lie Algebra is closed by definition and is fully determined by the so-called structure constants f_{abc} which satisfy the following relation

$$[t^a, t^b] = \sum_c \mathrm{i} f_{abc} t^c. \tag{6.1}$$

In view of quantum anomalies the symmetric structure constants d_{abc} play an essential role. They can be defined by the anticommutator. For SU(2) d_{abc} vanishes for all a, b and c.

Example 6.1. Consider the generators of SU(2) which form the Lie Algebra $\mathfrak{su}(2)$. From the defining representation we obtain $t^a = \frac{\sigma^a}{2},\ a = 1, 2, 3$ and $f_{abc} = \varepsilon_{abc}$ where σ^a are the Pauli matrices and ε is the Levi-Civita symbol. Given a SU(2) doublet $\phi_i,\ i = 1, 2$, the matching representation for the gauge transformation is given by the defining representation, and the transformation rule reads $\phi \to \phi' = \mathrm{e}^{\mathrm{i}\frac{\sigma^a}{2}\alpha^a}\phi$.

As in this example, we proceed in this work with the following convention for the normalization of the generators

$$\mathrm{Tr}[\mathbf{t}^a \mathbf{t}^b] = \frac{1}{2}\delta^{ab}. \tag{6.2}$$

From gauge invariance to local gauge invariance

From experimental side the Maxwell-Dirac equation has been confirmed to be a valid theory, explaining the interaction of electrons to photons. The inhomogeneous Maxwell equation is coupled to the Dirac source

$$\partial_\mu F^{\mu\nu} = \bar{\psi}\gamma^\nu\psi.$$

It is possible to motivate this result by requiring the invariance of the Lagrangian under local gauge transformations. The Dirac equation is already invariant under U(1) and by Noether's theorem there is a conserved charge - the electric charge. Going over to local gauge transformations $\alpha \to \alpha(x)$ does no longer leave the Dirac Lagrangian invariant. That is where the covariant derivatives comes in. The usual derivative is replaced by the covariant one which transforms under local gauge transformations as follows

$$\partial_\mu\phi(x) \to U^{-1}(\alpha(x))\partial_\mu U(\alpha(x))\phi(x) = \partial_\mu\phi(x) + U^{-1}(\alpha(x))\left(\partial_\mu U(\alpha(x))\right)\phi(x),$$

where we dropped the R, i.e. U stands for a suited representation $R(U)$ matching the dimension of $\phi(x)$. The corresponding covariant derivative is defined as

$$D_\mu := \partial_\mu - \mathrm{i}A_\mu \tag{6.3}$$

In this way the kinetic part can be made invariant under local gauge transformations. More precisely, the second part in (6.3) lives in the Lie algebra and can be canceled by a Lie-algebra-valued field where one chooses the gauge transformation of $\mathbf{A}_\mu$ such that

$$A_\mu \to U(\alpha(x))A_\mu U(\alpha(x))^{-1} + \mathrm{i}U(\alpha(x))\left(\partial_\mu U^{-1}(\alpha(x))\right) \Rightarrow D_\mu \to U^{-1}D_\mu U. \tag{6.4}$$

The local gauge invariance can be regarded as a way of introducing an interaction between gauge fields and matter fields, but in a controlled way. By this we mean that the interaction results from an invariance - the local gauge invariance which turns out to have very nice properties when it comes to quantum theories.

7. Gauge Dependence in Spontaneously Broken Gauge Theories

7.1. Faddeev-Popov Quantization

Defining a perturbative QFT via (2.8) does not work out for gauge theories where the problem is that the gauge-field propagator is undefined unless the gauge is fixed. As has been shown by Faddeev and Popov [FP67], fixing the gauge must be compensated by a term which would, if missing, imply gauge dependence of the theory. The result for a gauge invariant quantization reads

$$Z(A,\chi) = \int \mathcal{D}A\mathcal{D}\chi\ \delta(F(A)-\Phi)\det\left(\frac{\delta F}{\delta\alpha}\right)\exp\left(\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L}(A,\chi)\right).$$

Here, $F(A)$ denotes a gauge-fixing function of the gauge-field A which is set to be equal to Φ, and α is the parameter of the gauge transformation. Φ is arbitrary, and wlog the generating functional is integrated over a Gaussian weight $\mathrm{e}^{-\frac{1}{2}\Phi^2}$. Re-expressing the appearing determinant by a Gaussian Grassmann-valued path integral one arrives at

$$Z \to \int \mathcal{D}\bar{u}\mathcal{D}u\mathcal{D}A\mathcal{D}\chi\ \exp\left(\mathrm{i}\int \mathrm{d}^4x\ \mathcal{L} - \frac{1}{2}F^2 + \bar{u}\frac{\delta F}{\delta\alpha}u\right). \tag{7.1}$$

Thus, ghosts $u, \bar{u}$ are necessary to guarantee gauge independence, but may be neglected if they decouple from the theory, though, this is not the case for non-Abelian gauge theories. We note that the gauge transformation $\delta_\alpha F = \frac{\delta F}{\delta\alpha}\alpha$ is going to be replaced by a BRST variation $\frac{\delta F}{\delta\alpha}u = \delta_{\mathrm{B}}F$.

7.2. Spontaneous Symmetry Breaking and the Higgs Mechanism

The validity of the LSZ reduction formula (2.11) relies on the vanishing of vevs of single fields. The Higgs potential is an example for a potential which exhibits a non-vanishing vev. In this case the field cannot be interpreted as a physical field and one has to

reparametrize the field around the vev to identify the real physical field [EB64; Hig64]. If there is no unique vev, we are free to choose a vev and the symmetry is broken.
In the following section we discuss an SU(2) Higgs model coupled to gauge fields giving them masses via the Higgs mechanism. Our main concern is to investigate the underlying quantum symmetry. Based on gauge invariance, we construct the BRST transformations which will then be used later on. Besides that, we discuss the rigid symmetry of the model which is just as important as the BRST invariance in view of renormalizability.

7.2.1. SU(2) Higgs Model

Consider a Higgs doublet φ_i transforming as fundamental representation of SU(2). We sketch a calculation of an isomorphism, namely $\mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{SO}(4)$, and the trick is a simple parametrization where the components of the doublets are expanded in the basis of Pauli spin matrices which, with a closer look, reveals the connection. The statement and several steps of the calculation have been taken from [Van10]. The SO(4) appears in all invariant complex doublet products such as φ^2, φ^4 and $\partial_\mu \varphi \partial^\mu \varphi$. Coupling the Higgs to gauge-fields requires a covariant derivative and it turns out that only one of the two SU(2) can be promoted to a local invariance. On the other hand, the rigid SO(4) symmetry is preserved and plays a crucial role for the proof of renormalization. Consider only the kinetic part as the same statement follows trivially for the potential. The first step is a suited parametrization of the doublet

$$\varphi = \begin{pmatrix} \tilde{\sigma} + \mathrm{i}\chi^3 \\ \mathrm{i}\chi^1 - \chi^2 \end{pmatrix} = \left(\tilde{\sigma}\sigma^0 + \mathrm{i}\boldsymbol{\chi} \cdot \boldsymbol{\sigma}\right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \sigma^0 = \mathbb{1}_{2\times 2}$$

The Higgs potential reads $V(\varphi) = \mu^2 \varphi^2 + \lambda \varphi^4$ and for values of $\mu^2 < 0$ the potential implies symmetry breaking. There are infinite possible choices for a vev and a common choice is $\langle \tilde{\sigma} \rangle = v$, $\langle \chi^i \rangle = 0$ which is equivalent to

$$\varphi = \left((v + \sigma)\sigma^0 + \mathrm{i}\boldsymbol{\chi} \cdot \boldsymbol{\sigma}\right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \langle \sigma \rangle = 0, \qquad \langle \chi^i \rangle = 0, \tag{7.2}$$

where we have identified possible candidates for physical fields, namely σ and χ^i. The Goldstone theorem [GSW62] states that for every broken continuous symmetry there is a Goldstone boson. In our case all generators break the symmetry $t^a \langle \varphi \rangle \neq 0 \; \forall a$ and it turns out that the $\boldsymbol{\chi}$s are massless would-be Goldstone bosons giving the gauge-fields $\mathbf{A}$ mass.
The Higgs field is coupled to the gauge bosons via the covariant derivative (6.3) and we replace the kinetic part as follows $\partial_\mu \varphi \partial^\mu \varphi \to D_\mu \varphi D^\mu \varphi$. We expand the covariant

derivative and collect the terms in front of the Pauli matrices and the one. Using the representation (7.2), we obtain in a first step

$$D_\mu(v+\sigma+\mathrm{i}\boldsymbol{\chi}\cdot\boldsymbol{\sigma}) = \overbrace{\partial_\mu\sigma + \frac{g}{2}\mathbf{A}_\mu\cdot\boldsymbol{\chi}}^{D_\mu\sigma:=} + \mathrm{i}\boldsymbol{\sigma}\cdot\overbrace{\left(\partial_\mu\boldsymbol{\chi} - \frac{g}{2}\mathbf{A}(v+\sigma) + \frac{g}{2}\left(\mathbf{A}_\mu\times\boldsymbol{\chi}\right)\right)}^{D_\mu\boldsymbol{\chi}:=},$$

and it follows that

$$\Rightarrow D_\mu\varphi = \left(D_\mu\sigma + \mathrm{i}\boldsymbol{\sigma}\cdot D_\mu\boldsymbol{\chi}\right)\cdot\begin{pmatrix}1\\0\end{pmatrix}.$$

Since $D_\mu\sigma = (D_\mu\sigma)^*$ and $D_\mu\boldsymbol{\chi} = (D_\mu\boldsymbol{\chi})^*$ the kinetic part is easily evaluated using properties of the spin algebra[1]

$$\left(D_\mu\varphi\right)^\dagger D^\mu\varphi = \mathbf{s}^T\cdot\left[(D_\mu\sigma)^2 + (D_\mu\boldsymbol{\chi})^2\right]\sigma^0\cdot\mathbf{s},\quad \mathbf{s} = \begin{pmatrix}1\\0\end{pmatrix}. \tag{7.3}$$

The form of (7.3) reveals a redundancy in the choice of the symmetry vector and $\mathbf{s}$ can be chosen arbitrarily provided that it is appropriately normalized. Consequently, we can rewrite (7.3) by the trace

$$\left(D_\mu\varphi\right)^\dagger D^\mu\varphi = \frac{1}{2}\,\mathrm{Tr}\left[\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right)\right], \tag{7.4}$$

where we set $\Phi = (v+\sigma)\sigma^0 + \mathrm{i}\boldsymbol{\chi}\cdot\boldsymbol{\sigma}$. The representation makes the isomorphism obvious. We have the usual SU(2) invariance $\varphi\to U(\boldsymbol{\alpha})\varphi$ which corresponds to the left action of $U(\boldsymbol{\alpha})$ on Φ, i.e. $U_L : \Phi\to U(\boldsymbol{\alpha})\Phi$. On the other hand, we can transform Φ from the right $U_R : \Phi\to\Phi U(\boldsymbol{\alpha})$. U_L, U_R are inequivalent and only U_L can be promoted to a non-local gauge transformation $U(\alpha(x))$ if we demand that the covariant derivative transforms in the adjoint of SU(2). The local invariance is checked explicitly. Given a gauge-field $\mathbf{A}_\mu$ the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}g\mathbf{A}_\mu$ transforms according to (6.4), therefore

$$\begin{aligned} U_L : \left(D_\mu\varphi\right)^\dagger D^\mu\varphi &\to \frac{1}{2}\,\mathrm{Tr}\left[\left(UD_\mu U^{-1}U\Phi\right)^\dagger UD_\mu U^{-1}U\Phi\right] \\ &= \frac{1}{2}\left[\left(D_\mu\Phi\right)^\dagger U^\dagger U D^\mu\Phi\right] \\ &= \left(D_\mu\varphi\right)^\dagger D^\mu\varphi. \end{aligned}$$

Since the transformation rule for D_μ is fixed, a local U_R transformation does not leave (7.4) invariant. Nevertheless, U_R is a rigid invariance and according to the isomorphism,

[1] $(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})\,I + \mathrm{i}(\mathbf{a}\times\mathbf{b})\cdot\sigma,\ \left[\sigma,\sigma^0\right] = 0.$

the full SO(4) rigid invariance can be written as

$$U(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2):\ \Phi \to U(\boldsymbol{\alpha}_1)\Phi U(\boldsymbol{\alpha}_2),\quad (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)\in\mathbb{R}^6$$
$$D_\mu\varphi D^\mu\varphi \to D_\mu\varphi D^\mu\varphi. \tag{7.5}$$

From the structure in (7.3) it is clear that an SO(4) rotation of $(D_\mu\sigma, D_\mu\boldsymbol{\chi})\in\mathbb{R}^4$ leaves the Lagrangian invariant. To complete the isomorphism one would have to show that there is a bijective map between the six angles of SO(4) and $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)$.
We come back to rigid symmetry which plays an important role in sections 7.5.5 where we construct a basis of integrated local insertions which must be invariant under rigid SO(4). The calculation is dramatically simplified by demanding a necessary condition for SO(4) invariance instead of requiring SO(4) invariance from the beginning. The trick we propose is to transform Φ in the adjoint representation of SU(2):

$$\Phi \to U(\boldsymbol{\alpha})\Phi U(\boldsymbol{\alpha})^{-1},\ \text{i.e.}\ \boldsymbol{\alpha}_2=-\boldsymbol{\alpha}_1, \tag{7.6}$$

which obviously leaves (7.4) invariant. Translating this transformation to $\sigma,\boldsymbol{\chi}$, it follows that σ does not transform while $\boldsymbol{\chi}$ transforms under the adjoint representation which makes totally sense because $\boldsymbol{\chi}$ is Lie-algebra-valued. How does this help us? All the fields transform similar, except for the Higgs σ which transforms trivially. Further, we know how to construct all invariant tensors in this case. Let T_R^α denote Lie-algebra-valued fields, for instance, a gauge-field A_μ^a or a would-be Goldstone boson χ^a, i.e. $\mathbf{T}_R=(\mathbf{A},\boldsymbol{\chi},\ldots)$, then invariant tensors $M^{\alpha_1\ldots\alpha_n,m}$ read

$$M^{\alpha_1\ldots\alpha_n,m}=\sigma^m\,\mathrm{Tr}\left[T^{\alpha_1}\ldots T^{\alpha_n}\right], \tag{7.7}$$

where the invariance follows from the transformation rules for the adjoint representation, i.e. $T_R^\alpha\to U_A^{\alpha\beta}T_R^\beta=U_RT_R^\alpha U_R^{-1}$. We make use of the fact that (7.7) are all possible invariant tensors, but we do not proof this statement.
From a practical point of view it is useful to formulate symmetries in functional form. To this end, consider the transformation rule of the gauge-field under a local infinitesimal gauge transformation (6.4). Setting $\boldsymbol{\alpha}(x)=\boldsymbol{\alpha}$, one obtains the rigid transformation rule

$$\mathbf{A}_\mu(x)\to\mathbf{A}_\mu(x)+\mathrm{i}\left[\boldsymbol{\alpha},\mathbf{A}_\mu(x)\right]$$

which is non-trivial in the case of non-Abelian gauge groups. Since $\boldsymbol{\alpha}$ is arbitrary, the rigid transformation can also be written as

$$\delta^a_{\mathrm{rig}}=\mathrm{i}\left[t^a,\circ\right] \tag{7.8}$$

Thus, $\alpha_a \delta^a_{\mathrm{rig}} \mathbf{A}_\mu(x)$ reproduces the desired result. For more than one field one must make sure that the transformation acts on the field it is supposed to. This can be achieved by a functional derivative. Let $\mathbf{W}$ denote the rigid invariance operator

$$\mathbf{W} = \mathrm{i} \int \mathrm{d}^4 x \, \left[\mathbf{A}_\mu, \frac{\delta}{\delta \mathbf{A}_\mu} \right] := \mathrm{i} \int \mathrm{d}^4 x \, [\mathbf{A}_\mu, \mathbf{t}^a] \frac{\delta}{\delta A^a_\mu}, \tag{7.9}$$

where the i is convention. One verifies that $W^a A^b_\mu$ yields the result corresponding to $(\delta^a \mathbf{A}_\mu)^b$. The generalization to more Lie-algebra-valued fields is straightforward. Let Δ be part of the basis of integrated local insertions. Rigid invariance means $\mathbf{W}\Delta = 0$ and the general solution is given by (7.7). This concludes the part of rigid invariance, and we give the general form of $\mathbf{W}$ in section 7.5.5.

Finally, we derive the infinitesimal non-Abelian gauge transformations which are needed to construct the corresponding BRST transformations (see section 7.3). Consider

$$\begin{aligned} U_L : \Phi \to U(\alpha(x))\Phi &= \left(\sigma^0 + \mathrm{i}\boldsymbol{\alpha}(x) \cdot \frac{\boldsymbol{\sigma}}{2} \right) \Phi + \mathcal{O}(\alpha^2) \\ &= \Phi + \delta_\alpha \Phi. \end{aligned} \tag{7.10}$$

We plug in the explicit form of Φ in (7.10) and we collect all terms proportional to $\sigma^0, \boldsymbol{\sigma}$. Defining

$$\delta_\alpha v := 0, \ \delta_\alpha \sigma := -\frac{1}{2} g \boldsymbol{\alpha} \cdot \boldsymbol{\chi}, \ \delta_\alpha \boldsymbol{\chi} := \frac{g}{2} \left((v + \sigma)\boldsymbol{\alpha} - (\boldsymbol{\alpha} \times \boldsymbol{\chi}) \right),$$

we have that $\delta_\alpha \Phi = (\delta_\alpha \sigma)\, \sigma^0 + (\delta_\alpha \boldsymbol{\chi}) \cdot \boldsymbol{\sigma}$.

7.2.2. Mixing Fields and the R_ξ Gauges

Our construction of the theory involves mixing of fields which follows from the kinetic part of ϕ (7.4). To see this, consider (7.3) which is simpler to derive vertices. In $(D_\mu \chi)^2$ we encounter after partial integration

$$\frac{g}{2} v \, (\partial^\mu \mathbf{A}_\mu) \, \boldsymbol{\chi}. \tag{7.11}$$

The mixing is not a problem, but unwanted and it is well-known how to deal with it. Within the R_ξ gauges ('t Hooft gauges [Hoo71b; Hoo71a]) we can choose a linear gauge-fixing function $\mathbf{F}$ which eliminates any kind of field mixing. Given the gauge-fixing Lagrangian $\mathcal{L}_{\mathrm{fix}} = \frac{1}{2} \gamma_{ab} F^a F^b$ (7.1), we choose

$$\gamma_{ab} = \frac{1}{\xi} \delta_{ab}, \ \mathbf{F} = (-\partial^\mu \mathbf{A}_\mu + \frac{1}{2} g \xi v \boldsymbol{\chi}).$$

From $\mathcal{L}_{\text{fix}}$ we obtain, apart from other vertices, a mixing field $\mathcal{L}_{\text{fix}} \supset -\frac{1}{2} g v \left(\partial^\mu \mathbf{A}_\mu\right) \cdot \boldsymbol{\chi}$ which exactly cancels the mixing field (7.11).

7.3. BRST Transformation

The physics should not depend on different gauges and as a consequence of the fixing of the gauge one has to introduce ghost fields. However, explicit gauge invariance is violated as one can verify by direct computation. Further, it is not immediately clear that the Faddeev-Popov quantization leads to a physical S matrix and the question of unitarity should be revisited. Thereby related, there should be a mechanism relating non-physical fields such as the $\boldsymbol{\chi}$s to non-physical degree of freedom (dof) of particles. This connection is given by the BRST invariance [BRS74; Tyu75] which is the extension of gauge invariance for non-Abelian gauge theories.
We extend gauge transformations δ by BRST transformations δ_{B}

$$\delta_{\text{B}} \circ = \left[Q_{\text{BRST}}, \circ\right]_{\pm},$$

where Q_{BRST} is the BRST charge and $[\,,\,]_{\pm}$ is the super-bracket. The $\pm$ indicates that the transformation δ_{B} is Grassmann-valued which originates from the fact that Q_{BRST} has ghost-charge $+1$. Depending on whether δ_{B} acts on fermions or bosons, the bracket $[\,,\,]_{\pm}$ is either commutative or anticommutative, respectively. Since we will not use the explicit expression of Q_{BRST} we only mention the graded Leibniz rule

$$\delta_{\text{B}} \left(AB\right) = \left(\delta_{\text{B}} A\right) B + (-1)^{|A|} A \delta_{\text{B}} B, \tag{7.12}$$

which follows from the rules of the super-bracket and $|A| = 1(0)$ if Grassmann-valued (not Grassmann-valued). If we know the action of δ_{B} on elementary fields, we can derive the action of δ_{B} on products of fields with the help of (7.12). The usual construction is as follows. One replaces the gauge parameter in the gauge transformation by a ghost-field, changing the transformations to global, but non-linear ones. In the second step one requires BRST invariance of the full Lagrangian under the constraint that δ_{B} is nilpotent, i.e. $\delta_{\text{B}}^2 = 0$. As we will show, this fixes the BRST variation of ghost and anti-ghost fields. For the gauge and matter fields we obtain

$$\begin{aligned}
\delta_\alpha \mathbf{A}_\mu = \left(D_\mu \boldsymbol{\alpha}\right) &\to \delta_{\text{B}} A_\mu^a := \left(D_\mu \mathbf{u}\right)^a, \\
\delta_\alpha \sigma = -\frac{1}{2} g \boldsymbol{\alpha} \cdot \boldsymbol{\chi} &\to \delta_{\text{B}} \sigma := -\frac{1}{2} g \mathbf{u} \cdot \boldsymbol{\chi}, \\
\delta_\alpha \boldsymbol{\chi} = \frac{g}{2} \left((v+\sigma)\boldsymbol{\alpha} - (\boldsymbol{\alpha} \times \boldsymbol{\chi})\right) &\to \delta_{\text{B}} \boldsymbol{\chi} := \frac{g}{2} \left((v+\sigma)\mathbf{u} - (\mathbf{u} \times \boldsymbol{\chi})\right).
\end{aligned}$$

7.3.1. Gauge-Fixing and Ghost Lagrangian

The BRST transformations are more restrictive than the usual gauge transformations in the sense that gauge invariant parts are BRST invariant

$$\delta_B \left((D_\mu \phi)^\dagger D^\mu \phi \right) = 0, \ \delta_{\rm B} \left(F^{\mu\nu} F_{\mu\nu} \right) = 0,$$

but we further require that the whole action is invariant under BRST variation. Applying the Leibniz rule (7.12) to the gauge-fixing yields $\delta_{\rm B}\mathcal{L}_{\rm fix} = \gamma_{ab} F^a \delta_{\rm B} F^b \neq 0$ and BRST invariance requires $\delta_{\rm B}\mathcal{L}_{\rm fix} = -\delta_{\rm B}\mathcal{L}_{\rm ghost}$. From Faddeev-Popov quantization (see section 7.1) follows that $\mathcal{L}_{\rm ghost} = -\bar{u}_a \delta_{\rm B} F^a$. Requiring BRST nilpotency leads us to the following ansatz

$$\delta_{\rm B} \bar{u}^a = \gamma_{ab} F^b. \tag{7.13}$$

In case $\delta_{\rm B}^2 F^a = 0$ we have succeeded and as promised, the requirement of nilpotency fixes the BRST transformation of the ghost field $\delta_{\rm B} u^a$ as discussed below. Actually, the definition (7.13) implies breaking of nilpotency of the anti-ghost field which we will simply accept for the moment, then, with the help of auxiliary fields, we shall restore nilpotency later.
Given the R_ξ gauges, the explicit form of $\delta_{\rm B}\mathbf{F}$ reads

$$\delta_{\rm B}\mathbf{F} = \partial^\mu \delta_{\rm B} \mathbf{A}_\mu - \frac{1}{2} g v \xi \delta_{\rm B} \boldsymbol{\chi} = \partial^\mu D_\mu \mathbf{u} - \frac{1}{4} g^2 \xi v \left(\mathbf{u}(v+\sigma) + \mathrm{i} \left[\mathbf{u}, \boldsymbol{\chi} \right] \right).$$

We need the nilpotency on $\mathbf{F}$, but this is equivalent to the nilpotency on $\mathbf{A}$ and $\boldsymbol{\chi}$. Yet, we have not defined the BRST variation of ghost fields and we choose it such that nilpotency is guaranteed. This is achieved for

$$\delta_{\rm B} \mathbf{u} = \frac{\mathrm{i}}{2} \left\{ \mathbf{u}, \mathbf{u} \right\} \Rightarrow \delta_{\rm B} u^a = -\frac{1}{2} \varepsilon_{abc} u^b u^c. \tag{7.14}$$

Proof. We show that $\delta_{\rm B}^2 \boldsymbol{\chi}, \delta_{\rm B}^2 \mathbf{A}_\mu = 0$. Consider $\delta_{\rm B}^2 \boldsymbol{\chi}$ first.

$$\begin{aligned} &\frac{4}{g} \delta_{\rm B}^2 \boldsymbol{\chi} = 2\delta_{\rm B} \left(\mathbf{u}(v+\sigma) - (\mathbf{u} \times \boldsymbol{\chi}) \right) \\ &= - (\mathbf{u} \times \mathbf{u})(v+\sigma) + \mathbf{u}\ (\mathbf{u} \cdot \boldsymbol{\chi}) + ((\mathbf{u} \times \mathbf{u}) \times \boldsymbol{\chi}) + (v+\sigma)(\mathbf{u} \times \mathbf{u}) - (\mathbf{u} \times (\mathbf{u} \times \boldsymbol{\chi})) \end{aligned}$$

Making use of the following vector identities $(\mathbf{u} \times \mathbf{u}) \times \boldsymbol{\chi} = 2 (\boldsymbol{\chi} \cdot \mathbf{u}) \mathbf{u}$ and $\mathbf{u} \times (\mathbf{u} \times \boldsymbol{\chi}) = (\mathbf{u} \cdot \boldsymbol{\chi}) \mathbf{u}$, where one has to pay attention to the Grassmannian character of $\mathbf{u}$, we obtain

$$\frac{4}{g} \delta_{\rm B}^2 \boldsymbol{\chi} = \mathbf{u} (\mathbf{u} \cdot \boldsymbol{\chi}) + 2 (\boldsymbol{\chi} \cdot \mathbf{u}) \mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\chi}) \mathbf{u} = 0.$$

The statement for $\delta_{\mathrm{B}}^2 \mathbf{A}_\mu$ follows by direct computation

$$\delta_{\mathrm{B}}^2 A_\mu^a = -\frac{1}{2}\partial_\mu (\mathbf{u}\times\mathbf{u}) + \frac{1}{2}(\mathbf{u}\times\mathbf{u})\times\mathbf{A} + \mathbf{u}\times(\partial_\mu\mathbf{u}) - \mathbf{u}\times(\mathbf{u}\times\mathbf{A}) = 0.$$

□

For the case where structure constants are not given by the Levi-Civita symbol, the BRST variation of the ghost is still given by the super bracket (7.14). Actually, it is possible to give a more general prove of BRST invariance for simple Lie groups which is based on the super Jacobi identity instead of our vector identities.

7.3.2. The SU(2) Higgs Model Summarized

In the last sections we introduced non-Abelian gauge theories and SSB. To sum everything up, the model is described by the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{fix}} + \mathcal{L}_{\mathrm{ghost}} + \mathcal{L}_{\mathrm{mat}}$$

Besides the gauge $\mathcal{L}_{\mathrm{YM}}$ and matter fields $\mathcal{L}_{\mathrm{mat}}$, we additionally have ghosts $\mathcal{L}_{\mathrm{ghost}}$ and the gauge-fixing $\mathcal{L}_{\mathrm{fix}}$ which are necessary for self-consistent QFT and the individual parts are taken from the previous sections

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\operatorname{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right], \tag{7.15}$$

$$\mathcal{L}_{\mathrm{fix}} = \frac{2}{\xi}\operatorname{Tr}\left[\mathbf{F}^2\right] = \frac{2}{\xi}\operatorname{Tr}\left[\left(-\partial^\mu\mathbf{A}_\mu + \frac{1}{2}g\xi v\boldsymbol{\chi}\right)^2\right], \tag{7.16}$$

$$\mathcal{L}_{\mathrm{ghost}} = 2\operatorname{Tr}\left[\bar{\mathbf{u}}\left(\partial^\mu D_\mu\mathbf{u} - \frac{1}{4}g^2\xi v\left(\mathbf{u}(v+\sigma) + \mathrm{i}\left[\mathbf{u},\boldsymbol{\chi}\right]\right)\right)\right], \tag{7.17}$$

$$\mathcal{L}_{\mathrm{mat}} = (D_\mu\varphi)^\dagger D^\mu\varphi - V(\varphi^\dagger\varphi). \tag{7.18}$$

7.4. BRST Nilpotency, Extended BRST Invariance and the Nielsen Identities

We mentioned that the BRST variation is yet not nilpotent because of $\delta_{\mathrm{B}}^2\bar{\mathbf{u}} = \frac{1}{\alpha}\delta_{\mathrm{B}}\mathbf{F} \neq 0$. With regard to unitarity this is no problem because only nilpotency on physical fields is required, but nevertheless it is possible to restore nilpotency and we need this step for a proper derivation of the extended BRST transformations. To this end, we express the gauge-fixing Lagrangian $\mathcal{L}_{\text{fix}}$ as

$$\mathcal{L}_{\text{fix}} = \alpha \operatorname{Tr}[\mathbf{b}\mathbf{b}] + 2 \operatorname{Tr}[\mathbf{F}\mathbf{b}],$$

which can be seen by simply integrating out the fields $\mathbf{b}$ and using $G^{\mathbf{b}}_{ab} = \frac{1}{\alpha}\delta_{ab}$ for the $\mathbf{b}$ propagator. The field $\mathbf{b}$, having no physical relevance, is an auxiliary fields and to recover the BRST invariance we define the following transformation rules under BRST variation

$$\delta_{\mathrm{B}}\mathbf{b} = 0, \;\; \delta_{\mathrm{B}}\bar{\mathbf{u}} = \mathbf{b}, \tag{7.19}$$

but this time we have, in contrast to (7.13), nilpotency on the anti-ghost field and we arrived at what we wanted to show, namely

$$\delta^2_{\mathrm{B}}\{\mathbf{u};\bar{\mathbf{u}};\mathbf{A};\mathbf{b};\sigma;\boldsymbol{\chi}\} = 0 \;\Rightarrow \delta^2_{\mathrm{B}} = 0. \tag{7.20}$$

7.4.1. Extended BRST Invariance

The Nielsen Identities (NI) were discovered by Nielsen [Nie75] when analyzing the effective potential. He paved the way for investigating the validity of what is thought of the most fundamental property of the SM, the gauge invariance. The set of identities are expressed as extended BRST transformations or extended ST identities. They are very useful to prove gauge (in)dependence of quantities, for instance, with the help of the NI one can show the gauge independence of the physical mass [GG00; BLS95], the gauge independence of the phenomenon of SSB [Nie75; DFP99] or even the gauge independence of the S matrix [Kum01], therefore, serving for verifying whether quantities may serve as observables.

The NI involve new interaction terms which do not show up in physical processes. These additional contributions induce unphysical effects and the NI should be seen as a tool rather than a fundamental symmetry.

The first part is devoted to give a proper introduction and derivation of the NI. We follow reference [DFP99] giving the proof that the NI can be implemented for the case of arbitrary R_ξ gauges. In the second part we show that the renormalization can be carried out where the ST operator is replaced by the extended one and we show that divergences occurring in physical amplitudes can be absorbed by appropriate counter terms.

The whole idea behind the extended BRST transformation is the observation that transformations like

$$\mathcal{L} \to \mathcal{L} + \chi u b, \tag{7.21}$$

do not change the physics. Here, u is a ghost field, χ a Grassmann-valued parameter and b a scalar field. The transformation (7.21) implies a transformation of the generating functional $Z[j] \to Z^\chi[j]$. Since χ is Grassmann-valued, the expansion

$$Z^\chi[j] = Z[j] + \chi \frac{\partial}{\partial \chi} Z^\chi[j]\bigg|_{\chi=0}$$

is no approximation. Physical amplitudes are free of unphysical asymptotic fields which in the path integral sense means that unphysical fields can be integrated out. Excluding asymptotic ghosts implies the vanishing of the χ-dependent part of Z^χ, thus $Z = Z^\chi$ up to unphysical amplitudes. As a matter of fact, the statement follows from ghost conservation, i.e. # ghosts in $|\text{in}\rangle$ states equals # ghosts in $\langle\text{out}|$ states. We proof this statement and show that

$$\chi \frac{\partial}{\partial \chi} Z^\chi[j]\bigg|_{\chi=0} = \int D[\phi] \left(\mathrm{i}\chi \int \mathrm{d}x\, u(x) b(x)\right) \mathrm{e}^{\mathrm{i}S} \overset{?}{=} 0. \tag{7.22}$$

As already mentioned, the proof follows simply by integrating out the ghost fields which is a special case of ghost-charge conservation. Consider the theory without having performed (7.21). Coupling sources $j, \bar{j}$ to the ghost fields and integrating them out yields

$$\frac{\int D[u] D[\bar{u}]\, \mathrm{e}^{\mathrm{i}\hat{S} + \mathrm{i}\int \mathrm{d}^4x\, \bar{u}\, \hat{O}\, u + \bar{j} u + \bar{u} j}}{\int D[u] D[\bar{u}]\, \mathrm{e}^{\mathrm{i}\hat{S} + \mathrm{i}\int \mathrm{d}^4x\, \bar{u}\, \hat{O}\, u}} = \mathrm{e}^{\mathrm{i}\tilde{S} + \mathrm{i}\int \mathrm{d}^4x\, \bar{j}\, \hat{O}^{-1}\, j},$$

where we made use of the fact that in the usual action ghosts appear only as a ghost and anti-ghost pair. $\tilde{S}$ denotes the action without any ghost parts. In the next step we differentiate the lhs with respect to $\bar{j}$ and put $j, \bar{j}$ to zero. The rhs vanishes

$$\frac{\int D[u] D[\bar{u}]\; u\, \mathrm{e}^{\mathrm{i}S}}{\int D[u] D[\bar{u}]\; \mathrm{e}^{\mathrm{i}S}} = \hat{O}^{-1} j\; \mathrm{e}^{\mathrm{i}\tilde{S} + \int \mathrm{d}^4x\, \bar{j}\, \hat{O}^{-1}\, j}\bigg|_{j,\bar{j}=0} = 0,$$

and consequently equation (7.22) is proven and we have shown that the transformation (7.21) does not alter physical amplitudes.

A very natural way of introducing the extended BRST transformation, which also works out of the box when SSB is present, is to simply require nilpotency under extended BRST transformation, but keeping the BRST transformation rules for the fields. A non-trivial change of the transformation rule of course implies a change of the action. Besides nilpotency, one requires invariance of the so-obtained transformed action under extended BRST transformation. It will be shown that the physical content is untouched and that the difference between both theories is built in the style of (7.21). Following reference [DFP99], we rewrite the sum of gauge-fixing and ghost Lagrangian as a BRST variation

$$\begin{aligned}\mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{fix}} &= \alpha \operatorname{Tr}[\mathbf{b}^2] - 2 \operatorname{Tr}[\bar{\mathbf{u}}\delta_{\mathrm{B}}\mathbf{F}] + 2 \operatorname{Tr}[\mathbf{F}\mathbf{b}] \\ &= \delta_{\mathrm{B}} \operatorname{Tr}\left[\alpha \bar{\mathbf{u}}\mathbf{b} + 2\bar{\mathbf{u}}\mathbf{F}\right].\end{aligned} \tag{7.23}$$

This result implies $\delta_{\mathrm{B}}\left(\mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{fix}}\right) = 0$ provided δ_{B} is nilpotent, (7.20), which we prove below (proposition 7.1). We notice that the gauge dependence of the Lagrangian is given by a BRST transformation

$$\frac{\partial}{\partial \alpha}\mathcal{L} = \delta_{\mathrm{B}} \operatorname{Tr}[\bar{\mathbf{u}}\mathbf{b}], \quad \frac{\partial}{\partial \xi}\mathcal{L} = 2vg\, \delta_{\mathrm{B}} \operatorname{Tr}[\bar{\mathbf{u}}\boldsymbol{\chi}]. \tag{7.24}$$

Physical states are invariant under δ_{B} and the gauge dependence of the Lagrangian is proportional to δ_{B} indicating the unphysical nature of gauge dependence. The fact that the gauge dependence is only present in $\mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{fix}}$ (7.24) and that this term is a BRST variation (7.23), allows to define gauge-parameter transformations while maintaining (extended) BRST invariance. In this sense, let us define a new set of parameters by the following rules[2]

$$\begin{aligned}\delta_{\mathrm{BE}}\alpha = \beta, &\quad \delta_{\mathrm{BE}}\beta = 0, \\ \delta_{\mathrm{BE}}\xi = \eta, &\quad \delta_{\mathrm{BE}}\eta = 0,\end{aligned} \tag{7.25}$$

where β, η are Grassmann-valued parameters (or global ghosts). The transformation rules for gauge-parameters α, ξ and partners χ, η preserve nilpotency. Next we define a new Lagrangian $\mathcal{L}^{\beta\eta}$ which differs from $\mathcal{L}$ by the replacement

$$\delta_{\mathrm{B}} \operatorname{Tr}\left[\alpha \bar{\mathbf{u}}\mathbf{b} + 2\bar{\mathbf{u}}\mathbf{F}\right] \to \delta_{\mathrm{BE}} \operatorname{Tr}\left[\alpha \bar{\mathbf{u}}\mathbf{b} + 2\bar{\mathbf{u}}\mathbf{F}\right],$$

and we claim that the physics does not change. First we prove that $\mathcal{L}^{\beta\eta}$ is extended BRST invariant which follows from extended BRST nilpotency

$$\delta_{\mathrm{BE}}^2\{\mathbf{u}; \bar{\mathbf{u}}; \mathbf{A}; \mathbf{b}; \boldsymbol{\chi}; \sigma; \alpha; \beta; \xi; \eta\} = 0. \tag{7.26}$$

Proposition 7.1. Let A, B be fields with grading $|A|, |B|$, respectively, i.e. $AB = (-1)^{|A||B|}BA$. Further, let $\delta_{\mathrm{BE}}^2\{A; B\} = 0$. Then all polynomials of these fields are nilpotent $\delta_{\mathrm{BE}}^2\{AA; AB; BA; AA; \ldots\} = 0$.

[2] Pairs of parameters/fields which are related via BRST variation as in (7.25) are known in the literature as BRST doublets.

Proof. It is enough to show that $\delta^2_{\mathrm{BE}}(AB) = 0$

$$\delta_{\mathrm{BE}}(AB) = (\delta_{\mathrm{BE}}A)\,B + (-1)^{|A|}A\,(\delta_{\mathrm{BE}}B)\,,$$

and

$$\begin{aligned}\delta^2_{\mathrm{BE}}(AB) &= \left(\delta^2_{\mathrm{BE}}A\right)B + A\left(\delta^2_{\mathrm{BE}}B\right) + (-1)^{|A|+1}(\delta_{\mathrm{BE}}A)(\delta_{\mathrm{BE}}B) + (-1)^{|A|}(\delta_{\mathrm{BE}}A)(\delta_{\mathrm{BE}}B)\\ &= \left(\delta^2_{\mathrm{BE}}A\right)B + A\left(\delta^2_{\mathrm{BE}}B\right)\end{aligned}$$

Therefore $\delta^2_{\mathrm{BE}}\{A;B\} = 0 \Rightarrow \delta^2_{\mathrm{BE}}(AB) = \delta^2_{\mathrm{BE}}(BA) = 0$ and the statement for higher polynomials follows by induction. □

The field content of $\mathcal{L}_{\mathrm{BE}}$ is nilpotent and therefore $\delta^2_{\mathrm{BE}}\,\mathrm{Tr}\,[\alpha\bar{u}b - 2\bar{u}F] = 0$. Apart from the ghost and gauge-fixing sector nothing has changed and because of gauge invariance extended BRST invariance holds in this sector and in total we have shown $\delta_{\mathrm{BE}}\mathcal{L}^{\beta\eta} = 0$.

Proposition 7.2. The Lagrangian $\mathcal{L}^{\beta\eta}$ has the same physical content as $\mathcal{L}$.

Proof. The difference of both Lagrangians is given by

$$\begin{aligned}\mathcal{L}^{\beta\eta} - \mathcal{L} &= \mathrm{Tr}\,[(\delta_{\mathrm{BE}}\alpha)\,\bar{u}b + 2vg\,(\delta_{\mathrm{BE}}\xi)\,\bar{u}\chi]\\ &= \beta\,\mathrm{Tr}\,[\bar{u}b] + 2vg\eta\,\mathrm{Tr}\,[\bar{u}\chi]\end{aligned} \tag{7.27}$$

and both contributions have the structure of (7.21). Again, we are not interested in external ghosts and because of ghost-charge conservation we conclude that both contributions (7.27) to the generating functional do not contribute to physical amplitudes. There is one subtlety, namely one can construct a pathological example where individual contributions like (7.27), (7.21) do not contribute, but that the combination, i.e. a higher-order effect, does. For instance, this is the case for $\mathcal{L} \to \mathcal{L}^{\alpha_1\alpha_2} = \mathcal{L} + \alpha_1\bar{u}b + \alpha_2 ub$. To check this perform an expansion of $Z^{\alpha_1\alpha_2}[j]$ ($\mathcal{L}^{\alpha_1\alpha_2}$ as input Lagrangian) in α_1 and α_2

$$\begin{aligned}Z^{\alpha_1\alpha_2}[j] = Z[j] &+ \underbrace{\alpha_1\frac{\partial}{\partial\alpha_1}Z^{\alpha_1\alpha_2}[j]\Big|_{\alpha_1,\alpha_2=0}}_{=0}\\ &+ \underbrace{\alpha_2\frac{\partial}{\partial\alpha_2}Z^{\alpha_1\alpha_2}[j]\Big|_{\alpha_1,\alpha_2=0}}_{=0} - \alpha_1\alpha_2\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}Z^{\alpha_1\alpha_2}[j]\Big|_{\alpha_1,\alpha_2=0}.\end{aligned}$$

Z is the generating functional of the former theory, the two terms in between vanish because we restrict ourselves to physical amplitudes, but the last term is non-vanishing.

To prevent such cases one must make sure that there are only ghosts or anti-ghosts in (7.21), but no mixing which may lead to ghost-charge conservation by accident. In view of the extended BRST invariance this cannot happen because the prescription (7.25) always leads to anti-ghost fields as can be seen from (7.27). □

Having clarified that both Lagrangians $\mathcal{L}^{\beta\eta}, \mathcal{L}$ lead to the same physical amplitudes, we observe that $\mathcal{L}^{\beta\eta}, \mathcal{L}$ is not invariant under $\delta_{\mathrm{B}}, \delta_{\mathrm{BE}}$, respectively. Consequently, the BRST variation may not leave $\mathcal{L}$ invariant to maintain the physical content and that is the point where we get more information about the unphysical sector compared to the usual ST identities, more precisely, we get information about gauge dependence of n-point functions.

7.4.2. The Nielsen Identities

We derive the NI which express by a set of vertex identities the invariance of the theory under extended BRST transformations. From the invariance of the generating functional $Z^{\beta\eta}$ ($\mathcal{L}^{\beta\eta}$ as input Lagrangian) under BRST transformations $\delta_{\mathrm{B}} Z^{\beta\eta}[j] = 0$, we find the infinitesimal version

$$\int D[\phi] \int \mathrm{d}^4 x \left(\left(\delta_{\mathrm{B}} \mathcal{L}^{\beta\eta} \right)(x) + \sum_i (-1)^{|\psi_i|} \left(j_i \delta_{\mathrm{B}} \psi_i \right)(x) \right) \mathrm{e}^{\mathrm{i} S^{\beta\eta} + \mathrm{i} \sum_i \int \mathrm{d}^4 x \, j_i \psi_i} = 0,$$

where the fields are given by $\{\psi_i\} = \{\mathbf{A}_\mu, \boldsymbol{\chi}, \sigma, \mathbf{u}, \bar{\mathbf{u}}\}$ and by definition $\delta_{\mathrm{B}} j = 0$ such as $|j_i| = |\psi_i|$. Evaluating the BRST transformation, we obtain

$$\begin{aligned}
\delta_{\mathrm{B}} \mathcal{L}^{\beta\eta} &= \delta_{\mathrm{B}} \left(\mathcal{L}^{\beta\eta} - \mathcal{L} \right) \\
&= \delta_{\mathrm{B}} \left(\beta \operatorname{Tr}[\bar{\mathbf{u}}\mathbf{b}] + 2vg\eta \operatorname{Tr}[\bar{\mathbf{u}}\boldsymbol{\chi}] \right) \\
&= -\beta \delta_{\mathrm{B}} \operatorname{Tr}[\bar{\mathbf{u}}\mathbf{b}] - 2vg\eta \, \delta_{\mathrm{B}} \operatorname{Tr}[\bar{\mathbf{u}}\boldsymbol{\chi}] .
\end{aligned}$$

In the first line we used BRST invariance of $\mathcal{L}$ and from the first to the second line (7.27). The BRST variation $\delta_{\mathrm{B}} \mathcal{L}^{\beta\eta}$ is exactly the gauge dependence of the action (7.24) and we replace it as follows by partial derivatives

$$\begin{aligned}
\delta_{\mathrm{B}} Z^{\beta\eta}[j] = &\int D[\phi] \left(-\beta \frac{\partial}{\mathrm{i}\partial\alpha} - \eta \frac{\partial}{\mathrm{i}\partial\xi} + \int \mathrm{d}^4 x \, \sum_i (-1)^{|\psi_i|} j_i \delta_{\mathrm{B}} \psi_i \right) \mathrm{e}^{\mathrm{i} S^{\beta\eta} + \mathrm{i} \sum_i \int \mathrm{d}^4 x \, j_i \psi_i} \\
= &\, 0.
\end{aligned}$$

The BRST variations of fields are non-linear and we cannot easily switch to the generating functional of connected Green's functions. A possible solution to the problem is to couple the non-linear transformations $\delta_{\mathrm{B}}\phi_i$ to so-called anti-fields ϕ_i^* where $\phi_i \in \{\psi_i\}$

$$Z^{\beta\eta}[j] \to Z^{\beta\eta}[j,\phi^*] = \int D[\phi]\; \mathrm{e}^{\mathrm{i}S^{\beta\eta}+\mathrm{i}\sum_i \int \mathrm{d}^4x\; j_i\phi_i + \mathrm{i}\sum_j \int \mathrm{d}^4x\; \phi_j^*\delta_{\mathrm{B}}\phi_j}.$$

In our case only the anti-ghost transformation $\delta_{\mathrm{B}}\bar{\mathbf{u}}$ is linear in the fields (7.13) and we have $\{\phi_i\} = \{\mathbf{A}_\mu, \boldsymbol{\chi}, \sigma, \mathbf{u}\}$. The quantum action must be real and has to have vanishing ghost-charge. Therefore, the grading of the anti-fields are related via $|\phi_i| = |\phi_i^*| + 1$. Replacing $\delta_{\mathrm{B}}\phi_i$ by $\frac{\delta}{\mathrm{i}\delta\phi_i^*}$, we obtain the extended ST identities [DFP99]

$$\left(-\beta\frac{\partial}{\mathrm{i}\partial\alpha} - \eta\frac{\partial}{\mathrm{i}\partial\xi} + \int \mathrm{d}^4x\, \left(\sum_i j_i(-1)^{|\phi_i|}\frac{\delta}{\mathrm{i}\delta\phi_i^*} - 2\,\mathrm{Tr}\left[\mathbf{j}_{\bar{u}}\left(\delta_{\mathrm{B}}\bar{\mathbf{u}}\right)\right]\right)\right) Z^{\beta\eta}[j,\phi^*] = 0. \tag{7.28}$$

We replace Z by Z_c via $Z = \mathrm{e}^{\mathrm{i}Z_c}$ (2.12) which works out because there are no second (functional) derivatives and we obtain formally the same result for the generating functional of connected Green's function. Then, we perform a Legendre transform with respect to (wrt) all sources except for anti-fields

$$\Gamma = Z_c - \int \mathrm{d}^4x\, \sum_i j_i\psi_i, \quad \frac{\delta\Gamma}{\delta\psi_i} = -(-1)^{|\psi_i|}j_i, \quad \frac{\delta Z_c}{\delta j_i} = \psi_i,$$

and we arrive at what is known as the NI for the vertex function

$$\mathcal{S}(\Gamma) = \tilde{\mathrm{Tr}}\left[\int \mathrm{d}^4x\, \sum_i \frac{\delta\Gamma}{\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*} + \mathbf{b}\frac{\delta\Gamma}{\delta\bar{\mathbf{u}}}\right] + \beta\frac{\partial\Gamma}{\partial\alpha} + \eta\frac{\partial\Gamma}{\partial\xi} = 0, \tag{7.29}$$

where $\{\phi^i\} = \{\mathbf{A}_\mu, \boldsymbol{\chi}, \sigma, \mathbf{u}\}$ and

$$\tilde{\mathrm{Tr}} = \begin{cases} 2\,\mathrm{Tr} & \phi^i = \{\mathbf{A}_\mu, \mathbf{u}, \boldsymbol{\chi}\} \\ 1 & \phi^i = \sigma \end{cases}$$

distinguishes between the cases whether the fields are Lie-algebra-valued or not. Our convention for Lie-algebra-valued functional derivatives is simply $\frac{\delta}{\delta\mathbf{A}_\mu} = \mathbf{t}^a\frac{\delta}{\delta A_\mu^a}$ which explains the factor of 2 because of normalization (6.2).

7.5. Renormalizability

7.5.1. From the Classical Action to Quantum Corrections

The proof of renormalizability rests on showing that symmetries are respected at quantum level. To simplify things we make use of several relations, some of them listed now and the rest given later. Besides the NI, there is the so-called (extended) ST operator which significance will become clear when studying counter terms. The extended ST operator is a functional and is defined as

$$\mathcal{S}_F = \tilde{\mathrm{Tr}}\left[\sum\int \mathrm{d}^4x\,\frac{\delta F}{\delta\phi^*}\frac{\delta}{\delta\phi}+\frac{\delta F}{\delta\phi}\frac{\delta}{\delta\phi^*}+\mathbf{b}\frac{\delta}{\delta\bar{\mathbf{u}}}\right]+\beta\frac{\partial}{\partial\alpha}+\eta\frac{\partial}{\partial\xi}.$$

$\mathcal{S}_F$ does not coincide with the ST identity, i.e. $\mathcal{S}_\Gamma\Gamma\neq\mathcal{S}(\Gamma)$, but is related to the ST identities. The ST operator fulfills the following identities [PS95]

$$\mathcal{S}_F\mathcal{S}(F)=0,\ \ \forall F\quad\text{and}\quad \mathcal{S}_F^2=0,\ \text{iff}\ \mathcal{S}(F)=0 \tag{7.30}$$

We sketch the proof of the first identity which mainly relies on the proper treatment of grading of fields and therefore is independent of the specific functional F. The difficult part is when acting on the non-linear terms in Γ of $\mathcal{S}(\Gamma)$

$$\left(\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta}{\delta\phi_i^*}+\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta}{\delta\phi_i}\right)\frac{\delta\Gamma}{\delta\phi_j}\frac{\delta\Gamma}{\delta\phi_j^*}+\underset{x\leftrightarrow y}{i\leftrightarrow j}=\frac{\delta\Gamma}{\delta\phi_i}\left(\underset{\clubsuit}{\frac{\delta^2\Gamma}{\delta\phi_i^*\delta\phi_j}}\frac{\delta\Gamma}{\delta\phi_j^*}+\frac{\delta^2\Gamma}{\delta\phi_i^*\delta\phi_j^*}\frac{\delta\Gamma}{\delta\phi_j}\right) \tag{7.31}$$

$$+\frac{\delta\Gamma}{\delta\phi_i^*}\left(\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\frac{\delta\Gamma}{\delta\phi_j^*}+\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j^*}\frac{\delta\Gamma}{\delta\phi_j^*}\right) \tag{7.32}$$

$$+\underset{x\leftrightarrow y}{i\leftrightarrow j} \tag{7.33}$$

where the $x\leftrightarrow y$; $i\leftrightarrow j$ means that for every pair $\{(x,y),(i,j)\}$ we encounter the same object, but with exchanged coordinates and indices $\{(y,x),(j,i)\}$. This is true for $x\neq y$ and $i\neq j$ and follows from the fact that the coordinates x,y are integrated out and the indices i,j are summed. We now show that the four terms (7.31),(7.32) are canceled by the interchanged counter parts (7.33). Consider ♣

$$\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\frac{\delta\Gamma}{\delta\phi_j^*}+\underset{x\leftrightarrow y}{i\leftrightarrow j}.$$

The gradings of fields ϕ_i and anti-fields ϕ_i^* are related by $|\phi_i|=|\phi_i^*|+1$ and the vertex function has grading $|\Gamma|=0$. Applying the grading rules yields

$$\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\frac{\delta\Gamma}{\delta\phi_j^*}=(-1)^{|\phi_j^*||\phi_i|}\frac{\delta\Gamma}{\delta\phi_j^*}\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j} \tag{7.34}$$

$$\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta\Gamma}{\delta\phi_j^*}=(-1)^{|\phi_i^*||\phi_j^*|}\frac{\delta\Gamma}{\delta\phi_j^*}\frac{\delta\Gamma}{\delta\phi_i^*} \tag{7.35}$$

and we obtain

$$\begin{aligned}\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\frac{\delta\Gamma}{\delta\phi_j^*} &\overset{(7.34),(7.35)}{=} (-1)^{|\phi_j^*||\phi_i|+|\phi_i^*||\phi_j^*|}\frac{\delta\Gamma}{\delta\phi_j^*}\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\\ &=(-1)^{|\phi_j^*||\phi_i|+|\phi_i^*||\phi_j^*|+|\phi_i||\phi_j|}\frac{\delta\Gamma}{\delta\phi_j^*}\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta^2\Gamma}{\delta\phi_j\delta\phi_i}\\ &=(-1)^{|\phi_j^*||\phi_i|+|\phi_i^*||\phi_j^*|+|\phi_i^*||\phi_j|+|\phi_i||\phi_j|}\frac{\delta\Gamma}{\delta\phi_j^*}\frac{\delta^2\Gamma}{\delta\phi_j\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*}\end{aligned}$$

where in the second line we changed the order of the functional derivatives. The prefactor is simplified by using $|\phi_i|\left(\left|\phi_j^*\right|+|\phi_j|\right)=|\phi_i|$, $|\phi_i^*|\left(\left|\phi_j^*\right|+|\phi_j|\right)=|\phi_i^*|$ and $|\phi_i|+|\phi_i^*|=1$, and the final result reads

$$\frac{\delta\Gamma}{\delta\phi_i^*}\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\frac{\delta\Gamma}{\delta\phi_j^*}=-\frac{\delta\Gamma}{\delta\phi_j^*}\frac{\delta^2\Gamma}{\delta\phi_j\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*}$$

which is what had to be shown.
The linear and non-linear cross terms in $\mathcal{S}_\Gamma\mathcal{S}(\Gamma)$ are easily verified, for instance

$$\begin{aligned}\left(\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta}{\delta\phi_i^*}+\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta}{\delta\phi_i^*}\right)\beta\frac{\partial\Gamma}{\partial\alpha}+\beta\frac{\partial}{\partial\alpha}\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*} &= \frac{\partial}{\partial\alpha}\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*}\beta+\beta\frac{\partial}{\partial\alpha}\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*}\\ &=\left(1+(-1)^{|\beta|(|\phi_i|+|\phi_i^*|)}\right)\beta\frac{\partial}{\partial\alpha}\frac{\delta\Gamma}{\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*}=0\end{aligned}$$

where in the first line we used $|\alpha|=0$ and $|\phi_i|=|\phi_i^*|+1$ and the second line follows from $|\beta|=1$.
The proof for the second identity is similar but more difficult. One uses the same properties and one then extracts $\mathcal{S}(F)$ in $\mathcal{S}_F^2$ wherefrom the second identity follows. The second relation is nothing but nilpotency and can be interpreted as the analogue to δ_{B} since the classical action of $\mathcal{S}_S$ on fields equals the action of δ_{B} on those fields. $\mathcal{S}_\Gamma$ is more than just δ_{B} in the sense that it serves as a symmetry for a QFT, besides it can also act on anti-fields which is not defined for δ_{B}.

7.5.2. The Gauge Dependence of Symmetric Insertions

This sections purpose is to simplify the functional dependence of the basis of integrated symmetric insertions (see section 7.5.5). We start by taking care of the functional dependence of counter terms on BRST doublets (7.25) or equally on the gauge parameter closely following [PS85] and extending the arguments to more than one gauge parameter.

- The problem consists of solving $\mathcal{S}_\Gamma\Delta=0$ where for the moment the interpretation of Δ is irrelevant. Later we find that $\Delta = \Gamma^{n,\text{div}}$ are the counter terms.

- From the QAP follows that the canonical dimension of Δ is equal or smaller four and Δ is spanned by the space of integrated local functions [PS95] and we restrict the case to vanishing Faddeev-Popov charge $Q_{\phi\pi} = 0$.

Since β, η are Grassmann-valued, the most general expression for Δ is

$$\Delta = \beta\Delta_{-,\beta} + \eta\Delta_{-,\eta} + \Delta_0 =: \mathbf{x} \cdot \mathbf{\Delta}_- + \Delta_0,$$

where we defined

$$\mathbf{x} = \begin{pmatrix} \beta \\ \eta \end{pmatrix}, \quad \mathbf{\Delta}_- = \begin{pmatrix} \Delta_{-,\beta} \\ \Delta_{-,\eta} \end{pmatrix}.$$

Remark 7.1. Δ cannot have contributions proportional to $\beta\eta$ because of rigid invariance, ghost-charge and dimension (see table 7.1). Only external sources have negative $Q_{\phi\pi}$ except for the anti-ghost field $\bar{u}$. The only possible contribution is $\mathbf{u}^*\beta\eta$ satisfying ghost-charge and dimension where $\mathbf{u}^*$ is the anti-field of the ghost, but there is no non-trivial rigidly invariant term of this form and $\mathrm{Tr}[\mathbf{u}^*]\beta\eta = 0$. Consequently, we do not have such a contribution.

$\mathbf{\Delta}_-, \Delta_0$ are independent of β, η, have canonical dimension four and ghost-charge -1 and 0, respectively.
The problem can be highly simplified by using nilpotency of the ST operator and the fact that β, η are Grassmann-valued. Let $\bar{\mathcal{S}}_\Gamma$ denote a modified ST operator missing the parts $\beta\partial_\alpha + \eta\partial_\xi$. Applying the ST operator on Δ, we can write

$$\begin{aligned}\mathcal{S}_\Gamma\Delta &= -\beta\left(\bar{\mathcal{S}}_\Gamma\Delta_{-.\beta} - \partial_\alpha\Delta_0\right) - \eta\left(\bar{\mathcal{S}}_\Gamma\Delta_{-.\eta} - \partial_\xi\Delta_0\right) - \beta\eta\left(\partial_\xi\Delta_{-,\beta} - \partial_\alpha\Delta_{-,\eta}\right) + \bar{\mathcal{S}}_\Gamma\Delta_0 \\ &\overset{!}{=} 0\end{aligned}$$

$\bar{\mathcal{S}}_\Gamma$ does not generate contributions proportional to β, η and this way we know that the factors in front of these parameters must be zero, since β, η are arbitrary. From the terms linear in β, η we obtain

$$\left.\begin{aligned}\bar{\mathcal{S}}_\Gamma\Delta_{-,\beta} &= \partial_\alpha\Delta_0 \\ \bar{\mathcal{S}}_\Gamma\Delta_{-,\eta} &= \partial_\xi\Delta_0\end{aligned}\right\} \quad \bar{\mathcal{S}}_\Gamma\mathbf{\Delta}_- = \mathbf{\nabla}\Delta_0. \tag{7.36}$$

From the term independent of β, η we obtain $\bar{\mathcal{S}}_\Gamma\Delta_0 = 0$ and the quadratic term yields

$$\partial_\xi\Delta_{-,\beta} - \partial_\alpha\Delta_{-,\eta} = 0 \Leftrightarrow \mathbf{\nabla} \times_2 \mathbf{\Delta}_- = 0,$$

where $\times_2$ is the crossproduct in 2 dimensions. The rotation of $\mathbf{\Delta}_-$ vanishes which implies the existence of a potential

$$\boldsymbol{\nabla} \times_2 \mathbf{\Delta}_- = 0 \Rightarrow \exists\phi \text{ with } \mathbf{\Delta}_- = \boldsymbol{\nabla}\phi \tag{7.37}$$

Combining this result with (7.36) yields

$$0 = \bar{\mathcal{S}}_\Gamma \mathbf{\Delta}_- - \boldsymbol{\nabla}\Delta_0 = \boldsymbol{\nabla}\left(\bar{\mathcal{S}}_\Gamma\phi - \Delta_0\right)$$

$\bar{\mathcal{S}}_\Gamma$ is nilpotent and the general solution is

$$\Delta_0 = \bar{\mathcal{S}}_\Gamma\phi + \Lambda_0, \text{ with } \left.\begin{matrix}\boldsymbol{\nabla}\Lambda_0 = 0\\ \bar{\mathcal{S}}_\Gamma\Lambda_0 = 0\end{matrix}\right\} \Rightarrow \mathcal{S}_\Gamma\Lambda_0 = 0, \tag{7.38}$$

where we used that $\mathcal{S}_\Gamma = \bar{\mathcal{S}}_\Gamma + \mathbf{x}\cdot\boldsymbol{\nabla}$. Plugging in $\Delta = \mathbf{x}\cdot\mathbf{\Delta}_- + \Delta_0$ the results for $\mathbf{\Delta}_-$ (7.37) and Δ_0 (7.38) yields

$$\Delta = \mathcal{S}_\Gamma\phi + \Lambda_0. \tag{7.39}$$

At first sight the result does not seem to be that enlightening since such a structure is expected anyway, but now we know about the functions ϕ and Λ_0. Λ_0 is independent of α, β, ξ, η and that the terms proportional to β, η are absent in ϕ.

7.5.3. The Antighost Equation

In this section we derive the so-called antighost equation which further simplifies the functional dependence of symmetrical insertions and we go into the renormalization. Besides the quantum symmetries one has to define a quantum version of the gauge fixing, i.e. we need to define how the gauge fixing behaves beyond tree level. A common choice is to demand that the gauge fixing does not renormalize which we assume throughout this work. Given the vertex functional Γ, we demand that the gauge fixing does not take any quantum corrections. More precisely, we demand that [DFP99]

$$\frac{\delta\Gamma}{\delta\mathbf{b}} = \alpha\mathbf{b} + \mathbf{F} + \frac{\beta}{2}\bar{\mathbf{u}}$$

is valid at any stage of the perturbative expansion. To proceed, it is advisable to separate the gauge fixing from Γ and to go over to $\bar{\Gamma}$ which is defined to be independent of b

$$\Gamma =: \bar{\Gamma} + \mathrm{Tr}\left[\int \mathrm{d}^4x\, \alpha\mathbf{b}^2 + 2\mathbf{b}\mathbf{F} + \beta\bar{\mathbf{u}}\mathbf{b} + \ldots (\mathbf{b} \text{ independent})\right], \tag{7.40}$$

where the dots indicate a term specified later. In view of renormalization (7.40) implies the following relations (see (2.18) for conventions)

$$Z_\alpha = Z_b^{-1}, \quad Z_b^{\frac{1}{2}} = Z_A^{\frac{1}{2}} = Z_\chi^{\frac{1}{2}} Z_g Z_v Z_\xi, \quad Z_\beta Z_u^{\frac{1}{2}} = Z_b^{-\frac{1}{2}},$$

where we chose the convention to renormalize the anti-ghost as the ghost, i.e. $\bar{\mathbf{u}}_0 = Z_u^{\frac{1}{2}} \bar{\mathbf{u}}_R$, $\mathbf{u}_0 = Z_u^{\frac{1}{2}} \mathbf{u}_R$. The ST identities and the rigid invariance are global symmetries. From the ST identities and the gauge fixing (7.40) we derive the antighost equation (see [MSS95] for pure Yang-Mills model) which is, in contrast to the previous symmetries, a local symmetry. The antighost equation follows by taking the functional derivative of the ST identities (7.29) wrt $\mathbf{b}$. Consider the part $\mathcal{S}^{\mathbf{A},\mathbf{A}^*}(\Gamma)$ which is defined as the part of $\bar{\mathcal{S}}_{\bar{S}}$ (see equation (7.29)) involving functional derivatives wrt to $\mathbf{A}$ and $\mathbf{A}^*$

$$\begin{aligned}
\frac{\delta}{\delta \mathbf{b}(x)} \mathcal{S}^{\mathbf{A},\mathbf{A}^*}(\Gamma) &= \frac{\delta}{\delta \mathbf{b}(x)} 2\,\mathrm{Tr} \int \mathrm{d}^4 y\, \frac{\delta \Gamma}{\delta \mathbf{A}^\mu(y)} \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu(y)} = \mathbf{t}^a 2\,\mathrm{Tr} \int \mathrm{d}^4 y\, \frac{\delta^2 \Gamma}{\delta b^a(x) \delta \mathbf{A}^\mu(y)} \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu(y)} \\
&= \mathbf{t}^a 2\,\mathrm{Tr} \int \mathrm{d}^4 y\, \frac{\delta F^a(x)}{\delta \mathbf{A}^\mu(y)} \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu(y)} = -\mathbf{t}^a 2\,\mathrm{Tr} \int \mathrm{d}^4 y\, \partial_x^\nu \frac{\delta A^a_\nu(x)}{\delta \mathbf{A}^\mu(y)} \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu(y)} \\
&= -\mathbf{t}^a \partial_x^\nu 2\,\mathrm{Tr} \int \mathrm{d}^4 y\, \delta(x-y) \delta^{ab} \mathbf{t}^b \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu(y)} = -\partial_x^\nu \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu(x)}
\end{aligned}$$

where we used the explicit form of the gauge fixing (7.40). Similarly, one derives

$$\frac{\delta}{\delta \mathbf{b}(x)} \mathcal{S}^{\boldsymbol{\chi},\boldsymbol{\chi}^*}(\Gamma) = \frac{1}{2} \xi g v \frac{\delta \Gamma}{\delta \chi^*}, \qquad \frac{\delta}{\delta \mathbf{b}(x)} \mathcal{S}^{\mathbf{u},\mathbf{u}^*}(\Gamma) = 0, \qquad \frac{\delta}{\delta \mathbf{b}(x)} \mathcal{S}^{\bar{\mathbf{u}}}(\Gamma) = \frac{\delta \Gamma}{\delta \bar{u}} - \frac{\beta \mathbf{b}}{2},$$

and for the part involving partial derivatives wrt α and ξ yields

$$\frac{\delta}{\delta \mathbf{b}(x)} \mathcal{S}^{\alpha,\xi}(\Gamma) = \beta \mathbf{b} + \frac{1}{2} \eta g v \boldsymbol{\chi}.$$

Using $\mathcal{S}(\Gamma) = 0$ and combining the previous results we obtain the antighost equation for our SU(2) model

$$\mathcal{G}\Gamma := -\partial^\mu \frac{\delta \Gamma}{\delta \mathbf{A}^*_\mu} + \frac{1}{2} \xi g v \frac{\delta \Gamma}{\delta \boldsymbol{\chi}^*} + \frac{\delta \Gamma}{\delta \bar{\mathbf{u}}} = -\frac{1}{2} \left(\beta \mathbf{b} + \eta g v \boldsymbol{\chi} \right).$$

In the next step we specify the missing pieces in (7.40). Define $\bar{\Gamma}$ as

$$\Gamma - \bar{\Gamma} = \mathrm{Tr} \left[\int \mathrm{d}^4 x\, \alpha \mathbf{b}^2 + 2\mathbf{b}\mathbf{F} + \beta \bar{\mathbf{u}} \mathbf{b} + \eta g v \bar{\mathbf{u}} \boldsymbol{\chi} \right]. \tag{7.41}$$

In this way the antighost equation can be transformed to a homogeneous functional equation

$$\mathcal{G}\bar{\Gamma} = 0.$$

The solution to such a functional differential equation is obtained by similar strategies as in the case of partial differential equations. In our case we apply the same method as for linear first-order partial differential equations. Since $\mathcal{G}$ is linear we can decompose it as follows

$$\mathcal{G}_1(\bar{\Gamma}) = 0, \mathcal{G}_1 := -\partial^\mu \frac{\delta}{\delta \mathbf{A}_\mu^*} + \frac{\delta}{\delta \bar{\mathbf{u}}},$$
$$\mathcal{G}_2(\bar{\Gamma}) = 0, \mathcal{G}_2 := \frac{1}{2}\xi g v \frac{\delta}{\delta \boldsymbol{\chi}^*} + \frac{\delta}{\delta \bar{\mathbf{u}}},$$

and the general solution is given by combining both solutions obtained from $\mathcal{G}_1$, $\mathcal{G}_2$. From $\mathcal{G}_1$ we get the functional dependence $\Gamma = \Gamma(\boldsymbol{\rho}_1^{*,\mu}, \ldots)$ with $\boldsymbol{\rho}_1^{*,\mu} = \mathbf{A}^{*,\mu} - \partial^\mu \bar{\mathbf{u}}$ since $\mathcal{G}_1 \boldsymbol{\rho}_1^{*,\mu} = 0$. $\mathcal{G}_2$ restricts $\bar{\Gamma} = \bar{\Gamma}(\boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*)$ with $\boldsymbol{\rho}_2^* = \boldsymbol{\chi}^* - \frac{1}{2}\xi g v \bar{\mathbf{u}}$ and we have $\mathcal{G}\bar{\Gamma} = 0$.

7.5.4. Combining the Antighost Equation and the Slavnov-Taylor Identities

The antighost equation from the last section restricted the functional dependence of the reduced vertex function $\bar{\Gamma}$ on the fields $\mathbf{A}^*, \boldsymbol{\chi}^*$ and $\bar{\mathbf{u}}$ to $\boldsymbol{\rho}_1^*$ and $\boldsymbol{\rho}_2^*$. To take full advantage of this reduction we need to rewrite the ST identities in the new basis. With the help of the definition (7.41) we give the ST identity for $\bar{\Gamma}$ corresponding to (7.29). The gauge parameter dependence transforms as

$$\frac{\partial \Gamma}{\partial \alpha} = \frac{\partial \bar{\Gamma}}{\partial \alpha} + \operatorname{Tr} \int \mathbf{b}^2, \qquad \frac{\partial \Gamma}{\partial \xi} = \frac{\partial \bar{\Gamma}}{\partial \xi} + v g \operatorname{Tr} \int \mathbf{b}\boldsymbol{\chi},$$

and the extra contributions on the rhs are canceled against the $\Gamma^{\bar{\mathbf{u}}}$ term

$$\operatorname{Tr} \int \mathbf{b} \frac{\delta \Gamma}{\delta \bar{\mathbf{u}}} = \operatorname{Tr} \int \mathbf{b} \frac{\delta \bar{\Gamma}}{\delta \bar{\mathbf{u}}} - \operatorname{Tr} \int \left(\beta \mathbf{b}^2 + v g \eta \mathbf{b} \boldsymbol{\chi}\right).$$

A new contribution can only enter from derivatives wrt $\mathbf{A}^\mu$ and $\boldsymbol{\chi}$

$$\frac{\delta \Gamma}{\delta \mathbf{A}^\mu} = \frac{\delta \bar{\Gamma}}{\delta \mathbf{A}^\mu} + \partial_\mu \mathbf{b}, \qquad \frac{\delta \Gamma}{\delta \boldsymbol{\chi}} = \frac{\delta \bar{\Gamma}}{\delta \boldsymbol{\chi}} + \frac{1}{2}\left(g v \eta \bar{\mathbf{u}} + v \xi g \mathbf{b}\right).$$

Collecting all terms we arrive at

$$\mathcal{S}(\Gamma) = \bar{\mathcal{S}}(\bar{\Gamma}) + \tilde{\mathrm{Tr}} \int \mathbf{b} \left(-\partial^\mu \frac{\delta\bar{\Gamma}}{\delta \mathbf{A}^*_\mu} + \frac{1}{2}\xi g v \frac{\delta\bar{\Gamma}}{\delta \boldsymbol{\chi}^*} + \frac{\delta\bar{\Gamma}}{\delta \bar{\mathbf{u}}} \right) + \tilde{\mathrm{Tr}} \int \frac{1}{2} g v \eta \bar{\mathbf{u}} \frac{\delta\bar{\Gamma}}{\delta \boldsymbol{\chi}^*}, \tag{7.42}$$

with

$$\bar{\mathcal{S}}(\bar{\Gamma}) = \tilde{\mathrm{Tr}} \left[\int \mathrm{d}^4 x \sum_{\phi_i \in \{\mathbf{A}^\mu, \mathbf{u}, \boldsymbol{\chi}, \sigma\}} \frac{\delta\bar{\Gamma}}{\delta\phi_i} \frac{\delta\bar{\Gamma}}{\delta\phi_i^*} \right] + \beta \frac{\partial\bar{\Gamma}}{\partial\alpha} + \eta \frac{\partial\bar{\Gamma}}{\partial\xi}. \tag{7.43}$$

The equation in the parenthesis in (7.42) is the homogeneous antighost equation and vanishes for $\bar{\Gamma}$. In our last step we eliminate the last term on the rhs of (7.42) by going over to

$$\begin{aligned} &\frac{\delta}{\delta \mathbf{A}^{*,\mu}} \to \frac{\delta}{\delta \boldsymbol{\rho}_1^{*,\mu}}, \ \frac{\delta}{\delta \boldsymbol{\chi}^*} \to \frac{\delta}{\delta \boldsymbol{\rho}_2^*}, \\ &\bar{\Gamma}(\dots, \mathbf{A}^{*,\mu}, \boldsymbol{\chi}^*, \bar{\mathbf{u}}, \dots) \to \bar{\Gamma}(\dots, \boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*, \dots). \end{aligned}$$

$\boldsymbol{\rho}_2^*$ depends on ξ and additionally we have to transform the variation wrt ξ

$$\frac{\partial\bar{\Gamma}(\dots, \boldsymbol{\chi}^*, \bar{\mathbf{u}}, \xi, \dots)}{\partial\xi} \to \frac{\mathrm{d}\bar{\Gamma}(\dots, \boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*(\boldsymbol{\chi}, \bar{\mathbf{u}}, \xi), \xi, \dots)}{\mathrm{d}\xi} = \frac{\partial\bar{\Gamma}}{\partial\xi} + \tilde{\mathrm{Tr}} \int \frac{\mathrm{d}\boldsymbol{\rho}_2^*}{\mathrm{d}\xi} \frac{\delta\bar{\Gamma}}{\delta\boldsymbol{\rho}_2^*}.$$

Evaluating the last term we obtain

$$\tilde{\mathrm{Tr}} \int \frac{\mathrm{d}\boldsymbol{\rho}_2^*}{\mathrm{d}\xi} \frac{\delta\bar{\Gamma}}{\delta\boldsymbol{\rho}_2^*} = -\frac{1}{2} g v \tilde{\mathrm{Tr}} \int \bar{\mathbf{u}} \frac{\delta\bar{\Gamma}}{\delta\boldsymbol{\chi}^*},$$

and this result, multiplied with η from the left, cancels the previous term and we obtain

$$\bar{\mathcal{S}}(\bar{\Gamma}(\underbrace{\mathbf{A}_\mu, \boldsymbol{\chi}, \sigma, \mathbf{u}}_{\{\phi_i\}}, \underbrace{\boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*, \sigma^*, \mathbf{u}^*}_{\{\phi_i^*\}})) = 0.$$

$\bar{\mathcal{S}}(\bar{\Gamma})$ is given by (7.43), but with $\{\phi_i^*\} = \{\boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*, \sigma^*, \mathbf{u}^*\}$.

7.5.5. The Slavnov-Taylor Operator and the Basis of Symmetric Insertions

The symmetries introduced in the last sections restrict the divergences of our theory. Assume our theory has been successfully renormalized in $n-1$ loop-order and the induction begins at tree level

$$\Gamma^0 = S,$$
$$\Gamma^{\text{ren}} = \sum_{k=0}^{n-1} \Gamma^k \quad \text{is finite.} \tag{$\spadesuit$}$$

Using the claim ♠, the counter terms (divergences) at order n fulfill the ST identity $\bar{\mathcal{S}}_{\bar{S}}\Gamma^{\text{div}} = 0$ which follows from collecting all terms of order n in $\bar{S}\left(\bar{\Gamma}\right) = 0$[3]. The counter terms can also be characterized otherwise - in fact, one speaks of symmetric insertions. The QAP relates the change of a (renormalized) parameter in Γ to an insertion $\lambda\partial_\lambda\Gamma = \Delta_\lambda \cdot \Gamma$, where Δ has canonical dimension four and zero ghost-charge. Let ϵ denote generically a variation of all parameters in our theory, then the QAP yields $\Gamma_\epsilon = \Gamma + \epsilon\Delta \cdot \Gamma$. A change in parameters does not spoil our symmetries, i.e. we still have that $\bar{\mathcal{S}}_{\bar{S}}\Gamma_\epsilon = 0$ and consequently we derive $\bar{\mathcal{S}}_{\bar{S}}\Delta \cdot \Gamma = 0$ and we identify as possible candidates $\Delta \cdot \Gamma \stackrel{?}{=} \Gamma^{\text{div}}$ for invariant counter terms [PS85].
We follow the same strategy as in reference [PS85] in order to obtain the integrated basis of local symmetric insertions, but applied to our SU(2) SSB model (See also appendix of [MSS95] for a similar strategy).
Here is a good place for recapitulating what has been done in the last sections and to explain what we investigate now. From our analysis leading to (7.39) we learned that neither β nor η appear explicitly, but only in connection with the ST operator. In section 7.5.3 we studied the consequences of having a non-renormalized gauge fixing and BRST invariance, and the result has been expressed as a functional differential equation restricting the functional dependence of the vertex function. In the previous section 7.5.4 we combined all results and we found that for symmetric insertions one only has to consider the reduced ST operator $\bar{\mathcal{S}}$ and the reduced anti-field dependence $\{\phi_i^*\} = \{\boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*, \sigma^*, \mathbf{u}^*\}$. The linearized reduced ST operator therefore reads

$$\bar{\mathcal{S}}_{\bar{S}} = \tilde{\text{Tr}}\left[\int \mathrm{d}^4x \sum_{\substack{i \\ \phi_i \in \{\mathbf{A}^\mu, \boldsymbol{\chi}, \sigma, \mathbf{u}\} \\ \phi_i^* \in \{\boldsymbol{\rho}_1^{*,\mu}, \boldsymbol{\rho}_2^*, \sigma^*, \mathbf{u}^*\}}} \frac{\delta\bar{S}}{\delta\phi_i}\frac{\delta}{\delta\phi_i^*} + \frac{\delta\bar{S}}{\delta\phi_i^*}\frac{\delta}{\delta\phi_i}\right] + \beta\frac{\partial}{\partial\alpha} + \eta\frac{\partial}{\partial\xi}. \tag{7.44}$$

$\bar{S}$ is the classical solution to $\bar{\Gamma}$, i.e. it equals the usual action S, but is missing the gauge fixing and the extended BRST induced couplings (7.41), thus

$$\bar{S} = S_{\text{YM}} + S_{\text{mat}} + S_{\text{ghost}} + \sum_{\substack{\phi_i \in \{\mathbf{A}^\mu, \boldsymbol{\chi}, \sigma, \mathbf{u}\} \\ \phi_i^* \in \{\mathbf{A}^{*,\mu}, \boldsymbol{\chi}^*, \sigma^*, \mathbf{u}^*\}}} \int \mathrm{d}^4x \; \phi_i^* \delta_{\text{B}}\phi_i$$

[3]The ST identities are non-linear, thus, expanding up to a given loop-order, the result can be written as the "linearized" ST operator $\bar{\mathcal{S}}_{\bar{S}}$.

	$\mathbf{A}^\mu$	$\boldsymbol{\chi}$	σ, v	$\bar{\mathbf{u}}$	$\mathbf{u}$	$\mathbf{b}$	$\boldsymbol{\rho}_1^{*,\mu}$	$\boldsymbol{\rho}_2^*$	σ^*	$\mathbf{u}^*$	α, ξ	β, η	g
Dimension D	1	1	1	2	0	2	3	3	3	4	0	0	0
Ghostcharge $Q_{\phi\pi}$	0	0	0	-1	1	0	-1	-1	-1	-2	0	1	0

Table 7.1.: The table shows the field content, the corresponding canonical dimensions D and the ghost-charge $Q_{\phi\pi}$. The individual values for the dimension are obtained by power counting. There is a convention (see [MSS95]) concerning the ghosts because ghosts always appear as a pair.

and the individual actions are the spacetime integrated versions of (7.15), (7.17), (7.18).

In section 7.2.1 we discussed rigid invariance and we mentioned that it is easier not to require SO(4) invariance (see equation (7.5)) from the beginning which led us to a necessary condition for SO(4) invariance. Including the full field content, the rigid invariance operator (not including SO(4) invariance, but (7.6)) reads

$$\mathbf{W} = \mathrm{i} \int \mathrm{d}^4 x \left(\left[\mathbf{A}_\mu, \frac{\delta}{\delta \mathbf{A}_\mu} \right] + \left[\boldsymbol{\rho}_1^{*,\mu}, \frac{\delta}{\delta \boldsymbol{\rho}_1^{*,\mu}} \right]_+ + \left[\boldsymbol{\chi}, \frac{\delta}{\delta \boldsymbol{\chi}} \right] + \left[\boldsymbol{\rho}_2^*, \frac{\delta}{\delta \boldsymbol{\rho}_2^*} \right]_+ \right.$$
$$\left. + \left[\mathbf{u}, \frac{\delta}{\delta \mathbf{u}} \right]_+ + \left[\mathbf{u}^*, \frac{\delta}{\delta \mathbf{u}^*} \right] \right),$$

and structures annihilated by $\mathbf{W}$ are constructed in the way of (7.7).
We are now going to write down any integrated local Lorentz scalar Δ respecting all symmetries , i.e.

$$\begin{aligned} \bar{\mathcal{S}}_{\bar{S}} \Delta &= 0 && \text{BRST invariance and gauge fixing-condition} \\ \mathbf{W}\Delta &= 0 && \text{rigid invariance} \\ Q_{\phi\pi}\Delta &= 0 && \text{vanishing Faddeev-Popov charge} \\ D(\Delta) &\le 4 && \text{Canonical dimension equal or less 4} \end{aligned}$$

and the idea is the following. We eliminate step by step field dependences in such an order that the result does not interfere with further dependences. The order is crucial as it will become clear. As a summary over fields, their canonical dimension and ghost-charge, consider Table 7.1.

From now on any field φ or coupling g has to be understood as a renormalized field, coupling in $(n-1)$-loop order, i.e. $\varphi_{\text{ren}}^{n-1}, g_{\text{ren}}^{n-1}$, respectively. Recall the general solution (7.39) $\Delta = \mathcal{S}_S \phi + \Lambda_0$ and we start with ϕ. We need to list all possible integrated local composite field operators respecting the symmetries. ϕ has ghost-charge -1 because $\mathcal{S}_S\phi$ has $Q_{\phi\pi}=0$. Starting with the anti-field u^*, the only way to increase $Q_{\phi\pi}$ is by ghost fields u. Since u^* has $Q_{\phi\pi}=-2$ (Table 7.1), we append one ghost fields u

$$\phi = f_{u^*}\,\mathrm{Tr}\int \mathbf{u}^*\mathbf{u} \; + \; \mathbf{u}^* \text{ independent.}$$

For all the other anti-fields we proceed the same way. The left possibilities are combinations with $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \sigma^*$:

$$\phi = f_{u^*}\,\mathrm{Tr}\int \mathbf{u}^*\mathbf{u} + f_{\rho_1^{*,\mu}}\,\mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\mu}\mathbf{A}_\mu + f_{\rho_2^*}\,\mathrm{Tr}\int \boldsymbol{\rho}_2^*\boldsymbol{\chi} + f_{\sigma_\sigma^*}\int \sigma^*\sigma + f_{\sigma_v^*}\int \sigma^*,$$

where $\boldsymbol{\rho}_1^{*,\mu}$ can only be connected to $\mathbf{A}_\mu$ because of Lorentz invariance and $\boldsymbol{\rho}_2^*$ is fixed by rigid invariance[4]. The last contribution with σ^* shall be understood in the way that once the canonical dimension is four and once three. According to our discussion the constants may depend on α, ξ.

Λ_0 does not have to be a ST variation, but, clearly, we can take over the result for ϕ and declare the proportionality constants to be independent of α, ξ and this contribution turns out to be valid for Λ_0. We now prove the statement that anti-fields in Λ_0 do only appear as a ST variation in the same way as in ϕ, but with gauge parameter independent constants.

The strategy is to eliminate step by step the dependence of anti-fields starting with a specific one which does not interfere with what is following. The order we choose is highly important as it will become clear.

We start with the dependence of the ghost anti-field $\mathbf{u}^*$. Since $\mathbf{u}^*$ has dimension four (Table 7.1), the only way to get vanishing $Q_{\phi\pi}$ is to append two ghost fields $\mathbf{u}$, i.e.

$$\Lambda_0 \propto \mathrm{Tr}\int \mathbf{u}^*\mathbf{u}\mathbf{u} \; + \mathbf{u}^* \text{ indep.}$$

This contribution is rewritten as $\mathrm{Tr}\int \mathbf{u}^*\{\mathbf{u},\mathbf{u}\}$ and appearing constants can be absorbed. The super bracket is then replaced by the BRST variation giving

$$\Lambda_0 \propto -\,\mathrm{Tr}\int \mathbf{u}^*\delta_{\mathrm{B}}\mathbf{u} \; + \mathbf{u}^* \text{ indep.} = \bar{\mathcal{S}}_{\bar{S}}\,\mathrm{Tr}\int \mathbf{u}^*\mathbf{u} \; + \mathbf{u}^* \text{ indep..}$$

[4] σ, v are not Lie-algebra-valued which would be necessary to get a non-vanishing contraction with $\boldsymbol{\rho}_2^*$.

The last step deserves some comments and intermediate steps. Applying the ST operator yields

$$\bar{\mathcal{S}}_{\bar{S}}\,\mathrm{Tr}\int \mathbf{u}^*\mathbf{u} = 2\,\mathrm{Tr}\int\left(\frac{\delta\bar{S}}{\delta\mathbf{u}}\frac{\delta}{\delta\mathbf{u}^*}+\frac{\delta\bar{S}}{\delta\mathbf{u}^*}\frac{\delta}{\delta\mathbf{u}}\right)\mathrm{Tr}\int\mathbf{u}^*\mathbf{u} = \mathrm{Tr}\int\frac{\delta\bar{S}}{\delta\mathbf{u}}\mathbf{u}+\mathrm{Tr}\int\frac{\delta\bar{S}}{\delta\mathbf{u}^*}\mathbf{u}^*. \tag{7.45}$$

Since we are only interested in the $\mathbf{u}^*$ dependence, we throw away anything which does not involve $\mathbf{u}^*$. The $\mathbf{u}^*$ dependence can only enter via $\frac{\delta\bar{S}}{\delta\mathbf{u}}$, more precisely from the part of the Lagrangian where we couple the anti-fields to the BRST variation of the fields

$$\begin{aligned}\frac{\delta\bar{S}}{\delta\mathbf{u}} &= \mathbf{t}^a\frac{\delta}{\delta u^a}2\,\mathrm{Tr}\int\mathbf{u}^*\frac{\mathrm{i}}{2}\{\mathbf{u},\mathbf{u}\}\ +\mathbf{u}^*\text{ indep.}\\ &= \mathrm{i}^2\mathbf{t}^a u^{*,c}\epsilon_{cad}u^d\ +\mathbf{u}^*\text{ indep.}\end{aligned}$$

The first contribution on the rhs of (7.45) yields

$$\begin{aligned}\mathrm{Tr}\int\frac{\delta\bar{S}}{\delta\mathbf{u}}\mathbf{u} = -\int\frac{1}{2}u^{*,c}\epsilon_{cad}u^du^a &= -2\,\mathrm{Tr}\int\mathbf{u}^*\frac{\mathrm{i}}{2}\{\mathbf{u},\mathbf{u}\}+\ \mathbf{u}^*\text{ indep.}\\ &= -2\,\mathrm{Tr}\int\mathbf{u}^*\delta_{\mathrm{B}}\mathbf{u}+\ \mathbf{u}^*\text{ indep.}\end{aligned}$$

The second contribution simplifies to

$$\mathrm{Tr}\int\frac{\delta\bar{S}}{\delta\mathbf{u}^*}\mathbf{u}^* = \mathrm{Tr}\int\left(\delta_{\mathrm{B}}\mathbf{u}\right)\mathbf{u}^*,$$

and in total we obtain $\bar{\mathcal{S}}_{\bar{S}}\,\mathrm{Tr}\int\mathbf{u}^*\mathbf{u} = -\,\mathrm{Tr}\int\mathbf{u}^*\delta_{\mathrm{B}}\mathbf{u}+$ $\mathbf{u}^*$ independent. At this point it is important that $\bar{\mathcal{S}}_{\bar{S}}\left[\mathbf{u}^*\text{ independent}\right]$ stays $\mathbf{u}^*$ independent, which is the case because $\mathbf{u}^*$ can only be induced by a derivative with respect to $\mathbf{u}$ which in turn follows from the specific coupling of the anti-field $\mathbf{u}^*$ to $\mathbf{u}$ ($\mathbf{u}^*\{\mathbf{u},\mathbf{u}\}\subset\mathcal{L}$). The derivative with respect to $\mathbf{u}$ is coupled to the derivative with respect to $\mathbf{u}^*$ via $\mathrm{Tr}\int\frac{\delta\bar{S}}{\delta\mathbf{u}}\frac{\delta}{\delta\mathbf{u}^*}\subset\bar{\mathcal{S}}_{\bar{S}}$, therefore, $\mathbf{u}^*$ is only induced by itself.

The overall constant must be gauge-parameter independent (see section 7.5.2) and therefore the most general contribution to Λ_0 in (7.39) involving $\mathbf{u}^*$ is a ST variation[5].

We go on with $\boldsymbol{\rho}_1^{*,\mu}$. Because of dimension, $Q_{\phi\pi}$, Lorentz and rigid invariance the most general ansatz is

$$\Lambda_0 = \bar{\mathcal{S}}_{\bar{S}}\left[\tilde{f}_{\mathbf{u}^*}\,\mathrm{Tr}\int\mathbf{u}^*\mathbf{u}\right]+\tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}}\,\mathrm{Tr}\int\boldsymbol{\rho}_1^{*,\mu}\mathbf{F}_\mu(\mathbf{A},\mathbf{u})+\qquad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu}\text{ indep.},$$

[5]If the parameter were gauge dependent we could not pull $\mathcal{S}_S$ in front.

where $\mathbf{F}_\mu$ is a linear combination of $\mathbf{A}^\mu\mathbf{u}$ and $\partial^\mu\mathbf{u}$. We must solve the equation $\bar{\mathcal{S}}_{\bar{S}}\,\mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\mu}\mathbf{F}_\mu(\mathbf{A},\mathbf{u}) = 0$ for $\mathbf{F}$ (notation from [PS85])

$$\bar{\mathcal{S}}_{\bar{S}}\,\mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\mu}\mathbf{F}_\mu(\mathbf{A},\mathbf{u}) = \mathrm{Tr}\int \left[\frac{\delta\bar{S}}{\delta\mathbf{A}_\mu}\mathbf{F}_\mu - \boldsymbol{\rho}_1^{*,\mu}\mathcal{S}_{\bar{S}}\mathbf{F}_\mu\right] \tag{7.46}$$

We collect all contributions involving $\boldsymbol{\rho}_1^{*,\mu}$ and we must be careful about not inducing anti-fields from elsewhere, i.e. we need $\bar{\mathcal{S}}_{\bar{S}}\left[\boldsymbol{\rho}_1^{*,\mu}\text{ indep.}\right] = \boldsymbol{\rho}_1^{*,\mu}$ independent. We do neither catch contributions involving $\mathbf{u}^*$ since the $\mathbf{u}^*$ contribution is a ST variation and $\bar{\mathcal{S}}^2_{\bar{S}} = 0$, nor do we catch contributions from $\mathbf{u}^*, \boldsymbol{\rho}_1^{*,\mu}$ independent terms[6].
We need to extract the $\boldsymbol{\rho}_1^{*,\mu}$ dependent part of $\frac{\delta\bar{S}}{\delta\mathbf{A}_\mu}$

$$\begin{aligned}\frac{\delta}{\delta\mathbf{A}_\mu}2\,\mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\nu}\delta_{\mathrm{B}}\mathbf{A}_\nu &= \frac{\delta}{\delta\mathbf{A}_\mu}2\,\mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\nu}\mathrm{D}_\nu\mathbf{u} = \frac{\delta}{\delta\mathbf{A}_\mu}2\,\mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\nu}\mathrm{i}\left[\mathbf{u},\mathbf{A}_\nu\right]\\ &= \mathrm{i}\frac{\delta}{\delta\mathbf{A}_\mu}2\,\mathrm{Tr}\int \left\{\boldsymbol{\rho}_1^{*,\nu},\mathbf{u}\right\}\mathbf{A}_\nu = \mathrm{i}\left\{\boldsymbol{\rho}_1^{*,\mu},\mathbf{u}\right\}.\end{aligned}$$

Inserting into (7.46) yields the $\boldsymbol{\rho}_1^{*,\mu}$ dependent part

$$\begin{aligned}\mathrm{Tr}\int \left[\frac{\delta\bar{S}}{\delta\mathbf{A}_\mu}\mathbf{F}_\mu - \boldsymbol{\rho}_1^{*,\mu}\bar{\mathcal{S}}_{\bar{S}}\mathbf{F}_\mu\right] &= \mathrm{Tr}\int \left[\mathrm{i}\left\{\boldsymbol{\rho}_1^{*,\mu},\mathbf{u}\right\}\mathbf{F}_\mu - \boldsymbol{\rho}_1^{*,\mu}\bar{\mathcal{S}}_{\bar{S}}\mathbf{F}_\mu\right] + \quad \boldsymbol{\rho}_1^{*,\mu}\text{ indep.}\\ &= \mathrm{Tr}\int \boldsymbol{\rho}_1^{*,\mu}\left[\mathrm{i}\left\{\mathbf{u},\mathbf{F}_\mu\right\} - \bar{\mathcal{S}}_{\bar{S}}\mathbf{F}_\mu\right] + \quad \boldsymbol{\rho}_1^{*,\mu}\text{ indep.},\end{aligned}$$

and the problem reduces to solving the equation

$$\hat{\mathbf{Q}}\mathbf{F}_\mu = 0, \text{ with } \hat{\mathbf{Q}} = \mathrm{i}\left[\mathbf{u},\circ\right]_\pm - \bar{\mathcal{S}}_{\bar{S}},$$

where the sign $\pm$ denotes the super bracket. This equation is easily solved with one additional information, namely that $\hat{\mathbf{Q}}$ is nilpotent. This can be checked by direct computation, but it follows from the nilpotency of the ST operator. The symmetries restrict the general solution to

$$\mathbf{F}_\mu = \partial^\mu\mathbf{u} + \mathrm{const}_1 \times \mathbf{A}^\mu\mathbf{u} + \mathrm{const}_2 \times \mathbf{u}\mathbf{A}^\mu.$$

[PS85] claim that there is only one solution and that due to nilpotency the solution is given by $\mathbf{F}_\mu = \hat{\mathbf{Q}}\mathbf{A}_\mu$. An overall constant is irrelevant for this discussion, and one is left with two independent (relative) constants, but only one equation and at first glance there could be more than one solution. We take over this result and we give our proof

[6] $\bar{\mathcal{S}}_{\bar{S}}$ cannot induce terms proportional to $\boldsymbol{\rho}_1^{*,\mu}$ when acting on something independent of $\mathbf{u}^*, \boldsymbol{\rho}_1^{*,\mu}$, because $\boldsymbol{\rho}_1^{*,\mu}$ appears in the Lagrangian only as $\boldsymbol{\rho}_1^{*,\mu}\delta_{\mathrm{B}}\mathbf{A}_\mu$. Therefore, to induce $\boldsymbol{\rho}_1^{*,\mu}$ a derivation with respect to $\mathbf{A}_\mu, \mathbf{u}$ is necessary, but such terms do not show up since they are connected to derivatives with respect to $\boldsymbol{\rho}_1^{*,\mu}$, $\mathbf{u}^*$, respectively, and we act on something independent of that.

below in remark (7.2). Consequently, the solution is again a ST variation

$$\tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}} \operatorname{Tr} \int \boldsymbol{\rho}_1^{*,\mu} \mathbf{F}_\mu(\mathbf{A}, \mathbf{u}) = \tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}} \operatorname{Tr} \int \boldsymbol{\rho}_1^{*,\mu} \hat{\mathbf{Q}} \mathbf{A}_\mu = \bar{\mathcal{S}}_{\bar{S}} \left[\tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}} \operatorname{Tr} \int \boldsymbol{\rho}_1^{*,\mu} \mathbf{A}_\mu \right] + \quad \boldsymbol{\rho}_1^{*,\mu} \text{ indep.}$$

In the last step we pulled $\hat{\mathbf{Q}}$ out of the trace and integral which is basically the same step as in (7.46), but in the opposite direction.

Remark 7.2. We give the proof that $\hat{\mathbf{Q}}\mathbf{F} \overset{!}{=}$ implies that the solution $\mathbf{F}$ lies in the Lie algebra. To that end, consider the image of $\hat{\mathbf{Q}}$ which has the following form

$$\operatorname{Im}\left[\hat{\mathbf{Q}}\right] = \alpha' \mathbb{1} + \beta'_a \mathbf{t}^a. \tag{7.47}$$

Proof. (7.47) is true because $\mathbb{1}, \mathbf{t}^a$ span a basis of 2×2 matrices. On the other hand one observes that $\hat{\mathbf{Q}}$ preserves this structure, i.e.

$$\hat{\mathbf{Q}} \left(\alpha \mathbf{1} + \beta_a \mathbf{t}^a \right) = \alpha' \mathbf{1} + \beta'_a \mathbf{t}^a,$$

Let $\mathfrak{g}$ be the vector space spanned by $\mathbf{t}$, i.e. $\mathfrak{g}$ forms the $\mathfrak{su}(2)$ Lie algebra and let $\mathfrak{g}'$ be a vector space spanned by $\mathbf{t}$ and the one-element $\mathbb{1}$. From the definition of $\hat{\mathbf{Q}}$ it follows that both vector spaces form an invariant subspace under the action of $\hat{\mathbf{Q}}$[7], i.e.

$$\begin{aligned} \hat{\mathbf{Q}} &: \mathfrak{g} \to \mathfrak{g}, \\ \hat{\mathbf{Q}} &: \mathfrak{g}' \to \mathfrak{g}'. \end{aligned} \tag{7.48}$$

From $\operatorname{Im}\left[\hat{\mathbf{Q}}\right] = 0$ we conclude that the coefficient in front of $\mathbb{1}$, i.e. α' must vanish independently of β'. This means iff α' is non-vanishing then α must be zero. We apply the statement to $\mathbf{F}_\mu$. Rewriting $\mathbf{F}_\mu$ we obtain wlog

$$\mathbf{F}_\mu = \partial^\mu \mathbf{u} + \text{const}_1 \times [\mathbf{u}, \mathbf{A}_\mu] + \text{const}_2 \times [\mathbf{u}, \mathbf{A}_\mu]_+$$

Clearly $[\mathbf{u}, \mathbf{A}_\mu]$ is not in the Lie algebra, but lives in $\mathfrak{g}'$. The relevant part in $\mathfrak{g}'$ is given by $\operatorname{Tr}[\mathbf{u}\mathbf{A}_\mu]\,\mathbb{1}$[8]. It is easily verified that $\alpha' = \operatorname{Tr}\left[\hat{\mathbf{Q}} \operatorname{Tr}[\mathbf{u}\mathbf{A}_\mu]\,\mathbb{1}\right] \neq 0$:

$$\begin{aligned} &\{\mathbf{u}, \operatorname{Tr}[\mathbf{u}\mathbf{A}_\mu]\,\mathbb{1}\} = 0, \\ &\Rightarrow \hat{\mathbf{Q}} \operatorname{Tr}[\mathbf{u}\mathbf{A}_\mu]\,\mathbb{1} = \operatorname{Tr}\left[(\delta_{\mathrm{B}}\mathbf{u})\,\mathbf{A}_\mu + \mathbf{u}\,(\delta_{\mathrm{B}}\mathbf{A}_\mu)\right]\mathbb{1} \end{aligned}$$

[7] The argument is that the Lie bracket closes and the same holds for the superbracket $[\circ;\circ]_\pm$ which takes into account Grassmann valued fields. Further, the ST variation $\mathcal{S}_{\bar{S}}$ acting on fields is proportional to the BRST variation and the BRST variation on Lie-algebra-valued fields lies again in the Lie algebra.

[8] For $\mathfrak{su}(2)$ we have $\{\mathbf{u}, \mathbf{A}_\mu\} = 2\operatorname{Tr}[\mathbf{u}\mathbf{A}_\mu]\mathbb{1}$. In the general case one would have to project out $\mathfrak{g}$, for instance, by taking the trace.

and finally the non-vanishing of

$$\begin{aligned}\operatorname{Tr}\left[(\delta_{\mathrm{B}}\mathbf{u})\,\mathbf{A}_\mu+\mathbf{u}\,(\delta_{\mathrm{B}}\mathbf{A}_\mu)\right]&=\operatorname{Tr}\left[\frac{\mathrm{i}}{2}\left\{\mathbf{u},\mathbf{u}\right\}\mathbf{A}_\mu\right]+\operatorname{Tr}\left[\mathbf{u}\mathbf{D}_\mu\mathbf{u}\right]\\&=\operatorname{Tr}\left[\frac{3\mathrm{i}}{2}\left\{\mathbf{u},\mathbf{u}\right\}\mathbf{A}_\mu\right]+\operatorname{Tr}\left[\mathbf{u}\partial_\mu\mathbf{u}\right]\neq 0,\end{aligned}$$

implies $\alpha=0$. Consequently, $\mathbf{F}_\mu=\partial^\mu\mathbf{u}+\text{const}\times[\mathbf{u},\mathbf{A}_\mu]$ is the most general solution proving there is only one solution given by $\mathbf{F}_\mu=\hat{\mathbf{Q}}\mathbf{A}_\mu$. □

We go on with the SSB sector starting with σ^*. Symmetries restrict the dependency to

$$\begin{aligned}\Lambda_0=\bar{\mathcal{S}}_{\bar{S}}&\left[\tilde{f}_{\mathbf{u}^*}\operatorname{Tr}\int\mathbf{u}^*\mathbf{u}+\tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}}\operatorname{Tr}\int\boldsymbol{\rho}_1^{*,\mu}\mathbf{A}_\mu\right]-\tilde{f}_{\sigma^*}\operatorname{Tr}\int\sigma^*\mathbf{u}\boldsymbol{\chi}\\&+\quad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu},\sigma^*\text{ indep.},\end{aligned}$$

where the minus sign is just convenience. Making use of $\delta_{\mathrm{B}}\sigma=-\operatorname{Tr}[\mathbf{u}\boldsymbol{\chi}]$ allows us to rewrite the last term as

$$-\tilde{f}_{\sigma^*}\operatorname{Tr}\int\sigma^*\mathbf{u}\boldsymbol{\chi}=\tilde{f}_{\sigma^*}\operatorname{Tr}\int\sigma^*\delta_{\mathrm{B}}\sigma=\bar{\mathcal{S}}_{\bar{S}}\left[\tilde{f}_{\sigma^*}\operatorname{Tr}\int\sigma^*\sigma\right]+\quad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu},\sigma^*\text{ indep.}$$

In the last step we used that $\frac{\delta\bar{S}}{\delta\sigma}$ is indepedent of σ^* and moreover that $\frac{\delta\bar{S}}{\delta\sigma}$ can only induce $\boldsymbol{\rho}_2^*$ and neither $\mathbf{u}^*$ nor $\boldsymbol{\rho}_1^{*,\mu}$. Yet, this result is not complete because we have missed $\text{const}\times\mathcal{S}_{\bar{S}}\int\sigma^*$ which must be taken into account with an independent proportionality constant and in analogy to ϕ we call those constants $\tilde{f}_{\sigma^*_\sigma},\tilde{f}_{\sigma^*_v}$.
The left anti-field dependence is $\boldsymbol{\rho}_2^*$ and the most general ansatz in this case reads

$$\begin{aligned}\Lambda_0=&\bar{\mathcal{S}}_{\bar{S}}\left[\tilde{f}_{\mathbf{u}^*}\operatorname{Tr}\int\mathbf{u}^*\mathbf{u}+\tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}}\operatorname{Tr}\int\boldsymbol{\rho}_1^{*,\mu}\mathbf{A}_\mu+\tilde{f}_{\sigma^*_\sigma}\operatorname{Tr}\int\sigma^*\sigma+\tilde{f}_{\sigma^*_v}\operatorname{Tr}\int\sigma^*\sigma\right]\\&+\tilde{f}_{\boldsymbol{\rho}_2^*}\operatorname{Tr}\int\boldsymbol{\rho}_2^*\mathbf{F}(\boldsymbol{\chi},\sigma,v,\mathbf{u})+\quad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu},\sigma^*,\boldsymbol{\rho}_2^*\text{ indep.}\end{aligned}$$

The last term must vanish after ST variation and we conclude

$$\begin{aligned}\bar{\mathcal{S}}_{\bar{S}}\Lambda_0&=\bar{\mathcal{S}}_{\bar{S}}\tilde{f}_{\boldsymbol{\rho}_2^*}\operatorname{Tr}\int\boldsymbol{\rho}_2^*\mathbf{F}(\boldsymbol{\chi},\sigma,v,\mathbf{u})+\quad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu},\boldsymbol{\rho}_2^*\text{ indep.}\\&=\operatorname{Tr}\int\boldsymbol{\rho}_2^*\hat{\mathbf{Q}}\mathbf{F}(\boldsymbol{\chi},\sigma,v,\mathbf{u})+\quad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu},\boldsymbol{\rho}_2^*\text{ indep.}\\&\overset{!}{=}0+\quad\mathbf{u}^*,\boldsymbol{\rho}_1^{*,\mu},\boldsymbol{\rho}_2^*\text{ indep.}\end{aligned}$$

The operator $\hat{\mathbf{Q}}$ is obtained by collecting all terms proportional to $\boldsymbol{\rho}_2^*$ and we need to evaluate the $\boldsymbol{\rho}_2^*$ dependent part of

$$\frac{\delta \bar{S}}{\delta \boldsymbol{\chi}} = \frac{\delta}{\delta \boldsymbol{\chi}} 2\,\mathrm{Tr} \int \boldsymbol{\rho}_2^* \delta_{\mathrm{B}} \boldsymbol{\chi} + \quad \boldsymbol{\rho}_2^* \text{ indep.} = \frac{\delta}{\delta \boldsymbol{\chi}} \mathrm{i}\,\mathrm{Tr} \int \boldsymbol{\rho}_2^* \left[\mathbf{u}, \boldsymbol{\chi}\right] + \quad \boldsymbol{\rho}_2^* \text{ indep.}$$

Therefore $\hat{\mathbf{Q}} = \frac{\mathrm{i}}{2}\left[\mathbf{u}, \circ\right]_{\pm} - \bar{\mathcal{S}}_{\bar{S}}$. Looking for the kernel, one solution is given by $\mathbf{F} = \hat{\mathbf{Q}}\boldsymbol{\chi}$ due to nilpotency of $\hat{\mathbf{Q}}$, but again there may be others and the situation is a bit more complicated. The most general structure is given by

$$\mathbf{F} = \left[\boldsymbol{\chi}, \mathbf{u}\right] + \text{const} \times \left\{\boldsymbol{\chi}, \mathbf{u}\right\} + \text{const} \times \sigma \mathbf{u} + \text{ const} \times \mathbf{u}.$$

Applying remark 7.2 reduces the problem to $\mathbf{F} = \left[\boldsymbol{\chi}, \mathbf{u}\right] + \text{const} \times \sigma \mathbf{u} + \text{ const} \times \mathbf{u}$, but this time we are left with two constants. We must now argue why again there is a unique solution and this time we give the answer by direct computation. Applying $\hat{\mathbf{Q}}$ on $\mathbf{F}$ we obtain

$$\begin{aligned}
\hat{\mathbf{Q}}\mathbf{u} &= \mathrm{i}\left\{\mathbf{u}, \mathbf{u}\right\} \\
\hat{\mathbf{Q}}\sigma \mathbf{u} &= -\left(\delta_{\mathrm{B}}\sigma\right)\mathbf{u} + \sigma\left(\delta_{\mathrm{B}}\mathbf{u}\right) = -\,\mathrm{Tr}\left[\boldsymbol{\chi}\mathbf{u}\right]\mathbf{u} + \sigma \frac{\mathrm{i}}{2}\left\{\mathbf{u}, \mathbf{u}\right\} \\
\hat{\mathbf{Q}}\left[\boldsymbol{\chi}, \mathbf{u}\right] &= \frac{\mathrm{i}}{2}\left\{\mathbf{u}, \left[\boldsymbol{\chi}, \mathbf{u}\right]\right\} - \left\{\delta_{\mathrm{B}}\boldsymbol{\chi}, \mathbf{u}\right\} + \left[\boldsymbol{\chi}, \delta_{\mathrm{B}}\mathbf{u}\right]
\end{aligned}$$

Some further manipulations reveal

$$\begin{aligned}
\frac{\mathrm{i}}{2}\left\{\mathbf{u}, \left[\boldsymbol{\chi}, \mathbf{u}\right]\right\} &= \frac{\mathrm{i}^3}{2}\mathbf{u} \times \left(\mathbf{u} \times \boldsymbol{\chi}\right) = \mathrm{i}\,\mathrm{Tr}\left[\mathbf{u}\boldsymbol{\chi}\right]\mathbf{u}, \\
\left[\boldsymbol{\chi}, \delta_{\mathrm{B}}\mathbf{u}\right] &= \frac{\mathrm{i}}{2}\left[\boldsymbol{\chi}, \left\{\mathbf{u}, \mathbf{u}\right\}\right] = \frac{\mathrm{i}^3}{2}\boldsymbol{\chi} \times \left(\mathbf{u} \times \mathbf{u}\right) = 2\mathrm{i}\,\mathrm{Tr}\left[\mathbf{u}\boldsymbol{\chi}\right]\mathbf{u},
\end{aligned}$$

and we can read off the basis elements of the image of $\hat{\mathbf{Q}}$:

$$\begin{aligned}
\left\langle \hat{\mathbf{Q}}\mathbf{u} \right\rangle &= \left\{\mathbf{u}, \mathbf{u}\right\} \\
\left\langle \hat{\mathbf{Q}}\sigma\mathbf{u} \right\rangle &= \sigma\left\{\mathbf{u}, \mathbf{u}\right\}, \mathrm{Tr}[\mathbf{u}\boldsymbol{\chi}]\mathbf{u} \\
\left\langle \hat{\mathbf{Q}}\left[\boldsymbol{\chi}, \mathbf{u}\right] \right\rangle &= \left\{\mathbf{u}, \mathbf{u}\right\}, \sigma\left\{\mathbf{u}, \mathbf{u}\right\}, \mathrm{Tr}[\mathbf{u}\boldsymbol{\chi}]\mathbf{u}
\end{aligned}$$

One parameter is needed to cancel the term $\left\{\mathbf{u}, \mathbf{u}\right\}$ in the first and third line and another one is needed to cancel terms beween the second and the third line. Therefore, there only one unique solution which is given by $\mathbf{F} = \hat{\mathbf{Q}}\boldsymbol{\chi} \neq 0$. Because the result for $\mathbf{F}$ is a $\hat{\mathbf{Q}}$ variation the result for Λ_0 is a ST variation and we have

$$\Lambda_0 =$$

$$\bar{\mathcal{S}}_{\bar{S}}\left[\tilde{f}_{\mathbf{u}^*}\operatorname{Tr}\int \mathbf{u}^* u + \tilde{f}_{\boldsymbol{\rho}_1^{*,\mu}}\operatorname{Tr}\int \boldsymbol{\rho}_1^{*,\mu}\mathbf{A}_\mu + \tilde{f}_{\sigma_\sigma^*}\operatorname{Tr}\int \sigma^*\sigma + \tilde{f}_{\sigma_v^*}\operatorname{Tr}\int \sigma^* + \tilde{f}_{\boldsymbol{\rho}_2^*}\operatorname{Tr}\int \boldsymbol{\rho}_2^*\boldsymbol{\chi}\right]$$
$$+ \quad \mathbf{u}^*, \boldsymbol{\rho}_1^{*,\mu}, \sigma^*, \boldsymbol{\rho}_2^* \text{ independent.}$$

We have eliminated any dependence on anti-fields, and up to now all the contributions in ϕ and Λ_0 are $\bar{\mathcal{S}}_{\bar{S}}$ exact. This result can be merged to an ST exact term where the proportionality constants have a gauge dependent (f_{ϕ^*}) and a gauge independent ($\tilde{f}_{\phi^*}$) part.

We argue that there are no further exact terms. Assume there is another contribution $\bar{\mathcal{S}}_{\bar{S}}\rho$. Because of ghost-charge $Q_{\phi\pi}$, ρ must be composed of fields and anti-fields[9], but there cannot be more exact anti-field contributions and the only possible contributions allowed by ghost-charge, Lorentz invariance and rigid invariance are listed in ϕ, Λ_0.

Having eliminated the anti-field dependence, the missing contributions to Λ_0 are gauge invariant terms. Since we have eliminated any dependence on the gauge-fixing and ghost part, only the Yang-Mills part $\mathcal{L}_{\mathrm{YM}}$ and the matter part $\mathcal{L}_{\mathrm{mat}}$ are possible and we have found the most general structure of Δ

$$\Delta = \bar{\mathcal{S}}_{\bar{S}}\left[g_{\mathbf{u}^*}\operatorname{Tr}\int \mathbf{u}^*\mathbf{u} + g_{\boldsymbol{\rho}_1^{*,\mu}}\operatorname{Tr}\int \boldsymbol{\rho}_1^{*,\mu}\mathbf{A}_\mu + g_{\sigma_\sigma^*}\int \sigma^*\sigma + g_{\sigma_v^*}\int \sigma^* + g_{\boldsymbol{\rho}_2^*}\operatorname{Tr}\int \boldsymbol{\rho}_2^*\boldsymbol{\chi}\right]$$
$$+ g_{\mathrm{YM}}\mathcal{S}_{\mathrm{YM}} + g_{\mathrm{mat}}\mathcal{S}_{\mathrm{mat}},$$

where the constants g may depend on the gauge. To proof renormalizability we must show that a suited redefinition of fields and couplings cancels Δ. In reference [SV94] they prove the renormalizability of our SU(2) Higgs model, but imposig the usual BRST invariance instead of extendend BRST invariance. Their result for Δ is formally equal to ours and only the ST operator is replaced by the usual one. Let ρ denote the exact contributions to Δ, then the difference between both theories is given by $\delta\Delta = \beta\frac{\partial\rho}{\partial\alpha} + \chi\frac{\partial\rho}{\partial\xi}$[10]. A closer look at these terms reveals that they have the structure (7.21), i.e. they are ghost-charge violating and therefore do not contribute to physical amplitudes. Consequently, they must not be renormalized and the theory is renormalizable.

We skip the proof of renormalizability for the usual BRST invariance which is lengthy and which is worked out in full detail in [SV94]. Instead, we sketch steps of the proof, but in the case of pure Yang-Mills theory. Setting all kind of matter dependence to zero, the divergences Δ take the form

$$\bar{\mathcal{S}}_{\bar{S}}\left[g_{\mathbf{u}^*}\operatorname{Tr}\int \mathbf{u}^*\mathbf{u} + g_{\boldsymbol{\rho}^{*,\mu}}\operatorname{Tr}\int \boldsymbol{\rho}^{*,\mu}\mathbf{A}_\mu\right] + g_{\mathrm{YM}}\mathcal{S}_{\mathrm{YM}} \tag{7.49}$$

[9] A contribution with $\bar{\mathbf{u}}$ is possible, but this case is already gone in $\boldsymbol{\rho}_1^*, \boldsymbol{\rho}_2^*$ because of the antighost equation.

[10] The theories differ in their ST operator and the difference $\bar{\mathcal{S}}_{\bar{S}}^{\text{extended BRST}} - \bar{\mathcal{S}}_{\bar{S}}^{\text{usual BRST}}$ is given by $\beta\frac{\partial}{\partial\alpha} + \chi\frac{\partial}{\partial\xi}$.

Evaluating the ST operator one obtains for $\mathbf{u}^*$

$$\begin{aligned}\bar{\mathcal{S}}_{\bar{S}}\operatorname{Tr}\int \mathbf{u}^*\mathbf{u} &= \operatorname{Tr}\int \frac{\delta\bar{S}}{\delta\mathbf{u}}\mathbf{u} + \operatorname{Tr}\int \mathbf{u}^*\frac{\delta\bar{S}}{\delta\mathbf{u}^*}\\ &= \operatorname{Tr}\int \boldsymbol{\rho}^*_\mu D^\mu\mathbf{u} - \operatorname{Tr}\int \mathbf{u}^*\delta_{\mathrm{B}}\mathbf{u}\end{aligned}$$

and for ρ^*_μ

$$\begin{aligned}\bar{\mathcal{S}}_{\bar{S}}\operatorname{Tr}\int \boldsymbol{\rho}^*_\mu\mathbf{A}^\mu &= \operatorname{Tr}\int \frac{\delta\bar{S}}{\delta\mathbf{A}_\mu}\mathbf{A}_\mu - \operatorname{Tr}\int \boldsymbol{\rho}^*_\mu\frac{\delta\bar{S}}{\delta\boldsymbol{\rho}^*_\mu}\\ &= \operatorname{Tr}\int \frac{\delta S_{\mathrm{YM}}}{\delta\mathbf{A}_\mu}\mathbf{A}_\mu + \mathrm{i}\operatorname{Tr}\int \boldsymbol{\rho}^*_\mu\left[\mathbf{u},\mathbf{A}_\mu\right] - \operatorname{Tr}\int \boldsymbol{\rho}^*_\mu\delta_{\mathrm{B}}\mathbf{A}^\mu\\ &= \operatorname{Tr}\int \frac{\delta S_{\mathrm{YM}}}{\delta\mathbf{A}_\mu}\mathbf{A}_\mu - \operatorname{Tr}\int \boldsymbol{\rho}^*_\mu\partial^\mu\mathbf{u}\end{aligned}$$

The gauge-fixing is now simpliy $\mathcal{L}_{\text{fix}} = \frac{2}{\xi}\operatorname{Tr}\left(\partial^\mu\mathbf{A}^\mu\right)^2$ and since the gauge-fixing does not renormalize it follows that Z_ξ renormlizes as Z_A, i.e. $Z_\xi = Z_A$. From the antighost equation we deduce $Z_u = Z_{A^*} = Z_{\rho^*}$ and from the ST identities we conclude $Z_{A^*}^{\frac{1}{2}}Z_A^{\frac{1}{2}} = Z_{u^*}^{\frac{1}{2}}Z_u^{\frac{1}{2}}$. In total we have three independent renormalization constants Z_A, Z_u and Z_g where g is the gauge coupling. Having renormalized the theory in $n-1$ loop, we redefine the fields in n-loop as follows

$$\mathbf{A}_\mu^{n-1,\text{ren}} = \sqrt{\frac{1}{Z_A^{n-1}}}\mathbf{A}_\mu = \sqrt{\frac{Z_A^n}{Z_A^{n-1}}}\mathbf{A}_\mu^{n,\text{ren}} = \left(1+\frac{\delta Z_A^n}{2}\right)\mathbf{A}_\mu^{n,\text{ren}},$$

where we used the formal expansion in $Z-1$, i.e $Z^n = 1 + Z^n - 1 = 1 + \delta Z^1 + \delta Z^2 + \cdots + \delta Z^n$.

$$\sqrt{\frac{Z_A^n}{Z_A^{n-1}}} = \sqrt{1+\frac{\delta Z_A^n}{Z_A^{n-1}}} = 1 + \frac{\delta Z_A^n}{2} + \mathcal{O}(\hbar^{n+1})$$

Let us start with the Yang-Mills Field.

$$\operatorname{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}\right] = \operatorname{Tr}\left[2\left(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu\right)\left(\partial^\mu\mathbf{A}^\nu\right) - 4g\left[\mathbf{A}_\mu,\mathbf{A}_\nu\right]\partial^\mu\mathbf{A}^\nu + g^2\left[\mathbf{A}_\mu,\mathbf{A}_\nu\right]\left[\mathbf{A}^\mu,\mathbf{A}^\nu\right]\right]$$

We define the renormalization of the gauge coupling according to (2.19), then we have that

$$\begin{aligned}\operatorname{Tr}\left[\mathbf{F}^{n-1}_{\mu\nu}\mathbf{F}^{n-1}_{\mu\nu}\right] = \operatorname{Tr}\Big[&\mathbf{F}^n_{\mu\nu}\mathbf{F}^n_{\mu\nu} + \delta Z_A^n 2\left(\partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu\right)\left(\partial^\mu\mathbf{A}^\nu\right)\\ &- 4\delta Z_g g\left[\mathbf{A}_\mu,\mathbf{A}_\nu\right]\partial^\mu\mathbf{A}^\nu + \left(2\delta Z_g - \delta Z_A\right)g^2\left[\mathbf{A}_\mu,\mathbf{A}_\nu\right]\left[\mathbf{A}^\mu,\mathbf{A}^\nu\right]\Big]\end{aligned}$$

and we obtain the following system of linear equations

$$g_{\mathrm{YM}} + \delta Z_A^n + 2g_{\boldsymbol{\rho}^*} \quad = 0 \tag{7.50}$$
$$g_{\mathrm{YM}} + \delta Z_g^n + 3g_{\boldsymbol{\rho}^*} \quad = 0 \tag{7.51}$$
$$g_{\mathrm{YM}} + 2\delta Z_g^n - \delta Z_A^n + 4g_{\boldsymbol{\rho}^*} \quad = 0 \tag{7.52}$$

The divergences (7.49) do not respect manifestly the covariant derivative which is the result of different field powers

$$D_\mu^{n-1} = \partial_\mu - \mathrm{i}g^{n-1}A_\mu^{n-1} = D_\mu^n - (\delta Z_g^n - \delta Z_A^n)\mathrm{i}g^n A_\mu^n$$

taking this into account and rescaling $\mathcal{L}_{\mathrm{ghost}} + \mathcal{L}_{\mathrm{fix}}$, we obtain

$$\begin{aligned} \boldsymbol{\rho}_\mu^{*,n-1}\delta_{\mathrm{B}}A^{n-1,\mu} =& \boldsymbol{\rho}_\mu^{*,n}\delta_{\mathrm{B}}A^{n,\mu} + \delta Z_g \rho_\mu^{*,n}\partial^\mu \mathbf{u}^n - (\delta Z_g^n - \delta Z_A^n + \delta Z_u)\mathrm{i}g\boldsymbol{\rho}_\mu^{*,n}\left[\mathbf{A}^{n,\mu}, \mathbf{u}^n\right] \\ \mathbf{u}^{*,n-1}\delta_{\mathrm{B}}\mathbf{u}^{n-1} =& \sqrt{Z_{u^*}}Z_u\frac{Z_g}{\sqrt{Z_A}^3}\mathbf{u}^{*,n}\delta_{\mathrm{B}}\mathbf{u}^n = (1 + \delta Z_g^n - \delta Z_A^n + \delta Z_u^n)\mathbf{u}^{*,n}\delta_{\mathrm{B}}\mathbf{u}^n \end{aligned}$$

where we made use of the antighost equation, i.e. we did not consider the ghost Lagrangian $\mathcal{L}_{\mathrm{ghost}}$ separately. By comparing the powers in $\boldsymbol{\rho}^*\partial\mathbf{u}$, $\boldsymbol{\rho}^*\mathbf{A}\mathbf{u}$ and $\mathbf{u}^*\mathbf{u}^2$ we obtain the following system of equations

$$\delta Z_u^n \quad = \quad g_{u^*} - g_{\rho^*} \tag{7.53}$$
$$\delta Z_g^n - \delta Z_A^n + \delta Z_u^n \quad = \quad g_{u^*} \tag{7.54}$$

The system is overdetermined, but actually solvable, for instance, combining the equations (7.50), (7.51) is consistent with the combination of (7.53), (7.54) and the counter terms are uniquely defined by (7.50), (7.51), (7.53).

7.6. Applications of the Extended BRST Symmetry

7.6.1. Gauge Independence of Physical Poles and the Connection to the Complex Mass Scheme

The CMS requires the renormalization condition (2.23) such that expressions stay gauge invariant and well-defined beyond tree level. This is motivated by the fact that the complex pole is gauge invariant, thus serving as an observable. In this section we give a quick calculation of this property for the SU(2) Higgs model and with the help of the

NI following references [GG00], [BLS95]. Our starting point are the NI (7.29)

$$\mathcal{S}(\Gamma) = \tilde{\mathrm{Tr}}\left[\int \mathrm{d}^4x \sum_{\phi_i=\mathbf{A},\boldsymbol{\chi},\sigma,\mathbf{u}} \frac{\delta\Gamma}{\delta\phi_i}\frac{\delta\Gamma}{\delta\phi_i^*} + \mathbf{b}\frac{\delta\Gamma}{\delta\bar{\mathbf{u}}}\right] + \beta\frac{\partial\Gamma}{\partial\alpha} + \eta\frac{\partial\Gamma}{\partial\xi} = 0.$$

Taking the partial derivative wrt one of the extended BRST partners, say β, and setting all ghost fields and $\mathbf{b}$ to zero, we obtain

$$\frac{\partial\Gamma}{\partial\alpha} + \tilde{\mathrm{Tr}}\left[\int \mathrm{d}^4x \sum_{\phi_i=\mathbf{A},\boldsymbol{\chi},\sigma} \frac{\delta\Gamma}{\delta\phi_i}\frac{\partial}{\partial\beta}\frac{\delta\Gamma}{\delta\phi_i^*}\right],$$

where any ghost-charge-violating contribution has been dismissed because there will be no more functional derivatives wrt fields carrying non-vanishing ghost-charge. We are interested in the gauge dependence of the two-point function of gauge bosons, thus we simply take the functional derivatives with respect to gauge bosons $\mathbf{A}$ twice and set all fields to zero (which is not indicated explicitly)

$$\begin{aligned}&-\partial_\alpha\Gamma_{\mathbf{A}^\mu\mathbf{A}^\nu} = \\ &\tilde{\mathrm{Tr}}\left[\int \mathrm{d}^4x \sum_{\phi_i=\mathbf{A},\boldsymbol{\chi},\sigma} \Gamma_{\phi_i\mathbf{A}^\mu\mathbf{A}^\nu}\Gamma_{\beta\phi_i^*} + \Gamma_{\phi_i}\Gamma_{\beta\phi_i^*\mathbf{A}^\mu\mathbf{A}^\nu} + \Gamma_{\phi_i\mathbf{A}^\mu}\Gamma_{\beta\phi_i^*\mathbf{A}^\nu} + \Gamma_{\phi_i\mathbf{A}^\nu}\Gamma_{\beta\phi_i^*\mathbf{A}^\mu}\right].\end{aligned}$$

The physical part of the propagator is its transverse part. Defining $t^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$ we find that

$$-t^{\mu\nu}\partial_\alpha\Gamma_{\mathbf{A}^\mu\mathbf{A}^\nu} =: -\partial_\alpha\Gamma^{\mathrm{T}}_{\mathbf{A}\mathbf{A}} = 2\tilde{\mathrm{Tr}}\left[\int \mathrm{d}^4x\, \Gamma^{\mathrm{T}}_{\mathbf{A}\mathbf{A}}\Gamma^{\mathrm{T}}_{\beta\mathbf{A}^*\mathbf{A}}\right]. \tag{7.55}$$

We list arguments and assumptions leading to that result. One imposes that tadpoles are absent, thus $\Gamma_\phi = 0$. From Lorentz invariance follows that for a scalar ϕ, $\Gamma_{\phi\mathbf{A}_\mu}$ is proportional to p^μ and projected with t^μ the contributions vanishes. The last missing piece is that $\Gamma_{\beta\phi^*}$ should vanish. This is the case as discussed in [GG00], but there is one important point we must mention. $\Gamma_{\beta\phi^*}$ is divergent and we have not renormalized it. In the proof of renormalization we did only consider physical amplitudes, thus we did not take care of divergences in unphysical amplitudes, but gauge dependence is of unphysical nature and the formula above are, a priori, undefined unless appropriately regularized. Therefore, we assume that all divergences have been regularized and that the NI have been restored, for instance, by algebraic means. The location of the pole for vector bosons is defined by

$$\Gamma^{\mathrm{T}}_{\mathbf{A}\mathbf{A}}(\mu^2) = 0.$$

Evaluating the Fourier transform of (7.55) at the pole μ^2 yields what we wanted to demonstrate, namely the gauge independence of the pole

$$\partial_\alpha \Gamma^{\mathrm{T}}_{\mathbf{AA}}(p^2)\big|_{p^2=\mu^2} = 0 \;\Rightarrow\; \partial_\alpha \mu^2 = 0. \tag{7.56}$$

7.6.2. Gauge Independence of the S matrix

Renormalization renders the n-point Green's functions finite, and n-fold residues are well-defined, consequently, the S matrix is well-defined via the LSZ reduction. The question remains whether unitarity and gauge independence survive renormalization. There is a proof of gauge independence of the S matrix in the framework of SSB [CT74]. They have explicitly calculated the variation of Green's functions with respect to gauge-parameters. Further, they have shown that if the S matrix is properly defined via the LSZ reduction formula the gauge dependence drops out. We now show how to give the proof, but with the help of the extended BRST symmetry. The proof has already been carried out in [Kum01] and we apply their techniques, but more detailed. We extend the proof to the case of SSB, i.e. for more than one gauge-parameter, which is a trivial extension.

We start with the extended ST identities (7.28) and from now on the generating functional is understood as the one from the extended theory, i.e. $Z = Z^{\beta\eta}$. Setting all non-physical sources to zero we have

$$\beta\frac{\partial Z}{\partial \alpha} + \eta\frac{\partial Z}{\partial \xi} = \sum_\phi (-1)^{|\phi|} j_\phi \frac{\delta Z}{\delta \phi^*}.$$

If we set all sources to zero, we conclude that $Z[0]$ is gauge independent

$$\frac{\partial Z}{\partial \alpha}\bigg|_{j_\phi=0} = \frac{\partial Z}{\partial \xi}\bigg|_{j_\phi=0} = 0.$$

Then we expand Z in β and η where to zeroth order we retrieve the usual ST identities. In the order $\mathcal{O}(\beta)$ we have

$$\begin{aligned}
\beta\left(\frac{\partial Z}{\partial \alpha}\bigg|_{\beta,\eta=0}\right) &= \sum_\phi (-1)^{|\phi|} j_\phi \frac{\delta}{\delta \phi^*}\beta\left(\frac{\partial Z}{\partial \beta}\bigg|_{\beta,\eta=0}\right) = \sum_\phi \beta(-1)^{|\phi|+1} j_\phi \frac{\delta}{\delta \phi^*}\left(\frac{\partial Z}{\partial \beta}\bigg|_{\beta,\eta=0}\right),\\
\Rightarrow \frac{\partial Z}{\partial \alpha}\bigg|_{\beta,\eta=0} &= \sum_\phi (-1)^{|\phi|+1} j_\phi \frac{\delta}{\delta \phi^*}\,\frac{\partial Z}{\partial \beta}\bigg|_{\beta,\eta=0}.
\end{aligned}$$

The same calculation for the order $\mathcal{O}(\eta)$ yields

$$\left.\frac{\partial Z}{\partial \xi}\right|_{\beta,\eta=0} = \sum_{\phi}(-1)^{|\phi|+1} j_\phi \frac{\delta}{\delta\phi^*}\left.\frac{\partial Z}{\partial \eta}\right|_{\beta,\eta=0}. \tag{7.57}$$

In section 7.4.1 we argued that $\frac{\partial Z}{\partial \eta}$ vanishes for the case of physical amplitudes, but this time $\frac{\partial Z}{\partial \eta}$ is followed by a functional derivative with respect to an anti-field. The combination of derivatives wrt β, η and ϕ^* respects ghost-charge conservation (see $Q_{\phi\pi}$ in table 7.1) and $\frac{\delta}{\delta\phi^*}\left.\frac{\partial Z}{\partial \eta}\right|_{\beta,\eta=0}$ does actually contribute.
With definition (2.7) we can to study the gauge dependence of Green's functions

$$\frac{\partial G_{\omega_1,\dots,\omega_n}}{\partial\{\alpha;\xi\}} = \frac{1}{Z[0]}\prod_{i=1}^{n}\frac{\delta}{\mathrm{i}\delta j_{\omega_i}(x_i)}\left.\frac{\partial Z[j]}{\partial\{\alpha;\xi\}}\right|_{j=0},$$

where we used the gauge independence of $Z[0]$. Defining new generating functionals $\frac{\partial Z[j]}{\partial\{\beta;\eta\}} = Z^{\{\beta;\eta\}}$, the gauge dependence takes the compact expression

$$\frac{\partial G_{\omega_1,\dots,\omega_n}}{\partial\{\alpha;\xi\}} = \sum_{k=1}^{n}(-1)^{1+\sum_{m=k}^{n}|\phi_m|} G^{\{\beta;\eta\}}_{\omega_1,\dots,\omega_k^*,\dots,\omega_n}. \tag{$\spadesuit$}$$

where one of the external currents j_{ω_k} is replaced by an external anti-field. The anti-field ϕ^* itself is determined by the would-be current j_{ω_k}, i.e. $\phi^* = \phi^*_{\omega_k}$ which we simply indicate as ω_k^*.

Proof. Assume that $\spadesuit$ holds up to n external currents labeled by $j_{\omega_2}\dots j_{\omega_{n+1}}$. Because j is put to zero, one of the derivatives with respect to j_{ω_i} must hit j_ϕ in (7.57) and

$$\frac{\partial G_{\omega_1,\dots,\omega_{n+1}}}{\partial\{\alpha;\xi\}} = \frac{(-\mathrm{i})^{n+1}}{Z[0]}\frac{\delta^{n+1}}{\delta j_{\omega_1}\dots j_{\omega_{n+1}}}\sum_{\phi}(-1)^{1+|\phi|} j_\phi \left.\frac{Z^{\{\beta;\eta\}}}{\delta\phi^*}\right|_{j=0}$$

can be separated in the cases where $\frac{\delta}{\delta j_{\omega_1}}$ hits j_ϕ and where it does not. The latter allows to make use of the induction hypothesis

$$\begin{aligned}\frac{(-\mathrm{i})^{n+1}}{Z[0]}\frac{\delta}{\delta j_{\omega_1}}\sum_{k=2}^{n}(-1)^{1+\sum_{m=k}^{n+1}|\phi_m|}&\left.\frac{Z^{\beta\chi}}{\delta j_{\omega_2}\dots\delta\phi^*_{\omega_k}\dots\delta j_{\omega_{n+1}}}\right|_{j=0}\\ &= \sum_{k=2}^{n}(-1)^{1+\sum_{m=k}^{n}|\phi_m|} G^{\{\beta;\eta\}}_{\omega_1,\dots,\omega_k^*,\dots,\omega_n}.\end{aligned}$$

The former case must be evaluated directly, but poses no more problem. Each time a functional derivative $\frac{\delta}{\delta j_{\omega_k}}$ passes by j_ϕ a factor $(-1)^{|\phi_{\omega_k}||\phi|}$ arises, where the ϕ turns into ϕ_{ω_1}. Then one has to bring $\frac{\delta}{\delta\phi^*_{\omega_1}}$ to the left and when passing by $\frac{\delta}{\delta j_{\omega_k}}$ a factor $(-1)^{|\phi_{\omega_k}||\phi^*_{\omega_1}|}$ arises. In total the prefactor reads $(-1)^{1+\sum_{k=1}^{n}|\phi_{\omega_k}|}$, where we used that $|\phi|+|\phi^*|=1$ and combining both results yields ♠ for up to $n+1$ external currents. □

Example 7.1. To get some intuition we give an example on the formula ♠. Consider the gauge boson-propagator in the R_ξ gauges (set $\alpha=\xi$). The tree level two-point function is given by

$$\frac{-\mathrm{i}}{p^2-M^2}\left[g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi M^2}\right]$$

Applying formula ♠ to $G_{A^\mu A^\nu}$ reads $\frac{\partial G_{A_\mu A_\nu}}{\partial\alpha}=-G^{\{\beta;\eta\}}_{A_\mu,A^*_\nu}-G^{\{\beta;\eta\}}_{A^*_\mu A_\nu}$ and due to the symmetry in μ,ν we are left with $\frac{\partial G_{A_\mu A_\nu}}{\partial\alpha}=-2G^{\{\beta;\eta\}}_{A_\mu,A^*_\nu}$. We give a small calculation on how to evaluate $G^{\{\beta;\eta\}}_{A_\mu,A^*_\nu}$, but without working out the details. To this end, consider the Green's function obtained from the generating functional $Z^{\beta\chi}$

$$G^{\{\beta;\eta\}}_{A^a_\mu,A^{*,b}_\nu}=\frac{1}{Z[0]}\int D[\phi]\;A^a_\mu\,(D_\nu u)^b\left(\mathrm{i}\int\mathrm{Tr}\,[\bar{\mathbf{u}}\mathbf{b}]\right)\mathrm{e}^{\mathrm{i}S[\phi]}\bigg|_{\text{connected}}.$$

At tree level only $\partial_\nu u^b=(D_\nu u)^b+\mathcal{O}(g)$ contributes. Setting all anti-fields to zero and replacing fields by functional derivatives yields the following structure

$$\frac{\delta}{\delta A^a_\mu}\partial_\nu\frac{\delta}{\delta j_{u^b}}\left(\int Tr\left[\frac{\delta}{\delta\mathbf{j}_{\bar{u}}}\frac{\delta}{\delta\mathbf{j}_b}\right]\right)\frac{1}{Z[0]}\int D[\phi]\;\mathrm{e}^{\mathrm{i}S[\phi]},$$

where we already indicated by contraction which propagators will appear. Once the interaction is turned off the lowest contribution is given by $G_{A_\mu b}\partial_\nu G_{\bar{u}u}$. We are working with auxiliary fields and we do have a non-vanishing mixing field Feynman propagator $G_{A_\mu b}$. On the other hand, when b is integrated out, there is a real interaction coupling between A and $\bar{u}$ originating from the extended BRST symmetry. The propagator is calculated by standard techniques, for instance, one can start from the generating functional of the free theory and integrate out the fields.

$$\frac{\delta^2 Z}{\delta j_{A^a_\mu}\delta j_{b^b}}\bigg|_{j=0}\propto\frac{\delta^2}{\delta j_{A^a_\mu}\delta j_{b^b}}\int D[\phi]\;\mathrm{e}^{\mathrm{i}\int b^a\partial^\alpha A^a_\mu+\frac{\mathrm{i}}{2}\int A^a_\alpha G^{-1}_{A^\alpha A^\beta}A^b_\beta+\mathrm{i}\int\frac{\alpha}{2}b^{a\,2}+\mathrm{i}\int j_{A^a_\mu}A^a_\mu+j_{b^a}b^a}\bigg|_{j=0}$$

This would work out easily if there was not the gauge-fixing $\mathbf{b}\partial\mathbf{A}$, but it can be treated as interaction, i.e. the fields are replaced by derivatives and the derivatives are pulled out of the path-integral. Afterwards, the Gaussian integrals can be carried out and one arrives at

$$\left.\frac{\delta^2 Z}{\delta j_{A^a_\mu}\delta j_{b^b}}\right|_{j=0} \propto \frac{\delta^2}{\delta j_{A^a_\mu}\delta j_{b^b}} \mathrm{e}^{\mathrm{i}\int \frac{\delta}{\mathrm{i}\delta j_b}\partial^\alpha \frac{\delta}{\mathrm{i}\delta j_{A^\alpha}}} \left.\mathrm{e}^{\frac{\mathrm{i}}{2}\int j_\alpha G_{A^\alpha A^\beta} j_\beta + \frac{\mathrm{i}}{2\xi}\int j_b^2}\right|_{j=0}.$$

Evaluating the functional derivatives and making use of the symmetry in both Lorentz indices and space time inversion yields

$$\begin{aligned}\left.\frac{\delta^2 Z}{\delta j_{A^a_\mu}\delta j_{b^b}}\right|_{j=0} &= \frac{\delta^2}{\delta j_{A^a_\mu}\delta j_{b^b}} \mathrm{e}^{\mathrm{i}\int \frac{1}{\xi} j_b \partial^\alpha \int G_{A^\alpha A^\beta} j_\beta} \left.\mathrm{e}^{\frac{\mathrm{i}}{2}\int j_\alpha G_{A^\alpha A^\beta} j_\beta + \frac{\mathrm{i}}{\xi}\int j_b^2}\right|_{j=0} \\ &\propto \frac{1}{\xi}\partial^\nu G_{A^\nu A^\mu},\end{aligned}$$

and we arrive at what we wanted to show, namely

$$\begin{aligned} G_{A^\mu b}\, p^\nu G_{\bar{u}u} \propto & \frac{1}{\xi} p_\alpha G_{A^\alpha A^\mu}\, p^\nu \frac{1}{p^2 - \xi M^2} \\ &= \frac{1}{\xi}\frac{p^\mu p^\nu}{(p^2 - M^2)(p^2 - \xi M^2)} - \frac{1-\xi}{\xi}\frac{p^2 p^\mu p^\nu}{(p^2 - M^2)(p^2 - \xi M^2)^2} \\ &= \frac{p^\mu p^\nu}{(p^2 - M^2)(p^2 - \xi M^2)}\left(1 - \frac{M^2(1-\xi)}{p^2 - \xi M^2}\right) \propto \frac{\partial}{\partial \xi} G_{A^\mu A^\nu}\end{aligned}$$

The purpose of the example is twofold. It demonstrates explicitly that there are Feynman rules which describe the change in the gauge. Secondly, we observe that the change in the gauge of a two-point function is given by the two-point function itself connected via a ghost-charge-violating coupling to the ghost two-point function. Both properties are going to be important to understand qualitatively what is going to happen when truncating gauge variations of n-point functions. We now investigate the behavior of the gauge dependence near the on-shell phase-space, but at this point we stop following reference [Kum01] because they give an argument concerning a "convenient" choice of polarization spin vectors which we think is not a freedom[11], but turns out to be necessary as we shall demonstrate (7.65). Instead, we recapitulate and work out the ideas of [CT74]. In what follows we assume there are no degenerate masses, i.e. we choose $\xi \neq 1$ which will simplify the discussion. The case $\xi = 1$ does work out, but is more difficult. Further, we drop the explicit indication of any kind of grading.

The LSZ reduction relates the S matrix to the residua of an n-point function and the n-point function is said to be truncated which, roughly speaking, means that the external legs (whole 2-point functions) are removed. Therefore, we are especially interested

[11] At the end of their calculation they choose the basis such that the would-be gauge dependence of the S matrix drops out and claim that this is a "suited" choice one can make. This in turn implies that another choice would actually violate gauge independence of the S matrix.

in the residue of

$$\frac{\partial G_{\omega_1\omega_2}}{\partial\{\alpha;\xi\}} = G^{\{\beta;\eta\}}_{\omega_1^*\omega_2} + G^{\{\beta;\eta\}}_{\omega_1\omega_2^*}, \tag{7.58}$$

and we factorize out of $G^{\{\beta;\eta\}}_{\omega_1^*\omega_2}$ the desired 2-point function. Define

$$G^{\{\beta;\eta\}}_{\omega_1^*\omega_2} := \sum_{\omega_k} Y_{\omega_1\omega_k} G_{\omega_k,\omega_2}, \tag{7.59}$$

where the sum runs over all possible intermediate field configuration. For the case of no auxiliary fields (assume they have been integrated out), on-shell 2-point functions are diagonal in the sense that there is no field mixing in the on-shell 2-point function and one obtains

$$\lim_{p^2\to m^2_{\omega_1}} \left(p^2 - m^2_{\omega_1}\right) G_{\omega_1,\omega_2} = \delta_{\omega_1,\omega_2} \lim_{p^2\to m^2_{\omega_1}} \left(p^2 - m^2_{\omega_1}\right) G_{\omega_1,\omega_1}.$$

As a consequence, the sum in (7.59) will no longer run over different fields, but solely over polarization states. To further simplify the notation we drop the explicit field label and we define $G_{ij} := G_{\omega^i\omega^j}$. The indices i, j are wild cards for polarization indices, for instance, in the case of gauge bosons these indices are Lorentz indices or in the case of fermions they are given by spinor indices.
We put (7.58) on-shell and plug in the definition (7.59), which brings us to

$$\begin{aligned}
\lim_{p^2\to m^2_{\omega_1}} \left(p^2 - m^2_{\omega_1}\right) \frac{\partial G_{\omega_1^i,\omega_1^j}}{\partial\{\alpha,\xi\}} &= \sum_k y_{ik}\mathrm{Res}_{m^2_{\omega_1}}(G_{kj}) + y_{jk}\mathrm{Res}_{m^2_{\omega_1}}(G_{ki}) \\
&= \sum_k y_{ik}\mathrm{Res}_{m^2_{\omega_1}}(G_{kj}) + \mathrm{Res}_{m^2_{\omega_1}}(G_{ik})\tilde{y}_{kj},
\end{aligned}$$

where we used $G^T = G$ (zero grading) and we defined $y = Y|_{p^2=m^2}$ and $\tilde{y}^T = y$. From the Källén-Lehmann representation (2.10) follows that any two-point function near the resonance has the following structure

$$G_{ij} = \left(R^{\frac{1}{2}} \frac{1}{p^2 - m^2} R^{\frac{1}{2}}\right)_{ij} = \sum_k R^{\frac{1}{2}}_{ik} \frac{1}{p^2 - m^2} R^{\frac{1}{2}}_{kj}, \quad \text{for } p^2 \approx m^2, \tag{7.60}$$

where the separation in $R^{1/2}$ is convention. The i, j are again the components of polarization states and in general the residue $R^{\frac{1}{2}}$ is not diagonal. The Green's function G itself does not depend on the basis, but depends only on the polarization sum[12] and

[12]This follows from the fact that only the polarization sum enters the Feynman rules.

with a suited choice of polarization vectors $R^{\frac{1}{2}}$ is quasi diagonal in the sense

$$G_{ij} = \sum_r e_i^{(r)} \frac{R_r}{p^2 - m^2} e_j^{(r)} =: \sum_r e_i^{(r)} G_r e_j^{(r)}.$$

In general, a gauge transformation will not only transform R_r and leave the basis $e^{(r)}$ as it is, but $e^{(r)}$ must be transformed as well to guarantee independence of the choice of the basis. As we shall see, this implies that the polarization vectors are gauge dependent which is obvious from the point of view of canonical QFT.
Let $z = (\alpha, \xi)$ denote the gauge dependence and consider a small variation of $z \to z + \delta z$, then

$$G_{ij}^z \to G_{ij}^{z+\delta z} = \sum_r \left(e_i^{(r)} + \delta e_i^{(r)} \right) \frac{R_r + \delta R_r}{p^2 - m^2} \left(e_j^{(r)} + \delta e_j^{(r)} \right) \tag{7.61}$$

is the most general transformation rule respecting the gauge independence of the physical mass $\delta_z m^2 = 0$. The shifted eigenvectors fulfill the polarization sum P_{ij} and are orthogonal

$$\begin{aligned} P_{ij} &= \sum_r \left(e_i^{(r)} + \delta e_i^{(r)} \right) \left(e_j^{(r)} + \delta e_j^{(r)} \right), \\ \delta_{rs} &= \left(e^{(r)} + \delta e^{(r)}, e^{(s)} + \delta e^{(s)} \right). \end{aligned}$$

From the gauge dependence of two-point functions (7.58) we derive

$$\begin{aligned} G_{ij}^{z+\delta z} &= G_{ij}^z + \delta G = G_{ij}^z + \frac{\partial G_{ij}}{\partial z} \delta z \\ &= G_{ij}^z + Y_{ik} G_{kj}^z \delta z + G_{ik}^z \tilde{Y}_{kj} \delta z \\ &= \sum_k \left((1 + \delta y) R^{\frac{1}{2}} \right)_{ik} \frac{1}{p^2 - m^2} \left(R^{\frac{1}{2}} (1 + \delta \tilde{y}) \right)_{kj}, \quad \text{for } p^2 \approx m^2, \end{aligned} \tag{7.62}$$

where from the second to the third line we used the polarization basis independent representation of G (7.60). Further, we defined $y_{ij} \delta z = \delta y_{ij}$ and $\tilde{y}_{ij} \delta z = \delta \tilde{y}_{ij}$.
Contracting both sides of (7.61) with $\mathbf{e} + \delta \mathbf{e}$, we obtain

$$\left(e_i^{(r)} + \delta e_i^{(r)} \right) G_{ij}^{z+\delta z} \left(e_i^{(s)} + \delta e_i^{(s)} \right) = \delta_{rs} G_r^z \left(1 + \frac{\delta R_r}{R_r} \right),$$

and on the other hand contracting both sides of (7.62) yields

$$\begin{aligned} \left(e_i^{(r)} + \delta e_i^{(r)} \right) G_{ij}^{z+\delta z} \left(e_i^{(s)} + \delta e_i^{(s)} \right) = \delta_{rs} G_r^z &+ \left(\delta \mathbf{e}^{(r)}, \mathbf{e}^{(s)} \right) G_s^z + \left(\mathbf{e}^{(r)}, \delta \mathbf{e}^{(s)} \right) G_r^z \\ &+ \left(\mathbf{e}^{(r)}, \delta \tilde{\mathbf{y}}^{(s)} \right) G_r^z + \left(\delta \mathbf{y}^{(r)}, \mathbf{e}^{(s)} \right) G_s^z, \end{aligned}$$

where we defined $\left(\delta\mathbf{y}^{(r)}\right)_i = e_k^{(r)}\delta y_{ki}$ and $\left(\delta\tilde{\mathbf{y}}^{(r)}\right)_i = \delta\tilde{y}_{ik}e_k^{(r)}$. Both results must be equal and combining them results in

$$\left(\delta\mathbf{e}^{(r)},\mathbf{e}^{(s)}\right)G_s^z + \left(\mathbf{e}^{(r)},\delta\mathbf{e}^{(s)}\right)G_r^z + \left(\mathbf{e}^{(r)},\delta\tilde{\mathbf{y}}^{(s)}\right)G_r^z + \left(\delta\mathbf{y}^{(r)},\mathbf{e}^{(s)}\right)G_s^z = \delta_{rs}\frac{\delta R_r}{R_r}G_r^z. \tag{7.63}$$

This equation is solved as follows. First we note that $\left(\delta\mathbf{y}^{(r)},\mathbf{e}^{(s)}\right), \left(\mathbf{e}^{(r)},\delta\tilde{\mathbf{y}}^{(s)}\right)$ are not independent of each other and because of $\tilde{y}^{\mathrm{T}} = y$ they are related by $\left(\delta\mathbf{y}^{(r)},\mathbf{e}^{(s)}\right) = \left(\mathbf{e}^{(s)},\delta\tilde{\mathbf{y}}^{(r)}\right)$. Then, in general $G_r^z \neq G_s^z$ and the components in front of G_r^z, G_s^z must vanish separately. The solution reads

$$\left(\delta\mathbf{y}^{(r)},\mathbf{e}^{(s)}\right) = \left(\frac{\delta R_r}{2R_r}\mathbf{e}^{(r)} - \delta\mathbf{e}^{(r)},\mathbf{e}^{(s)}\right) \quad \forall r,s. \tag{7.64}$$

In the case $G_r^z = G_s^z$, i.e. the case where the residues are independent of the polarization state, one has a certain freedom in the definition of $\delta\mathbf{e}$ and one can define (7.64) to hold[13]. Equation (7.64) is valid for all $\mathbf{e}^{(s)}$ and we have that

$$\left(\delta\mathbf{y}^{(r)} - \frac{\delta R_r}{2R_r}\mathbf{e}^{(r)} + \delta\mathbf{e}^{(r)},\mathbf{e}^{(s)}\right) = 0, \quad \forall s \quad \Rightarrow \delta\mathbf{y}^{(r)} - \frac{\delta R_r}{2R_r}\mathbf{e}^{(r)} + \delta\mathbf{e}^{(r)} = 0. \tag{7.65}$$

Note that usually $\mathbf{e}^{(r)}$ do not span a basis of the vector space they are living in and (7.65) has to be taken with a pinch of salt. The point is that (7.65) will be contracted with a vector living in the subspace spanned by these vectors[14] and therefore this result is true up to unphysical polarizations.

We are now able to prove the gauge independence of the S matrix which is done simply by direct computation

$$\frac{\partial S}{\partial\{\alpha,\xi\}} = \left(\prod_{i=1}^{n}\lim_{k_i^2\to m_i^2}\frac{\mathrm{i}}{\sqrt{R_i}}(k_i^2 - m_i^2)\right)\sum_{i=1}^{n}(\mathcal{R}_i + \mathcal{E}_i + \mathcal{G}_i),$$

where the three parts $\mathcal{R},\mathcal{E},\mathcal{G}$ are the gauge-transformed residue, polarization vector and Green's function, respectively. To keep the notation compact we introduce the product

$$\{\mathbf{e}_{\alpha_1}\ldots\mathbf{e}_{\alpha_n}, G_{\alpha_1\ldots\alpha_n}\} = \left(\prod_{k=1}^{n}\sum_{i_k}\right)e_{\alpha_1^{i_1}}\ldots e_{\alpha_n^{i_n}}G_{\alpha_1^{i_1}\ldots\alpha_n^{i_n}},$$

[13]This freedom does not alter the result for the S matrix. For that case one can check that the whole equation (7.63) as it is here reappears.

[14]More concretely, (7.65) is going to be contracted with a two-point function. Since only the polarization sum enters the two-point function, (7.65) will necessarily hit $\mathbf{e}^{(r)}$.

then the components of δS read

$$
\begin{aligned}
\mathcal{R}_i =& \frac{-1}{2R_{\omega_i}} \frac{\partial R_{\omega_i}}{\partial z} \left\{ \left(\prod_{k=1}^{n} \mathbf{e}^{\omega_k} \right), G_{\omega_1,\ldots,\omega_n} \right\}, \\
\mathcal{E}_i =& \left\{ \mathbf{e}^{\omega_1} \ldots \frac{\partial \mathbf{e}^{\omega_i}}{\partial z} \ldots \mathbf{e}^{\omega_n}, G_{\omega_1,\ldots,\omega_n} \right\}, \\
\mathcal{G}_i =& \left\{ \left(\prod_{k=1}^{n} \mathbf{e}^{\omega_k} \right) (-1)^{1+\sum_{m=i}^{n} |\phi_m|}, G^{\{\beta;\eta\}}_{\omega_1,\ldots,\omega_i^*,\ldots,\omega_n} \right\},
\end{aligned}
$$

and let us define

$$
\mathbf{R}_k := \sum_r \left(\mathbf{e}^{(\omega_k)}_{\omega_k}, \mathbf{e}^{(r)}_{\omega_k} \right) R^{(r)} \mathbf{e}^{(r)}_{\omega_k},
$$

then the truncated result for $\mathcal{R}, \mathcal{E}$ takes the form

$$
\begin{aligned}
&\left(\prod_{k=1}^{n} \lim_{p_k^2 \to m_k^2} (p_k^2 - m_k^2) \right) \mathcal{R}_i = \\
&\qquad \left\{ \left(\prod_{k<i} \mathbf{R}_k \right) \frac{-1}{2R_{\omega_i}} \frac{\partial R_{\omega_i}}{\partial z} \sum_r \left(\mathbf{e}^{(\omega_i)}_{\omega_i}, \mathbf{e}^{(r)}_{\omega_i} \right) R^{(r)} \mathbf{e}^{(r)}_{\omega_i} \left(\prod_{k>i} \mathbf{R}_k \right), G^{\mathrm{amp}}_{\omega_1\ldots\omega_n} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\prod_{k=1}^{n} \lim_{p_k^2 \to m_k^2} (p_k^2 - m_k^2) \right) \mathcal{E}_i = \\
&\qquad \left\{ \left(\prod_{k<i} \mathbf{R}_k \right) \sum_r \left(\frac{\partial \mathbf{e}^{(\omega_i)}_{\omega_i}}{\partial z}, \mathbf{e}^{(r)}_{\omega_i} \right) R^{(r)} \mathbf{e}^{(r)}_{\omega_i} \left(\prod_{k>i} \mathbf{R}_k \right), G^{\mathrm{amp}}_{\omega_1\ldots\omega_n} \right\}
\end{aligned}
$$

where

$$
G^{\mathrm{amp}}_{\omega_1\ldots\omega_n} = \prod_k \lim_{p_k^2 \to m_k^2} (p_k^2 - m_k^2)\, G_{\omega_1\ldots\omega_n}
$$

From the discussion of the generating functional $Z^{\{\beta;\eta\}}$ follows that any n-point function originating from $Z^{\{\beta;\eta\}}$ underlies the same Feynman rules as if it would originate from Z, except for the one extended BRST induced coupling. Due to this additional coupling the external anti-fields may couple in a complicated way to the n-point function $\mathcal{G}$, but when truncating the only surviving structure is the one where the anti-field is coupled to the two-point function via (7.59). That is true because first of all a two-point function is needed for giving a contribution to the residue. On the other hand, the two-point function cannot simply turn into an anti-field with different ghost-charge, but the extended BRST induced coupling can make this happen. Since this coupling can only occur once ($\beta^2 = \eta^2 = 0$), when truncating the anti-fields from $G^{\{\beta;\eta\}}$ the amputated result underlies the usual Feynman rules and is therefore proportional to G^{amp}. Formally this means

$$\lim_{p_k^2 \to m_k^2} (p_k^2 - m_k^2) \sum_j e^{(\omega_k)}_{\omega_k^j} G^{\beta\chi}_{\omega_1,\ldots,\omega_k^{*,j},\ldots,\omega_n} = \sum_r \left(\mathbf{y}^{(\omega_k)}_{\omega_k}, \mathbf{e}^{(r)}_{\omega_k}\right) R^{(r)} \sum_i e^{(r)}_{\omega_k^i} G^{\mathrm{amp}}_{\omega_1,\ldots,\omega_k^i,\ldots,\omega_n},$$

and the truncated result for $\mathcal{G}_i$ reads

$$\left(\prod_{k=1}^{n} \lim_{p_k^2 \to m_k^2} (p_k^2 - m_k^2)\right) \mathcal{G}_i = \left\{ \left(\prod_{k<i} \mathbf{R}_k\right) \sum_r \left(\mathbf{y}^{(\omega_i)}_{\omega_i}, \mathbf{e}^{(r)}_{\omega_i}\right) R^{(r)} \mathbf{e}^{(r)}_{\omega_i} \left(\prod_{k>i} \mathbf{R}_k\right), G^{\mathrm{amp}}_{\omega_1\ldots\omega_n} \right\}.$$

In our final step we assemble the results for $\mathcal{R}, \mathcal{E}$, and $\mathcal{G}$. Let $\delta S = \frac{\partial S}{\partial z}\delta z$, then

$$\begin{aligned}
\delta S = &\left(\prod_{k=1}^{n} \frac{\mathrm{i}}{\sqrt{R^{(\omega_k)}}}\right) \times \\
&\sum_i \left\{ \left(\prod_{k<i} \mathbf{R}_k\right) \underbrace{\left(-\frac{\delta R^{(\omega_i)}}{2R^{(\omega_i)}} \mathbf{e}^{(\omega_i)}_{\omega_i} + \delta \mathbf{e}^{(\omega_i)}_{\omega_i} + \delta \mathbf{y}^{(\omega_i)}_{\omega_i}, \mathbf{e}^{(r)}_{\omega_i}\right)}_{\overset{(7.65)}{=} 0} R^{(r)} \mathbf{e}^{(r)}_{\omega_i} \left(\prod_{k>i} \mathbf{R}_k\right), G^{\mathrm{amp}}_{\omega_1,\ldots,\omega_n} \right\} \\
= &\; 0,
\end{aligned}$$

which concludes the proof of the gauge independence of the S matrix.

8. Summary

The subject of this thesis is the proper treatment of unstable particles in perturbative QFT within the complex mass scheme (CMS), and we investigated unitarity and gauge (in-)dependence. The CMS provides a straightforward method to consistently implement unstable particles in perturbative calculations which is an analytic continuation of matrix elements to complex masses with an appropriate renormalization condition.
The Cutkosky cutting rules are violated, and it is no longer clear how perturbative unitarity is implemented which results from the fact that the topological derivation does no longer work because the $+i\epsilon$ prescription is needed. We do not claim that Cutkosky's result could not have been adapted, but we chose another approach. We derived an extended largest time equation (LTE) which could then be used to obtain a diagrammatical solution for the imaginary part of scattering amplitudes also when unstable particles are present. The result has been applied to several examples which were then compared to what one would have expected of unitarity and the results were approved. Our derivation of an extended LTE is based on the decomposition theorem and we showed that a similar decomposition can be achieved for the CMS propagator. As a result, one finds that the would-be cuts $\Delta^{\pm}$ of unstable particles are smoothed versions of the stable ones. In case of stable particles the LTE coincides with the Cutkosky cutting rules, but including unstable particles one has additional contributions which can be interpreted as contributions where the energy flow is backward. Performing an expansion solely of $\Delta^{\pm}$ of unstable particles in $\frac{\Gamma}{M}$ does indeed yield cutting rules where unstable pseudo cuts can be replaced by higher-order cuts through stable particles only. In this way, we demonstrated how to eliminate any appearing pseudo-cut replacing it with real cuts. Finally, we recovered the perturbative statement of Veltman's result, namely that a QFT is unitary only if unstable particles are excluded from asymptotic states.
In the second part we discussed an extended BRST invariance which is the basis for the Nielsen identities (NI). Having demonstrated how the extended BRST invariance results from the usual one simply by demanding nilpotency, we discussed quantum extensions and renormalization, i.e. we went further into the question whether the extended BRST symmetry is fitting for a quantum symmetry. We derived all possible symmetric contributions which are candidates for counter-terms for the example of an SU(2) Higgs

model. The calculation is done systematically as it is done in the case of the usual BRST invariance provided that certain simplifications of the so-called Slavnov-Taylor operator are made. Such simplifications were introduced by Piguet. We adopted his strategy and we have extended the arguments for the case of spontaneous symmetry breaking using the example of the SU(2) Higgs model. We verified that a renormalizable gauge theory being equipped with the extended BRST invariance is renormalizable at least for the physical sector. Studying gauge dependence relies on the unphysical sector which has to be renormalized, for instance, by algebraic means, thus allowing to use the NI beyond tree level and analyzing in general gauge (in-)dependence of n-point functions. As an application of the extended BRST we discussed the gauge independence of the renormalization condition of the CMS and of the S matrix.

Zusammenfassung

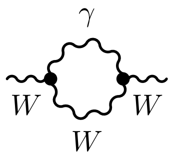
In dieser Arbeit wird die Beschreibung instabiler Teilchen in perturbativer QFT mit Hilfe des komplexen Massenschemas (CMS) behandelt. Dabei wurden die Schwerpunkte auf Unitarität und Eichunabhängigkeit gelegt. Das CMS stellt eine hinreichend einfache Methode dar, instabile Teilchen in Eichtheoreien im Rahmen der Störungstheorie beschreiben zu können. Es handelt sich um eine analytische Fortsetzung von Matrixelementen zu komplexen Massen mit entsprechenden Renormierungsbedingungen.
Die Vorraussetzungen für die Cutkosky Cutting Regeln sind verletzt und es ist nicht mehr klar wie perturbative Unitarität realisiert ist, da die topologische Ableitung der Regeln von der Kausalitätsvorschrift Gebrauch macht. Wir behaupten nicht, dass es nicht möglich sei jene Regeln anzupassen, jedoch haben wir uns für einen alternativen Zugang entschieden. Ausgehend von der Largest Time Equation (LTE) wurde eine erweiterte LTE, unter Einbezug von instabilen Teilchen, abgeleitet, welche schliesslich genutzt werden konnte, um diagrammatische Lösungen für den Imaginärteil von Amplituden zu erhalten. Das Ergebnis wurde mit dem verglichen was von Unitarität erwartet würde und konnte bestätigt werden. Die erweiterte LTE basiert auf dem Zerlegungstheorem für Feynman-Propagatoren und wir haben gezeigt, dass eine ähnliche Zerlegung für CMS Propagatoren existiert. Im Falle instabiler Teilchen findet man als Ergebnis anstatt scharfer Cuts, ausgeschmierte Funktionen. Im Falle stabiler Teilchen stimmen die LTE und die Cutkosky Cutting Regeln überein, werden allerdings instabile Teilchen eingeschlossen, so ergeben sich unter anderem auch Beiträge, die im Falle stabiler Teilchen kinematisch verboten wären, beziehungsweise eine verkehrte Energieflussrichtung aufweisen. Wird eine Entwicklung in $\frac{\Gamma}{M}$ unternommen, so entstehen aus den Lösungen der erweiterten LTE Cutting Regeln, wobei relevante Beiträge von instabilen Pseudo-Cuts als echte Cuts interpretiert werden können. So haben wir gezeigt, dass jeder zusätzliche Beitrag, der nicht wegdiskutiert werden kann letztendlich als Cut stabiler Teilchen interpretiert werden kann. Das entspricht der Aussage Veltmans, nämlich dass instabile Teilchen nicht als asymptotische Zustände in Erscheinung treten.
Im zweiten Teil dieser Arbeit haben wir eine erweiterte BRST Invarianz untersucht, welche die Basis für die Nielsen Identitäten (NI) darstellt. Wir haben diskutiert, wie die erweiterte BRST Invarianz implementiert wird und wie sie sich natürlich ergibt, wenn Nilpotenz der erweiterten BRST Variation verlangt wird. Wir haben weiterhin bespro-

chen ob eine solche Invarianz als Quantensymmtrie tauglich ist. Für ein SU(2) Higgs Modell haben wir alle möglichen symmetrischen Beiträge abgeleitet, welche mögliche Kandidaten für Counterterme darstellen. Die Berechnung der Terme funktioniert systematisch, wie auch im Falle der gewöhnlichen BRST Invarianz, vorrausgesetzt, dass gewisse Vereinfachungen am sogenannten Slavnov-Taylor Operator vorgenommen werden. Solche Vereinfachungen wurden von Piguet demonstriert. Wir haben seine Vorgehensweise übernommen und die gegebenen Argumente für den Fall von spontaner Symmetriebrechung am Beispiel des SU(2) Higgs Modells verallgemeinert. Es wurde verifiziert, dass eine renormierbare Eichtheorie, welche um die erweiterte BRST Invarianz erweitert wird, renormierbar bleibt, zumindest für den physikalischen Sektor. Zur Untersuchung von Eichabhängigkeit muss der unphysikalische Sektor z.B. algebraisch renormiert werden, sodass die Relationen der NI auch in höheren Ordnungen verwendet werden können, um auch in aller Allgemeinheit Eichabhängigkeit zu studieren. Als Anwendung haben wir die Definition der physikalischen Masse diskutiert und die Eichunabhängigkeit der S Matrix.

Appendices

A. Landau singularity in the scalar two-point function

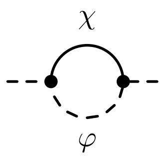
The CMS imposes, similar to the on-shell scheme, that the renormalized self-energy is zero at the complex mass μ. We used the renormalization condition to motivate the behavior of the renormalized self-energy near $p^2 = \mu^2$ (4.1), i.e. we assumed that Σ_R stays small when evaluated near μ which would formally follow by Taylor expansion, but a Taylor expansion is not always possible and the statement (4.1) is no longer correct. This happens, for instance, for charged unstable particles when massless particles are present. An example is the following diagram



(A.1)

which is a contribution to the W-boson self-energy. The W-boson is charged and unstable and the photon γ is massless. We give a quick calculation showing that the statement (4.1) must be generalized for such cases.

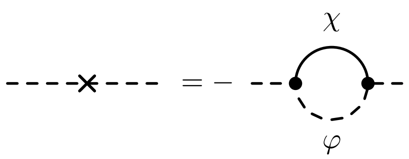
The polarizations in (A.1) make the discussion unnecessarily complicated and instead we consider the scalar analogue



$$= \frac{\mathrm{i}g^2}{16\pi^2}\left(\Delta - \int_0^1 \mathrm{d}x\ \log\left(p^2x^2 - x(p^2-\mu^2) - \mathrm{i}\varepsilon\right)\right) \tag{A.2}$$

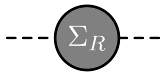
.

This result for the scalar two-point function has been taken from [Den93]. The UV divergence has been extracted via dimensional regularization and resides in $\Delta = \frac{2}{\varepsilon} - \gamma_E + \log 4\pi$ where the limit $\varepsilon \to 0$ equals $d \to 4^-$. Assuming this is the only contribution to the self energy, we need to add a counter-term canceling any contribution on the complex pole



$$\Bigg|_{p^2=\mu^2} = \frac{\mathrm{i}g^2}{16\pi^2}\left(-\Delta - 2 + \log\mu^2\right) \tag{A.3}$$

.
and the CMS renormalized self-energy reads

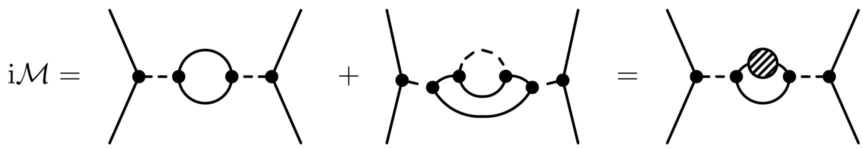


$$= \frac{\mathrm{i}g^2}{16\pi^2} \frac{p^2 - \mu^2}{p^2} \left(\log \mu^2 - \log \left(\mu^2 - p^2 \right) \right) . \tag{A.4}$$

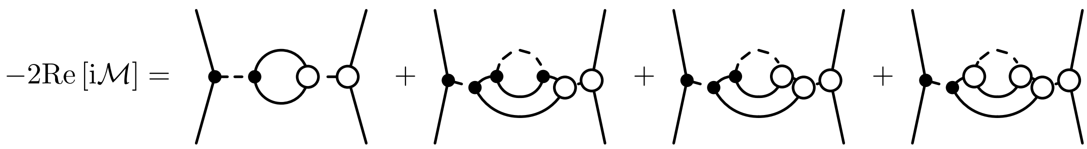
(A.4) behaves as requested for $p^2 \to \mu^2$, but cannot be Taylor expanded around $p^2 = \mu^2$. For $p^2 \approx \mu^2$ we find the behavior $\alpha^2 \log \alpha$ which is not what one would expect from naive Taylor expansion (4.1), but obeys (A.4).

B. The LTE and nested two-point functions

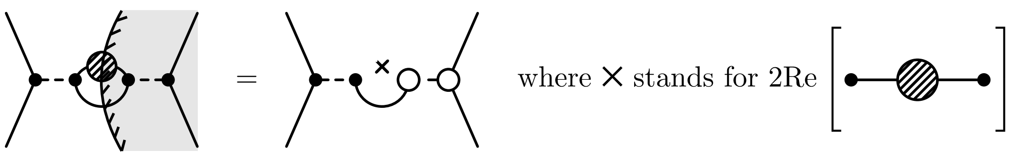
One observes that nested two-point functions in an amplitude reappear as cut two-point functions in the LTE relations of that amplitude. We discuss this property of the LTE by an example. Consider the following amplitude



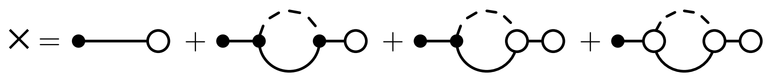
where we built in a nested two-point function. The example reflects the problem we have faced, namely in the first one-loop contribution we are going to find a pseudo-cut, but we cannot simply replace it by the first order approximation because the whole amplitude is in two-loop order. Consequently, we have to take along higher-order contributions. Computing the LTE of $\mathrm{i}\mathcal{M}$ away from the s-channel resonance and keeping only the relevant contributions, we obtain



This result can be written as



and in the last step one has to compute × with the help of the LTE



This is what we wanted to demonstrate, i.e. the higher-order contributions can be summarized to a cut 2-point function.

Acronyms

1PI one-particle irreducible
BPHZL Bogoliubov, Parasiuk, Hepp, Zimmermann and Lowenstein
BRST Becchi, Rouet, Stora and Tyutin
CCR Canonical Commutation Relations
CMS Complex Mass Scheme
dof degree of freedom
DS Dyson-Schwinger
EOM equations of motion
GSW Glashow Salam Weinberg
iff if and only if
LTE Largest Time Equation
LSZ Lehmann Symanzik Zimmermann
lhs left-hand side
LHSUE left-hand side of the unitarity equation
NI Nielsen Identities
QAP Quantum Action Principle
QCD Quantum Chromodynamics
QED Quantum Electrodynamics
QFT Quantum Field Theory
QM Quantum Mechanics
rhs right-hand side
RHSUE right-hand side of the unitarity equation
SM Standard Model
SSB Spontaneous Symmetry Breaking
ST Slavnov-Taylor
vev vacuum expectation values
wlog Without loss of generality
wrt with respect to

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