The Noncommutative Standard Model

Construction Beyond Leading Order in θ
and
Collider Phenomenology

Dissertation zur Erlangung des
naturwissenschaftlichen Doktorgrades
der Julius-Maximilians-Universität Würzburg

vorgelegt von

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Würzburg 2007

Eingereicht am: 27. Juli 2007
bei der Fakultät für Physik und Astronomie

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der Dissertation.

1. Prüfer: Prof. Dr. Reinhold Rückl
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3. Prüfer: Prof. Dr. Thomas Trefzger
im Promotionskolloquium.

Tag des Promotionskolloquiums: 31. August 2007

To teach how to live without certainty,
and yet without being paralyzed by hesitation,
is perhaps the chief thing that philosophy,
in our age, can still do for those who study it.

Bertrand Russell: A History of Western Philosophy

Zusammenfassung

Trotz seiner präzisen Übereinstimmung mit dem Experiment ist die Gültigkeit des Standardmodells (SM) der Elementarteilchenphysik bislang nur bis zu einer Energieskala von einigen hundert GeV gesichert. Abgesehen davon erweist sich schon das Einbinden der Gravitation in einer einheitlichen Beschreibung aller fundamentalen Wechselwirkungen als ein durch gewöhnliche Quantenfeldtheorie nicht zu lösendes Problem. Das Interesse an Quantenfeldtheorien auf einer nichtkommutativen Raumzeit wurde durch deren Vorhersage als niederenergetischer Limes von Stringtheorien erweckt. Unabhängig davon, kann die Nichtlokalität einer solchen Theorie den Rahmen zur Einbeziehung der Gravitation in eine vereinheitlichende Theorie liefern. Die Hoffnung besteht, dass die Energieskala $\Lambda_{\rm NC}$, ab der solche Effekte sichtbar werden können und für die es keinerlei theoretischen Vorhersagen gibt, schon bei der nächsten Generation von Beschleunigern erreicht wird. Auf dieser Annahme beruht auch die vorliegende Arbeit, im Rahmen deren eine mögliche Realisierung von Quantenfeldtheorien auf nichtkommutativer Raumzeit auf ihre phänomenologischen Konsequenzen hin untersucht wurde.

Diese Arbeit ist durch fehlende LHC (Large Hadron Collider) Studien für nichkommutative Quantenfeldtheorien motiviert. Im ersten Teil des Vorhabens wurde der hadronische Prozess $pp \to Z\gamma \to \ell^+\ell^-\gamma$ am LHC sowie die Elektron-Positron Paarvernichtung in ein Z-Boson und ein Photon am ILC (International Linear Collider) auf nichtkommutative Signale hin untersucht.

Das dieser Arbeit zugrunde liegende Modell besteht in einer Erweiterung des SM auf nichtkommutativer Raumzeit, welche auf zwei grundlegende Bausteine aufbaut: die Einführung eines deformierten Produktes, dem sogenannten Moyal-Weyl $\star$-Produkt und den Seiberg-Witten Abbildungen. Letztere bilden die üblichen Eich- und Materiefelder sowie die Eichparameter auf die entsprechenden nichtkommutativen Größen ab. Die Seiberg-Witten Abbildungen werden als Lösungen inhomogener Differentialgleichungen, der sogenannenten Eichäquivalenzbedingungen, Ordnung für Ordnung im nichtkommutativen Parameter θ erhalten. Dadurch wird der Forderung Rechnung getragen, dass nichtkommutative Eichtransformationen durch die entsprechenden kommutativen Eichtransformationen induziert werden. Somit kann mit Hilfe des Moyal-Weyl $\star$-Produktes und der Seiberg-Witten Abbildungen eine Erweiterung des SM auf nichtkommutative Raumzeit als effektive Theorie hinsichtlich des Entwicklungsparameters θ konstruiert werden. Die phänomenlogischen Untersuchungen wurden im Rahmen dieses Modells in erster Ordnung des nichtkommutativen Parameters θ durchgeführt.

Eine nichtkommutative Raumzeit führt zur Brechung der Rotationsinvarianz bezüglich der Strahlrichtung der einlaufenden Teilchen. Im differentiellen Wirkungsquerschnitt für Streuprozesse äußert sich dieses als eine azimuthale Abhängigkeit, die weder im SM noch in anderen Modellen jenseits des SM auftritt. Diese klare, für nichtkommutative Theorien typische Signatur kann benutzt werden, um nichtkommutative Modelle von anderen Modellen, die neue Physik beschreiben, zu unterscheiden. Auch hat es sich erwiesen, dass die azimuthale Abhängigkeit des Wirkungsquerschnittes am besten dafür geeignet ist, um die Sensitivität des LHC und des ILC auf der nichtkommutativen Skala Λ_{NC} zu bestimmen.
Im phänomenologischen Teil der Arbeit wurde herausgefunden, dass Messungen am LHC für den Prozess $pp \to Z\gamma \to \ell^+\ell^-\gamma$ nur in bestimmten Fällen auf nichtkommutative Effekte sensitiv sind. Für diese Fälle wurde für die nichtkommutative Energieskala Λ_{NC} eine Grenze von $\Lambda_{\mathrm{NC}} \gtrsim 1.2\,\mathrm{TeV}$ bestimmt. Diese ist um eine Größenordnung höher als die Grenzen, die von bisherigen Beschleunigerexperimenten hergeleitet wurden. Bei einem zukünftigen Linearbeschleuniger, dem ILC, wird die Grenze auf Λ_{NC} im Prozess $e^+e^- \to Z\gamma \to \ell^+\ell^-\gamma$ wesentlich erhöht (bis zu 6 TeV). Abgesehen davon ist dem ILC gerade der für den LHC kaum zugängliche Parameterbereich der nichtkommutativen Theorie erschlossen, was die Komplementarität der beiden Beschleunigerexperimente hinsichtlich der nichtkommutativen Parameter zeigt.
Der zweite Teil der Arbeit entwickelte sich aus der Notwendigkeit heraus, den Gültigkeitsbereich der Theorie zu höheren Energien hin zu erweitern. Dafür haben wir den neutralen Sektor des nichtkommutativen SM um die nächste Ordnung in θ ergänzt. Es stellte sich wider Erwarten heraus, dass die Theorie dabei um einige freie Parameter erweitert werden muss. Die zusätzlichen Parameter sind durch die homogenen Lösungen der Eichäquivalenzbedingungen gegeben, welche Ambiguitäten der Seiberg-Witten Abbildungen darstellen. Die allgemeine Erwartung war, dass die Ambiguitäten Feldredefinitionen entsprechen und daher in den Streumatrixelementen verschwinden müssen. In dieser Arbeit wurde jedoch gezeigt, dass dies ab der zweiten Ordnung in θ nicht der Fall ist und dass die Nichteindeutigkeit der Seiberg-Witten Abbildungen sich durchaus in Observablen niederschlägt. Die Vermutung besteht, dass jede neue Ordnung in θ neue Parameter in die Theorie einführt.
Wie weit und in welche Richtung die Theorie auf nichtkommutativer Raumzeit entwickelt werden muss, kann jedoch nur das Experiment entscheiden.

Abstract

Despite its precise agreement with the experiment, the validity of the standard model (SM) of elementary particle physics is ensured only up to a scale of several hundred GeV so far. Even more, the inclusion of gravity into an unifying theory poses a problem which cannot be solved by ordinary quantum field theory (QFT). String theory, which is the most popular ansatz for a unified theory, predicts QFT on noncommutative space-time as a low energy limit. Nevertheless, independently of the motivation given by string theory, the nonlocality inherent to noncommutative QFT opens up the possibility for the inclusion of gravity.

There are no theoretical predictions for the energy scale Λ_{NC} at which noncommutative effects arise and it can be assumed to lie in the TeV range, which is the energy range probed by the next generation of colliders. Within this work we study the phenomenological consequences of a possible realization of QFT on noncommutative space-time relying on this assumption.

The motivation for this thesis was given by the gap in the range of phenomenological studies of noncommutative effects in collider experiments, due to the absence in the literature of Large Hadron Collider (LHC) studies regarding noncommutative QFTs. In the first part we thus performed a phenomenological analysis of the hadronic process $pp \to Z\gamma \to \ell^+\ell^-\gamma$ at the LHC and of electron-positron pair annihilation into a Z boson and a photon at the International Linear Collider (ILC).

The noncommutative extension of the SM considered within this work relies on two building blocks: the Moyal-Weyl $\star$-product of functions on ordinary space-time and the Seiberg-Witten maps. The latter relate the ordinary fields and parameters to their noncommutative counterparts such that ordinary gauge transformations induce noncommutative gauge transformations. This requirement is expressed by a set of inhomogeneous differential equations (the gauge equivalence equations) which are solved by the Seiberg-Witten maps order by order in the noncommutative parameter θ. Thus, by means of the Moyal-Weyl $\star$-product and the Seiberg-Witten maps a noncommutative extension of the SM as an effective theory as expansion in powers of θ can be achieved, providing the framework of our phenomenological studies.

A consequence of the noncommutativity of space-time is the violation of rotational invariance with respect to the beam axis. This effect shows up in the azimuthal dependence of cross sections, which is absent in the SM as well as in other models beyond the SM. Thus, the azimuthal dependence of the cross section is a typical signature of noncommutativity and can be used in order to dis-

criminate it against other new physics effects. We have found this dependence to be best suited for deriving the sensitivity bounds on the noncommutative scale $\Lambda_{\rm NC}$.

By studying $pp \to Z\gamma \to \ell^+\ell^-\gamma$ to first order in the noncommutative parameter θ, we show in the first part of this work that measurements at the LHC are sensitive to noncommutative effects only in certain cases, giving bounds on the noncommutative scale of $\Lambda_{\rm NC} \gtrsim 1.2\,{\rm TeV}$. Our result improved the bounds present in the literature coming from past and present collider experiments by one order of magnitude. In order to explore the whole parameter range of the noncommutativity, ILC studies are required. By means of $e^+e^- \to Z\gamma \to \ell^+\ell^-\gamma$ to $\mathcal{O}(\theta)$ we have shown that ILC measurements are complementary to LHC measurements of the noncommutative parameters. In addition, the bounds on $\Lambda_{\rm NC}$ derived from the ILC are significantly higher and reach $\Lambda_{\rm NC} \gtrsim 6\,{\rm TeV}$.

The second part of this work arose from the necessity to enlarge the range of validity of our model towards higher energies. Thus, we expand the neutral current sector of the noncommutative SM to second order in θ. We found that, against the general expectation, the theory must be enlarged by additional parameters. The new parameters enter the theory as ambiguities of the Seiberg-Witten maps. The latter are not uniquely determined and differ by homogeneous solutions of the gauge equivalence equations. The expectation was that the ambiguities correspond to field redefinitions and therefore should vanish in scattering matrix elements. However, we proved that this is not the case, and the ambiguities do affect physical observables. Our conjecture is, that every order in θ will introduce new parameters to the theory. However, only the experiment can decide to what extent efforts with still higher orders in θ are reasonable and will also give directions for the development of theoretical models of noncommutative QFTs.

Contents

Chapter 1

Introduction

"It is possible that the usual four-dimensional continuous space-time does not provide a suitable framework within which interacting fields and matter can be described." It was in 1947 when Snyder made this statement in the article which is generally considered to mark the beginnings of Quantum Field Theories (QFT) on noncommutative space-time [1, 2]. It was the result of an exchange among Heisenberg, Pauli, Oppenheimer and Peierls [3, 4], initiated by the former already in 1930 and driven by the motivation to find a natural cut-off for the divergencies which plagued quantum electrodynamics (QED). For this purpose, Snyder, a student of Oppenheimer, constructed a Lorentz invariant noncommutative structure of space-time introducing a minimal length-scale, which might be used to regularize the infinities in QED, as it was hoped. Nevertheless, this ansatz did not succeed whereas renormalization proved to be the right cure for the UV divergencies in QFT. Thus, the idea of noncommutative space-time was abandoned for the time being. On the other hand, noncommutativity was pursued on the mathematical side, where especially the work of Alain Connes on noncommutative geometry in the 1980's stands out, providing the mathematical tools for further studies on noncommutative space-time. In particle physics the interest in quantum field theories on noncommutative space-time declined, though not entirely, and was renewed only in 1999 by the work of Seiberg and Witten on string theory [5]. They showed that the dynamics of the endpoints of an open string on a D-brane in the presence of a magnetic background field can be described by a Yang Mills theory on noncommutative space-time. Since string theory is nowadays the most popular ansatz for an ultimate theory capturing all laws of nature up to the Planck scale, the impact of this result on the particle physics community resulted in an outburst of publications on theories on noncommutative space-time within the last decade.

Nevertheless, the motivation for studying physics on noncommutative space-time can also be provided independently of string theory. Whatever the theory describing physics at the Planck scale is, we know that it is certainly not the Standard Model (SM) of particle physics, even though its predictive power has been experimentally verified to astonishing accuracy within the past decades. One of its major drawbacks is its incompatibility with general relativity. Thus, QFT and the SM have to be altered on the road towards the Planck scale in order to incorporate gravity. Since gravity alters the geometry of ordinary space-time, we expect that its quantization occurs at or before the Planck scale. Doplicher et al. show that space-time noncommutativity prevents the gravitational collapse allowing thus to incorporate space-time fluctuations into quantum field theory [6] .

We can look at noncommutative quantum field theory from another point of view, and take it as an intermediate regime between ordinary QFT and the physics at the Planck scale, whatever this might be. The Planck scale of 10^{19} GeV will remain out of direct reach probably forever. Apart from the glimpses due to the study of very energetic but rare cosmic rays, in a bottom-up approach we can only infer on the physics at the Planck scale from physics at experimentally accessible scales. The start of the Large Hadron Collider (LHC) experiment in Geneva in 2008 will open a window towards the TeV range, allowing for a (still) short range vista on the physics beyond the electroweak scale. We hope that it will show us the next step towards the development of a more satisfying model. This step might contain, amongst others, supersymmetry, extra dimensions and/or noncommutative quantum field theory. All or one of these might represent the right formalism describing the *terra incognita* in between Planck scale physics and the SM. So much the better, that all of them are predicted by string theory.

The aim of this work derives from the motivation presented in the previous paragraph. We will perform a phenomenological analysis of the effects of a noncommutative space-time employing observables at the upcoming LHC experiment and subsequently derive bounds on the energy scale at which noncommutativity of space-time might occur. In order to define the noncommutative energy scale, we need to define noncommutative space-time first, by promoting ordinary space-time coordinates to noncommuting operators:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \neq 0 \tag{1.1}$$

The object $\theta^{\mu\nu}$ has the dimension of a length squared and thus we can extract the noncommutative energy scale $1/\Lambda_{\mathrm{NC}}^2 \propto \theta^{\mu\nu}$. Even if noncommutativity of space-time has not been observed yet, it can be allowed as long as the characteristic length scale at which noncommutative effects set in is small enough compared to the length scales of present experiments. There are no theoretical predictions on Λ_{NC}, such that only experiments can determine or at least constrain it. The bounds on Λ_{NC} derived in the literature are very different, ranging over many orders of magnitude, and depend on the model assumed. The constraints coming from high energy scattering experiments were mostly derived from QED processes within the simplest, but theoretically unsatisfactory model of QED on noncommutative space-time. The experiment which will dominate the phenomenological landscape of particle physics beginning with 2008 is the LHC, a proton-proton collider reaching center of mass energies of 14 TeV. For constraining the noncommutative scale, phenomenological studies of the SM on noncommutative space-time at the LHC are indispensable. Apart from partial results of this work, which have already been published [7] and the recent thesis [8], no phenomenological studies for the LHC are present in

the current literature. This work contributes to the completion of this gap by studying the process $pp \to Z\gamma \to \ell^+\ell^-\gamma$ at the LHC.
We have worked within a θ expanded approach which gives a satisfactory description of the SM on noncommutative space-time. Nevertheless, since it is an effective theory, during our phenomenological analysis, the need of going towards higher orders in the expansion parameter θ has crystallized. This leads us to the second part of this thesis, namely the development of the model up to second order in the noncommutativity parameter θ. In doing so, we discovered the surprising fact that some ambiguities, which occur when building the model and which were previously believed to cancel in observables, as they did for the $\mathcal{O}(\theta)$ analysis, in fact survive. We demonstrate in an exemplary calculation the dependency of the cross section on these ambiguities [9].
The thesis is structured as follows: First we will present the theoretical background of field theories on noncommutative space-time in chapter 2. The first tool needed in doing so is a deformed product of functions, allowing us to work on the ordinary space-time coordinates and still reproducing the noncommutativity of space-time. We will show that models built only by means of this product exhibit a series of problems which can be cured by the introduction of the Seiberg-Witten maps. These can be obtained as power series in the noncommutative parameter θ as solution to differential equations. These so called gauge equivalence equations ensure that ordinary gauge transformations induce noncommutative gauge transformations. We will present the solution to $\mathcal{O}(\theta)$. Some of the important problems of noncommutative quantum field theories are reviewed with emphasis on the renormalizability of such theories, and in the end, we give an overview on the current bounds on the noncommutative energy scale Λ_{NC}.
In chapter 3 we introduce one extension of the SM on noncommutative space-time (NCSM) as an effective theory, that means as expansion in powers of θ. We are concerned in the matter and the kinetic sector of the NCSM up to $\mathcal{O}(\theta)$. We will derive Feynman rules and calculate the cross section for $q\bar{q} \to Z\gamma$, showing by means of this process the most important consequences of the noncommutativity on observables. These effects are then used in order to derive sensitivity bounds on the noncommutative scales from Monte Carlo simulations of $pp \to e^+e^-\gamma$ at the LHC and $e^+e^- \to Z\gamma$ at the ILC. We will show the complementarity of these two collider experiments. We also calculate the constraints coming from past and present collider experiments, like LEP and Tevatron. During the phenomenological analysis we were confronted with the limited range of validity of our model due to the expansion up to $\mathcal{O}(\theta)$. Thus, the need for going to higher orders in θ arises.
This will be the subject of chapter 4 where we solve the consistency and gauge equivalence equations up to $\mathcal{O}(\theta^2)$ and give the full solution, i.e. the Seiberg-Witten maps, up to this order. The Seiberg-Witten maps are not unique since homogeneous solutions to the mentioned differential equations can always be added. We will disprove the common belief that these ambiguities all correspond to field redefinitions and thus should cancel in scattering matrix elements. For this purpose, the neutral current sector of the NCSM at $\mathcal{O}(\theta^2)$ is studied. We thus show, that the NCSM at $\mathcal{O}(\theta^2)$ exhibits more parameters than expected.

We will end with a phenomenological outlook on the NCSM.

Chapter 2

Noncommutative Quantum Field Theory

This chapter is meant to provide the theoretical basis for the model we will study in the remainder of this work. We will define noncommutative space-time and see how QFT can be realized on such a deformed space-time. We will review some problems of these theories and for some we will present a solution which finally leads to a concrete realization of the SM on noncommutative space-time, which is the subject of this work.
This chapter is not intended to give a complete overview of QFT on noncommutative space-time. Nevertheless, some aspects are discussed which are not directly connected to the main topic of this work, like UV/IR-Mixing and the renormalizability of NCQFTs, which nevertheless constitute essential facets of the theory and should not be left unmentioned.

2.1 Noncommutative Space-Time

Noncommutative space-time is a deformation of space-time that can be realized by representing ordinary space-time coordinates x^μ by Hermitian operators $\hat{x}^\mu$ that do not commute:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \,. \tag{2.1}$$

In this work, we assume for simplicity that

$$[\theta^{\mu\nu}, \hat{x}^\rho] = 0 \,. \tag{2.2}$$

A priori θ has an arbitrary complicated dependency on $\hat{x}$. Nevertheless, we can assume a constant θ. In the literature two other cases have also been studied, where θ depends linearly and quadratically on $\hat{x}$. Thus, noncommutativity with a Lie algebra structure

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\lambda^{\mu\nu}_\rho \hat{x}^\rho \tag{2.3}$$

and noncommutative space-time with quantum group structure

$$[\hat{x}^\mu, \hat{x}^\nu] = \left(\frac{1}{q}\hat{R}^{\mu\nu}_{\kappa\rho} - \delta^\mu_\rho \delta^\nu_\kappa\right) \hat{x}^\kappa \hat{x}^\rho \tag{2.4}$$

can be defined. We assume that the canonical noncommutativity (2.1) is a reasonable approximation and we will adopt it throughout this work. Thus, we will introduce the following parametrization

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} = \mathrm{i}\frac{1}{\Lambda_{\mathrm{NC}}^2} C^{\mu\nu} = \mathrm{i}\frac{1}{\Lambda_{\mathrm{NC}}^2} \begin{pmatrix} 0 & E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} , \tag{2.5}$$

with the constant symmetric 4×4 matrix $C^{\mu\nu}$. In analogy to the electromagnetic field strength tensor we have denoted the time-like components of $C^{\mu\nu}$ by $\vec{E}$ and the space-like components by $\vec{B}$. As we will see, $\vec{E}$ and $\vec{B}$ will play different rôles, theoretically as well as phenomenologically.

Building quantum field theories on the noncommutative space-time (2.1) starting from the noncommuting operators $\hat{x}^\mu$ is a bold venture. The construction of quantum field theories on noncommutative space-time can be done more straightforwardly, if we take into account that experiments do not measure space-time coordinates themselves, but particles and fields, and that in the corresponding mathematical framework providing the calculation of observables we only encounter functions of the space-time coordinates and not the coordinates themselves. Therefore, we may seek for a way to express the commutator (2.1) of the noncommuting objects $\hat{x}^\mu$ by means of ordinary coordinates x^μ and a deformed product. Thus, we are looking for a homomorphism between the associative algebra $(\hat{\mathcal{A}}, \cdot)$ generated by $\hat{x}^\mu$ which defines the noncommutative space-time and the algebra $(\mathcal{A}, \star)$ of functions of the ordinary space-time coordinates and a deformed product $\star$, just like noncommutative geometry is constructed in algebraic geometry.

2.2 Moyal-Weyl $\star$-Product

The framework of Weyl's quantization procedure [10] provides a formalism for associating with the algebra of noncommuting coordinates $(\hat{\mathcal{A}}, \cdot)$ an algebra of functions of commuting variables with deformed product $(\mathcal{A}, \star)$. We define a map $W : \mathcal{A} \to \hat{\mathcal{A}}$ by which an element from $\hat{\mathcal{A}}$ is assigned to a function $f(x^0, \ldots, x^{n-1}) \equiv f(x)$ from $\mathcal{A}$:

$$W(f) = \hat{f} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \mathrm{d}^n k \, e^{\mathrm{i}k_\nu \hat{x}^\nu} \tilde{f}(k), \tag{2.6}$$

with $\tilde{f}(k)$ the Fourier transform of $f(x)$:

$$\tilde{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \mathrm{d}^n x \, e^{-\mathrm{i}k_\nu x^\nu} f(x). \tag{2.7}$$

The multiplication of two operators $W(f)$ and $W(g)$ obtained from (2.6) yields another operator $W(f \star g)$:

$$W(f) \cdot W(g) = \hat{f} \cdot \hat{g} = W(f \star g) \ , \tag{2.8}$$

with $f \star g \in (\mathcal{A}, \star)$, a classical function which is well defined, as we will now show. Inserting (2.6) in (2.8) we obtain:

$$W(f \star g) = W(f)W(g) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \, \mathrm{d}^n p \, \mathrm{e}^{\mathrm{i}k_\mu \hat{x}^\mu} \mathrm{e}^{\mathrm{i}p_\nu \hat{x}^\nu} \tilde{f}(k)\tilde{g}(p). \tag{2.9}$$

In the case of canonical noncommutativity (2.1), the product of the two exponentials in the above formula will give an exponential of a linear combination of the $\hat{x}^\mu$ after applying the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^{\widehat{A}}\mathrm{e}^{\widehat{B}} = \mathrm{e}^{\widehat{A}+\widehat{B}+\frac{1}{2}[\widehat{A},\widehat{B}]+\frac{1}{12}\left([\widehat{A},[\widehat{A},\widehat{B}]]+[[\widehat{A},\widehat{B}],\widehat{B}]\right)+\ldots} \tag{2.10}$$

and considering the commutator relation (2.2), which thus makes all terms including more than one commutator in (2.10) vanish:

$$\mathrm{e}^{\mathrm{i}k_\mu \hat{x}^\mu} e^{\mathrm{i}p_\nu \hat{x}^\nu} = \mathrm{e}^{\mathrm{i}(k_\nu + p_\nu)\hat{x}^\nu - \frac{\mathrm{i}}{2}k_\mu p_\nu \theta^{\mu\nu}}. \tag{2.11}$$

We obtain $f \star g$ by comparing (2.9) with (2.6) and replacing the operator $\hat{x}^\mu$ by the coordinate x^μ:

$$(f \star g)(x) = \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \, \mathrm{d}^n p \, \mathrm{e}^{\mathrm{i}(k_\nu + p_\nu)x^\nu - \frac{\mathrm{i}}{2}k_\mu \theta^{\mu\nu} p_\nu} \tilde{f}(k)\tilde{g}(p). \tag{2.12}$$

Thus, the Moyal-Weyl $\star$ product [11] is obtained:

$$(f \star g)(x) = \exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)\bigg|_{y\to x}. \tag{2.13}$$

Using this prescription for the $\star$-product, we now calculate the $\star$-commutator of the ordinary coordinate functions $[x^\mu \stackrel{\star}{,} x^\nu]$ and obtain, remembering the antisymmetry of $\theta^{\mu\nu}$:

$$[x^\mu \stackrel{\star}{,} x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = x^\mu x^\nu + \frac{\mathrm{i}}{2}\theta^{\mu\nu} - x^\nu x^\mu - \frac{\mathrm{i}}{2}\theta^{\nu\mu} = \mathrm{i}\theta^{\mu\nu}. \tag{2.14}$$

This reproduces exactly the commutator (2.1):

$$[x^\mu \stackrel{\star}{,} x^\nu] = [\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \ , \tag{2.15}$$

and shows how the noncommutativity encoded in the operators $\hat{x}^\mu$ is shifted into the $\star$-product of functions on ordinary space-time. Thus, we are now able to start the construction of QFT on noncommutative space-time still dealing with ordinary space-time coordinates or more precisely, with functions on the ordinary space-time, but with a deformed product instead of the ordinary one. Before going on in doing so, we need to give some important properties of the $\star$-product. Under the integral the $\star$-product of *two* functions is equivalent to the ordinary product

$$\int \mathrm{d}^4 x \ (f \star g)(x) = \int \mathrm{d}^4 x \ (g \star f)(x) = \int \mathrm{d}^4 x \ f(x)g(x), \tag{2.16}$$

but this is not the case for the $\star$-product of *three or more* functions, where only one $\star$-product can be replaced by the usual $\cdot$-product:

$$\int \mathrm{d}^4x\ (f \star g \star h)(x) = \int \mathrm{d}^4x\ ((f \star g) \cdot h)(x) = \\ = \int \mathrm{d}^4x\ (f \cdot (g \star h))(x) \neq \int \mathrm{d}^4x\ f(x)g(x)h(x)\,. \quad (2.17)$$

Furthermore, we have invariance under cyclical permutation of the functions under the integral:

$$\int \mathrm{d}^4x\ (f \star g \star h)(x) = \int \mathrm{d}x\ ((f \star g) \cdot h)(x) = \\ = \int \mathrm{d}^4x\ (h \cdot (f \star g))(x) = \int \mathrm{d}^4x\ (h \star f \star g)(x)\,. \quad (2.18)$$

2.3 $U(N)$-Theories on Noncommutative Space-Time

Equipped with the Moyal-Weyl $\star$-product, we are now able to make a first step towards the construction of QFT on noncommutative space-time, starting with QED.

2.3.1 NCQED

As suggested by equation (2.15), we will replace in the action all ordinary products between the fields with the Moyal-Weyl $\star$-product defined in (2.13):

$$\mathcal{S}_{\text{NCQED}} = \int \mathrm{d}^4x\ \left(\bar{\psi} \star (\mathrm{i}\not{D}) \star \psi - m\bar{\psi} \star \psi - \frac{1}{4e^2} F_{\mu\nu} \star F^{\mu\nu}\right) \quad (2.19)$$

with the covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu \quad (2.20)$$

and the field-strength tensor

$$F_{\mu\nu} = \mathrm{i}[D_\mu \overset{\star}{,} D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu \overset{\star}{,} A_\nu]\ . \quad (2.21)$$

We have absorbed the coupling constant e into the definition of the gauge field for reasons of consistency with later notations. The NCQED-Lagrangian in (2.19) is invariant under the noncommutative gauge transformations:

$$\psi \to \psi' = \mathrm{e}_\star^{\mathrm{i}\lambda}\psi\,, \quad (2.22a)$$
$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\mathrm{e}_\star^{-\mathrm{i}\lambda}\,, \quad (2.22b)$$
$$A_\mu \to A'_\mu = \mathrm{e}_\star^{\mathrm{i}\lambda}(A_\mu - \mathrm{i}\partial_\mu\lambda)\mathrm{e}_\star^{-\mathrm{i}\lambda}\,, \quad (2.22c)$$

where the $\star$ in the exponentials indicates a formal power series where in each term the ordinary product is replaced by the $\star$-product.

Due to the trace property (2.16), bilinear terms with the $\star$-product resemble under the integral the ones with the ordinary product, such that propagators and mass terms remain the same in NCQED as in ordinary QED. The novelty brought in by the $\star$-product manifests itself in the interaction terms by means of modified SM vertices on the one hand and new, SM forbidden ones, on the other. Translating the $\star$-product (2.13) into momentum space, the term $\bar{\psi}\not{A} \star \psi$ in (2.19) acquires a momentum dependent phase factor $\mathrm{e}^{\mathrm{i}p\theta k}$, with $p\theta k \equiv p_\mu \theta^{\mu\nu} k_\nu$. From the definition of $F_{\mu\nu}$ one of the striking features brought in by noncommutativity becomes clear: the nonvanishing commutator $[A_\mu \stackrel{\star}{,} A_\nu]$ gives rise to a SM-forbidden triple photon interaction. Thus, any abelian theory on ordinary space-time gets a nonabelian character when formulated on noncommutative space-time, having - as in QCD - the coupling constant fixed by gauge invariance. This causes one of the problems which makes such "naive" theories on noncommutative space-time fail as extensions of the SM.

2.3.2 Problems

Theories equipped only with the $\star$-product suffer from some major problems. We will review two of them since their solution motivates the model which is the subject of this work. As it was shown in [12, 13], NCQED as presented above suffers from the quantization of the charge to $\{0, \pm 1\}$. This is because the matter fields have only three possibilities for the representation in which they can live. These are the fundamental representation with the corresponding charge $Q = 1$:

$$\psi \rightarrow \psi' = U \star \psi \,, \tag{2.23}$$
$$D_\mu \psi = \partial_\mu \psi - \mathrm{i} A_\mu \star \psi \,, \tag{2.24}$$

the antifundamental representation with $Q = -1$:

$$\psi \rightarrow \psi' = \psi \star U^{-1} \,, \tag{2.25}$$
$$D_\mu \psi = \partial_\mu \psi + \mathrm{i} \psi \star A_\mu \,, \tag{2.26}$$

and the adjoint representation with $Q = 0$:

$$\chi \rightarrow \chi' = U \star \chi \star U^{-1} \,, \tag{2.27}$$
$$D_\mu \chi = \partial_\mu \chi - \mathrm{i}[A_\mu \stackrel{\star}{,} \chi] \,. \tag{2.28}$$

Only these transformations are compatible with the transformation of the gauge field

$$A_\mu \rightarrow A'_\mu = U \star A \star U^{-1} + \mathrm{i} U \star \partial_\mu U^{-1} \,, \tag{2.29}$$

which leaves the kinetic term

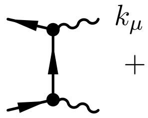
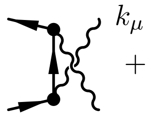
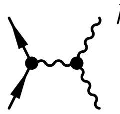
$$\mathcal{L}_{\mathrm{gauge}} = -\frac{1}{4g^2} \mathrm{Tr} F_{\mu\nu} \star F^{\mu\nu} \tag{2.30}$$

invariant. The photon can not couple to matter fields carrying charges other than $\{0, \pm 1\}$ since the corresponding minimal coupling

$$D_\mu \psi^{(n)} = \partial_\mu \psi^{(n)} - \mathrm{i} q^{(n)} A_\mu \star \psi^{(n)} \tag{2.31}$$

does not transform covariantly under $\psi^{(n)} \to U^{(n)} \star \psi^{(n)}$ and (2.29), with $U^{(n)} = \mathrm{e}^{\mathrm{i}q^{(n)}\lambda}$. The gauge transformation of the photon field would have to depend on the charge n, which would lead to a multitude of photon fields.

The charge quantization problem can be viewed in a more intuitive way [7]. Consider the pair annihilation of fermions. The Ward identity (and thereby gauge invariance) demands the cancellation of the on shell amplitudes when one of the photon polarization vectors is replaced by the corresponding momentum. The situation is illustrated below:

$g^2 \cdot$  $+ g^2 \cdot$  $+ g\, g_{\gamma\gamma\gamma} \cdot$  $\stackrel{!}{=} 0.$

The cancellation can only take place if the three photon coupling $g_{\gamma\gamma\gamma}$ in the third diagram equals the fermion-photon coupling g. The situation is similar to a non-abelian gauge theory, where gauge invariance fixes the three and four gauge boson couplings. Yet, this becomes a problem for the noncommutative version of an abelian gauge theory: fixing $g_{\gamma\gamma\gamma}$ to $1 \cdot e$ as in the case of e^+e^- annihilation is not compatible with the weak hypercharges $Y(L_e, e_R, \nu_{e,R}, L_{u,d}, u_R, d_R) = (-1, -2, 0, 1/3, 4/3, -2/3)$ leading to the fractional charge of quarks.

The next major drawback of the "naive" approach for constructing gauge theories on noncommutative space-time is that is does not allow $SU(N)$ gauge theories on noncommutative space-time. The reason is obvious, when looking at the $\star$-commutator between two gauge fields:

$$[A^a_\mu T^a \stackrel{\star}{,} A^b_\nu T^b] = \frac{1}{2}\{A^a_\mu \stackrel{\star}{,} A^b_\nu\}[T^a, T^b] + \frac{1}{2}[A^a_\mu \stackrel{\star}{,} A^b_\nu]\{T^a, T^b\}\,. \tag{2.32}$$

The first term is proportional to the ordinary commutator of generators and remains thus in the Lie algebra. This is not true for the second term which contains an anti-commutator of generators. Its coefficient, zero in the commutative case, is nonzero due to the $\star$-product. Thus, the commutation relation of gauge fields (and analogously gauge parameters) closes only in the fundamental representation of $U(N)$ and it is not possible to describe $SU(N)$ on noncommutative space-time, and therefore the construction of the SM with the gauge group $U(1)_Y \times SU(2)_L \times SU(3)_S$ is prohibited.

One way to solve the charge quantization problem was proposed in [14, 15]. A $U(1) \times U(2) \times U(3)$ gauge symmetry on noncommutative space-time is broken to the symmetry of the SM by introducing two new scalars, the so called Higgsac's. The fractional charges of the quarks are explained automatically in this model. In the next section we will present another solution to this problem, which at the same time will cure the second problem.

The model proposed in [14, 15] solves also the problem related to the difficulty of constructing noncommutative $SU(N)$ gauge theories. They circumvent it in the sense that the gauge group they start from has a $U(1) \times U(2) \times U(3)$ gauge symmetry. However, we will present another solution to these problems, which allows the $U(1)_Y \times SU(2)_L \times SU(3)_S$ gauge group on noncommutative space-time and solves the charge quantization problem, without changing the SM particle content.

2.4 Seiberg-Witten Maps and $SU(N)$

The way out of the trouble with the charge quantization and with the commutator (2.32) not closing in the Lie algebra is to go to the corresponding universal enveloping algebra, an associative algebra that can always be built and that is spanned by products of generators. We take instead of the ordinary, Lie algebra valued gauge fields and parameters A and λ corresponding objects from the enveloping algebra $\hat{A}$ and $\hat{\lambda}$ and expand them

$$\begin{aligned} \hat{\lambda} &= \lambda_a T^a + \lambda^1_{ab} :T^aT^b: + \lambda^2_{abc} :T^aT^bT^c: \dots\,, \\ \hat{A} &= A_a T^a + A^1_{ab} :T^aT^b: + A^2_{abc} :T^aT^bT^c: \dots\,, \end{aligned} \tag{2.33}$$

in a basis of symmetrised products

$$\begin{aligned} :T^a: &= T^a\,, \\ :T^aT^b: &= \frac{1}{2}\left\{T^a, T^b\right\} = \frac{1}{2}\left(T^aT^b + T^bT^a\right)\,,\ \dots \end{aligned} \tag{2.34}$$

Of course, this introduces infinitely many degrees of freedom. But, in the end, this will not be really the case, because, as we will immediately show, the expansion coefficients can be made to depend on the ordinary gauge fields and parameters. The map between the ordinary objects and the hatted ones is provided in the same work of Seiberg and Witten [5], that proved the equivalence of commutative and noncommutative gauge theories as a description of the low energy limit for certain string theories and led to the sudden revival of the interest in NCQFT after all.
In [5], the authors show that the dynamics of bosonic open string endpoints on a D-brane in the presence of a magnetic field can be described by a noncommutative Yang Mills theory. They find that the corresponding effective action can be described by an ordinary Yang Mills theory as well as by a noncommutative Yang Mills theory[1]. Thus, they must be related by a change of variables:

$$\begin{pmatrix} A \\ \lambda \end{pmatrix} \to \begin{pmatrix} \hat{A}(A) \\ \hat{\lambda}(\lambda, A) \end{pmatrix}\,. \tag{2.35}$$

Note that we must allow the gauge parameter $\hat{\lambda}$ to depend on the gauge field A, because this would otherwise imply that an ordinary abelian gauge group is isomorphic to its noncommutative counterpart, which is not abelian any more. Yet, this is not possible, since an abelian and a nonabelian group can never be isomorphic to each other. This means, that if two gauge fields $\hat{A}(A)$ and $\hat{A}'(A)$ belonging to the same gauge orbit and connected by a noncommutative gauge transformation $\hat{\lambda}$, then the corresponding mapped gauge fields, A and A', should

[1] They differ by the choice of the regularization: Pauli-Villars regularization for the commutative and point-splitting regularization for the noncommutative theory.

be connected by an ordinary gauge transformation λ. The mapping between $\hat{\lambda}$ and λ also depends on A, otherwise the commutative and noncommutative gauge groups would be equivalent, which, as just mentioned, cannot be.
The task is now to find, including also matter fields,

$$\begin{aligned} \hat{A} &= \hat{A}(A,\theta)\,, && (2.36a)\\ \hat{\lambda} &= \hat{\lambda}(\lambda,A,\theta)\,, && (2.36b)\\ \hat{\psi} &= \hat{\psi}(\psi,A,\theta)\,, && (2.36c)\end{aligned}$$

such, that the so-called gauge equivalence conditions hold:

$$\begin{aligned} \hat{A}(A,\theta)+\hat{\delta}_{\hat{\lambda}}\hat{A}(A,\theta) &= \hat{A}(A+\delta_\lambda A,\theta)\ , && (2.37a)\\ \hat{\psi}(\psi,A,\theta)+\hat{\delta}_{\hat{\lambda}}\hat{\psi}(\psi,A,\theta) &= \hat{\psi}(\psi+\delta_\lambda\psi,A+\delta_\lambda A,\theta)\ , && (2.37b)\end{aligned}$$

with the infinitesimal commutative gauge transformations

$$\begin{aligned} \delta_\lambda A_\mu &= \partial_\mu\lambda-\mathrm{i}\,[A_\mu,\lambda]\ , && (2.38a)\\ \delta_\lambda\psi &= \mathrm{i}\lambda\psi\ , && (2.38b)\end{aligned}$$

and the infinitesimal noncommutative gauge transformations

$$\begin{aligned} \hat{\delta}_{\hat{\lambda}}\hat{A}_\mu &= \partial_\mu\hat{\lambda}(\lambda,A,\theta)-\mathrm{i}\left[\hat{A}_\mu(A,\theta)\stackrel{\star}{,}\hat{\lambda}(\lambda,A,\theta)\right]\ , && (2.39a)\\ \hat{\delta}_{\hat{\lambda}}\hat{\psi}(\psi,A,\theta) &= \mathrm{i}\hat{\lambda}(\lambda,A,\theta)\star\hat{\psi}(\psi,A,\theta)\ . && (2.39b)\end{aligned}$$

We may also require that the commutator of two infinitesimal gauge transformations $\hat{\delta}_\lambda\hat{\psi}$ closes to another gauge transformation just as in the commutative case:

$$\left(\hat{\delta}_{\lambda_1}\hat{\delta}_{\lambda_2}-\hat{\delta}_{\lambda_2}\hat{\delta}_{\lambda_1}\right)\hat{\psi}=\hat{\delta}_{\mathrm{i}[\lambda_1,\lambda_2]}\hat{\psi}\,, \tag{2.40}$$

where we have omitted the hat over the gauge parameter in the subscript of λ and we will continue to do so ($\hat{\delta}_{\hat{\lambda}}\equiv\hat{\delta}_\lambda$), since also the noncommutative gauge transformation actually depends on the commutative one trough the mapping (2.36b). Equation (2.40) can be expanded to

$$\delta_{\lambda_1}\hat{\lambda}(\lambda_2,A,\theta)-\delta_{\lambda_2}\hat{\lambda}(\lambda_1,A,\theta)-\mathrm{i}\left[\hat{\lambda}(\lambda_1,A,\theta)\stackrel{\star}{,}\hat{\lambda}(\lambda_2,A,\theta)\right]=\hat{\lambda}(\lambda_3,A,\theta)\,, \tag{2.41}$$

where we have factorized the field $\hat{\psi}$ and kept in mind that $\hat{\lambda}$ also depends on A and thus $\delta_{\lambda'}\hat{\lambda}(\lambda,A,\theta)\neq 0$. With $\lambda_3\equiv-\mathrm{i}[\lambda_1,\lambda_2]$ we abbreviate the commutative gauge transformation resulting from the commutative consistency equation

$$(\delta_{\lambda_1}\delta_{\lambda_2}-\delta_{\lambda_2}\delta_{\lambda_1})\psi=[\lambda_1,\lambda_2]\psi\equiv\delta_{-\mathrm{i}[\lambda_1,\lambda_2]}\equiv\delta_{\lambda_3}\psi\,. \tag{2.42}$$

After first solving the consistency equation (2.40), the solution for $\hat{\lambda}$ can be plugged into (2.37), which can be then solved for $\hat{A}$ and $\hat{\psi}$.
The solution of the gauge equivalence and the consistency equation can be found order by order in θ. For the Moyal-Weyl $\star$-product we take its Taylor expansion

and expand the gauge and matter fields and the gauge parameter in powers of θ:

$$\hat{\lambda}(\lambda, A, \theta) = \lambda + \sum_{n=1}^{\infty} \lambda^n(\lambda, A, \theta) \,, \tag{2.43a}$$

$$\hat{A}_\mu(A, \theta) = A_\mu + \sum_{n=1}^{\infty} A_\mu^n(A, \theta) \,, \tag{2.43b}$$

$$\hat{\psi}(\psi, A, \theta) = \psi + \sum_{n=1}^{\infty} \psi^n(\psi, A, \theta) \,. \tag{2.43c}$$

Note that the expansion in (2.43) is not to be understood as the expansion in the basis of the enveloping algebra (2.33). We obtain for the consistency equation at first order in θ an inhomogeneous linear equation:

$$\delta_{\lambda_1}\lambda^1(\lambda_2, A, \theta) - \delta_{\lambda_2}\lambda^1(\lambda_1, A, \theta) - \mathrm{i}\left[\lambda^1(\lambda_1, A, \theta), \lambda_2\right] - \mathrm{i}\left[\lambda_1, \lambda^1(\lambda_2, A, \theta)\right] - \lambda^1(\lambda_3, A, \theta) = -\frac{1}{2}\theta^{\mu\nu}\left\{\partial_\mu\lambda_1, \partial_\nu\lambda_2\right\} \,, \tag{2.44}$$

where all terms involving the unknown λ^1 have been brought on one side and the inhomogeneity on the other. For solving this equation, we make an ansatz containing all possible terms involving one gauge parameter, gauge fields and partial derivatives. We only have to account for the hermiticity of the non-commutative gauge parameter. The special solution for the gauge parameter at $\mathcal{O}(\theta)$ obtained from (2.44) is thus:

$$\lambda^1(\lambda, A, \theta) = \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu\lambda, A_\nu\right\} \,. \tag{2.45}$$

The gauge equivalence equations (2.37) can be written as follows:

$$\hat{\delta}_\lambda \hat{A}_\mu(A, \theta) = \delta_\lambda \hat{A}_\mu(A, \theta) \,, \tag{2.46a}$$

$$\hat{\delta}_\lambda \hat{\psi}(\psi, A, \theta) = \delta_\lambda \hat{\psi}(\psi, A, \theta) \,, \tag{2.46b}$$

where the commutative gauge transformation δ_λ acts on the arguments of $\hat{A}_\xi(A, \theta)$ via the chain rule. Expanded in powers of θ, equations (2.46) read in first order:

$$\delta_\lambda A_\xi^1 - \mathrm{i}[\lambda, A_\xi^1] = \partial_\xi\lambda^1(\lambda) - \mathrm{i}[A_\xi, \lambda^1(\lambda)] + \frac{1}{2}\theta^{\mu\nu}\left\{\partial_\mu A_\xi, \partial_\nu\lambda\right\} \,, \tag{2.47a}$$

$$\delta_\lambda\psi^1 - \mathrm{i}\lambda\psi^1 = \mathrm{i}\lambda^1\psi - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\lambda\partial_\nu\psi \,. \tag{2.47b}$$

Inserting the solution (2.45) for λ^1 and making the corresponding ansatz for A_ξ^1 and ψ^1, we obtain the special solutions in $\mathcal{O}(\theta)$:

$$A_\xi^1(A, \theta) = \frac{1}{4}\theta^{\mu\nu}\left\{F_{\mu\xi} + \partial_\mu A_\xi, A_\nu\right\} \,, \tag{2.48a}$$

$$\psi^1(\psi, \theta) = \frac{1}{2}\theta^{\mu\nu}\left(A_\mu\partial_\nu\psi + \frac{\mathrm{i}}{2}A_\mu A_\nu\psi\right) \,. \tag{2.48b}$$

These solutions together with the solution for the gauge parameter (2.45) represent the Seiberg-Witten maps up to first order in the noncommutative parameter θ as found in [5] (for the gauge field and parameter) and in [16] (also for the matter field). Nevertheless, these are not the full solutions of the corresponding equations, they have to be completed by the solutions of the homogeneous part of equations (2.44) and (2.47).
Again, accepting only Hermitian expressions for the gauge parameter, we find one solution to the homogeneous consistency equation

$$\delta_{\lambda_1}\lambda^1(\lambda_2, A, \theta) - \delta_{\lambda_2}\lambda^1(\lambda_1, A, \theta) - \mathrm{i}\left[\lambda^1(\lambda_1, A, \theta), \lambda_2\right] - \mathrm{i}\left[\lambda_1, \lambda^1(\lambda_2, A, \theta)\right] \\ - \lambda^1(\lambda_3, A, \theta) = 0\,, \quad (2.49)$$

which we parametrize by a real but otherwise arbitrary coefficient c_λ^1:

$$\lambda^1_{c^1_\lambda}(\lambda, A, \theta) = \mathrm{i}\, c^1_\lambda \theta^{\mu\nu}\left[\partial_\mu\lambda, A_\nu\right]\,. \quad (2.50)$$

The freedom of adding the homogeneous solution to the special solution (2.45) is thus a freedom in the Seiberg-Witten map. Of course, we must consistently plug the homogeneous solution for the gauge parameter into the gauge equivalence equations (2.47) and seek for the emerging contributions to the solutions for A^1_ξ and ψ^1. We thus obtain:

$$A^1_{\xi, c^1_\lambda} = \mathrm{i} c^1_\lambda\, \theta^{\mu\nu}[D_\xi A_\mu, A_\nu]\,, \quad (2.51a)$$
$$\psi^1_{c^1_\lambda} = -\,c^1_\lambda\, \theta^{\mu\nu} A_\mu A_\nu \psi\,, \quad (2.51b)$$

with the covariant derivative $D_\xi A_\mu = \partial_\xi A_\mu - \mathrm{i}[A_\xi, A_\mu]$. The gauge and the matter field have also their own ambiguities. Independently of the freedom in the Seiberg-Witten map of λ^1 we find the homogeneous solutions to the gauge equivalence equations (2.47):

$$A^1_{\xi, c^1_A} = -2\mathrm{i}\, c^1_A\, \theta^{\mu\nu} D_\sigma F_{\mu\nu}\,, \quad (2.52a)$$
$$\psi^1_{c^1_\psi} = \frac{c^1_\psi}{2}\, \theta^{\mu\nu} F_{\mu\nu}\psi\,, \quad (2.52b)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]$ the ordinary field strength tensor and the covariant derivative acting on it in the usual way:

$$D_\xi F_{\mu\nu} = \partial_\xi F_{\mu\nu} - \mathrm{i}[A_\xi, F_{\mu\nu}]\,. \quad (2.53)$$

The detailed discussion of these ambiguities and their physical (and non-physical) meaning is the subject of chapter 4. For now it suffices to acknowledge that in first order in θ they correspond to field redefinitions, as it will be proven in section 4.3. We summarize the full solution for the Seiberg-Witten maps for the gauge parameter, gauge and matter fields to first order in θ including all ambiguities:

$$\lambda^1(\lambda, A, \theta) = \theta^{\mu\nu}\Big(\frac{1}{4}\{\partial_\mu\lambda, A_\nu\} + \mathrm{i} c^1_\lambda\left[\partial_\mu\lambda, A_\nu\right]\Big)\,, \quad (2.54a)$$

$$A^1_\xi(A,\theta) = \theta^{\mu\nu}\Big(\frac{1}{4}\{F_{\mu\xi}+\partial_\mu A_\xi, A_\nu\} + \mathrm{i}\, c^1_\lambda\,[D_\xi A_\mu, A_\nu] - 2\mathrm{i}\, c^1_A\, D_\xi F_{\mu\nu}\Big)\,, \tag{2.54b}$$

$$\psi^1(\psi,A,\theta) = \theta^{\mu\nu}\Big(\frac{1}{2}A_\mu\partial_\nu\psi + \frac{\mathrm{i}}{4}A_\mu A_\nu\psi - c^1_\lambda\, A_\mu A_\nu\psi + \frac{c^1_\psi}{2}F_{\mu\nu}\psi\Big)\,. \tag{2.54c}$$

It can now be seen how the problems of the "naive" NCQED model in section 2.3.2 are solved by means of the Seiberg-Witten maps. It is clear from (2.54) that the fields and gauge parameter are enveloping algebra valued, and a commutator like (2.32) closes only in the enveloping algebra.
Of course, in reproducing a version of the full SM on noncommutative space-time the Higgs field ϕ and thus its Seiberg-Witten map $\hat{\phi}$ are also needed. It is not trivially given by the corresponding Seiberg-Witten map for matter fields. Yukawa terms which enter the noncommutative Lagrangian resemble the form $\bar{\hat{\Psi}}_L \star \hat{\phi} \star \hat{\psi}_R$, with $\hat{\Psi}_L$ denoting a left-handed doublet and $\hat{\psi}_R$ a right-handed singlet. Since ϕ does not commute with the generators of $U(1)$ and $SU(3)$ in the noncommutative case, the Higgs field must transform from both sides each with the appropriate gauge group in order to preserve gauge invariance [16]:

$$\hat{\delta}_{\lambda,\lambda'}\hat{\phi}(\phi,A,A',\theta) = \mathrm{i}\hat{\lambda}(\lambda,A,\theta)\star\hat{\phi} - \mathrm{i}\hat{\phi}\star\hat{\lambda}'(\lambda',A',\theta)\,. \tag{2.55}$$

The solution is given by a hybrid Seiberg-Witten map, reading to $\mathcal{O}(\theta)$:

$$\phi^1(\phi,A,A',\theta) = \frac{1}{2}\theta^{\mu\nu}A_\nu\Big(\partial_\mu\phi - \frac{\mathrm{i}}{2}(A_\mu\phi + \phi A'_\mu)\Big) - \frac{1}{2}\theta^{\mu\nu}\Big(\partial_\mu\phi - \frac{\mathrm{i}}{2}(A_\mu\phi+\phi A'_\mu)\Big)A'_\nu\,. \tag{2.56}$$

In this thesis we are only concerned with the matter and gauge field interactions in the fermionic massless limit, thus we will not pursue the Higgs field, its Seiberg-Witten map or finding the ambiguities therein.

2.5 Other Problems

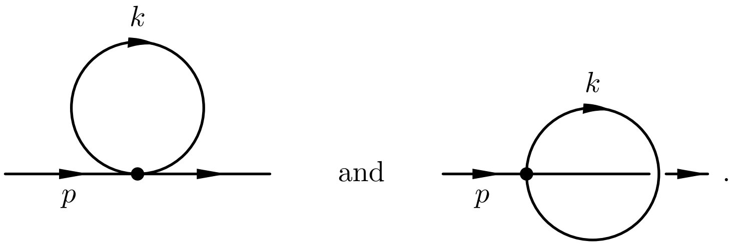
Theories on noncommutative space-time are still plagued with problems for which satisfactory solutions haven't been found yet, like unitarity. Nevertheless, other problems of great importance, like renormalization and the so called UV/IR mixing phenomenon innate to QFT's on noncommutative space-time seem to have found at least some partial answers. Thus, we will close this introductory chapter by shortly reviewing these problems and the proposed solutions as far as available.

2.5.1 UV/IR Mixing and Renormalization

Already a simple theory, like the four dimensional Euclidean scalar field theory ϕ^4 appears to be nonrenormalizable in the noncommutative case. The interaction term in the ϕ^4-action equipped with the Moyal-Weyl $\star$-product

$$\mathcal{S} = \int \mathrm{d}^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big) \tag{2.57}$$

enters the theory on noncommutative space-time accompanied by infinitely many derivatives rendering the theory highly nonlocal. We will see that the nonlocal behaviour is the cause for UV/IR mixing in the ϕ^4 theory. The Feynman rules derived from the action above will contain momentum dependent phase factors and, due to the noncommutativity, the interaction is not totally symmetric under the exchange of the momenta, but only under their cyclic permutation [17]. This leads to the distinction between so called planar and non-planar diagrams which does not exist for ordinary ϕ^4 theory. As an example, we take a look at the two tadpole diagrams



The quartic ϕ-interaction following from (2.57) receives a momentum dependent phase factor:

$$V(p_1,p_2,p_3,p_4) = \mathrm{e}^{\mathrm{i}\sum_{i<j=1}^{4}(-1)^{i+j+1}p_i\theta p_j} \,. \tag{2.58}$$

The planar amplitude on the left side will equal the commutative one, since the order of the momenta entering the the vertex is such, that the factors in the phase of the vertex formula given by (2.58) will add up to zero:

$$\Gamma_{1,\text{planar}} = \frac{\lambda}{3(2\pi)^4}\int\frac{\mathrm{d}^4k}{k^2+m^2}\,. \tag{2.59}$$

This is similar to the one loop mass correction of the commutative theory and is quadratically divergent at high energies. For the non-planar amplitude on the right hand side, the lines enter the vertex in a different order and a momentum dependent phase factor remains, yielding

$$\Gamma_{1,\text{non-planar}} = \frac{\lambda}{6(2\pi)^4}\int\frac{\mathrm{d}^4k}{k^2+m^2}\mathrm{e}^{\mathrm{i}k\theta p}\,, \tag{2.60}$$

where we recall the notation $k\theta p \equiv k_\mu\theta^{\mu\nu}p_\nu$. This factor gives the expectation that in the high energy limit, $k\to\infty$, the integral, which otherwise is quadratically divergent like the planar one, might be finite due to the damping effect of the rapid oscillation of the phase factor. The quadratic divergence of the commutative case is recovered in the limit $\theta\to 0$, as expected. Nevertheless, the damping factor of the phase can be removed by letting $p\to 0$, and then we reobtain the quadratic divergence, but this time in the $p\to 0$ limit. This

UV/IR mixing was first noticed in [18], where, after introducing the Schwinger parameter α

$$\frac{1}{k^2+m^2} = \int_0^\infty \mathrm{d}\alpha\ \mathrm{e}^{-\alpha}(k^2+m^2)\,, \tag{2.61}$$

performing the k-integration and multiplying the integrands by the Pauli-Villars regulator $\exp\left(-\frac{1}{\Lambda^2\alpha}\right)$ to regulate the $\alpha \to 0$ divergence one obtains:

$$\Gamma_{1,\text{planar}} = \frac{\lambda}{48\pi^2}\Big(\Lambda^2 - m^2\ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}(1)\Big)\,, \tag{2.62}$$

$$\Gamma_{1,\text{non-planar}} = \frac{\lambda}{96\pi^2}\Big(\Lambda^2_{\text{eff}} - m^2\ln\left(\frac{\Lambda^2_{\text{eff}}}{m^2}\right) + \mathcal{O}(1)\Big)\,, \tag{2.63}$$

with

$$\Lambda^2_{\text{eff}} = \frac{1}{\frac{1}{\Lambda^2} + p\circ p} \quad \text{and} \quad p\circ p \equiv |p^\mu\theta_{\mu\nu}\theta^{\nu\rho}p_\rho|\,. \tag{2.64}$$

Thus, in the $\Lambda \to \infty$ limit, the non-planar one loop graph remains finite (regulated by the noncommutativity). But, in this limit, the effective cut-off $\Lambda^2_{\text{eff}} = \frac{1}{p\circ p}$ goes to infinity when either $\theta \to 0$ or $p \to 0$. The total $1PI$ quadratic effective action up to first order in the coupling λ is then given by [18]:

$$\mathcal{S}_{1PI} = \int \mathrm{d}^4p\ \frac{1}{2}\Big(p^2 + M^2 + \frac{\lambda}{96\pi^2}\Lambda^2_{\text{eff}} - \frac{\lambda}{96\pi^2}M^2\ln\left(\frac{\Lambda^2_{\text{eff}}}{M^2}\right)\Big)\phi(p)\phi(-p)\,, \tag{2.65}$$

where $M^2 = m^2 = \frac{\lambda}{48\pi^2}\Lambda^2 - \frac{\lambda}{48\pi^2}m^2\ln\left(\frac{\Lambda^2}{m^2}\right)$ is the renormalized mass coming from the planar graph.

Although the original UV divergence of the non-planar one loop diagrams has been regularized by the noncommutativity for generic external momenta, they diverge for exceptional external momenta, i.e. $p \to 0$, where the regulating phase becomes inefficient. In the $\Lambda \to \infty$ limit, the theory has a new IR divergence, arising from the UV region of the momentum integration. This is called UV/IR mixing. Inserting non-planar graphs as a subgraph into bigger graphs, the external momenta becomes internal and the exceptional momentum is realized by loop integration.

The break-through w.r.t. the UV/IR mixing problem arrived with the work [19], where the authors prove the renormalizability of the real four dimensional Euclidean scalar theory defined on the Moyal deformed space R^4_θ to all orders in perturbation theory. By adding a harmonic term to the Lagrangian the propagator of the free theory is modified giving rise to an infrared cut-off which allows to decouple the different scales of the theory. A detailed review of the UV/IR mixing problem and the Grosse-Wulkenhaar solution to it for the scalar field theory can be found in in [20].

In [21], an external gauge potential is minimally coupled to the scalar renormalizable ϕ^4 theory. Thus, the Yang Mills action receives an additional term, which can be identified as the gauge theory harmonic counter part of the oscillator term, giving hope for renormalizability of such gauge models. Encouraged by the results of [19], a promising candidate for renormalizable noncommutative $U(1)$ gauge theory was presented in [22, 23]. Here, the oscillator terms enters the theory as a gauge fixing term. Nevertheless, the transition from Euclidean to Minkowski space-time remains problematic [24].

While the UV/IR mixing is the impediment to the renormalizability of θ-unexpanded theories, θ-expanded noncommutative gauge theories, like the model we consider, seem to be renormalizable order by order in perturbation theory, albeit we must specify in which sense we understand renormalizability.

In [25] the renormalizability of the photon self-energy is proven to all orders in perturbation theory in the Seiberg-Witten map approach. The authors in [26, 27, 28] address the one loop renormalizability of noncommutative $SU(N)$ theories in the θ expanded approach using Seiberg-Witten maps. They demonstrate that only for a special choice of a free parameter ($a = 1, 3$) originating from a higher order noncommutative gauge interaction, noncommutative $SU(N)$ theories are renormalizable [26]. We will refer to their work in section 3.1.2, where we explicitly point out where the parameter a comes into play.

Thus, it seems that pure Yang Mills theories on noncommutative space-time pose no problems w.r.t. their renormalizability. However, this behaviour is spoiled when adding fermions. Multiplicatively renormalization, meaning that the renormalized theory is achieved by a redefinition of the appropriate quantities, has not been proven yet for gauge theories including Dirac fermions.

In [29, 30] noncommutative QED with fermions is shown to be (multiplicatively) nonrenormalizable in the θ-expanded approach, due to divergencies in the fermion four-point function. The counter-term which has to be added by hand does not correspond to a redefinition of the Seiberg-Witten maps.

For the purpose of this work, we do not meet this obstruction. We are on the safe side by considering tree level processes and making no use of four-fermion operators. Anyway, even if we had such operators, within an effective theory, at each order of the expansion parameter only a finite number of counter term would have to be added.

2.5.2 Unitarity

Unitarity of noncommutative theories becomes a nontrivial problem when considering time-like ($\theta^{0i} \neq 0$) noncommutativity. It has been shown that such theories are either gauge invariant but not unitary [31] *or* they are unitary but not gauge invariant [32, 33].

NCQFT with Seiberg-Witten maps are gauge invariant by construction. In order to check the unitarity of the theory, it must exist in all orders of θ. In [34], where the Seiberg-Witten maps have been computed to all orders in θ (up to a finite order in the gauge field), tree level unitarity for e^+e^- annihilation could be proven to hold, even if a smearing of momenta is found to be indispensable.

In this thesis we consider an effective theory using the θ expansion of the $\star$-product and of the Seiberg-Witten maps up to $\mathcal{O}(\theta^2)$, where unitarity is not an issue, as long as $|s\theta| < 1$.

2.5.3 Lorentz Violation

In general, when speaking of Lorentz invariance, one must specify whether *observer* or *particle* Lorentz transformations are meant. The first ones refer to coordinate changes relating physical observations made in two inertial frames characterized by different velocities and orientations. The latter ones designate transformations relating physical properties of two particles with different momenta within one specific inertial frame. In the presence of a background field, these two approaches are not equivalent any more, since the background field transforms as a tensor under observer Lorentz transformations and as a set of scalars under particle Lorentz transformations.

In theories on noncommutative space-time defined by the relation (2.1), particle Lorentz invariance is obviously lost, while $\theta^{\mu\nu}$ transforms like a Lorentz tensor. The Lorentz violation originates from the constants $\theta^{\mu\nu}$ in (2.1). θ^{0i} and $\epsilon^{ikj}\theta_{jk}$ are fixed three-vectors that define preferred directions in a given Lorentz frame. Thus, phenomena such as the diurnal variation of collider cross sections have to be taken into account, or the position of the experiment on earth. Then, when performing the same experiment in two different laboratories, the components of $\theta^{\mu\nu}$ will differ depending of the local coordinate frame. Thus, the coordinates of different experiments have to be translated to a common, slowly varying astronomical frame, e.g. the cosmical microwave background [35].

Of course, so far it seems that Lorentz invariance is an unbroken symmetry of nature. No experimental evidence for Lorentz violation is available, giving thus very stringent bounds on noncommutative theories.

One point of view of dealing with the Lorentz violation inherent to the noncommutativity of space-time is to simply ignore it, by considering that noncommutativity becomes relevant only for very short distances, whereas the existing precise tests of Lorentz invariance probe distances from the atomic scale to astronomical scales. Thus, effective theories as expansion in θ are examined and one can settle for reobtaining the Lorentz-symmetry conserving commutative case in the limit $\theta \to 0$. From another point of view, one can try to build Lorentz conserving noncommutative field theories ab initio. Thus, in [36] a model is proposed, where a Lorentz invariant discrete space-time is achieved by promoting θ to an operator living in the same algebra as the coordinates. The appropriate algebra is interpreted as the contraction of the Lorentz invariant algebra due to Snyder [1]. Another approach is to exploit quantum-group techniques to reinterpret noncommutative field theory as a twist deformed quantum field theory now invariant under the twist deformed Poincaré algebra [37].

We will adopt the first attitude being satisfied by considering an almost Lorentz invariant noncommutative theory which is restored in the limit of a vanishing θ.

Even though Lorentz violation is an intrinsic feature of canonical noncommutativity (2.1), the CPT symmetry seems to be accidentally realized [38, 39]. In contrast, all other combinations of the discrete symmetries C, P, T can be broken in general noncommutative theories.

2.6 Current Bounds on the Noncommutative Scale

One of the goals of this work is to derive bounds on the noncommutative scale $\Lambda_{\rm NC}$ from future colliders. Therefore, we will dedicate this section to the bounds which can presently be found in the literature inlcuding estimates for some planned experiments.

There are no theoretical predictions on the magnitude of $\Lambda_{\rm NC}$. A priori, the natural scale where noncommutative effects arise in quantum gravity is given by the Planck scale $m_p = 10^{19}$ GeV. If this were so, we can probably forget about direct observations of noncommutativity for ever. Nevertheless, we can imagine that this scale might be broken down to the TeV range (e.g. in scenarios with large extra dimensions) and thus be reachable by the next generation of colliders.

A quite comprehensive overview is given in [40], from which we will draw in the remainder of this section with emphasis on bounds coming from high energy scattering experiments.

Phenomenological studies subsequently giving bounds on the noncommutative scale have been performed mainly within the framework of the simplest noncommutative extension of the SM, the "naive" NCQED, and considering electrons to avoid the charge quantization problem discussed in section 2.3.2. Thus, the OPAL collaboration [41] finds $\Lambda_{\rm NC} > 140$ GeV. In [35] various NCQED processes (Møller and Bhabha scattering, pair annihilation and $\gamma\gamma \to \gamma\gamma$ scattering) are studied, revealing a complementary behaviour w.r.t. the noncommutative parameter space. The estimated bounds for $\Lambda_{\rm NC}$ at 95% confidence level from a future linear collider with $\sqrt{s} = 500$ GeV and $\mathcal{L} = 500\,\text{fb}^{-1}$ range from $\Lambda_{\rm NC} \gtrsim 500\,\text{GeV} - 1.7\,\text{TeV}$.

In [42] the C-violating decay of the neutral pion into three photons is studied. In the SM the decay occurs via weak interactions and is too small to be experimentally accessible, giving thus room for studying C-violating effects beyond weak interactions, like QED. In NCQED, $\pi^0 \to \gamma\gamma\gamma$ is possible and assuming a noncommutative scale of order 1 TeV, its branching ratio is much larger then its SM counterpart. Nevertheless, it still is far below the current experimental upper bound.

A completely new interaction channel is studied in [43]. The decay of an off-shell photon into a neutrino-antineutrino pair in stellar clusters is calculated in the Seiberg-Witten map approach of the NCSM. While in the SM the effective $\gamma\nu\bar{\nu}$-vertex is induced by a penguin diagram, in the NCSM this process is allowed at tree level. From demanding that the ratio of the noncommutative tree-level and the SM one-loop $\gamma \to \nu\bar{\nu}$ decay rates $\Gamma_{\rm NC}(\gamma \to \nu\bar{\nu})/\Gamma_{\rm SM}(\gamma \to \nu\bar{\nu})$ is less than one in order to satisfy the requirement that any new energy loss mechanism should not excessively exceed the standard neutrino losses, the rather low bound $\Lambda_{\rm NC} > 80$ GeV is obtained.

Within the framework of the renormalizable model [26] already mentioned in section 2.5.1, the authors derive bounds from the $Z\gamma\gamma$ decay, otherwise forbidden in the SM at tree level [44]. They find $\Lambda_{\rm NC} > 110$ GeV using existing experimental values for the partial decay width obtained from $e^+e^- \to \gamma\gamma$ annihilation (see references in [44]). From LHC experimental expectations a bound of $\Lambda_{\rm NC} \gtrsim 1\,\text{TeV}$ is found for $\vec{E}^2 + \vec{B}^2 \simeq 1$ and a value $K_{Z\gamma\gamma} = 0.5$ for the triple gauge boson coupling constant (see section 3.1.2 for details regarding the model).

Electroweak precision measurements, like those of the anomalous muon magnetic moment a_μ, can also constrain $\Lambda_{\rm NC}$. The noncommutative parameter enters a_μ by an additional diagram contributing to the fermion-photon vertex at one-loop level containing the SM forbidden three photon vertex [45]. Nevertheless, this leads only to a contribution $\sim \vec{B}$ to the magnetic moment of the muon, which is independent of the muon spin. Experiments measuring a_μ from the precession of the muon spin in an external magnetic field are thus not sensitive for this contribution. On the other hand, noncommutative effects enter also the muon decay, which is used in order to measure the muon spin. In the muon decay $\mu \to \nu_\mu e^- \bar{\nu}_e$ the fermion-W-boson vertices receive noncommutative corrections at one loop level, which alter the electron angular distribution [46]. Noncommutativity of order $\Lambda_{\rm NC} \simeq 1\,\text{TeV}$ may account for the discrepancy between the experimental value and the theoretical prediction $a_\mu^{exp} - a_\mu^{SM}$ of the anomalous magnetic moment.

In [47] it is shown how noncommutativity of the order of $\Lambda_{\rm NC} \simeq 2\,\text{TeV}$ can account for the CP violating observable ϵ_K in the K^0-meson system, while [48] discusses inclusive $b \to s\gamma$ decay.

Recently, W^+W^- production at the LHC was studied within the same framework of the θ-expanded NCSM as it is used in this work [8]. It was shown that if noncommutative effects occur at a scale of $\Lambda_{\rm NC} \simeq 700\,\text{GeV}$ they can be measured at the LHC by means of $pp \to W^+W^-$ with subsequent semileptonic decay.

The noncommutative parameter $\Lambda_{\rm NC}$ also receives constraints from measurements coming from entirely different experimental corners. We mention that the bounds from high precision atomic experiments vary from $\Lambda_{\rm NC} \gtrsim 200\,\text{TeV}$ (Lamb shift in the hydrogen atom) [49] to $\Lambda_{\rm NC} \gtrsim 10^{12} - 10^{14}\,\text{GeV}$ (clock comparison experiments)[50]. Astrophysical and cosmological bounds were derived to range from $\Lambda_{\rm NC} \gtrsim 1\,\text{TeV} - 10^8\,\text{TeV}$. For further details regarding bounds on $\Lambda_{\rm NC}$ we refer to the overview in [40] and references therein.

We have highlighted only some of the bounds on $\Lambda_{\rm NC}$ derived in the literature. Concluding, we note that the bounds are very different in magnitude. However, since the models considered by the various authors are different and since also the experimental setups probe different energy and length scales, this is not surprising.

We remark that apart from the already published parts of this work [7] and [8] no bounds coming from hadronic scattering experiments are currently present in the literature. This is the sector where the present work comes in: the first part is dedicated to deriving bounds on $\Lambda_{\rm NC}$ from proton-proton scattering into a Z and a photon at the LHC.

Chapter 3

The NCSM at $\mathcal{O}(\theta)$

This chapter presents a realization of the $U(1)_Y \times SU(2)_L \times SU(3)_S$-SM on noncommutative space-time as an effective theory (as an expansion in powers of θ), as it was proposed in [16]. The theoretical part discusses the model up to first order in θ, providing the tools for phenomenological studies. In the second part of this chapter we analyze the phenomenological consequences of space-time noncommutativity of mainly one process, $f\bar{f} \to Z\gamma$, up to first order in θ. Thus, at the LHC we will look for noncommutative signals in $pp \to \ell^+\ell^-\gamma$ scattering, with $q\bar{q} \to Z\gamma$ as a subprocess, and at the ILC we will investigate $e^+e^- \to Z\gamma \to \ell^+\ell^-\gamma$.

3.1 The model

The previous chapter showed that the restriction to $U(N)$-gauge theories on noncommutative space-time can be circumvented by extending the fields and gauge parameters to the enveloping algebra. $SU(N)$ gauge theories and therefore the $U(1)_Y \times SU(2)_L \times SU(3)_S$-SM can now be realized on noncommutative space-time. The crucial components are the Moyal-Weyl-$\star$ product carrying information about the underlying noncommutative manifold (reproducing the noncommutative algebra (2.1) on the commutative four dimensional manifold) on the one hand, and the Seiberg-Witten maps (2.54), on the other hand, accounting for the noncommutative gauge structure to be induced by ordinary gauge transformations.
In order to build the noncommutative SM as described above, we also need to deal with the tensor product of gauge groups for the noncommutative case. The most general expression for the noncommutative gauge parameter for the tensor product of two gauge groups $G \times G'$ has to satisfy the corresponding consistency equation (2.40) for each gauge group as well as new mixed consistency relations (for a detailed computation see the appendix of [39]). The freedom in the choice of the Seiberg Witten map can then be used to take the simplest and most natural approach. Therefore, in order to account for the structure group $U(1)_Y \times SU(2)_L \times SU(3)_S$ of the standard model, we take the tensor product and consider the "master" gauge potential:

$$A_\mu = g'\mathcal{A}_\mu Y + gB_{\nu a}T^a_L + g_s G_{\nu b}T^b_S \tag{3.1}$$

with Y, T_L^a, T_S^b the generators of the groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_S$ respectively. The corresponding gauge parameter is given by

$$\lambda = g'\alpha Y + g\alpha_a^L T_L^a + g_s \alpha_b^S T_S^b \tag{3.2}$$

and the Seiberg-Witten map for the gauge parameter by[1]:

$$\widehat{\lambda} = \lambda + \frac{1}{4}\theta^{\mu\nu}\{\partial_\mu \alpha, A_\nu\} + \mathcal{O}(\theta^2)\ . \tag{3.3}$$

Due to the anticommutator, this is not a naive sum of the noncommutative gauge parameter corresponding to each factor in $U(1)_Y \times SU(2)_L \times SU(3)_S$ and thus the gauge groups mix in higher order in θ.
The parts of the noncommutative action which are relevant to this work are the fermionic and the kinetic sector:

$$\begin{aligned} \mathcal{S}_{\text{fermion}} &= \int \mathrm{d}^4x \left(\widehat{\overline{\psi}}{}^i_L \star \mathrm{i}\widehat{\not{D}}\widehat{\psi}^i_L + \widehat{\overline{\psi}}{}^i_{u,R} \star \mathrm{i}\widehat{\not{D}}\widehat{\psi}^i_{u,R} + \widehat{\overline{\psi}}{}^i_{d,R} \star \mathrm{i}\widehat{\not{D}}\widehat{\psi}^i_{d,R}\right) && (3.4) \\ \mathcal{S}_{\text{gauge}} &= -\frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{Tr}(\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}) && (3.5) \end{aligned}$$

where in the first equation we implicitly sum over the three families i. We note that in the second equation there is a trace over generators with values in the enveloping algebra. Thus, the trace is not unique, as it is in the SM, where the fields are Lie algebra valued. This will lead to inequivalent realizations of the NCSM, as we will see in the next sections. For now we just remark that we can have a minimal extension of the SM (minimal NCSM), in the sense that it stays as close as possible to the SM, and within this framework no interactions among triple gauge bosons appear in the kinetic sector. On the other hand, by choosing another representation of the fields, SM forbidden interactions among neutral gauge bosons can be added. We have anticipated the results of section 3.1.2, in order to give the context for the following brief remark regarding the other parts of the Lagrangian, the Higgs and the Yukawa sector. A detailed discussion will only be dedicated to the fermionic and gauge Lagrangian.
Due to the Seiberg-Witten mapping, the Higgs part introduces gauge boson interactions proportional to their mass, such that even in the minimal NCSM, where triple neutral gauge boson interaction are absent as in the SM, we still obtain a mass dependent contribution to the ZZZ-coupling. On the other hand, the Yukawa part gives rise to mass dependent interactions among gauge bosons while the SM interactions proportional to masses always include the interaction with the Higgs field.
When deriving the Feynman rules for the interactions needed in this work, we will take the massless limit for fermions[2], omitting thus the mass dependent terms in the fermionic interactions. We also will not need the ZZZ-coupling,

[1]We omit the ambiguity for $\hat{\lambda}$ at $\mathcal{O}(\theta)$, since it occurs neither in the Feynman rules nor in physical observables.

[2]The energies at which the processes under consideration take place are so high that it allows for this approximation. Also, in the processes considered in this work, the top quark does not appear.

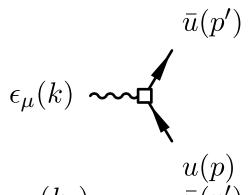
therefore we will discuss neither the Higgs nor the Yukawa Lagrangian in detail. One more feature specific to the NCSM should be mentioned: QCD and electroweak interactions are mixed giving rise to couplings like $q\bar{q}\gamma g$, $qqW^{\pm}g$ etc.
The Feynman rules at $\mathcal{O}(\theta)$ for the neutral current and the kinetic neutral gauge boson sector have first been derived in [51] as well as in [52], though without accounting for the ambiguities in the Seiberg-Witten maps. In the meantime the full Lagrangian and the complete set of Feynman rules for the electroweak and strong interactions (again, without the ambiguities) has been computed and is available in [53, 54]. In the next sections we will give the Feynman rules for the interactions relevant to this work with all ambiguities included.

3.1.1 Matter Sector (Neutral Currents)

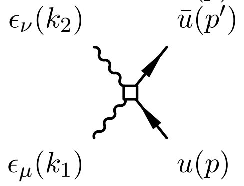
The fermionic part of the action can be calculated starting from

$$\mathcal{S}_{\text{fermionic}} = \int \mathrm{d}x\ \bar{\hat{\psi}} \star \mathrm{i}\hat{\not{D}}\hat{\psi} = \int \mathrm{d}x\ \left(\bar{\hat{\psi}} \star \mathrm{i}\not{\partial}\hat{\psi} + \bar{\hat{\psi}} \star \hat{\not{A}}\hat{\psi}\right) , \tag{3.6}$$

where we have already used the property (2.17) to eliminate one of the $\star$-products. The chiral structure of the fermionic Lagrangian is not affected by the Seiberg-Witten maps. Thus, in the above Lagrangian, we consider the fermions as pure vector currents. The necessary substitutions $\gamma_\mu \to g_V\gamma_\mu - g_A\gamma_\mu\gamma_5$ depending on the fermion flavor and the vector boson contracted with to γ_μ can be made when it is required for the calculation of scattering matrix elements.
Inserting the Seiberg Witten map (2.54) for A_μ and expanding the $\star$-product up to first order in θ the fermionic action can be computed to $\mathcal{O}(\theta)$. This calculation has already been done in [16] and the resulting Feynman rules have been constructed in [7, 51] as well as in [52, 53, 54]. Yet, the calculations in the literature do not include the ambiguities of the Seiberg Witten maps. The results presented in [7, 51, 52, 53, 54] use only one particular inhomogeneous solutions to the gauge equivalence and consistency equations (2.47) and (2.44) while the Feynman Rules presented below use the full solution of (2.44) and (2.47), including all ambiguities:



$$= \left\{ -\frac{g}{2}\left[k\theta^\mu \not{p}(1-4\xi^1_\Psi) + 2\ k\theta^\mu \not{k}(\xi^1_A - \xi^1_\Psi) - p\theta^\mu \not{k} - (k\theta p)\gamma_\mu\right] , \right. \tag{3.7}$$



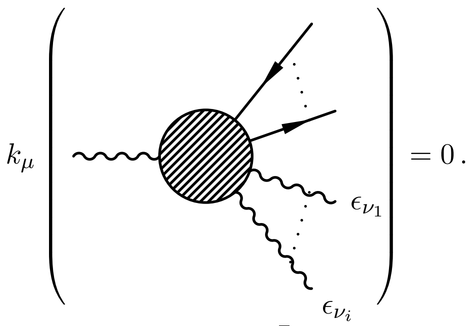
$$= \left\{ -\frac{g^2}{2}\left[k_2\theta^\mu\gamma^\nu - k_1\theta^\mu\gamma^\nu(1-4\xi^1_\Psi) - \theta^{\mu\nu}\not{k}_1 + (\mu \leftrightarrow \nu, k_1 \leftrightarrow k_2)\right] , \right. \tag{3.8}$$

with all momenta incoming. The energy scale at which the processes considered in the next section take place is large in comparison to the u- and d-quark masses, respectively. Therefore, mass term contributions are not considered

here. For the complete Feynman rules with mass terms, although without ambiguities see [53, 54].
Before turning to the gauge sector we consider the amplitude for pair annihilation using the Feynman rules from above and verify the Ward identity. The Ward identity must hold since it is the quantum field theoretical expression of local gauge symmetry, which of course must also be preserved in the noncommutative case. The mathematical expression for the Ward identity is given by

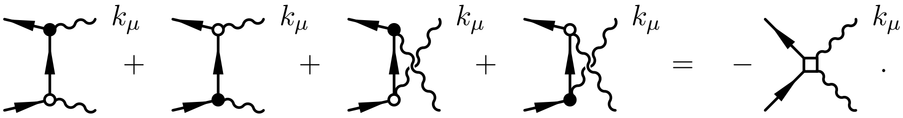
$$\frac{\partial}{\partial x^\mu} \langle 0|\mathrm{T}A^\mu(x)\Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_n(x_n)|0\rangle_{\text{amputated, on-shell}} = 0\,, \tag{3.9}$$

with the considered gauge field A_μ and the other matter or gauge fields Φ_i with physical polarizations. Its translation into momentum space can be pictured as follows:



(3.10)

We now consider the Ward identity for $f\bar{f} \to VV$. The diagrams which contribute to this process in the Seiberg-Witten map approach differ from the ones considered in the "naive" model by a contact diagram induced by the new $f\bar{f}VV$ interaction. After replacing one polarization vector with the corresponding momentum we find for on-shell particles:



Unlike the case of "naive" QED, an s-channel diagram is not required by gauge invariance. Therefore there are no constraints on the three photon couplings. In fact, as it will be immediately shown, the $\gamma\gamma\gamma$ coupling and the triple neutral gauge boson (TGB) couplings are in general not uniquely determined in the nonminimal NCSM (see section 3.1.2).

3.1.2 Gauge Sector

The noncommutative generalization of the kinetic term is given by:

$$\mathcal{S} = -\frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{Tr}(\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu})\,, \tag{3.11}$$

where the coupling constant has been explicitly extracted from the field strength $F_{\mu\nu}$. The noncommutative field strength is:

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - \mathrm{i}[\widehat{A}_\mu \stackrel{\star}{,} \widehat{A}_\nu]\,. \tag{3.12}$$

Starting from the Seiberg-Witten map for the gauge field (2.54b), the field strength to $\mathcal{O}(\theta)$ is given by:

$$
\begin{aligned}
F^1_{\rho\sigma} &= \frac{1}{2}\theta^{\mu\nu}\{F_{\rho\mu}, F_{\sigma\nu}\} - \frac{1}{4}\theta^{\mu\nu}\{A_\mu, (\partial_\nu + D_\nu)F_{\rho\sigma}\} \\
&+ c^1_\lambda\theta^{\mu\nu}[A_\mu A_\nu, F_{\rho\sigma}] + \frac{c^1_A}{2}\theta^{\mu\nu}[F_{\rho\sigma}, F_{\mu\nu}]\,,
\end{aligned} \tag{3.13}
$$

with

$$
D_\nu F_{\rho\sigma} = \partial_\nu F_{\rho\sigma} - \mathrm{i}[A_\nu, F_{\rho\sigma}]\,. \tag{3.14}
$$

Under the trace we can perform cyclical permutations of the fields. Partial integration, the antisymmetry of θ and the fact that surface terms vanish leads to

$$
\mathcal{S}^1 = -\frac{\theta^{\mu\nu}}{g^2}\int \mathrm{d}^4x\, \left(\mathrm{Tr}(F^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu}) - \frac{1}{4}\mathrm{Tr}(F^{\rho\sigma}F_{\rho\sigma}F_{\mu\nu})\right)\,. \tag{3.15}
$$

It is clear from (3.13) that the ambiguities cannot contribute to the kinetic term at $\mathcal{O}(\theta)$. They enter the Seiberg-Witten map for the field strength only with commutators of the form $[X_{\mu\nu}, F_{\rho\sigma}]$ and vanish under the trace, since $\mathrm{Tr}([A,B]B) = 0$.
We will rewrite (3.15). The first term can be written as follows:

$$
\begin{aligned}
&\theta^{\mu\nu}\mathrm{Tr}(F^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu}) \equiv \theta^{\mu\nu}F^{\rho\sigma,a}F^b_{\rho\mu}F^c_{\sigma\nu}\mathrm{Tr}(T^aT^bT^c) = \\
&= \frac{\theta^{\mu\nu}}{2}F^{\rho\sigma,a}F^b_{\rho\mu}F^c_{\sigma\nu}\mathrm{Tr}(T^a\{T^b,T^c\}) + \frac{\theta^{\mu\nu}}{2}F^{\rho\sigma,a}F^b_{\rho\mu}F^c_{\sigma\nu}\mathrm{Tr}(T^a[T^b,T^c])\,.
\end{aligned} \tag{3.16}
$$

$\theta^{\mu\nu}$ and $F^{\rho\sigma}$ are antisymmetric, therefore $\theta^{\mu\nu}F^{\rho\sigma,a}F^b_{\rho\mu}F^c_{\sigma\nu}$ is symmetric under $b \leftrightarrow c$. The expression under the trace in the second term is antisymmetric under $b \leftrightarrow c$ and thus the last term in (3.16) vanishes. A similar manipulation can be performed on the other term in (3.15) and we obtain:

$$
\mathcal{S}^1 = -\frac{\theta^{\mu\nu}}{g^2}\int \mathrm{d}^4x\, \left(\frac{1}{2}\mathrm{Tr}(F^{\rho\sigma}\{F_{\rho\mu}, F_{\sigma\nu}\}) - \frac{1}{8}\mathrm{Tr}(F_{\mu\nu}\{F^{\rho\sigma}, F_{\rho\sigma}\})\right)\,. \tag{3.17}
$$

So far no choice for the representation of the generators $\rho(T^a)$ was made. In the kinetic term of the standard model, where all the fields are Lie algebra valued, one always has the trace over two generators $\mathrm{Tr}(T^aT^b)$ which is uniquely determined ($\sim \delta^{ab}$) up to a normalization constant, independently of the representation $\rho(T^a)$. The θ expanded fields are enveloping algebra valued, which leads to the trace over three generators $\mathrm{Tr}(T^aT^bT^c)$ in (3.15). At this point, the model depends on the choice of the representation according to which the fields transform. Note that even if products $\rho(T^a)\rho(T^b)\rho(T^c)$ or anticommutators $\{\rho(T^a), \rho(T^b)\}$ are enveloping algebra valued, the generators $\rho(T^a)$ are still matrices from Lie algebra representations and remain thus traceless, a property which will become important below.
In order to account for the dependence on the representation, we write the gauge action in a more general form:

$$
\mathcal{S}_{\mathrm{gauge}} = -\frac{1}{2}\int \mathrm{d}^4x\, \sum_\rho c_\rho \mathrm{Tr}\big(\rho(\widehat{F}_{\mu\nu}) \star \rho(\widehat{F}^{\mu\nu})\big)\,, \tag{3.18}
$$

where c_ρ are weighting coefficients and the sum is over all unitary irreducible and nonequivalent representations of the group. Using (3.17) we obtain:

$$\mathcal{S}^1 = \frac{1}{4}(\sum_\rho c_\rho D_\rho^{abc})\theta^{\mu\nu} \int \mathrm{d}^4x \; F^{\rho\sigma,a} \left(\frac{1}{4} F^b_{\mu\nu} F^c_{\rho\sigma} - F^b_{\rho\mu} F^c_{\sigma\nu}\right) , \tag{3.19}$$

with

$$\frac{1}{2} D_\rho^{abc} \equiv \mathrm{Tr}\big(\rho(T^a)\{\rho(T^b), \rho(T^c)\}\big) , \tag{3.20}$$

a completely symmetric tensor in the representation ρ, which is proportional to the completely symmetric tensor d^{abc} in the adjoint representation: $D_\rho^{abc} = C(\rho) d^{abc}$. For $U(1)$ and $SU(2)$, d^{abc} and therefore D_ρ^{abc} vanishes, but it is nonzero for all $SU(N)$ with $N \geq 3$.

There are various natural choices for the representation: the matter field representations and the adjoint representation. The latter is rather suited for pure gauge interactions and it represents a minimal extension of the SM on noncommutative space-time, in the sense that it remains closest to the SM. In the adjoint representation with the field strength given by

$$F_{\mu\nu} = g' F_{Y\mu\nu} Y + g F^a_{L\mu\nu} T^a_L + g_S F^b_{S\mu\nu} T^b_S \tag{3.21}$$

and using $d^{abc} = 0$ for $U(1)$ and $SU(2)$, the gauge action at $\mathcal{O}(\theta)$ will contain only gluon interactions

$$\mathcal{S}^1 = \theta^{\mu\nu} \int \mathrm{d}^4x \; \frac{1}{4} \mathrm{Tr}(F^{S\rho\sigma} F_{S\mu\nu} F_{S\rho\sigma}) - \mathrm{Tr}(F^{S\rho\sigma} F_{S\rho\mu} F_{S\sigma\nu}) . \tag{3.22}$$

In particular, for this choice of the representation, no triple neutral gauge boson interactions appear. However, the inclusion of matter fields suggests the presence of other representations of $U(1)_Y \times SU(2)_L \times SU(3)_S$, and the gauge action (3.19) is not uniquely determined. The ambiguity lies in the choice of the real numbers c_ρ.

Any irreducible representation of the $U(1)_Y \times SU(2)_L \times SU(3)_S$ is a product of irreducible representations of $U(1)_Y$, $SU(2)_L$ and $SU(3)_S$. The generators of the $U(1)_Y \times SU(2)_L \times SU(3)_S$ are denoted by T^A, with $A = 1, \ldots, 12$:

$$\{\rho(T^A)\} = \{\rho_1(Y) \otimes \mathbf{1}_{\rho_2} \otimes \mathbf{1}_{\rho_3}, 1 \otimes \rho_2(T^a_L) \otimes \mathbf{1}_{\rho_3}, 1 \otimes \mathbf{1}_{\rho_2} \otimes \rho_3(T^l_S)\} , \tag{3.23}$$

with $a = 1, 2, 3$ and $l = 1, \ldots, 8$. From (3.18), using $\rho \equiv \rho_1 \otimes \rho_2 \otimes \rho_3$ and

$$F_{\mu\nu} \equiv F^A_{\mu\nu} T^A = f_{\mu\nu} Y + F^a_{L\mu\nu} T^a_L + F^l_{S\mu\nu} T^l_S \tag{3.24}$$

we obtain for the θ independent part of the action:

$$\begin{aligned} \mathcal{S}^{\theta\to 0}_{\mathrm{gauge}} = -\frac{1}{2} \int \mathrm{d}^4x \quad & \Big(\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2) d(\rho_3) \rho_1(Y)^2 \Big) f_{\mu\nu} f^{\mu\nu} \\ + & \Big(\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_3) \mathrm{Tr}\big(\rho_2(T^a_L)\rho_2(T^b_L)\big) \Big) F^a_{L\mu\nu} F^b_{L\mu\nu} \\ + & \Big(\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2) \mathrm{Tr}\big(\rho_3(T^l_S)\rho_3(T^k_S)\big) \Big) F^a_{S\mu\nu} F^b_{S\mu\nu} . \end{aligned} \tag{3.25}$$

We have used that

$$\begin{aligned}\mathrm{Tr}(\rho(T^1)\rho(T^i))_{i=2,3,4} &= \mathrm{Tr}\big(\rho_1(Y)\otimes \mathbf{1}_{\rho_2}\otimes \mathbf{1}_{\rho_3}\cdot 1\otimes \rho_2(T^a_L)\otimes \mathbf{1}_{\rho_3}\big)\\ &= \mathrm{Tr}\big(\rho_1(Y)\otimes \rho_2(T^a_L)\otimes \mathbf{1}_{\rho_3}\big)\\ &= \mathrm{Tr}\big(\rho_1(Y)\big)\mathrm{Tr}\big(\rho_2(T^a_L)\big)d(\rho_3) = 0\,, \end{aligned} \quad (3.26)$$

with $\mathrm{Tr}(\mathbf{1}_\rho) = d(\rho)$ the dimension of the representation and

$$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B). \quad (3.27)$$

Similarly

$$\mathrm{Tr}(\rho(T^i)\rho(T^j))_{i=1,2,3;j=5,..,12} = 0\,. \quad (3.28)$$

The expression (3.25) of course has to match the SM action

$$\mathcal{S}^{SM}_{\text{gauge}} = -\int \mathrm{d}^4x\, \left(\frac{1}{4g'^2} f_{\mu\nu}f^{\mu\nu} + \frac{1}{4g^2}F^a_{L\mu\nu}F^{a\mu\nu}_L + \frac{1}{4g_S^2}F^l_{S\mu\nu}F^{l\mu\nu}_S\right), \quad (3.29)$$

from which we obtain constraints on the otherwise free coefficients c_ρ:

$$\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2)d(\rho_3)\rho_1(Y)^2 = \frac{1}{2g'^2}\,, \quad (3.30a)$$

$$\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_3)\mathrm{Tr}\big(\rho_2(T^a_L)\rho_2(T^b_L)\big) = \frac{1}{2g^2}\delta^{ab}\,, \quad (3.30b)$$

$$\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2)\mathrm{Tr}\big(\rho_3(T^l_S)\rho_3(T^k_S)\big) = \frac{1}{2g_S^2}\delta^{lk}\,. \quad (3.30c)$$

The part of the action linear in θ is given by:

$$\begin{aligned}\mathcal{S}^1_{\text{gauge}} = \int \mathrm{d}^4x \quad & \kappa_1\theta^{\mu\nu}\left(\frac{1}{4}f_{\mu\nu}f_{\rho\sigma}f^{\rho\sigma} - f_{\rho\mu}f_{\sigma\nu}f^{\rho\sigma}\right) \quad (3.31)\\ + \quad & \kappa_2\theta^{\mu\nu}\left(\frac{1}{4}f_{\mu\nu}F^a_{L\rho\sigma}F^{a\rho\sigma}_L - f_{\rho\mu}F^a_{L\sigma\nu}F^{a\rho\sigma}_L + c.p.\right)\\ + \quad & \kappa_3\theta^{\mu\nu}\left(\frac{1}{4}f_{\mu\nu}F^l_{S\rho\sigma}F^{l\rho\sigma}_S - f_{\rho\mu}F^l_{S\sigma\nu}F^{l\rho\sigma}_S + c.p.\right)\,,\end{aligned}$$

where *c.p.* denotes cyclical permutations of the fields preserving the position of the indices, and the coefficients κ_i encode the dependency on the $c_\rho \equiv c_{\rho_1\otimes\rho_2\otimes\rho_3}$ and the irreducible representations ρ_1, ρ_2 and ρ_3:

$$\kappa_1 = \sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2)d(\rho_3)\rho_1(Y)^3\,, \quad (3.32a)$$

$$\kappa_2\delta^{ab} = \sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_3)\rho_1(Y)\mathrm{Tr}\big(\rho_2(T^a_L)\rho_2(T^b_L)\big)\,, \quad (3.32b)$$

$$\kappa_3\delta^{lk} = \sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2)\rho_1(Y)\mathrm{Tr}\big(\rho_3(T^l_S)\rho_3(T^k_S)\big)\,. \quad (3.32c)$$

Of course, other combinations of the generators appear in the trace when calculating products of three field strengths given by (3.24), but they all give no contributions. Traces over three generators each one belonging to different representations ρ_i vanish trivially, since the trace over a tensor product equals the product of traces over the individual components of the tensor product (see (3.27)) and the generators are traceless in every $SU(N)$ representation. There are still two combinations which have to be discussed separately (the indices of the field strength and θ are suppressed, the structure of the field products is the same as in (3.31)):

$$\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_3)\mathrm{Tr}\big(\rho_2(T_L^a)\{\rho_2(T_L^b),\rho_2(T_L^c)\}\big)(\theta F_L^a F_L^b F_L^c)\,, \tag{3.33}$$

$$\sum_{\rho_1,\rho_2,\rho_3} c_\rho d(\rho_2)\mathrm{Tr}\big(\rho_3(T_S^k)\{\rho_3(T_S^l),\rho_3(T_S^m)\}\big)(\theta F_L^a F_L^b F_L^c)\,. \tag{3.34}$$

The first one is proportional to the completely symmetric tensor d^{abc}, which vanishes for $SU(2)$, as already mentioned. In the second term, we have to account for the fact that any irreducible representation ρ_3 of $SU(3)$ should appear with its conjugate irreducible representation $\bar\rho_3$ and with the same weighting factor $c_{\rho_1\otimes\rho_2\otimes\rho_3^*} = c_{\rho_1\otimes\rho_2\otimes\rho_3}$, thus preserving the invariance under charge conjugation of the Lagrangian, also for the noncommutative case. The complex conjugate representation $\bar\rho_3(T^a)$ of a generator T^a is given by the negative complex conjugate of the representation $-(\rho_3(T^a))^*$, the minus sign ensuring the Lie algebra structure:

$$\Big[\bar\rho_3(T^a),\bar\rho_3(T^b)\Big] = \mathrm{i}f^{abc}\bar\rho_3(T^c)\,. \tag{3.35}$$

The trace in (3.34) is proportional to the completely symmetric tensor d_3^{klm}. The trace over the corresponding generators in the complex conjugate representation is proportional to $d_{\bar 3}^{klm}$. Since the generators T^A are Hermitian, it can be shown easily that

$$d_{\bar 3}^{klm} = -d_3^{klm}$$

and hence, (3.34) also vanishes.

From equation (3.31) we see that, for this choice of the representation, completely new interactions appear (forbidden in the SM), representing one of the most striking features of the nonminimal NCSM. Yet, the coupling constants of these interactions are not determined uniquely. The only constraint on the κ_i comes from demanding that the NCSM in the limit of the vanishing noncommutative parameter $\theta \to 0$ matches the SM, i.e. (3.30). To see how these constraints affect κ_i, we need to discuss the dependence of the κ_i on the representations of the matter fields. Considering the first generation of the standard model and the Higgs, we have six representations, summarized in table 3.1.2 which will generate six constants $c_1,\dots,c_6$.

In terms of these coefficients we can rewrite (3.30):

$$c_1 + \frac{c_2}{2} + \frac{4c_3}{3} + \frac{c_4}{3} + \frac{c_5}{6} + \frac{c_6}{2} = \frac{1}{2g'^2}\,, \tag{3.36a}$$

$$\frac{c_2}{2} + \frac{3c_5}{2} + \frac{c_6}{2} = \frac{1}{2g^2}\,, \tag{3.36b}$$

$$\frac{c_3}{2} + \frac{c_4}{2} + c_5 = \frac{1}{2g_S^2}\,, \tag{3.36c}$$

		$\mathrm{SU(3)_C}$	$\mathrm{SU(2)_L}$	$\mathrm{U(1)_Y}$	$\mathrm{U(1)_Q}$	$\mathrm{T_3}$
c_1	e_R	$\mathbf{1}$	$\mathbf{1}$	-1	-1	0
c_2	$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$
c_3	u_R	$\mathbf{3}$	$\mathbf{1}$	$2/3$	$2/3$	0
c_4	d_R	$\mathbf{3}$	$\mathbf{1}$	$-1/3$	$-1/3$	0
c_5	$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$1/6$	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$
c_6	$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$1/2$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

Table 3.1: Matter fields of the first generation and the corresponding coefficients c_i

and also (3.32):

$$\kappa_1 = -c_1 - \frac{c_2}{4} + \frac{8c_3}{9} - \frac{c_4}{9} + \frac{c_5}{36} + \frac{c_6}{4}\,, \tag{3.37a}$$

$$\kappa_2 = -\frac{c_2}{4} + \frac{c_5}{4} + \frac{c_6}{4}\,, \tag{3.37b}$$

$$\kappa_3 = \frac{c_3}{3} - \frac{c_4}{6} + \frac{c_5}{6}\,. \tag{3.37c}$$

Equation (3.31) is not in terms of the physical fields and we have to make a change of basis to obtain the physical mass eigenstates A_μ, Z_μ from the isospin eigenstates $A_{Y\mu}$, $A^a_{L\mu}$:

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \sin\theta_W & \cos\theta_W \\ \cos\theta_W & -\sin\theta_W \end{pmatrix} \begin{pmatrix} A_{Y\mu} \\ A^3_{L\mu} \end{pmatrix}\,, \tag{3.38}$$

with the weak mixing angle θ_W:

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{3.39}$$

In the definition of the field strength the corresponding coupling constant was absorbed in the gauge field, therefore, if we make the above change of basis, we actually have to write:

$$\frac{A_{Y\mu}}{g'} = \cos\theta_W A_\mu - \sin\theta_W Z_\mu\,, \tag{3.40a}$$

$$\frac{A^3_{L\mu}}{g} = \sin\theta_W A_\mu + \cos\theta_W Z_\mu\,. \tag{3.40b}$$

To obtain the action for the three photon interaction we have to consider the first two terms in (3.31) and make the replacements (3.40):

$$\mathcal{S}^1_{\gamma\gamma\gamma} = \int \mathrm{d}^4x\, \frac{e}{4} \sin 2\theta_w K_{\gamma\gamma\gamma} \left(A_{\mu\nu}A_{\rho\sigma}A^{\rho\sigma} - 4A_{\rho\mu}A_{\sigma\nu}A^{\rho\sigma}\right)\,, \tag{3.41}$$

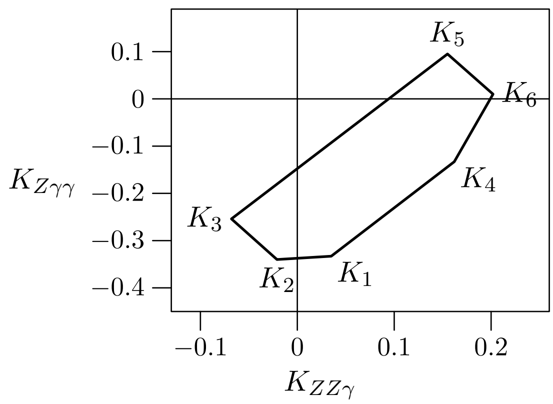


Figure 3.1: Allowed region for the values of the couplings $K_{Z\gamma\gamma}$ and $K_{ZZ\gamma}$

	K_0	K_1	K_2	K_3	K_4	K_5	K_6
$K_{Z\gamma\gamma}$	0.0	-0.333	-0.340	-0.254	-0.133	0.095	0.010
$K_{ZZ\gamma}$	0.0	0.035	-0.021	-0.068	0.162	0.155	0.202

Table 3.2: Some allowed values for the TGB coupling constants corresponding to the corners of the polygon in figure 3.1

with $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $e = gg'/\sqrt{g^2 + g'^2}$ and $K_{\gamma\gamma\gamma} = gg'(\kappa_1 + 3\kappa_2)/2$. In the same way one can obtain the interaction terms for $Z\gamma\gamma$, $ZZ\gamma$ etc. and the corresponding couplings, see [55, 56]. The constraints (3.36) can now be translated into constraints on the couplings $K_{\gamma\gamma\gamma}$, $K_{Z\gamma\gamma}$ etc. since they depend on the coefficients c_i through the κ_j. Equations (3.36) together with the condition[3] $c_i > 0$ will define an allowed region for the otherwise undetermined coupling constants. The three dimensional simplex corresponding to the conditions (3.36) was calculated in [55, 56]. The allowed region for the values which can be taken by e.g. $K_{Z\gamma\gamma}$ and $K_{ZZ\gamma}$ is depicted in figure 3.1. It is remarkable that both couplings cannot vanish simultaneously, a feature which distinguishes the nonminimal version of the NCSM from the minimal one, where these interactions are forbidden. In table 3.2 the values of TGB coupling constants corresponding to the corners of the polygon are assigned.
The Feynman rules for the three photon interaction can be derived from (3.41) and are given in appendix B.
Before closing the model building part of this chapter and moving on to the phenomenological analysis, we shortly discuss the renormalizability of the gauge sector of the NCSM, since new results on this question were recently published. After the calculations presented in this thesis were completed, it was shown

[3]The Hamiltonian must be bounded from below, therefore the coefficients c_i must be positive.

in [26] that the nonminimal NCSM is also renormalizable[4] if an appropriate finite term is added to (3.15). The counter term was shown to originate from a Lagrangian generalized to higher orders in the noncommutative field strength:

$$\mathcal{S}_\star = \mathrm{Tr} \int \mathrm{d}^4x \, \left[-\frac{1}{2} \hat{F}_{\rho\sigma} \star \hat{F}^{\rho\sigma} + b\theta^{\mu\nu} \hat{F}^{\rho\sigma} \star \hat{F}_{\rho\sigma} \star \hat{F}_{\mu\nu} + c\theta^{\mu\nu} \hat{F}^{\rho\sigma} \star \hat{F}_{\rho\mu} \star \hat{F}_{\sigma\nu} \right] . \tag{3.42}$$

This is the most general form of the action to third order in $F^{\mu\nu}$. One is free to choose $c = 0$ and thus the following action linear in θ is obtained:

$$\mathcal{S}^1_\star = -\,\theta^{\mu\nu} \mathrm{Tr} \int \mathrm{d}^4x \, \left(F^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} - \frac{a}{4} F^{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} \right) , \tag{3.43}$$

where the redefinition $a = 1 + 4b$ was used. a is a free parameter which can be determined from renormalizability requirements. This was done in [26], where two solutions for a could thus be found: $a = 1$ and $a = 3$. For $a = 1$ only the first term in (3.42) remains and thus reproduces the action (3.15), which we have used for our calculations and which was shown to be renormalizable at one-loop [28] for the minimal NCSM. For the nonminimal NCSM, the solution $a = 3$ is required, in order to make the theory one-loop renormalizable. Hence, we must add the counter term $F^{\rho\sigma} F_{\rho\sigma} F_{\mu\nu}/2$ to our action (3.15), stemming from the second term in (3.42), in order to ensure renormalizability.
Thus, it was proven that the gauge sector of noncommutative $SU(N)$ gauge theories are one-loop renormalizable. Nevertheless, the question regarding the renormalizability of theories including Dirac fermions remains still unanswered. For our calculations and subsequent phenomenological studies we have started from the action (3.15) which is renormalizable only for the minimal NCSM. For a renormalizable nonminimal NCSM the action (3.43) should be used. The parameter a alters the TGB vertex and it would be worthwhile to repeat our phenomenological analysis for the nonminimal NCSM with $a = 3$.
In any case, our calculations are consistent also in the nonminimal NCSM with $a = 1$ since our cross sections were calculated at tree-level. Neither do the processes under consideration require four fermion counter terms.

3.2 Phenomenology

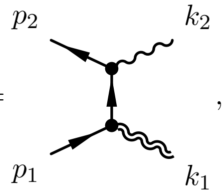
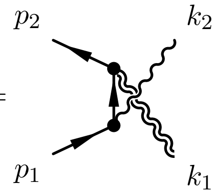
For the phenomenological studies at $\mathcal{O}(\theta)$ we have concentrated on the process $q\bar{q} \to Z\gamma$. A process involving two neutral gauge bosons offers the possibility to study the neutral gauge boson couplings presented in the previous chapter otherwise forbidden in the standard model or in the mNCSM. On the other hand, processes containing photons or Z-bosons in the final state are particularly appealing because they will provide a rather clean signature at the LHC. At the LHC the polarization of particles in the final state will not be observed, therefore we are interested in noncommutative signatures in unpolarized cross sections. As it was presented in [52], processes with a symmetric final state (e.g. $\gamma\gamma$) will have only minute observable noncommutative effects after summing all polarization states. It will be shown in the next section that due to

[4]The renormalizability of the minimal NCSM had already been proven earlier [28].

the axial coupling of the Z, the situation is different if the final state particles are a Z-boson and a photon.

3.2.1 Amplitudes

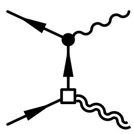
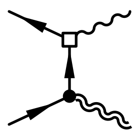
Within the standard model, only the t- and u-channel contribute to the amplitude for $q\bar{q} \to Z\gamma$:

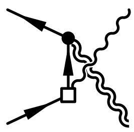
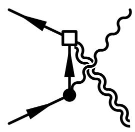
$A_t^{\mathrm{SM}} =$  , $A_u^{\mathrm{SM}} =$  , (3.44)

leading to

$$A^{\mathrm{SM}} = A_t^{\mathrm{SM}} + A_u^{\mathrm{SM}} = -\mathrm{i}C_A C_Z \bar{v}(p_2)\Gamma_5 \left(\not\varepsilon_2 \frac{1}{\not q_t} \not\varepsilon_1 + \not\varepsilon_1 \frac{1}{\not q_u} \not\varepsilon_2 \right) u(p_1) \ . \quad (3.45)$$

In the NCSM the above diagrams will receive $\mathcal{O}(\theta)$ corrections:

$A_{t,1}^{\mathrm{NC}} =$  , $A_{t,2}^{\mathrm{NC}} =$  , (3.46a)

$A_{u,1}^{\mathrm{NC}} =$  , $A_{u,2}^{\mathrm{NC}} =$  , (3.46b)

where the open box marks the first order correction to the vertices. Analytically, their sum reads:

$$\begin{aligned} A_{t+u}^{\mathrm{NC}} &= \sum_{i=1,2} (A_{t,i}^{\mathrm{NC}} + A_{u,i}^{\mathrm{NC}}) \\ &= \frac{1}{2} C_A C_Z \bar{v}(p_2)\Gamma_5 \Big[(p_1\theta p_2 + k_1\theta k_2) \not\varepsilon_2 \frac{1}{\not q_t} \not\varepsilon_1 + (p_1\theta p_2 - k_1\theta k_2) \not\varepsilon_1 \frac{1}{\not q_u} \not\varepsilon_2 \\ &\qquad - (k_1 + k_2)\theta\varepsilon_1 \not\varepsilon_2 - (k_1 + k_2)\theta\varepsilon_2 \not\varepsilon_1 \\ &\qquad + 4c_\psi^1 k_1\theta\varepsilon_1 \not\varepsilon_2 + 4c_\psi^1 k_2\theta\varepsilon_2 \not\varepsilon_1 \Big] u(p_1) \ , \end{aligned} \quad (3.46c)$$

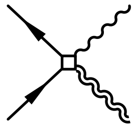
where all particles are on-shell and the equations of motion in the massless limit

$$\not p_1 u(p_1) = 0 \quad \text{and} \quad \bar{v}(p_2)\not p_2 = 0 \quad (3.46d)$$

have already been used.
The ambiguity c_ψ^1 of the matter field enters the formula for the summed t- and u-channel amplitude. As it will be discussed in detail in chapter 4, we expect that the ambiguities of the Seiberg-Witten maps have no physical consequences

since they correspond to reparametrizations of the fields. Therefore, at this point, before even testing the Ward identity, the need for another diagram in order to cancel the terms proportional to c_ψ^1 arises. And indeed, the four point amplitude

$$A_c^{\mathrm{NC}} = \quad , \tag{3.46e}$$



which has no tree-level pendant in the SM, will lead to the desired cancellation. Adding this amplitude

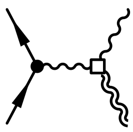
$$A_c^{\mathrm{NC}} = \frac{1}{2} C_A C_Z \bar{v}(p_2) \Gamma_5 \big[(k_1 - k_2)\theta\varepsilon_1 \not\varepsilon_2 + (k_2 - k_1)\theta\varepsilon_2 \not\varepsilon_1 + 2\varepsilon_1\theta\varepsilon_2 \not k_1 - 4c_\psi^1 k_1\theta\varepsilon_1 \not\varepsilon_2 - 4c_\psi^1 k_2\theta\varepsilon_2 \not\varepsilon_1 \big] u(p_1) \tag{3.46f}$$

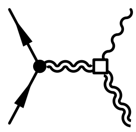
to (3.46c) we obtain a c_ψ^1 independent expression:

$$\begin{aligned} A_{t+u+c}^{\mathrm{NC}} &= \sum_{i=1,2} (A_{t,i}^{\mathrm{NC}} + A_{u,i}^{\mathrm{NC}}) + A_c^{\mathrm{NC}} \\ &= \frac{1}{2} C_A C_Z \bar{v}(p_2) \Gamma_5 \Big[(p_1\theta p_2 + k_1\theta k_2) \not\varepsilon_2 \frac{1}{\not q_t} \not\varepsilon_1 + (p_1\theta p_2 - k_1\theta k_2) \not\varepsilon_1 \frac{1}{\not q_u} \not\varepsilon_2 \\ &\qquad - 2k_2\theta\varepsilon_1 \not\varepsilon_2 - 2k_1\theta\varepsilon_2 \not\varepsilon_1 + 2\varepsilon_1\theta\varepsilon_2 \not k_1 \Big] u(p_1) \,. \end{aligned} \tag{3.46g}$$

We used the reparametrization invariance as a second cross check for the correctness of the calculations. The first cross check is of course the test of the Ward Identity. As already mentioned, the summed amplitude must vanish if one of the polarization vectors is replaced by the corresponding momentum. Doing such for the amplitudes under discussion, we see that the contact amplitude will cancel the other four: The contact amplitude is therefore requested by gauge invariance. There is no such requirement for the s-channel diagrams

$$A_{s,\gamma}^{\mathrm{NC}} = \quad , \quad A_{s,Z}^{\mathrm{NC}} = \quad , \tag{3.46h}$$





which are allowed in the nonminimal NCSM and have to be taken into consideration. They satisfy the Ward identity separately. Therefore, there is no restriction on the coupling constants $K_{Z\gamma\gamma}$ and $K_{ZZ\gamma}$. Unlike the SM QCD, where gauge invariance fixes the triple and quartic gauge boson couplings, the TGB couplings are free parameters of the theory, restricted only by (3.36).
Squaring the total amplitude and performing the traces which usually enter the formula for the squared amplitude, it can now be seen why the axial coupling of the Z-Boson plays an important rôle for noncommutative effects to survive after summing over the polarizations. Consider for example the process $q\bar{q} \to \gamma\gamma$ and ignore for a moment the s-channel diagrams: the contributing amplitudes are similar to those for the process with a Z and a photon in the final state, the only (crucial, as we will see) difference is the factor $\Gamma_5 = (g_V + \gamma_5 g_A)$ instead

of Qe in the vertex containing the Z-boson. The squared amplitude to $\mathcal{O}(\theta)$ is given by:

$$|A|^2 = |A^{\rm SM} + A^{\rm NC}|^2 = |A^{\rm SM}|^2 + 2\mathcal{R}e\left(A^{*\,\rm SM}A^{\rm NC}\right) + O(\theta^2) \qquad (3.47)$$

For a process with two photons in the final state, the noncommutative part is real, which makes the interference term with the SM amplitude purely imaginary. Therefore $\mathcal{R}e\left(A^{*\,\rm SM}A^{\rm NC}\right) = 0$ and noncommutative effects drop out in the unpolarized cross section. The only noncommutative signal which survives the sum over polarizations can only come from the Z-exchange diagram. There the width of the intermediate Z boson is the only source for a imaginary part, such that the interference term with the SM becomes real. Nevertheless, the energy scales at which the processes take place are very large compared to the Z-mass $\sqrt{\hat{s}} \gg m_Z$ and the width of the intermediate Z boson is negligible. The only other possibility to study noncommutative effects for $q\bar{q} \to \gamma\gamma$ at $\mathcal{O}(\theta)$ is to consider polarized cross sections. Unfortunately, we do not have the luxury to study polarization effects at the LHC, the machine at our disposal for the next 10 years, and we must therefore look for processes which provide noncommutative signals in the unpolarized cross section. As already mentioned, the axial coupling of the Z makes the process $q\bar{q} \to Z\gamma$ interesting for LHC-studies w. r. t. noncommutativity of space-time, since the trace over γ-matrices and a γ_5 matrix will yield terms proportional to $\mathrm{i}\epsilon^{\mu\nu\rho\sigma}$ making thus parts of the noncommutative amplitude imaginary and giving therefore contributions to the squared amplitude even after summing over polarizations.

3.2.2 Partonic Cross Section

The most striking effect of the space-time noncommutativity reveals itself in the azimuthal distribution of the cross section. This is not surprising since $\theta^{\mu\nu}$ can be viewed as a fixed background field with "magnetic"($\vec{B}$) and "electric" ($\vec{E}$) components which break rotational invariance with respect to the beam axis. This effect is depicted in figures 3.2, 3.3 and 3.4 where the differential cross section is plotted against the azimuthal angle ϕ.

Figure 3.2 shows the dependency on the TGB couplings for $u\bar{u}$ scattering into a Z and a photon. The values chosen for the coupling constants are taken within the allowed region shown in figure 3.1. We concentrated on the dependency of the sets $(K_{Z\gamma\gamma}, K_{ZZ\gamma})$ corresponding to the corner of the polygon and denoted by K_i, with $i = 1, \ldots, 6$, since due to the linear dependency on the TGB couplings, we expect maximal (or minimal) effects for these values. For illustration we have selected two corners of the polygon which – for $u\bar{u}$ scattering[5] – give the maximal and minimal deviation from the standard model. In particular, we can see that for values of the TGB couplings lying in the lower region of the polygon, the noncommutative effects almost vanish. Therefore, little sensitivity of the LHC for this region of TGB couplings values can be expected. We have also plotted the special case of vanishing TGB couplings corresponding to the minimal NCSM.

[5] As it will be shown later, for particles with opposite sign of the charge, other sets of the TGB couplings become important.

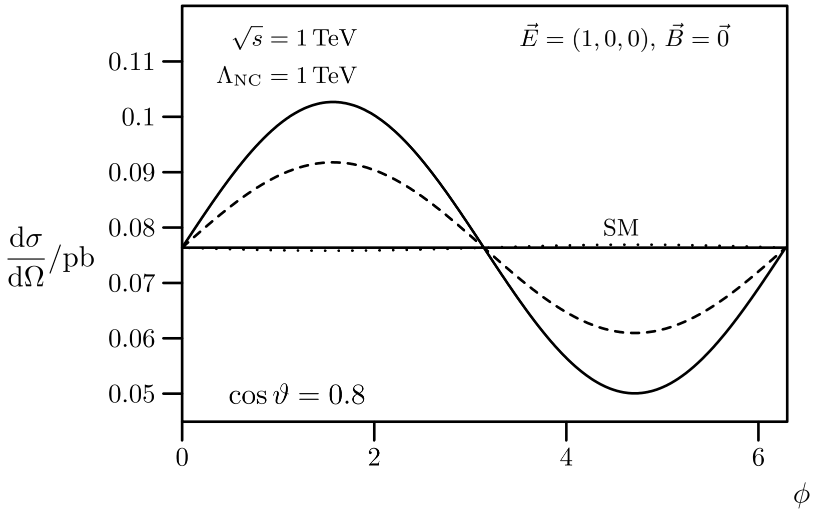


Figure 3.2: Azimuthal dependence of the cross section for different values for the TGB couplings: $K_1 = (-0.333, 0.035)$ (dotted) , $K_5 = (0.095, 0.155)$ (solid) and $K_0 = (0.0, 0.0)$ (dashed).

It is very important to note, that in the nonminimal version of the NCSM, the (differential) cross section depends not only on the absolute value of the charge of the scattered particles, as in the SM or in the minimal NCSM, but also on its sign. For illustration, we split the differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\rm SM} + \frac{d\sigma'}{d\Omega}_{\rm NC}(\theta, Q_f^4) + \frac{d\sigma''}{d\Omega}_{\rm NC}(\theta, |Q_f|^3, sign(Q_f), K_{Z\gamma\gamma}, K_{ZZ\gamma}) \,, \quad (3.48)$$

where the third term stems from the interference of the s channel ($\propto Q_f$) with the SM ($\propto Q_f^2$) diagrams, leading to a contribution $\propto Q_f^3$. Therefore, cancellations or enhancement of the deviation w. r. t. the SM cross section can occur depending on $sign(Q_f)$ of the scattered particles. We only discussed the case of $u\bar{u}$ scattering, because this will give the important contribution for pp scattering at the LHC, since the up-quark counts twice on the one hand, and on the other, due to $|Q_u| = 2/3$, its cross section is larger than the $d\bar{d} \to Z\gamma$ cross section. In section 3.2.6 we will present the dependency on the TGB couplings for e^+e^- scattering, where we will see, that this process will beautifully probe the region of the polygon exactly opposite to the one probed by the LHC.
As has been shown previously [52], the $\gamma\gamma \to f\bar{f}$ amplitude in the NCSM depends only on E_1 and E_2. In contrast, the $f\bar{f} \to Z\gamma$ amplitude depends also on B_1 and B_2, due to the axial $Zf\bar{f}$ couplings. We observe a sine- or cosine-like dependency (or a superposition of both) on the azimuthal angle ϕ for the entries in the matrix $C^{\mu\nu}$ corresponding to E_1, E_2, B_1 and B_2, stemming from contractions of the particle momenta

$$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, -1) \quad (3.49a)$$

$$k_1 = \omega\left(\frac{E_Z}{\omega}, \ -\sin\vartheta\cos\phi, \ -\sin\vartheta\sin\phi, \ -\cos\vartheta\right) \quad (3.49b)$$

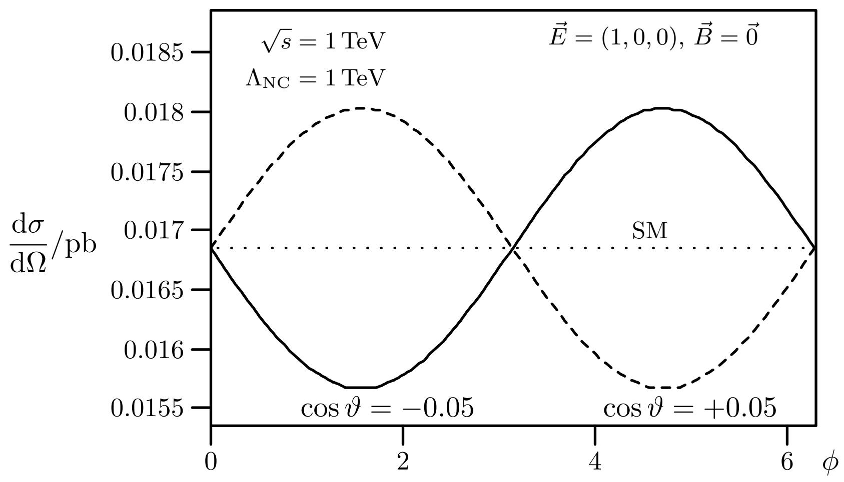


Figure 3.3: Azimuthal dependence of the cross section, with $K_5 = (0.095, 0.155)$ (the lower right corner of the polygon in figure 3.1), showing the antisymmetry in the partonic scattering angle $\cos\vartheta$.

$$k_2 = \omega\,(1,\ \sin\vartheta\cos\phi,\ \sin\vartheta\sin\phi,\ \cos\vartheta) \tag{3.49c}$$

with $C^{\mu\nu}$. $E_Z = \sqrt{s} - \omega$ is the Z boson energy while $\omega = (s - m_Z^2)/(2\sqrt{s})$ denotes the photon energy.

We do not obtain an azimuthal dependence on the components which are aligned with the beam axis, i.e. E_3 and B_3. Kinematics dictate the total independence of the cross section on the B_3 component. It does not appear in any of the combinations $p_1\theta k_{1/2}$ or $k_1\theta k_2$, since $\vec{p}_i||\vec{B}_3$, with $i = 1, 2$ and $\vec{B}_3 = (0, 0, B_3)$, and $\vec{k}_1||\vec{k}_2$ (back to back scattering). A dependence on E_3 appears because the vector bosons are not aligned with the beam axis x_3. E_3 enters the formula for the differential cross section without a factor $\sin\phi$ or $\cos\phi$, therefore the azimuthal distribution for nonvanishing E_3 shows only a constant shift w. r. t. to the SM cross section.

One important feature of the azimuthal dependence of the NCSM when at least one of the components E_1 or E_2 is nonzero is its antisymmetry w. r. t. the partonic scattering angle ϑ, see figure 3.3. Integration over the entire allowed range of the scattering angle will lead to an almost total cancellation of the noncommutative effects. This must be taken into account in the Monte Carlo simulation, by integrating only over one hemisphere $0 < \cos\vartheta_{Z/\gamma} < (1 - \epsilon)$ or $-(1 - \epsilon) < \cos\vartheta_{Z/\gamma} < 0$ respectively.

This is not the case for the B components. For nonvanishing $\vec{B}$ the azimuthal dependency is symmetric in $\cos\vartheta$. Nevertheless, the cross sections show a much weaker dependence on the components of $\vec{B}$ than on those of $\vec{E}$, as exemplified in figure 3.4. To be precise, this is true everywhere except sufficiently close to the polar angle $\vartheta = \pi/2$, where the dependence on $\vec{E}$ vanishes due to the antisymmetry of the $\mathcal{O}(\theta)$ contribution to $\mathrm{d}\sigma/\mathrm{d}\Omega$ in $\cos\vartheta$.

Before going on with the next section, another important remark has to be made. We are treating the NCSM as an effective theory, that means as a theory which is expanded in powers of $s/\Lambda_{\mathrm{NC}}^2$. Therefore, the validity of the theory

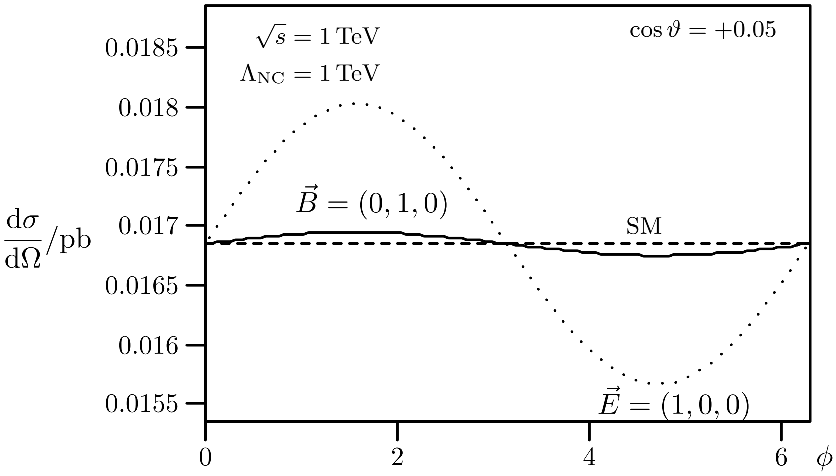


Figure 3.4: Azimuthal dependence of the cross section, with $K_5 = (0.095, 0.155)$ (the lower right corner of the polygon in figure 3.1), showing the different dependency of the E and B components of the noncommutative parameter θ on the partonic scattering angle $\cos\vartheta$.

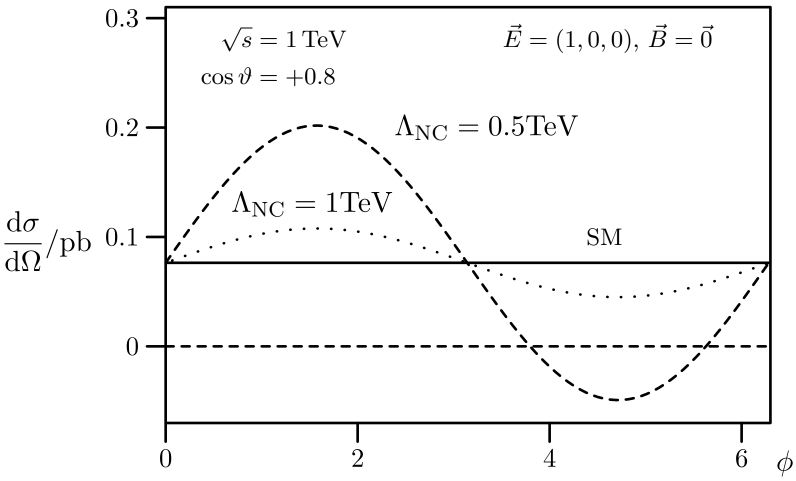


Figure 3.5: Azimuthal dependence of the cross section, with $K_5 = (0.095, 0.155)$. The validity of the NCSM at $\mathcal{O}(\theta)$ can be trusted as long as $\sqrt{s}/\Lambda_{NC} \lesssim 1$ (dotted). If $\sqrt{s}/\Lambda_{NC} > 1$ (dashed) we get unphysical (negative) cross sections.

is ensured up to scales $\Lambda_{NC} \lesssim \sqrt{s}$. Figure 3.5 shows the behaviour of the cross section for two different values of the ratio $\sqrt{s}/\Lambda_{NC}$. When $\sqrt{s}/\Lambda_{NC} > 1$ the interference between the noncommutative and the SM amplitude ($A^{*\,SM}A^{NC}$) dominates the purely SM contribution and the cross section can become negative, leading to unphysical results. We have to keep this in mind in the next section by demanding an upper cut on the partonic CMS scattering energy $\sqrt{s^*}$.

3.2.3 Hadronic Cross Section

The phenomenological interest of this work is dedicated mainly to the LHC, the experimental endeavor that will hopefully point out new directions that

elementary particle physics will take in the next years. The LHC is a proton proton collider which will run at center of mass energies of 14 TeV beginning with the year 2008. In this section we will concentrate on the process $pp \to Z\gamma \to e^+e^-\gamma$ in the NCSM up to $\mathcal{O}(\theta)$, which has $q\bar{q} \to Z\gamma$ considered in the previous section as a subprocess.
The model we will use in order to connect the partonic to the hadronic cross section is the QCD Parton Model and its powerful tool, the QCD factorization theorem. It essentially states that at high energies the hadronic cross section $\sigma(P_1, P_2)$ is obtained from the partonic cross sections $\hat{\sigma}_{ij}(p_1, p_2)$ by convoluting them with the Parton Distribution Functions (PDF) f_i, with i labelling the corresponding parton (u, d, $\bar{u}$, ...):

$$\sigma(P_1, P_2) \quad = \quad \sum_{i,j} \int \mathrm{d}x_1 \mathrm{d}x_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \hat{\sigma}_{ij}(p_1, p_2, \alpha_S(\mu^2), Q^2/\mu^2)\,. \tag{3.50}$$

P_1 and P_2 are the proton momenta, of which the partons carry a fraction x: $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$ respectively. The PDF are universal and can be determined experimentally. They are defined at the factorization scale μ. This is the scale w. r. t. which interactions either belong to the hadronic structure (if their transversal momentum $p_T < \mu$) and are therefore absorbed in the PDF or have to be assigned to the partonic cross section (if $p_T > \mu$). In other words, μ separates the scales at which the computable and the not computable (but experimental measurable and universal) interactions of the hadronic cross section take place. In general the factorization scale is taken to be equal to the hard scale Q^2 at which the partonic scattering takes place. For the analysis performed in this work we will use $\mu^2 = Q^2 = m_Z^2$. Furthermore, in all hadronic simulations we have used for the (anti)quark-PDF the CTEQ4M series in the standard $\overline{\mathrm{MS}}$ factorization scheme of the CTEQ collaboration [57].
The hadronic cross section is given by

$$\begin{aligned} &\sigma(p(P_1)\bar{p}(P_2) \to Z\gamma) = \\ &\quad = \quad \sum_q \int_0^1 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 f_q(x_1, m_Z^2) f_{\bar{q}}(x_2, m_Z^2)\, \hat{\sigma}(q(p_1)\bar{q}(p_2) \to Z\gamma) \end{aligned} \tag{3.51}$$

and can be pictured as in figure 3.6.
We will calculate the partonic cross section within the framework of the NCSM discussed in the previous section and will convolute the result with the PDF mentioned above. In fact, for the sake of consistency and a proper analysis we should of course consider noncommutative effects also in PDF. Nevertheless, the use of the usual PDF within the SM is justified by the fact that the physics in the proton takes place at low energy scales compared to the one of the hard scattering process considered. This is ensured by the factorization theorem according to which high energy processes encoded in the hard cross section are separated from the low energy processes absorbed in the PDF. Thus, for the value of the noncommutative scale Λ_{NC} assumed in our calculations, noncommutative effects are important for the hard processes but play a negligible rôle for the physics beyond the partonic scattering process we are interested in.

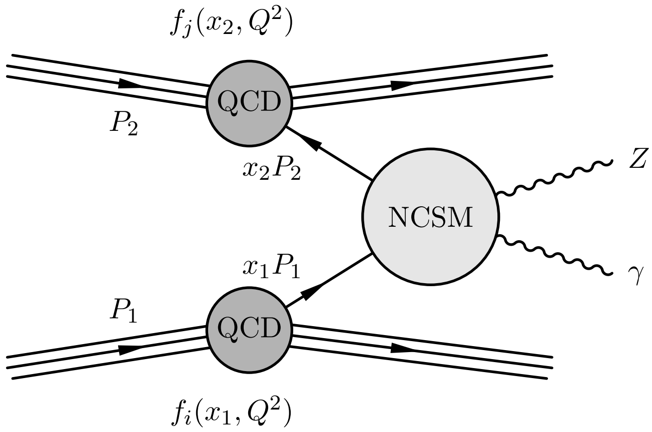


Figure 3.6: Illustration of the factorization theorem, as described by equation (3.51).

Observing the noncommutative effects of the partonic cross section analyzed in the previous section after the convolution with the PDF is not straightforward. The convolution with the parton distribution functions essentially means a boost of the partonic cross section in the direction of the beam axis on the one hand, and a variable partonic CMS energy $\sqrt{\hat{s}}$ on the other hand.
Usually, new physics effects attract attention to themselves and therefore are sought-after in the p_T distribution. This holds of course also for the NCSM and we have done the corresponding simulations. Nevertheless, the deviation from the SM p_T distribution is not significant, even after applying cuts on the azimuthal angle ϕ. These cuts are indeed necessary as can be seen in figures 3.2 - 3.4, integration over the full range of ϕ would lead to cancellation of the effects in the p_T distribution.
Nevertheless, in the case of the NCSM we have a much better observable, which can also be used in order to discriminate our model against other new physics models. From now on we will concentrate on the azimuthal dependence of the cross section.
We are interested in the hadronic process $pp \to Z\gamma$ at the LHC and $p\bar{p} \to Z\gamma$ at the Tevatron with subsequent decays of the Z into e^+e^- and $\mu^+\mu^-$. We will focus our analysis on the LHC, a short discussion regarding a similar Tevatron analysis will follow this section.
We calculated the hard $q\bar{q} \to \ell^+\ell^-\gamma$ cross sections as in the previous section and implemented the noncommutative corrections in the source code of the matrix element generator O'Mega [58, 59, 60]. The phase space generation and parton distribution functions were provided by WHiZard [61, 62] for Monte Carlo simulation.
An important issue which has to be discussed are the cuts which have to be applied on the process under study. We demand the following acceptance cuts on leptons and photons:

$$E(\ell^\pm), E(\gamma) \geq 10\,\mathrm{GeV}\,, \tag{3.52a}$$
$$\theta(\ell^\pm), \theta(\gamma) \geq 5^\circ\,, \tag{3.52b}$$

$$\theta(\ell^\pm), \theta(\gamma) \leq 175^\circ\,, \tag{3.52c}$$

$$p_T(\ell^\pm), p_T(\gamma) \geq 10\,\text{GeV}\,, \tag{3.52d}$$

which are typical for the detector geometry and performance. Besides these, we have to address the irreducible background for $p\bar{p} \to \ell^+\ell^-\gamma$, which can be suppressed by applying corresponding cuts. The hard scattering process $q\bar{q} \to Z\gamma$ with subsequent decays $Z \to \ell^+\ell^-$ has the Drell-Yan process $q\bar{q} \to \ell^+\ell^-$ with photon radiation and $q\bar{q} \to \gamma^*\gamma$ with $\gamma^* \to \ell^+\ell^-$ as irreducible backgrounds. Both have been taken into account in our calculation.
As discussed in section 3.2.1, the unpolarized azimuthal distribution in $q\bar{q} \to \gamma\gamma$ is flat. By off-shell extrapolation this will hold also approximatelly for $q\bar{q} \to \gamma^*\gamma$. We suppress this background by requiring

$$|M(\ell^+\ell^-) - M_Z| \leq \Gamma_Z\,. \tag{3.53}$$

In order to also reduce the radiative Drell-Yan events, we apply an angular separation cut of

$$\Delta R_{\ell^\pm\gamma} = \sqrt{\Delta\eta^2 + \Delta\phi^2} > 0.7 \tag{3.54}$$

(cf. [64, 65]).
An additional cut specific to the NCSM and discussed in section 3.2.2 has to be imposed. Since we are interested in the azimuthal distribution, the polar angle ϑ^* is integrated out. But we have seen that this implies the complete cancellation of the noncommutative effects, if only “electric” noncommutativity is allowed. Therefore we have to account for the antisymmetry w. r. t. the partonic scattering angle ϑ^* and integrate only over one hemisphere by requiring

$$0 < \cos\vartheta^*_\gamma < 0.9\,. \tag{3.55}$$

Of course this is not necessary if only “magnetic” noncommutativity is regarded. In fact, integration over the entire sphere doubles the magnitude the noncommutative effects, since in the case $\vec{E} = 0, \vec{B} \neq 0$ the cross section is symmetric w. r. t. $\cos\vartheta^*$. Yet, as we have seen in the previous chapter, the effects for nonvanishing “magnetic” components are so small, that we can not hope to get measurable effects in the hadronic observables. Therefore, we will always use (3.55) in the simulations.
Nevertheless, with all these cuts, a first Monte Carlo simulation of the azimuthal dependence of the cross section in the NCSM will not lead to any observable noncommutative effects, even for unreasonably small values of the noncommutative scale Λ_{NC}[6]. The partonic CMS energy $\sqrt{\hat{s}}$ is estimated to be 1/10 of the hadronic CMS energy for most of the processes. We therefore expect to see signs of noncommutativity for $\Lambda_{\text{NC}} \sim \sqrt{\hat{s}} \simeq \sqrt{s}/10 = 1.4\,\text{TeV}$. However, this is not the case. The reason is found to lie in the symmetric hadronic initial state of the process under study.
At the LHC, $q\bar{q}$- and $\bar{q}q$-collisions will occur with identical rates. The dominant E_1 and E_2 contributions to the deviation of the parton cross sections from the

[6] By “unreasonable” we mean values of Λ_{NC} lying very much below the partonic CMS energy.

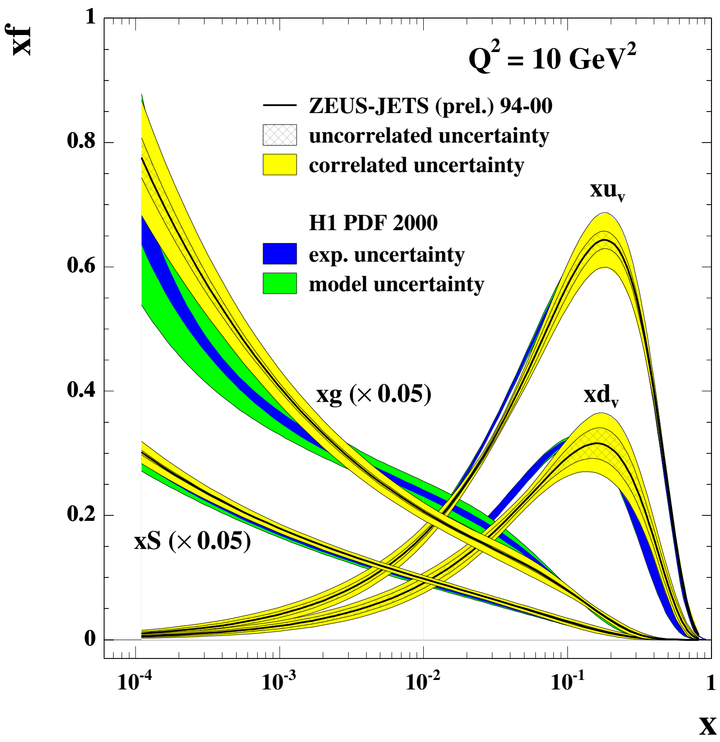


Figure 3.7: Parton Distribution Functions (plot obtained from `www-h1.desy.de`). The main part of the proton momentum is carried by the valence quarks.

SM prediction are antisymmetric in $\cos\vartheta^*$ and will therefore cancel for pp initial states unless additional cuts are applied that separate events originating from $q\bar{q}$ and $\bar{q}q$.

However, in the proton, the average momentum fraction of the light valence quarks is much higher than that of the anti-quarks which exist only in the sea, see figure 3.7. As a result, all $q\bar{q}$-events will be boosted strongly in the direction of the quark. The situation is illustrated in figure 3.8. The events which correspond to back-to-back scattering in the quark and antiquark center of mass system are boosted in the laboratory system.

Therefore, we can enrich our samples of signal events by requiring a minimal boost in the appropriate direction. Demanding the momenta of both the photon and the lepton pair to lie in the *same* hemisphere in the laboratory frame, that is

$$\cos\vartheta_\gamma \cdot \cos\vartheta_{\ell^+\ell^-} > 0\,, \tag{3.56}$$

will separate $q\bar{q}$ and $\bar{q}q$ events, the cut on the partonic scattering angle $\cos\vartheta^*$ now becomes effective and the expected signal is indeed produced, as displayed in figure 3.9. In the figure, we have chosen a rather low value of $\Lambda_{\rm NC} = 0.6\,$TeV for illustration.

We also must not forget that we have calculated the cross section up to $\mathcal{O}(\theta)$

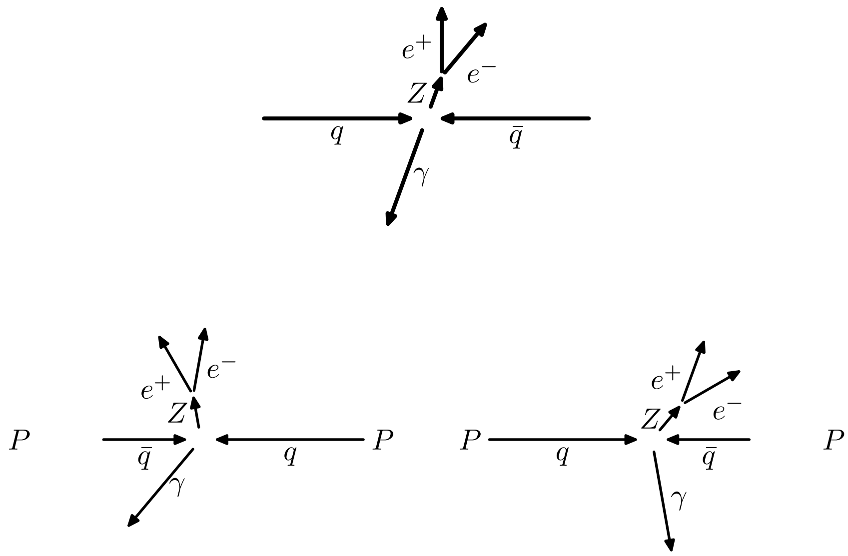


Figure 3.8: Events in the quark-antiquark center of mass system (upper figure) are boosted in the laboratory frame (lower figure).

and we consistently had to neglect $\mathcal{O}(\theta^2)$ contributions. Therefore, in this approximation, the deviations from the SM only come from the interference of A^{SM} with A^{NC} in (3.47). As a consequence, the $\mathcal{O}(\theta)$ cross section can become negative (see figure 3.5) in some regions of phase space for sufficiently large absolute values and appropriate signs of components of $\theta^{\mu\nu}$, such that the NC/SM interference dominates over the SM contribution. The relative size of the interference term is determined by $\hat{s}/\Lambda_{\mathrm{NC}}^2$, $\sqrt{\hat{s}}$ being the partonic CMS energy. There is a wide range of possible values of $\sqrt{\hat{s}}$ in high energy hadron collisions, but the most statistics will be collected at moderate values of $\sqrt{\hat{s}}$. Thus in $\mathcal{O}(\theta)$, values of Λ_{NC} that cause observable deviations at such moderate values of $\sqrt{\hat{s}}$ can lead to unphysical cross sections at the highest $\sqrt{\hat{s}}$ available. This problem is not specific to simulations in the NCSM, but common to all studies of new physics that can be parametrized by anomalous couplings. A pragmatic solution is to unitarize the contributions from new physics by applying appropriate form factors that cut off the unphysical effects. Since there are very few events to be expected at the highest CMS energies, the conclusions should not depend on the details of such form factors. Therefore, we have simply replaced $\mathrm{d}\sigma/\mathrm{d}\Omega^*$ by $\max(\mathrm{d}\sigma/\mathrm{d}\Omega^*, 0)$ everywhere in our simulations.
This solution is also legitimized by the expectation that higher orders in $\theta^{\mu\nu}$ will damp the large negative interference contributions. As we will see in the next chapter, results to second order in $\theta^{\mu\nu}$ support this expectation.
Therefore, we finally require a minimum and maximum total energy in the partonic CMS:

$$200\,\mathrm{GeV} \leq |M(\ell^+\ell^-\gamma)| \leq 1\,\mathrm{TeV}\,. \tag{3.57}$$

The lower cut enhaces the signal, while the upper cut reduces the influence of

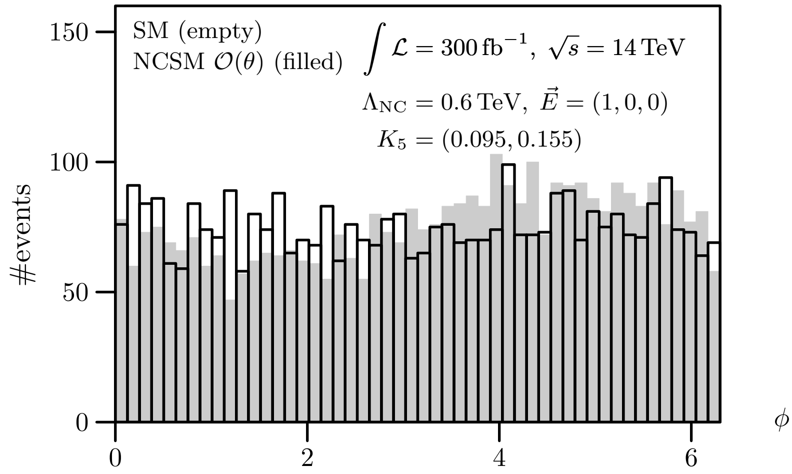


Figure 3.9: Monte Carlo simulation for the azimuthal dependence of the process $pp \to e^+e^-\gamma$ at the LHC.

the elimination of the unphysical parton cross sections discussed above.
The values K_5 for $K_{Z\gamma\gamma}$ and $K_{ZZ\gamma}$ used for the illustration in figure 3.9 and in the simulations of the next chapter correspond to one of the upper corners of the polygon in figure 3.1. The scattering amplitude for up quarks shows maximal noncommutative effects for this choice, whereas in the corresponding amplitude for the down-quarks the noncommutative contribution almost cancels out due to the opposite sign of the down-quarks charge. Therefore, summing up twice the up-quark amplitude and once the down-quark amplitude and considering that the absolute value of the deviations from the SM amplitude of the latter is anyway smaller than the corresponding deviation for the up-quark the result is dominated by the up-quark dependence. We have seen in the previous section, that for values of the TGB couplings lying in the lower part of the polygon of figure 3.1, the cross section for the up-quark shows no significative dependence on the noncommutative parameters. The down-quark noncommutative contribution, which is maximal for this choice of the TGB couplings, is not sufficiently large in order to give observable deviations from the SM in the cross section for pp-scattering. Therefore, if nature has chosen noncommutativity with TGB couplings lying in the lower region of the polygon 3.1, it cannot be detected with the LHC in $pp \to Z\gamma$.

3.2.4 Likelihood Analysis

The previous section showed that the azimuthal distribution of the cross section is best suited to deriving sensitivity bounds on the noncommutative parameter. In doing such, the least-squares method gives the desired results [63].
The parameters of interest are $\vec{E}$ and $\vec{B}$ (and by these implicitly the noncommutative energy scale Λ_{NC}) as well as the neutral gauge boson coupling constants (two for the process $pp \to Z\gamma$). The main interest nevertheless lies in determining sensitivity bounds for the noncommutative scale Λ_{NC}. Instead of making a

fit for the joint parameter set $(K_{Z\gamma\gamma}, K_{ZZ\gamma}, \vec{E}, \vec{B})$ we will perform the analysis for the parameter set $(\vec{E}, \vec{B})$ for different values of the coupling constants. We disregard the small contribution from E_3, since, by symmetry, it has no influence on the azimuthal distribution.
Usually, the aim of a likelihood analysis is to determine the best estimates $\mathbf{X}_0$ of the parameters $\mathbf{X}$ which describe the model and a measure of their reliabilities. In our case, the NCSM is parametrized by the components of the θ-Matrix $\mathbf{X} = (E_1, E_2, B_1, B_2)$, of which we already know the best estimates in the absence of noncommutative effects: $\mathbf{X}_0 = (0,0,0,0)$. For these values of the parameters, the probability density P that the NCSM is true given the data and its logarithm L reach the maximum:

$$\left.\frac{\partial L}{\partial X_i}\right|_{\{X_{0j}\}} = 0, \quad (i = 1, \ldots, 4)\,. \tag{3.58}$$

The quantity which we are interested in, and which we are left to determine, is the covariance matrix.
By expanding the multidimensional L in a Taylor series about the point $\mathbf{X} = (E_{01}, E_{02}, B_{01}, B_{02})$ we can obtain a measure of the reliability of the best estimates:

$$L = L(\mathbf{X_0}) + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \left.\frac{\partial^2 L}{\partial X_i \partial X_j}\right|_{\mathbf{X_0}} (X_i - X_{0i})(X_j - X_{0j}) + \ldots \tag{3.59}$$

Ignoring higher order terms in the expansion, the exponential of the above expression gives a good approximation to the probability density P:

$$P \propto \exp\left[\frac{1}{2}(\mathbf{X} - \mathbf{X_0})^T \boldsymbol{\nabla\nabla} L (\mathbf{X} - \mathbf{X_0})\right], \tag{3.60}$$

where $\boldsymbol{\nabla\nabla} L$ is a 4×4 symmetric matrix: $[\boldsymbol{\nabla\nabla} L]_{ij} = \partial^2 L / \partial X_i \partial X_j$. P represents a multivariate Gaussian with its maximum given by $\mathbf{X_0}$. It can be shown [63] that the covariance matrix $\boldsymbol{\sigma}^2$ is given by:

$$[\boldsymbol{\sigma}^2]_{ij} = \langle (X_i - X_{0i})(X_j - X_{0j}) \rangle = -[(\boldsymbol{\nabla\nabla} L)^{-1}]_{ij}\,. \tag{3.61}$$

The diagonal elements (σ^2_{ii}) give the (marginal) error-bars of the associated parameters, while the off-diagonal elements $(\sigma^2_{ij}, i \neq j)$ encode the correlations between the inferred values of X_i and X_j.
On the other hand, due to Bayes' theorem, the probability P that the NCSM is true given the data is directly related to the probability that we obtain a set of data assuming that the NCSM is true – the so called likelihood function. Assuming that the data are independent and that the noise associated with the experimental measurements can be represented as a Gaussian process, the logarithm of the likelihood function, and therefore L, can be written as:

$$L = \text{constant} - \frac{\chi^2}{2}, \quad \text{with} \quad \chi^2 = \sum_{k \text{ bins}} \frac{(N_{SM,k} - N_{NC,k})^2}{N_{SM,k}} = \sum_{k \text{ bins}} f_k\,, \tag{3.62}$$

where $N_{SM,k}$ represents the data and $N_{NC,k}$ gives the functional relationship between the parameters $\mathbf{X}$ and the ideal data, i.e. $\sigma_{NC}\mathcal{L}$.
The maximum of L will occur when χ^2 is smallest, therefore the best estimates of the parameters $\mathbf{X}$ are referred to as *least-squares* estimates. In general, in the denominator of (3.62) is given by the error-bars σ_k^2, but, in the limit of large counts N, these can be replaced by $N_{SM,k}$, making thus the denominator independent of the parameters $\mathbf{X}$. From (3.62) it can be seen that χ^2 is minimized, when $N_{NC} \to N_{SM}$, which happens when $\vec{E}, \vec{B} \to 0$. The 1 σ confidence level then sets the bounds on the noncommutative parameters:

$$[\boldsymbol{\sigma}^2]_{ij} = (2)[M^{-1}]_{ij}\,,$$

where M is the coefficient matrix if we write χ^2 as

$$\chi^2 = \vec{X}^T M \vec{X}. \tag{3.63}$$

There is no analytical expression for $N_{NC,k}$, since in calculating the hadronic cross section we must perform the convolution with the Parton Distribution Functions. $N_{NC,k}$ can only be obtained numerically from Monte Carlo simulations, as described in the previous section. But we can use the fact that the cross section and hence N, the event number, is linear in $\vec{E}$ and $\vec{B}$. From (3.62) it is clear that χ^2 is quadratic in these parameters and (3.62) describes a paraboloid.
The idea is now to simulate the "χ^2-data" and fit it to a paraboloid which has its minimum centered at $\vec{0}$. We proceed as follows: We generate histograms N_{NC} for the azimuthal dependence of the cross section for a set of values of $(\vec{E}, \vec{B}) \equiv \mathbf{X}_l$. For each histogram we calculate the corresponding χ^2_l by means of (3.62). The set of $\{\chi^2_l\}$ obtained in this way will then be fitted to the paraboloid, from which we can then determine the wanted covariance.
For the parabola-fit the error for each $\{\chi^2_l\}$ is needed, and can be calculated using error propagation (the subscript l has been omitted for convenience):

$$(\delta\chi^2)^2 = \sum_{bins}\left[\left(\frac{\partial f_k}{\partial N_{SM,k}}\right)^2 (\delta N_{SM,k})^2 + \left(\frac{\partial f_k}{\partial N_{NC,k}}\right)^2 (\delta N_{NC,k})^2\right]. \tag{3.64}$$

3.2.5 Bounds from the LHC

A priori, multidimensional fits must be done in order to take into account possible correlations among the parameters. Nevertheless, the only source of violation of rotational invariance is $\theta^{\mu\nu}$ itself. Therefore no correlations for the pairs (E_1, E_2), (B_1, B_2) are expected and even more, the corresponding error ellipses degenerate to circles. Therefore, neither correlations for (E_i, E_3) and (B_i, B_3) with $i = 1, 2$ will occur. Yet, the situation is different for the pairs (E_1, B_2) and (E_2, B_1): at hadron colliders, the partonic CMS of most events is boosted significantly along the beam axis[7]. As is well known from

[7]The boosted events are further enriched by the cuts chosen for the LHC.

electrodynamics, $\vec{E}$ and $\vec{B}$ are mixed by Lorentz boosts along the beam axis x_3:

$$\begin{aligned} E_1 &\to \gamma(E_1 - \beta B_2)\,, & B_1 &\to \gamma(B_1 + \beta E_2)\,, \\ E_2 &\to \gamma(E_2 + \beta B_1)\,, & B_2 &\to \gamma(B_2 - \beta E_1)\,, \\ E_3 &\to E_3\,, & B_3 &\to B_3 \end{aligned} \tag{3.65}$$

($\gamma = 1/\sqrt{1-\beta^2}$, $\beta = v/c$). The measurements of (E_1, B_2) and (E_2, B_1), respectively, are therefore highly correlated by kinematics.

As expected from the Lorentz boost (3.65), we find that in the laboratory frame measurements of E_1 are correlated with B_2 and measurements of E_2 with B_1. The corresponding 1σ and 3σ contours are depicted for $\Lambda_{\mathrm{NC}} = 500\,\mathrm{GeV}$ in the right most column of figure 3.10. We have verified that the strength of the correlation is essentially determined by the expectation value of the boost $\langle|\beta|\rangle$. Since the dependence on B_1 and B_2 in the partonic CMS is very weak, one expects very elongated ellipses in the laboratory frame, in agreement with our result. Due to statistical fluctuations, the fitted matrix X^2 can have a negative eigenvalue, as happened in the bottom right plot of figure 3.10. This sign is unphysical and, as expected, changes with the random number sequence used in the simulations. For all practical purposes, the error ellipses for (E_1, B_2) and (E_2, B_1) should be viewed as straight lines.

Having established the absence of correlations that are not of purely kinematical origin, one can avoid expensive non-linear 4-parameter fits in subsequent work that will take higher orders in $\theta^{\mu\nu}$ into account.

Nevertheless, the correlations among (E_1, B_2) and (E_2, B_1), respectively, inhibit in principle measurements of pure space- or pure time-like noncommutativity. Yet, there is a way out, as we will see. It was shown previously that the dependency on $\vec{E}$ is much stronger than on $\vec{B}$. Thus, if setting $\vec{E} \neq 0$, $\vec{B} = 0$, the Lorentz-boost will induce the corresponding B-component, and vice-versa, an E component will be induced when assuming pure B type noncommutativity. Therefore, when using the cuts specified in section 3.2.3 and integrating over *one* hemisphere, in both cases (pure $\vec{E}$ or pure $\vec{B}$) we will be sensitive only on the time-like directions, since the space-like is negligibly small. Nevertheless, a pure measurement on B can be achieved by the appropriate cuts, namely integration over the *whole* sphere. The fact that for $\vec{B} \neq 0$ the noncommutative effects are symmetric w. r. t. $\cos\vartheta_\gamma$ while for $\vec{E} \neq 0$ they are antisymmetric, will lead to the cancellation of E-type noncommutativity, such that only noncommutative effects in B direction will remain.

We derived the sensitivity bounds on Λ_{NC} using the method of the previous section. The fits were done by a `Fortran`-program, which we checked against `MINUIT` [66] for some one dimensional examples.

For the LHC, the results of the likelihood fits are shown in figure 3.10, setting $\Lambda_{\mathrm{NC}} = 500\,\mathrm{GeV}$ and $K_{Z\gamma\gamma} = K_{ZZ\gamma} = 0$. The bounds on the noncommutative scale were derived for an integrated luminosity of $100\,\mathrm{fb}^{-1}$ (with 100% detection efficiency) and are presented in table 3.3.

The results are not surprising, if we review the discussions of the previous sections. Therefore, at the LHC we have maximal sensitivity in E-direction for values of the TGB couplings lying in the upper region of the polygon 3.1. The

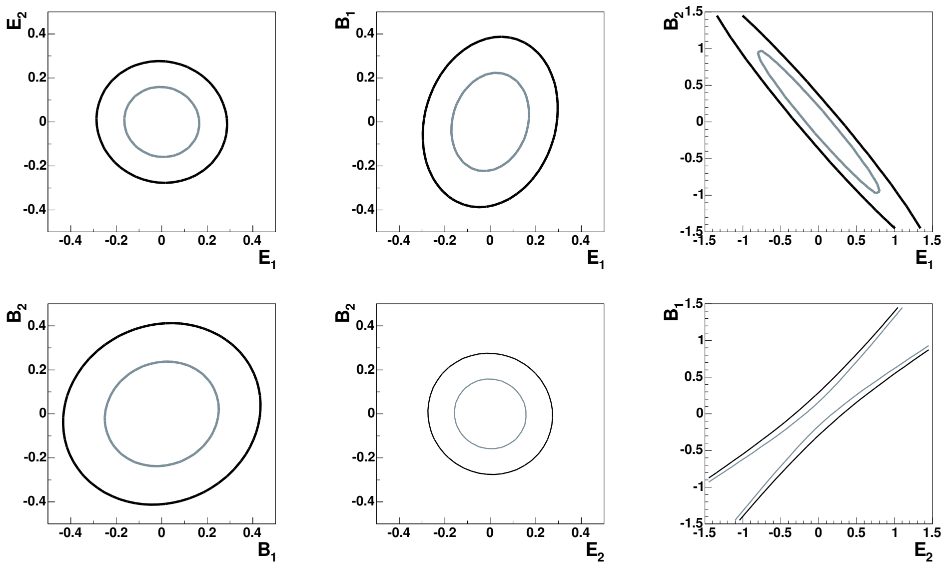


Figure 3.10: The 1σ (dark) and 3σ (light) exclusion contours for $\Lambda_{\rm NC} = 500\,\text{GeV}$, K_0 and $100\,\text{fb}^{-1}$ at the LHC discussed in the text.

$(K_{Z\gamma\gamma}, K_{ZZ\gamma})$	$\vert\vec{E}\vert^2 = 1, \vec{B} = 0$	$\vec{E} = 0, \vert\vec{B}\vert^2 = 1$
$K_0 \equiv (0,0)$ (mNCSM)	$\Lambda_{\rm NC} \gtrsim 1\,\text{TeV}$	-
$K_1 \equiv (-0.333, 0.035)$ (nmNCSM)	-	-
$K_5 \equiv (0.095, 0.155)$ (nmNCSM)	$\Lambda_{\rm NC} \gtrsim 1.2\,\text{TeV}$	-

Table 3.3: Bounds on $\Lambda_{\rm NC}$ from $pp \to Z\gamma \to e^+e^-\gamma$ at the LHC.

LHC is not sensitive to noncommutative effects for couplings lying in the lower region of the polygon, nor is it at all sensitive to space-like noncommutativity. The effects for this case are simply too small in order to detect them with energies and the statistics available, despite the cuts mentioned above.
Nevertheless, this analysis indicates that the current collider bounds [41] can be improved at the LHC by an order of magnitude, for certain choices of the TGB couplings and time-like noncommutativity. The bounds could even be enhanced by including the decay of the Z boson into muons and adding the mirrored histogram of the same Monte Carlo simulation yet integrating over the other hemisphere.
In order to obtain a realistic estimate of the sensitivity at the Tevatron and the LHC, one has to take into account backgrounds, detector effects and selection cuts. Clearly, a comprehensive analysis of all reducible backgrounds and detector effects is beyond the scope of a theoretical study and must eventually be performed by the experimental collaborations. However, the final states under consideration are simple enough for our phenomenological analysis based on simple cuts.
Moreover, experience at the Tevatron [64, 65] indicates that the combined detection efficiency for $\ell^+\ell^-\gamma$ can be assumed to be larger than 50%. All numerical results presented are obtained for a 100% efficiency. Smaller uniform efficiencies can easily be taken into account by scaling up the integrated luminosity accordingly.
For simplicity, we have assumed that the components of $\theta^{\mu\nu}$ remain aligned with the beam axis and the detector over the time of the measurement. This assumption is not justified, because we should expect that $\theta^{\mu\nu}$ is aligned with a fixed cosmic reference frame, that is determined by the dynamics of the underlying string theory. Therefore the alignment of the detector must be recorded with each event and the combined effect of the earth's rotation and revolution must be taken into account. This poses no principal difficulty and will not change our conclusions.

3.2.6 Bounds from the ILC

The LHC is the experiment providing experimental data within the next years, hence we dedicated most of our phenomenological studies to this experiment. Nevertheless, since a design effort is underway for the construction of the International Linear Collider (ILC), an electron-positron collider, starting from 2015, we have also performed an analysis of the NCSM for a linear collider. The ILC has definite advantages over the LHC, which not only will change the sensitivity bounds on the noncommutative scale quantitatively, but will also provide measurements for other sets of values of the TGB couplings. In this sense, experiments at the ILC are complementary to those of the LHC.
The complementarity of the two experiments w.r.t. the nmNCSM is due to the different charge of the scattered particles, more precisely due to the different signs of their charge. We have seen in section 3.2.2 and recall it here once more, that while the cross section of the minimal NCSM is independent of the sign of the charge Q the cross section within the nonminimal NCSM additionally

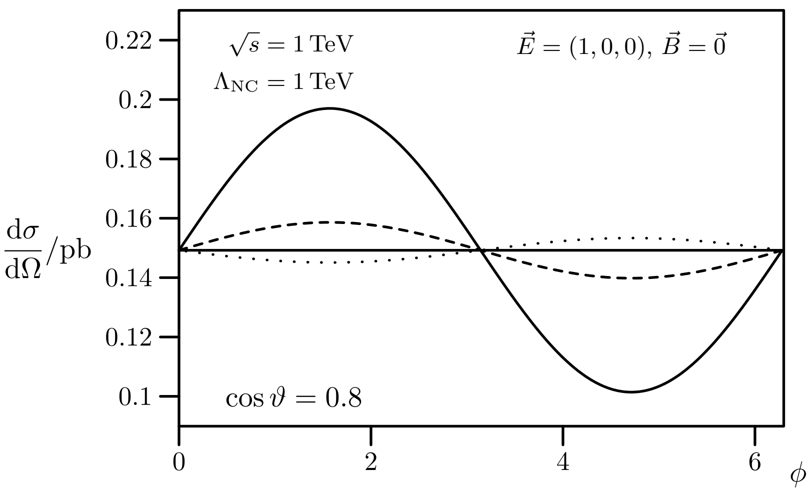


Figure 3.11: Azimuthal dependence of the cross section, with different values for the TGB couplings: $K_1 = (-0.333, 0.035)$ (solid) , $K_5 = (0.095, 0.155)$ (dotted) and $K_0 = (0.0, 0.0)$ (dashed).

depends on $sign(Q)$ via the s-channel amplitudes. These depend linearly on the charge of the particles, while the t-, u- and c-channels are always proportional to Q^2. Therefore, the interference terms $A_{\text{SM}}A^*_{\text{NC},s}$ depend on Q^3 and the $sign(Q)$ becomes relevant. Thus, for e^+e^- scattering, noncommutative effects are maximally enhanced by the s-channels contribution for K_1 and K_2[8], whereas these couplings lead to cancellations of the noncommutative effects for $u\bar{u}$ scattering resulting in minimal deviations of the NCSM w. r. t. the SM. On the other hand, the set of couplings K_5, which gave maximal effects for the LHC, will lead to an NCSM cross section comparable to the one where the couplings vanish. These dependencies are depicted in figure 3.2 and figure 3.11. We have performed Monte Carlo simulations for different values of the TGB couplings, similar to the LHC analysis.

The ILC is planned to initially reach a CMS energy of $\sqrt{s} = 500\,\text{GeV}$. Counting on an integrated luminosity of $\mathcal{L} = 500\,\text{fb}^{-1}$ corresponding to fours years of running, minus the first year of calibration, we will have a large number of events at our disposal, thus improving the statistics significantly. The better statistics enhances the sensitivity on Λ_{NC} by a factor $\sqrt[4]{N_{\text{ILC}}/N_{\text{LHC}}}$ where N is the number of events.

One important advantage of the ILC compared to the LHC with consequences on our analysis is the only mildly boosted initial state. We have an e^+e^- initial state, where just beam-strahlung has to be accounted for, which we have done, using CIRCE [67]. This will lead to a boost of the CMS of the electrons to the laboratory frame. Yet, compared to the LHC, this boost is negligibly small: $\beta_{\text{ILC}} = 0.14$ versus $\beta_{\text{LHC}} = 0.8$. We therefore have negligible correlations between E_1 and B_2 or E_2 and B_1, respectively, as can be seen from figure 3.15 Due to the high statistics and the sharp CMS energy of the initial particles

[8] The azimuthal dependence of the cross section for these two values of the TGB couplings is almost identical, therefore we have done the fits only for K_1.

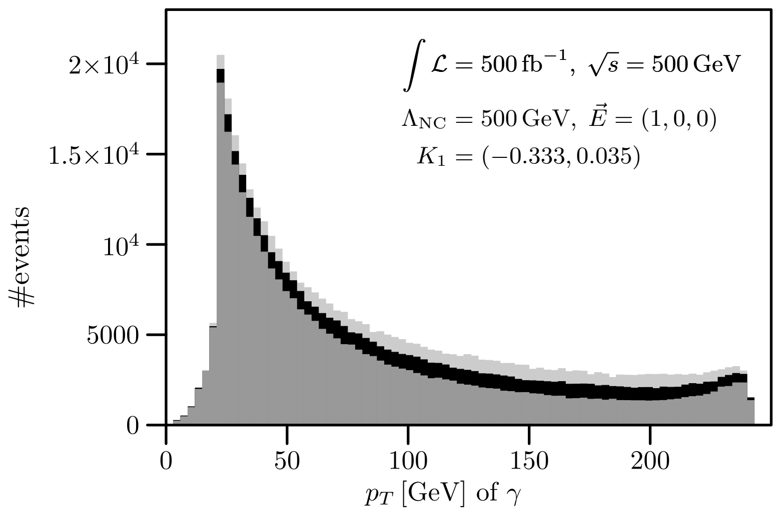


Figure 3.12: Monte Carlo simulation for the photon p_T distribution of the process $e^+e^- \to Z\gamma$ at the ILC showing above the black SM histogramm the NCSM distribution for $0.0 < \phi < \pi$ and beneath the NCSM distribution for $\pi < \phi < 2\pi$.

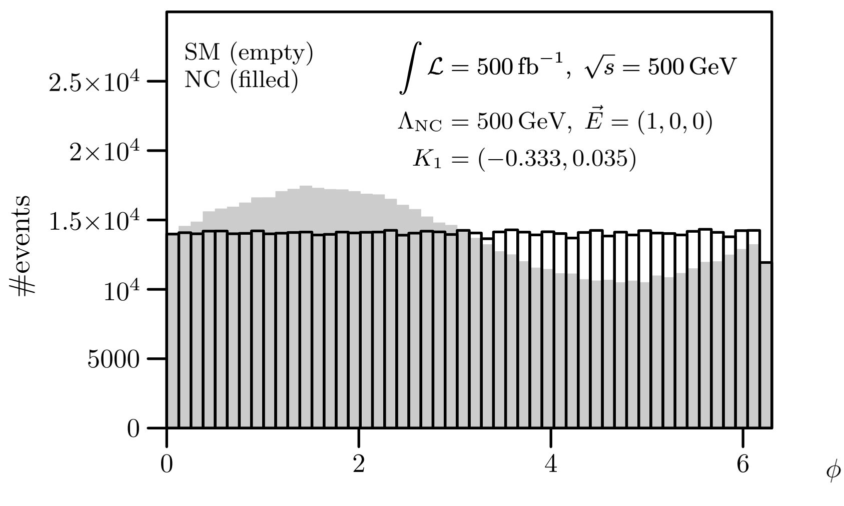


Figure 3.13: Monte Carlo simulation for the azimuthal dependence of the process $e^+e^- \to Z\gamma$ at the ILC.

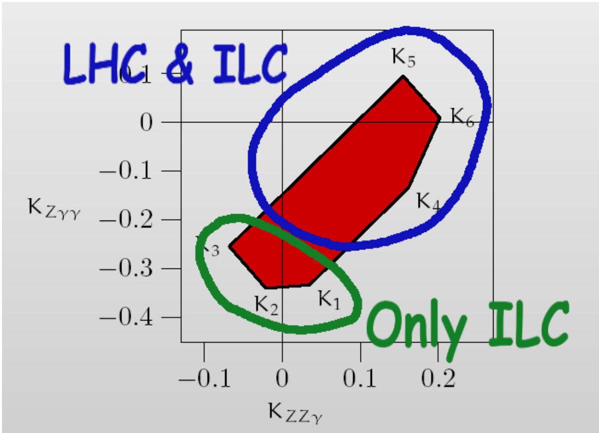


Figure 3.14: The allowed range for the values of TGB couplings is probed complementary by the LHC and ILC.

at the ILC, deviations of the NCSM from the SM can actually be seen also in the p_T distribution for reasonable values of $\Lambda_{\rm NC}$ (see figure 3.12). Of course, cuts w.r.t. the azimuthal angle ϕ have to be performed, otherwise the effect will cancel, since the events "missing" in one hemisphere (e. g. for $\pi < \phi < 2\pi$) are compensated by the "excess" of events in the other (see figure 3.13). Figure 3.13 shows this distribution exemplarily, where for the TGB couplings we have chosen the set of values, where we expect the largest deviation from the SM distribution, i. e. K_1. Nevertheless, we still have used the azimuthal dependency of the cross section in order to derive the bounds on $\Lambda_{\rm NC}$ from the ILC. This is exactly the set of TGB couplings, for which the LHC is less sensitive, while the TGB couplings leading to maximal deviations at the LHC, lead to minimal effects at the ILC. Here the complementarity of LHC and ILC measurements comes in, as illustrated in figure 3.14.

We derived bounds on $\Lambda_{\rm NC}$ considering time-like as well as space-like noncommutativity, like in the previous section. Unlike the LHC, the ILC is also sensitive on the B-type components of the noncommutative parameter, which could not be probed by the LHC, due to the poor statistics. By integrating over the whole range of the scattering angle $\cos\vartheta$, statistics is enhanced even more. In addition, the influence of the E-components, which might be introduced by the boost due to beamstrahlung, is canceled out. Anyway, this is almost redundant, since the correlation between E and B is minimal for the ILC.

The resulting bounds on $\Lambda_{\rm NC}$ for purely time-like ($\vec{E} \neq 0$) and purely space-like ($\vec{B} \neq 0$) noncommutativity, respectively are presented in table 3.4.

We have derived sensitivity bounds on $\Lambda_{\rm NC}$ for three cases: the mNCSM (i. e. vanishing TGB couplings), and for some sets of values ($K_{Z\gamma\gamma}$,$K_{ZZ\gamma}$). K_1 corresponds to maximal and K_5 to minimal deviation from the SM . The corresponding error ellipses are depicted in figure 3.15. We have plotted the ellipses only for the mNCSM. The shape of the ellipses does not change by changing the values of the TGB couplings, since these cannot introduce correlations among the entries of the matrix $\theta^{\mu\nu}$.

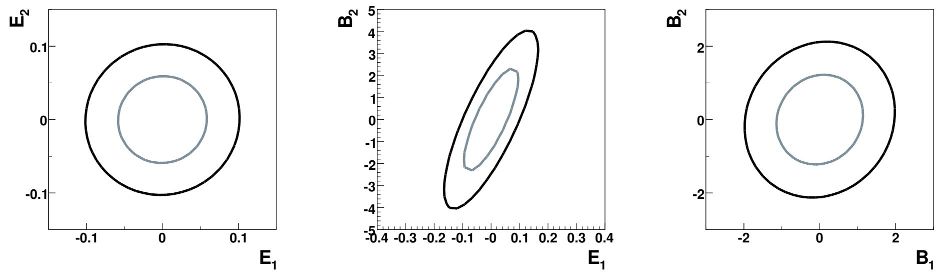


Figure 3.15: The 1σ (dark) and 3σ (light) exclusion contours for $\Lambda_{\rm NC} = 500\,{\rm GeV}$, K_0 and $500\,{\rm fb}^{-1}$ at the ILC for the mNCSM.

$(K_{Z\gamma\gamma}, K_{ZZ\gamma})$	$\lvert\vec{E}\rvert^2 = 1, \vec{B} = 0$	$\vec{E} = 0, \lvert\vec{B}\rvert^2 = 1$
$K_0 \equiv (0,0)$ (mNCSM)	$\Lambda_{\rm NC} \gtrsim 2\,{\rm TeV}$	$\Lambda_{\rm NC} \gtrsim 0.4\,{\rm TeV}$
$K_1 \equiv (-0.333, 0.035)$ (nmNCSM)	$\Lambda_{\rm NC} \gtrsim 5.9\,{\rm TeV}$	$\Lambda_{\rm NC} \gtrsim 0.9\,{\rm TeV}$
$K_5 \equiv (0.095, 0.155)$ (nmNCSM)	$\Lambda_{\rm NC} \gtrsim 2.6\,{\rm TeV}$	$\Lambda_{\rm NC} \gtrsim 0.25\,{\rm TeV}$
$K_3 \equiv (-0.254, -0.048)$ (nmNCSM)	$\Lambda_{\rm NC} \gtrsim 5.4\,{\rm TeV}$	$\Lambda_{\rm NC} \gtrsim 0.9\,{\rm TeV}$

Table 3.4: Bounds on $\Lambda_{\rm NC}$ from $pp \to Z\gamma \to e^+e^-\gamma$ at the LHC.

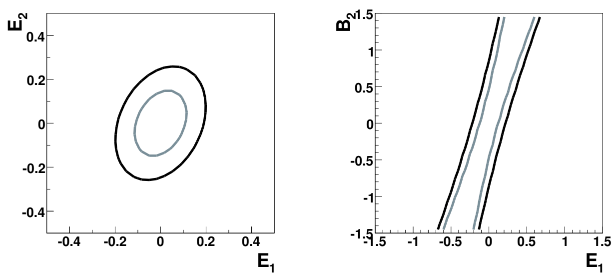


Figure 3.16: The 1σ (dark) and 3σ (light) exclusion contours for $\Lambda_{\text{NC}} = 50\,\text{GeV}$ and $15\,\text{fb}^{-1}$ at the Tevatron discussed in the text.

Thus, the ILC not only provides measurements for additional parameters (B-components) and probes the NCSM for different values of the TGB couplings, but it also improves the sensitivity bounds on Λ_{NC} considerably. Furthermore, we do not have to worry about getting negative cross sections and therefore unphysical results in our simulations. The phase space of the process is much simpler than in the case of the LHC, where convolutions with the broad parton distribution functions had to be taken into consideration[9]. The long tails of the PDF towards high energy leading to regions in phase space for which $\sqrt{\hat{s}}/\Lambda_{\text{NC}}$ becomes too large, as discussed in section 3.2.5, are absent in the case of the beam-strahlung distribution.

3.2.7 Bounds from LEP and Tevatron

Before closing the chapter on the phenomenological consequences of the NCSM at $\mathcal{O}(\theta)$, for the sake of completeness, we also include a brief analysis of the NCSM and the resulting bounds which can be derived from present experiments, like LEP or Tevatron. The OPAL collaboration has established bounds resulting from LEP in the framework of the "naive" version of NCQED [41], nevertheless we perform the analysis within the model which is subject of this thesis.
Thus, for the Tevatron, we have simulated $p\bar{p} \to \ell^+\ell^-\gamma$ at $\sqrt{s} = 1.96\,\text{TeV}$ using the same acceptance cuts (3.52). In the case of the Tevatron, the rôles of quarks and anti-quarks are reversed in the antiproton. Therefore, we demand in this case that the momenta of the photon and the lepton pair lie in *opposite* hemispheres. For sufficiently small Λ_{NC}, the azimuthal distribution of the γ is similar to the distribution plotted in figure 3.9. The error ellipses are similar to the ones obtained for the LHC or ILC, only the size and, due to the smaller boost, also the magnitude of the correlation between the boost-mixed components of θ vary (figure 3.16). From these we derive the sensitivity on the noncommutative scale which reaches $\Lambda_{\text{NC}} \gtrsim 130\,\text{GeV}$ at the Tevatron, comparable to the LEP bounds in [41].
At LEP we have simulated $e^+e^- \to Z\gamma$ at $\sqrt{s} = 200\,\text{GeV}$. Using an integrated luminosity of $\int \mathcal{L} = 3\,\text{fb}^{-1}$ and the same cuts as for the ILC analysis we obtain

[9]The convolution with photonic distribution function can only reduce the "partonic" center of mass energy $\sqrt{\hat{s}}$.

$\Lambda_{\rm NC} \gtrsim 770\,{\rm GeV}$ assuming K_1 for the TGB, the value which gives maximal noncommutative effects. This is a much better bound that the one obtained in [41], but we should not forget that we did not perform a full detector simulation, nor did we take into account a realistic detector efficiencies. On the other hand, bounds on $\Lambda_{\rm NC}$ obtained assuming the minimal version of the NCSM, i.e. vanishing TGB, are found to be $\Lambda_{\rm NC} \gtrsim 300\,{\rm GeV}$, which are closer to those obtained by the OPAL collaboration in [41]. The difference to the ILC bounds can be accounted for by scaling the luminosity and center of mass scattering energy.

3.2.8 Remarks

Before closing the chapter on the phenomenology of the NCSM we need to emphasise, that we did not account for the motion of the earth (or our galaxy) w.r.t. to the universe, as discussed in section 2.5.3. We have no information at all on the values of the components of $\theta^{\mu\nu}$, we do not know whether, how or how much they vary w.r.t. to the space-time coordinate system. We did not include in our analysis the position of the collider on the earth, which a complete phenomenological analysis should eventually include. Effects might cancel out and therefore tagged data taking w.r.t. the orientation of the earth is required. If nevertheless all noncommutative effects cancel, other processes will be needed to reveal a possible noncommutative nature of space-time. For example, $Z \to \gamma\gamma$ decay, as studied in [68], or higher orders in the noncommutative parameter have to be included in the calculation of processes. When the dependence on $\theta^{\mu\nu}$ in observables is realized by harmonic functions, the noncommutative effects average to zero at $\mathcal{O}(\theta)$ when integrating over the motion of the detector w.r.t a given frame. This cannot happen when noncommutative effects enter observables quadratically in $\theta^{\mu\nu}$.

Chapter 4

The Neutral Current Sector of the NCSM at $\mathcal{O}(\theta^2)$

The θ-expanded approach to the noncommutative extension of the SM studied within this work results in an effective theory, which means that its range of validity is determined by the size of the expansion parameter $\sqrt{s}/\Lambda_{\rm NC}$. In section 3.2.2 we have seen that the validity of the model breaks down if this ratio is chosen too large. This manifests itself in unphysical (negative) cross sections, as shown in figure 3.5 and has to be circumvented when performing phenomenological studies.

It is not always possible to control the magnitude of $\sqrt{s}/\Lambda_{\rm NC}$. As we have seen in the case of the LHC study, the broad distribution of the PDFs led to regions in phase space where the partonic scattering energy was so large that the model was not valid any more. To repair this unphysical behaviour in a consistent manner, higher orders in the expansion parameter are needed. Thus, as a first step, cross sections have to be computed up to $\mathcal{O}(\theta^2)$, which requires the corresponding Feynman rules and therefore the Seiberg-Witten maps up to this order.

Also, we have seen that some processes, like $pp \to \gamma\gamma$, show at $\mathcal{O}(\theta)$ small or no noncommutative signatures. Nevertheless, some of them, like the mentioned one, will be under intense study at the LHC due to their importance for the confirmation (or falsification) of other theoretical models, like the discovery or exclusion of the Higgs from $pp \to \gamma\gamma$. Since in such processes, the noncommutative signal enters the observable (the unpolarized cross section for this example) at $\mathcal{O}(\theta^2)$, the processes must of course be calculated to this order.

We will lay the foundation to this endeavor, and derive the Seiberg-Witten maps to $\mathcal{O}(\theta^2)$. It was shown previously that the Seiberg-Witten maps are not unique. To our surprise it proved that some of these ambiguities, which parametrize the solutions of the homogeneous consistency and gauge equivalence equations, respectively, do not always correspond to field redefinitions and thus have physical consequences.

Finding the general solution of the Seiberg-Witten map at $\mathcal{O}(\theta^2)$ required extensive manual and computerized calculations. For the main part of the computations `FORM` [69] was used.

4.1 General Solution of the Seiberg-Witten Maps up to $\mathcal{O}(\theta^2)$

Seiberg-Witten maps to second order in θ have been derived previously [70, 71]. In the first reference only one special solution of the consistency and gauge equivalence equations is given, whereas in the second reference some ambiguities are included, but not all. We will give the complete solution of the Seiberg-Witten maps up to second order in θ for the gauge parameter, gauge and matter field and show that, against our expectation, not all ambiguities correspond to field redefinitions, as it was stated in [71].
The approach is the same as for the $\mathcal{O}(\theta)$ calculation of the Seiberg-Witten maps done in section 2.4. The θ expansion (2.43) of the gauge parameter and the fields is inserted in the consistency and gauge equivalence equations (2.40, 2.41) and then the $\mathcal{O}(\theta^2)$ contributions are collected. We introduce an abbreviation for the homogeneous part of the consistency and gauge equivalence equations. At every oder in θ, the homogeneous part of the consistency equation has the same form

$$\begin{aligned}H_\lambda[\lambda^k(\lambda, A, \theta)] \equiv\, &\delta_{\lambda_1}\lambda^k(\lambda_2, A, \theta) - \delta_{\lambda_2}\lambda^k(\lambda_1, A, \theta)\\ &- \mathrm{i}\left[\lambda^k(\lambda_1, A, \theta), \lambda_2\right] - \mathrm{i}\left[\lambda_1, \lambda^k(\lambda_2, A, \theta)\right] - \lambda^k(\lambda_3, A, \theta)\,. \quad (4.1)\end{aligned}$$

For the homogeneous part of the gauge equivalence equations we have for every order in θ:

$$H_A[A^k(A, \theta)] \equiv \delta_\lambda A^k_\xi - \mathrm{i}[\lambda, A^k_\xi]\,, \quad (4.2a)$$

$$H_\psi[\psi^k(\psi, A, \theta)] \equiv \delta_\lambda \psi^k - \mathrm{i}\lambda\psi^k\,. \quad (4.2b)$$

Thus, for the consistency equation at $\mathcal{O}(\theta^2)$ we can write (omitting from now on the arguments A, ψ and θ in λ^k, A^k, ψ^k, with $k = 1, 2$)

$$\begin{aligned}H_\lambda[\lambda^2] = \mathrm{i}[\lambda^1(\lambda_1), \lambda^1(\lambda_2)] - \frac{1}{2}\theta^{\mu\nu}\Big(\left\{\partial_\mu\lambda^1(\lambda_1), \partial_\nu\lambda_2\right\} + \left\{\partial_\mu\lambda_1, \partial_\nu\lambda^1(\lambda_2)\right\}\Big)\\ - \frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\left[\partial_\mu\partial_\nu\lambda_1, \partial_\kappa\partial_\lambda\lambda_2\right]. \quad (4.3)\end{aligned}$$

For the gauge equivalence equations we obtain:

$$\begin{aligned}H_A[A^2] \;=\;& \partial_\xi\lambda^2(\lambda) - \mathrm{i}[A^1_\xi, \lambda^1(\lambda)] - \mathrm{i}[A_\xi, \lambda^2(\lambda)] \quad (4.4a)\\ &+\frac{1}{2}\theta^{\mu\nu}\Big(\left\{\partial_\mu A^1_\xi, \partial_\nu\lambda\right\} + \left\{\partial_\mu A_\xi, \partial_\nu\lambda^1(\lambda)\right\}\Big)\\ &+\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa A_\xi\partial_\nu\partial_\lambda\lambda\,,\\ H_\psi[\psi^2] \;=\;& \mathrm{i}\lambda^1(\lambda)\psi^1 + \mathrm{i}\lambda^2(\lambda)\psi - \frac{1}{2}\theta^{\mu\nu}\left(\partial_\mu\lambda^1(\lambda)\partial_\nu\psi + \partial_\mu\lambda\partial_\nu\psi^1\right) \quad (4.4b)\\ &-\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa\lambda\partial_\nu\partial_\lambda\psi\,.\end{aligned}$$

We will first give one special solution for each of these equations, before turning to their homogeneous solutions. We will see that the $\mathcal{O}(\theta)$ ambiguities of the Seiberg-Witten maps will also contribute to the $\mathcal{O}(\theta^2)$ Seiberg-Witten maps.

4.1.1 Special Solutions at $\mathcal{O}(\theta^2)$

We start, as in the $\mathcal{O}(\theta)$ case, by solving the inhomogeneous linear equation (4.3) for the gauge parameter. A special solution preserving hermiticity was found to be:

$$\begin{aligned}\lambda^2(\lambda) = \frac{\mathrm{i}}{32}\theta^{\kappa\lambda}\theta^{\mu\nu}\big(& -3A_\kappa A_\lambda \partial_\nu\lambda A_\mu - 4A_\kappa A_\nu \partial_\lambda \lambda A_\mu - 3A_\kappa \partial_\lambda\lambda A_\mu A_\nu \\ & - 2A_\lambda \partial_\nu \lambda A_\mu A_\kappa - 2A_\mu A_\kappa A_\lambda \partial_\nu \lambda - A_\mu A_\nu A_\kappa \partial_\lambda \lambda \\ & - 2A_\nu A_\kappa \partial_\lambda \lambda A_\mu - 4A_\nu \partial_\lambda\lambda A_\mu A_\kappa - 2\partial_\mu A_\kappa \partial_\lambda \partial_\nu \lambda \\ & - 2\partial_\lambda \lambda A_\mu A_\nu A_\kappa - \partial_\nu \lambda A_\mu A_\kappa A_\lambda + 2\partial_\lambda\partial_\nu\lambda\partial_\mu A_\kappa\big) \\ + \frac{1}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}\big(& 4A_\kappa \partial_\lambda\partial_\nu\lambda A_\mu + A_\lambda\partial_\nu\lambda\partial_\mu A_\kappa + 2A_\mu A_\kappa \partial_\lambda\partial_\nu\lambda \\ & - 2A_\mu \partial_\kappa A_\nu \partial_\lambda \lambda - \partial_\kappa A_\nu \partial_\lambda\lambda A_\mu + \partial_\mu A_\kappa A_\lambda \partial_\nu \lambda \\ & - \partial_\lambda\lambda A_\mu \partial_\kappa A_\nu + 2\partial_\nu\lambda\partial_\mu A_\kappa A_\lambda + 2\partial_\lambda\partial_\nu\lambda A_\mu A_\kappa\big)\,.\end{aligned} \tag{4.5}$$

This solution was checked against the ones presented in [70] and [71], we find that they are all equivalent, differing by homogeneous solutions to (4.3).
With the expression (4.5) for λ^2 and the first order special solutions for λ^1 and A^1_ξ, we can solve the gauge equivalence equation (4.4a) for the gauge field. We find the special Hermitian solution:

$$\begin{aligned}A^2_\xi = & \frac{\mathrm{i}}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(2[\partial_\nu\partial_\lambda A_\xi, \partial_\mu A_\kappa] + [\partial_\mu A_\kappa, \partial_\xi\partial_\lambda A_\nu]\big) \\ & + \frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ 2A_\mu A_\kappa \partial_\nu\partial_\lambda A_\xi - A_\mu A_\kappa \partial_\nu \partial_\xi A_\lambda + A_\mu A_\kappa \partial_\xi\partial_\lambda A_\nu \\ & \quad + 4A_\mu \partial_\nu A_\kappa \partial_\lambda A_\xi - 4A_\mu\partial_\nu A_\kappa \partial_\xi A_\lambda - 2A_\mu \partial_\kappa A_\nu \partial_\lambda A_\xi \\ & \quad - 2A_\nu \partial_\lambda A_\xi \partial_\mu A_\kappa + 3A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa + 4A_\kappa \partial_\nu\partial_\lambda A_\xi A_\mu \\ & \quad + 2A_\lambda \partial_\nu A_\xi \partial_\mu A_\kappa - A_\lambda \partial_\xi A_\nu \partial_\mu A_\kappa - A_\xi \partial_\mu A_\kappa \partial_\lambda A_\nu \\ & \quad - 4\partial_\mu A_\kappa A_\nu \partial_\lambda A_\xi + \partial_\mu A_\kappa A_\nu \partial_\xi A_\lambda + \partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu \\ & \quad - \partial_\mu A_\kappa \partial_\lambda A_\nu A_\xi + 2\partial_\nu A_\kappa \partial_\lambda A_\xi A_\mu - 3\partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu \\ & \quad + 2\partial_\nu A_\xi \partial_\mu A_\kappa A_\lambda - 2\partial_\kappa A_\nu \partial_\lambda A_\xi A_\mu + \partial_\kappa A_\nu \partial_\xi A_\lambda A_\mu \\ & \quad + 2\partial_\lambda A_\nu A_\xi \partial_\mu A_\kappa + 2\partial_\lambda A_\xi A_\mu \partial_\nu A_\kappa - 4\partial_\lambda A_\xi \partial_\mu A_\kappa A_\nu \\ & \quad - \partial_\xi A_\lambda A_\mu \partial_\nu A_\kappa - \partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu + 4\partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu \\ & \quad + 2\partial_\nu\partial_\lambda A_\xi A_\mu A_\kappa + \partial_\nu \partial_\xi A_\lambda A_\mu A_\kappa - \partial_\xi\partial_\lambda A_\nu A_\mu A_\kappa\big) \\ & + \frac{\mathrm{i}}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- 4A_\mu A_\nu A_\kappa \partial_\lambda A_\xi + 3A_\mu A_\nu A_\kappa \partial_\xi A_\lambda - 2\partial_\xi A_\lambda A_\mu A_\kappa A_\nu \\ & \quad + 4A_\mu A_\kappa A_\nu \partial_\lambda A_\xi - 2A_\mu A_\kappa A_\nu \partial_\xi A_\lambda - 4A_\mu A_\kappa A_\lambda \partial_\nu A_\xi \\ & \quad - 2A_\mu A_\kappa \partial_\nu A_\lambda A_\xi + 2A_\mu A_\kappa \partial_\lambda A_\nu A_\xi - 8A_\mu \partial_\nu A_\kappa A_\lambda A_\xi \\ & \quad - 4A_\nu A_\kappa \partial_\lambda A_\xi A_\mu + 4A_\nu A_\kappa \partial_\xi A_\lambda A_\mu + 8A_\nu A_\lambda A_\xi \partial_\mu A_\kappa \\ & \quad - 4A_\nu \partial_\lambda A_\xi A_\mu A_\kappa - 2A_\nu \partial_\xi A_\lambda A_\mu A_\kappa - 4A_\kappa A_\nu \partial_\lambda A_\xi A_\mu \\ & \quad - 2A_\kappa A_\nu \partial_\xi A_\lambda A_\mu - 4A_\kappa A_\lambda \partial_\nu A_\xi A_\mu + A_\kappa A_\lambda \partial_\xi A_\nu A_\mu \\ & \quad - 8A_\kappa \partial_\lambda A_\nu A_\xi A_\mu - 4A_\kappa \partial_\lambda A_\xi A_\mu A_\nu + A_\kappa \partial_\xi A_\lambda A_\mu A_\nu\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&- 4A_\lambda A_\nu A_\xi \partial_\mu A_\kappa - 4A_\lambda A_\xi A_\mu \partial_\nu A_\kappa + 4A_\lambda A_\xi A_\mu \partial_\kappa A_\nu \\
&+ 8A_\lambda A_\xi \partial_\mu A_\kappa A_\nu - 4A_\lambda \partial_\nu A_\xi A_\mu A_\kappa + 4A_\lambda \partial_\xi A_\nu A_\mu A_\kappa \\
&- 2A_\xi A_\mu A_\kappa \partial_\nu A_\lambda - 2A_\xi A_\mu A_\kappa \partial_\lambda A_\nu + 8A_\xi A_\mu \partial_\kappa A_\nu A_\lambda \\
&- 2A_\xi \partial_\mu A_\kappa A_\nu A_\lambda + 2A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + 2\partial_\mu A_\kappa A_\nu A_\lambda A_\xi \\
&+ 2\partial_\mu A_\kappa A_\lambda A_\nu A_\xi - 4\partial_\nu A_\kappa A_\lambda A_\xi A_\mu + 4\partial_\nu A_\lambda A_\xi A_\mu A_\kappa \\
&- 4\partial_\nu A_\xi A_\mu A_\kappa A_\lambda + 4\partial_\kappa A_\nu A_\lambda A_\xi A_\mu - 8\partial_\lambda A_\nu A_\xi A_\mu A_\kappa \\
&- 4\partial_\lambda A_\xi A_\mu A_\nu A_\kappa + 4\partial_\lambda A_\xi A_\mu A_\kappa A_\nu + 3\partial_\xi A_\nu A_\mu A_\kappa A_\lambda \big) \\
+\frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- 3A_\mu A_\nu A_\kappa A_\lambda A_\xi + 2A_\mu A_\kappa A_\nu A_\lambda A_\xi - 4A_\nu A_\kappa A_\lambda A_\xi A_\mu \\
&+ 4A_\nu A_\lambda A_\xi A_\mu A_\kappa - 4A_\nu A_\xi A_\mu A_\kappa A_\lambda + 4A_\kappa A_\nu A_\lambda A_\xi A_\mu \\
&- 4A_\kappa A_\lambda A_\nu A_\xi A_\mu - 2A_\kappa A_\lambda A_\xi A_\mu A_\nu - 8A_\lambda A_\nu A_\xi A_\mu A_\kappa \\
&- 4A_\lambda A_\xi A_\mu A_\nu A_\kappa + 4A_\lambda A_\xi A_\mu A_\kappa A_\nu \\
&- 3A_\xi A_\mu A_\nu A_\kappa A_\lambda + 2A_\xi A_\mu A_\kappa A_\nu A_\lambda \big) \,,
\end{aligned}
$$

where we have arranged the terms by the order of A, for later convenience. Again, this solution was compared to [70, 71]. It is equivalent up to homogeneous solutions to the solution found in the first reference, but not to the one in [70]. It was verified that the solution therein does not fulfill the corresponding gauge equivalence equation.
Plugging λ^2, λ^1 and ψ^1 into equation (4.4b), we obtain the special solution for the matter field:

$$
\begin{aligned}
\psi^2 = &-\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu A_\kappa \partial_\nu \partial_\lambda \psi \\
&+\frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- 2A_\mu \partial_\kappa A_\nu \partial_\lambda \psi - 2\partial_\mu A_\kappa A_\nu \partial_\lambda \psi \\
&\qquad + 2A_\mu A_\kappa \partial_\nu \partial_\lambda \psi + 4A_\mu \partial_\nu A_\kappa \partial_\lambda \psi - \partial_\mu A_\kappa \partial_\nu A_\lambda \psi \big) \\
&+\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- 2A_\mu \partial_\nu A_\kappa A_\lambda \psi + \partial_\mu A_\kappa A_\nu A_\lambda \psi \\
&\qquad - A_\mu A_\kappa A_\lambda \partial_\nu \psi + A_\mu A_\kappa A_\nu \partial_\lambda \psi - A_\mu A_\nu A_\kappa \partial_\lambda \psi \big) \\
&+\frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- 3A_\mu A_\nu A_\kappa A_\lambda \psi + 4A_\mu A_\kappa A_\nu A_\lambda \psi - 2A_\mu A_\kappa A_\lambda A_\nu \psi \big) \,.
\end{aligned}
\tag{4.7}
$$

Differences between the above special solution and the ones published in [70, 71] are due to different choices for the gauge parameter $\lambda^2(\lambda)$ at $\mathcal{O}(\theta^2)$ and to homogeneous solutions to the gauge equivalence equations (4.4b).

4.1.2 Contributions from $\mathcal{O}(\theta)$-Ambiguities to the $\mathcal{O}(\theta^2)$ Solutions

The ambiguities at first order will propagate in the solutions at the second order. Thus, the ambiguity c^1_λ at first order of the gauge parameter will contribute an additional term to the solution at second order

$$
\begin{aligned}
\lambda^2_{c^1_\lambda}(\lambda) = \frac{c^1_\lambda}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&+ A_\kappa A_\lambda \partial_\nu \lambda A_\mu - A_\kappa \partial_\lambda \lambda A_\mu A_\nu - \mathrm{i}(A_\mu \partial_\kappa A_\nu \partial_\lambda \lambda \\
&+ \partial_\mu A_\kappa A_\lambda \partial_\nu \lambda + \partial_\lambda \lambda A_\mu \partial_\kappa A_\nu + \partial_\nu \lambda \partial_\mu A_\kappa A_\lambda)\big)
\end{aligned}
\tag{4.8}
$$

$$+\frac{\mathrm{i}(c^1_\lambda)^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ A_\kappa A_\lambda \partial_\nu\lambda A_\mu + A_\kappa\partial_\lambda\lambda A_\mu A_\nu - A_\mu A_\nu A_\kappa \partial_\lambda\lambda - \partial_\nu\lambda A_\mu A_\kappa A_\lambda\big)\,.$$

Of course, c^1_λ will contribute to the solutions for the gauge and matter field, too, since λ^1 appears in the inhomogeneous term on the right hand side of the gauge equivalence equations at second order. Thus, we obtain the additional c^1_λ terms for the gauge field:

$$\begin{aligned} A^2_{\xi,c^1_\lambda} = &\frac{c^1_\lambda}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}\{A_\mu\partial_\kappa A_\nu, \partial_\lambda A_\xi\} - \mathrm{i}\{\partial_\mu A_\kappa A_\lambda, \partial_\nu A_\xi\}\big) \\ &+ [A_\mu A_\nu, \{A_\kappa, \partial_\lambda A_\xi\}] - A_\mu A_\nu A_\kappa \partial_\xi A_\lambda + \partial_\xi A_\nu A_\mu A_\kappa A_\lambda\big) \\ &+ \mathrm{i}[A_\nu A_\xi A_\mu, A_\kappa A_\lambda] + \mathrm{i}[A_\xi, A_\mu A_\nu A_\kappa A_\lambda]\big) \\ +&\frac{(c^1_\lambda)^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathrm{i}[A_\kappa\partial_\xi A_\lambda, A_\mu A_\nu] + \mathrm{i}[A_\kappa A_\lambda, \partial_\xi A_\nu A_\mu] \\ &+ \{A_\mu A_\nu A_\kappa A_\lambda, A_\xi\} - A_\kappa A_\lambda A_\xi A_\mu A_\nu\big) \end{aligned} \tag{4.9}$$

and for the matter field:

$$\begin{aligned} \psi^2_{c^1_\lambda} = &\frac{c^1_\lambda}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu A_\kappa D_\lambda\psi - \mathrm{i}[A_\mu, \partial_\kappa A_\nu]\partial_\lambda\psi\big) \\ &+ \frac{(c^1_\lambda)^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu A_\kappa A_\lambda\psi\big)\,. \end{aligned} \tag{4.10}$$

The $\mathcal{O}(\theta^2)$ Seiberg-Witten maps depend linearly and quadratically on the parameter c^1_λ. The quadratic term arises due to the product of the two $\mathcal{O}(\theta)$ Seiberg-Witten maps in the inhomogeneous part of (4.4a) and (4.4b). Since the gauge parameter, the gauge field and the matter field all depend on c^1_λ, quadratic terms will occur in the $\mathcal{O}(\theta^2)$ solutions.
The first order ambiguity of the gauge field, parametrized by c^1_A, will contribute linearly to the gauge field in second order:

$$\begin{aligned} A^2_{\xi,c^1_A} = &-\frac{\mathrm{i}c^1_A}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\{A_\lambda, \partial_\xi\partial_\mu\partial_\kappa A_\nu\} \\ &+\frac{c^1_A}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu\partial_\xi\partial_\kappa A_\lambda + 2A_\mu A_\kappa\partial_\nu\partial_\xi A_\lambda + 2A_\mu\partial_\nu A_\kappa\partial_\xi A_\lambda \\ &- 2A_\mu\partial_\kappa A_\lambda\partial_\nu A_\xi - 2A_\mu\partial_\nu\partial_\kappa A_\lambda A_\xi + 2A_\nu\partial_\xi A_\lambda\partial_\mu A_\kappa \\ &+ 2A_\nu\partial_\xi\partial_\kappa A_\lambda A_\mu + 2A_\kappa\partial_\lambda A_\xi\partial_\mu A_\nu + 2A_\kappa\partial_\nu\partial_\xi A_\lambda A_\mu \\ &- 2A_\kappa\partial_\xi\partial_\lambda A_\nu A_\mu - 2A_\lambda A_\xi\partial_\mu\partial_\kappa A_\nu - 2A_\xi\partial_\mu\partial_\kappa A_\nu A_\lambda \\ &+ 2\partial_\nu A_\kappa\partial_\xi A_\lambda A_\mu - 2\partial_\kappa A_\lambda\partial_\nu A_\xi A_\mu + 2\partial_\lambda A_\xi\partial_\mu A_\nu A_\kappa \\ &+ 2\partial_\xi A_\lambda\partial_\mu A_\kappa A_\nu - 2\partial_\nu\partial_\kappa A_\lambda A_\xi A_\mu + \partial_\xi\partial_\kappa A_\lambda A_\mu A_\nu \\ &- 2\partial_\xi\partial_\lambda A_\nu A_\mu A_\kappa\big) \\ &+\frac{\mathrm{i}c^1_A}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu\partial_\kappa A_\lambda A_\xi - A_\mu A_\nu A_\kappa\partial_\xi A_\lambda + 2A_\mu A_\kappa A_\lambda\partial_\nu A_\xi \\ &+ 2A_\mu A_\kappa\partial_\nu A_\lambda A_\xi + 2A_\mu\partial_\nu A_\kappa A_\lambda A_\xi - 2A_\nu A_\kappa\partial_\xi A_\lambda A_\mu \\ &+ 2A_\nu A_\xi\partial_\mu A_\kappa A_\lambda + 2A_\nu\partial_\kappa A_\lambda A_\xi A_\mu - A_\kappa A_\lambda A_\xi\partial_\mu A_\nu \end{aligned} \tag{4.11}$$

$$\begin{aligned}
&+ 2A_\kappa A_\lambda \partial_\nu A_\xi A_\mu + A_\kappa A_\lambda \partial_\xi A_\nu A_\mu + 2A_\kappa \partial_\nu A_\lambda A_\xi A_\mu \\
&- 2A_\kappa \partial_\lambda A_\xi A_\mu A_\nu - A_\kappa \partial_\xi A_\lambda A_\mu A_\nu + 2A_\lambda A_\xi A_\mu \partial_\kappa A_\nu \\
&- 2A_\lambda A_\xi \partial_\mu A_\nu A_\kappa + 2A_\lambda \partial_\xi A_\nu A_\mu A_\kappa + 2A_\xi A_\mu \partial_\kappa A_\nu A_\lambda \\
&- A_\xi \partial_\mu A_\nu A_\kappa A_\lambda + 2A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + 2\partial_\nu A_\kappa A_\lambda A_\xi A_\mu \\
&+ \partial_\kappa A_\lambda A_\xi A_\mu A_\nu - 2\partial_\lambda A_\xi A_\mu A_\nu A_\kappa + \partial_\xi A_\nu A_\mu A_\kappa A_\lambda \big)
\end{aligned}$$

$$+\frac{c_A^1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big([A_\mu A_\nu A_\kappa A_\lambda, A_\xi] + 2[A_\nu A_\kappa, A_\lambda A_\xi A_\mu]\big)\,.$$

We also have terms where the first order ambiguities from the gauge parameter mix with those from the gauge and matter field, respectively:

$$\begin{aligned}
A^2_{\xi,c^1_\lambda,c^1_A} = c^1_\lambda\, c^1_A\, \theta^{\mu\nu}\theta^{\kappa\lambda}\big(&\mathrm{i}[A_\mu A_\nu, \partial_\xi \partial_\kappa A_\lambda] + [A_\mu A_\nu, A_\kappa \partial_\xi A_\lambda] \\
&+ \big[A_\kappa A_\lambda, [A_\xi, \partial_\mu A_\nu]\big] - [A_\kappa A_\lambda, \partial_\xi A_\nu A_\mu] \\
&+ \mathrm{i}[A_\mu A_\nu, A_\kappa A_\lambda A_\xi] + \mathrm{i}[A_\xi A_\mu A_\nu, A_\kappa A_\lambda]\big)\,. \qquad (4.12)
\end{aligned}$$

The mixed $\mathcal{O}(\theta)$-ambiguities are caused by the presence of the same commutator-term in (4.4a) which led to the c^1_λ-quadratic term in (4.8).
The first order ambiguity c^1_ψ in the solution for the matter field leads to additional terms to the second order solution:

$$\begin{aligned}
\psi^2_{c^1_\psi} = \frac{c^1_\psi}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&A_\mu A_\nu A_\kappa A_\lambda \psi + \mathrm{i}A_\mu A_\nu \partial_\kappa A_\lambda \psi + 2\mathrm{i}A_\mu A_\kappa A_\lambda \partial_\nu \psi \\
&+ 2\mathrm{i}A_\mu A_\kappa \partial_\nu A_\lambda \psi + 2\mathrm{i}A_\mu \partial_\nu A_\kappa A_\lambda \psi \\
&- 2A_\mu \partial_\kappa A_\lambda \partial_\nu \psi + 2A_\lambda \partial_\mu \partial_\kappa A_\nu \psi\big)\,. \qquad (4.13)
\end{aligned}$$

Again, terms containing both first order ambiguities (c^1_λ, c^1_ψ) appear, due to the first term in the inhomogeneous part of (4.4b):

$$\psi^2_{c^1_\lambda,c^1_\psi} = c^1_\lambda\, c^1_\psi\, \theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathrm{i}A_\mu A_\nu A_\kappa A_\lambda \psi - A_\mu A_\nu \partial_\kappa A_\lambda \psi\big)\,. \qquad (4.14)$$

4.1.3 Homogeneous Solutions at $\mathcal{O}(\theta^2)$

Let us now consider the purely second order ambiguities, i.e. those parametrizing the solutions of the homogeneous equations at second order in θ. The homogeneous consistency and gauge equivalence equations can be compactly written as

$$H_\lambda[\lambda^2(\lambda, A, \theta)] = 0\,, \qquad (4.15a)$$

$$H_A[A^2(A, \theta)] = 0\,, \qquad (4.15b)$$

$$H_\psi[\psi^2(\psi, A, \theta)] = 0\,. \qquad (4.15c)$$

As before, we first consider the consistency equation. The homogeneous solution for the gauge parameter is lengthy and since its explicit form will not affect our final results, we just give it in the appendix A. We find a 15 parameter family for each value of c^1_λ, which again appears in the Seiberg-Witten maps for the gauge and matter field. Thus, adding this homogeneous solution to the special

one and plugging it into the gauge equivalence equation for the gauge and matter field, we obtain additional $c^2_{\lambda,i}$ contributions to A^2_ξ and ψ^2. Again, the corresponding expressions can be found in appendix A.
Now we focus our attention to the purely second order ambiguities for the gauge field. We find the following six-parameter family of Hermitian solutions in $\mathcal{O}(\theta^2)$:

$$A^2_{\xi,c^2_{A,1}} = c^2_{A,1}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\kappa F_{\mu\nu})F_{\lambda\xi}\,, \tag{4.16a}$$

$$A^2_{\xi,c^2_{A,2}} = c^2_{A,2}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\lambda\xi}(D_\kappa F_{\mu\nu})\,, \tag{4.16b}$$

$$A^2_{\xi,c^2_{A,3}} = c^2_{A,3}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\mu\kappa}(D_\xi F_{\nu\lambda})\,, \tag{4.16c}$$

$$A^2_{\xi,c^2_{A,4}} = c^2_{A,4}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\xi F_{\nu\lambda})F_{\mu\kappa}\,, \tag{4.16d}$$

$$A^2_{\xi,c^2_{A,5}} = c^2_{A,5}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\kappa\lambda}(D_\xi F_{\mu\nu})\,, \tag{4.16e}$$

$$A^2_{\xi,c^2_{A,6}} = c^2_{A,6}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\xi F_{\mu\nu})F_{\kappa\lambda}\,. \tag{4.16f}$$

These are all possible solutions to the homogeneous equation (4.15b), two more as in [70], where the ambiguities $A^2_{\xi,c^2_{A,1}}$ and $A^2_{\xi,c^2_{A,2}}$ were omitted. Exactly these ambiguities (together with the first order ambiguity $A^1_{\xi,c^1_{A,1}}$) will lead to the unexpected result in the next section.
The homogeneous solutions for the matter field lead to three ambiguities at second order:

$$\psi^2_{c^2_{\psi,1}} = \mathrm{i}c^2_{\psi,1}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\mu F_{\nu\kappa})D_\lambda\psi\,, \tag{4.17a}$$

$$\psi^2_{c^2_{\psi,2}} = -\frac{c^2_{\psi,2}}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\mu\nu}F_{\kappa\lambda}\psi\,, \tag{4.17b}$$

$$\psi^2_{c^2_{\psi,3}} = \frac{c^2_{\psi,3}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\mu\kappa}F_{\nu\lambda}\psi\,. \tag{4.17c}$$

These correspond to [70]. To our original result for (4.17a), given in appendix A, we added the other two homogeneous solutions in order to obtain the compact form (4.17a).

4.1.4 Remarks

We have shown how an enveloping algebra valued gauge theory up to $\mathcal{O}(\theta^2)$ can be constructed. The computations are lengthy and little transparent. An alternative way of obtaining the Seiberg-Witten maps even order by order up to an arbitrary high order n in θ was proposed in [72]. The starting point, provided again in the work of Seiberg and Witten [5], are the differential equations describing the change in the gauge parameter along a trajectory $\theta + \delta\theta$ in θ space:

$$\delta\hat{\lambda}(\theta) = \delta\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\hat{\lambda}(\theta) = -\frac{1}{4}\delta\theta^{\mu\nu}\{\hat{A}_\mu \stackrel{\star}{,} \partial_\nu\hat{\lambda}\}\,. \tag{4.18}$$

Parametrizing the path $\delta\theta$ in θ space by $t\theta$, the above equation can be rewritten:

$$\frac{\partial\hat{\lambda}(t)}{\partial t} = -\frac{1}{4}\theta^{\mu\nu}\{\hat{A}_\mu(t) \stackrel{\star^t}{,} \partial_\nu\hat{\lambda}(t)\}\,, \tag{4.19}$$

with $\star^t$ denoting the Moyal-Weyl $\star$-product (2.13) with $\theta \to t\theta$. Taking the Taylor expansion for $\hat{\lambda}(t)$

$$\hat{\lambda}(t) = \lambda + t\lambda^1 + t^2\lambda^2 + \ldots = \lambda + t\left.\frac{\partial\hat{\lambda}(t)}{\partial t}\right|_{t=0} + \frac{1}{2!}t^2\left.\frac{\partial^2\hat{\lambda}(t)}{\partial t^2}\right|_{t=0} + \ldots\,, \tag{4.20}$$

the gauge parameter to $\mathcal{O}(\theta)$ is simply given by

$$\lambda^1 = \left.\frac{\partial\hat{\lambda}}{\partial t}\right|_{t=0} = -\frac{1}{4}\theta^{\mu\nu}\{\hat{A}_\mu \stackrel{\star^t}{,} \partial_\nu\hat{\lambda}\}|_{t=0} = -\frac{1}{4}\theta^{\mu\nu}\{A_\mu, \partial_\nu\lambda\}\,. \tag{4.21a}$$

The next order can be obtained by differentiating the previous one w.r.t. to t:

$$\begin{aligned}\lambda^2 &= \frac{1}{2}\left.\frac{\partial^2\hat{\lambda}}{\partial t^2}\right|_{t=0} = \frac{1}{2}\left(\frac{\partial}{\partial t}\frac{\partial\hat{\lambda}}{\partial t}\right)_{t=0} = \\ &= -\frac{1}{8}\theta^{\mu\nu}\Big(\{A^1_\mu, \lambda\} + \{A_\mu \stackrel{\star^1}{,} \lambda\} + \{A_\mu, \lambda^1\}\Big)\,,\end{aligned} \tag{4.21b}$$

and so on:

$$\lambda^3 = \frac{1}{3!}\left.\frac{\partial^3\hat{\lambda}}{\partial t^3}\right|_{t=0} = \ldots\,, \tag{4.21c}$$

with $A^1_\xi = (\partial\hat{A}_\xi/\partial t)_{t=0}$ and the $\mathcal{O}(\theta)$ term in the Taylor expansion of the $\star$-product:

$$f(x)\star^1 g(x) = f(x)\left(\frac{\partial}{\partial t}\star^t\right)g(x)|_{t=0} = \frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x)\,. \tag{4.22}$$

Starting with the second order, these equations are not closed anymore, since they require derivatives of A_ξ w.r.t. t. Therefore, similar differential equations for the gauge field must be solved first. The higher order Seiberg-Witten maps for the matter fields can be derived analogously, also needing the higher orders in A_ξ.

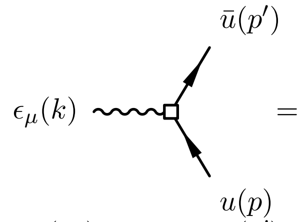
Thus, the Seiberg-Witten maps can be derived up to any power θ^n. Nevertheless, this approach has a drawback: only Seiberg-Witten maps corresponding to special solutions of the consistence and gauge equivalence equations are provided and is thus useless for our purpose of analyzing the relevance of the ambiguities of the Seiberg-Witten maps for observables. Since we will show that some of these ambiguities indeed have a physical meaning and appear in observables, this approach leads to incomplete results.

4.2 Feynman Rules

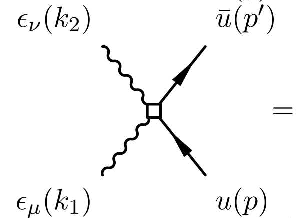
We now posses the complete expressions for the Seiberg-Witten maps for the gauge parameter, gauge and matter field and are ready to derive Feynman rules up to $\mathcal{O}(\theta^2)$. We will not resolve to derive the complete Lagrangian and to write it in a gauge covariant way. Instead, we will restrict ourselves to the matter and the kinetic part. We are interested only in neutral currents for the purpose of this work. We have chosen the process $f\bar{f} \to \gamma\gamma$ for illustration and hence, we will consider only the NCQED-Lagrangian. We go no further than tree level in perturbation theory. Thus, we are interested only in terms containing no more than two gauge fields in the matter part and only the cubic photon interaction in the kinetic part. This means that in the Seiberg-Witten maps of the gauge and matter field we will collect only the terms containing no more than two gauge fields and insert them into the Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\text{fermionic}} &= \bar{\hat{\psi}} \star \mathrm{i}\hat{\not{D}}\hat{\psi} = \left(\bar{\hat{\psi}} \star \mathrm{i}\not{\partial}\hat{\psi} + \bar{\hat{\psi}} \star \hat{\not{A}}\hat{\psi}\right) && \text{(4.23a)}\\
&= \mathrm{i}\big(\bar{\psi} + \bar{\psi}^1 + \bar{\psi}^2\big) \star \big(\not{\partial}\psi + \not{\partial}\psi^1 + \not{\partial}\psi^2\big)\\
&\quad + \big(\bar{\psi} + \bar{\psi}^1 + \bar{\psi}^2\big)\big(\not{A} + \not{A}^1 + \not{A}^2\big) \star \big(\psi + \psi^1 + \psi^2\big)\,,\\
\mathcal{L}_{\text{gauge}} &= -\frac{1}{2g^2}\mathrm{Tr}\big(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}\big)\,. && \text{(4.23b)}
\end{aligned}$$

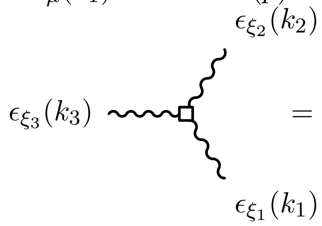
The Feynman rules are derived in the usual way. With all momenta incoming, we obtain the vertex factors up to $\mathcal{O}(\theta^2)$ as follows:



$$= \mathrm{i}g \cdot V_\mu(p', k, p)\,, \tag{4.24a}$$



$$= \mathrm{i}g^2 \cdot V^c_{\mu_2\mu_1}(p', k_2, k_1, p)\,, \tag{4.24b}$$



$$= \mathrm{i}g_{[\rho]} \cdot V^3_{\xi_1\xi_2\xi_3}(k_1, k_2, k_3)\,, \tag{4.24c}$$

with $g_{[\rho]}$ indicating the representation-dependence of the TGB coupling and

$$V^{(1)}_\mu(p', k, p) = \frac{\mathrm{i}}{2}\Big[k\theta^\mu\not{p}(1 - 4c^1_\psi) + 2\,k\theta^\mu\not{k}(c^1_A - c^1_\psi) - p\theta^\mu\not{k} - (k\theta p)\gamma_\mu\Big]\,, \tag{4.25a}$$

$$V^{(2)}_{\mu}(p',k,p) = \frac{1}{8}(k\theta p)\Big[k\theta^{\mu}\not{p}(1-16c^2_{\psi}) + 4k\theta^{\mu}\not{k}(c^1_A - 2c^2_{\psi}) - p\theta^{\mu}\not{k} - (k\theta p)\gamma_{\mu}\Big]\,, \tag{4.25b}$$

$$V^{c,(1)}_{\nu\mu}(p',k_2,k_1,p) = \frac{\mathrm{i}}{2}\Big[k_2\theta^{\mu}\gamma^{\nu} - k_1\theta^{\mu}\gamma^{\nu}(1-4c^1_{\psi}) - \theta^{\mu\nu}\not{k}_1 + (\mu\leftrightarrow\nu, k_1\leftrightarrow k_2)\Big]\,, \tag{4.25c}$$

$$\begin{aligned} V^{c,(2)}_{\nu\mu}(p',k_2,k_1,p) = \frac{1}{8}\Big[& k_1\theta k_2\, k_1\theta^{\mu}\gamma^{\nu}\,(8c^2_A - 4c^1_{\psi} + 8c^2_{\psi} - 1) \\ &+ k_1\theta p\, k_1\theta^{\mu}\gamma^{\nu}\,(16c^2_{\psi}-1) + 2\,k_2\theta p\, k_1\theta^{\mu}\gamma^{\nu}\,(4c^1_{\psi}-1) \\ &- k_1\theta k_2\, k_2\theta^{\mu}\gamma^{\nu} + 3\,k_1\theta p\, k_2\theta^{\mu}\gamma^{\nu} + 2\,k_2\theta p\, k_2\theta^{\mu}\gamma^{\nu} - 3\,k_1\theta k_2\, p\theta^{\mu}\gamma^{\nu} \\ &+ 4\,k_1\theta^{\mu}k_1\theta^{\nu}\not{k}_1(2c^2_A - c^1_A - c^1_{\psi}) + 2\,k_1\theta^{\mu}p\theta^{\nu}\not{k}_1(1-4c^1_{\psi}) + 2\,k_2\theta^{\mu}p\theta^{\nu}\not{k}_1 \\ &- 4\,\theta^{\mu\nu}k_1\theta p\not{k}_1 + (\mu\leftrightarrow\nu, k_1\leftrightarrow k_2)\Big] \\ &+ \text{equation of motion terms}\,, \end{aligned} \tag{4.25d}$$

$$\begin{aligned} &V^{3,(1)}_{\xi_1\xi_2\xi_3}(k_1,k_2,k_3) = \\ &\quad \theta_{\xi_1\xi_2}\,[(k_1k_3)k_{2,\xi_3} - (k_2k_3)k_{1,\xi_3}] + (k_1\theta k_2)\,[k_{3,\xi_1}g_{\xi_2\xi_3} - g_{\xi_1\xi_3}k_{3,\xi_2}] \\ &\quad + \Big[(k_1\theta)_{\xi_1}\,[k_{2,\xi_3}k_{3,\xi_2} - (k_2k_3)g_{\xi_2\xi_3}] - (\xi_1\leftrightarrow\xi_2) - (\xi_1\leftrightarrow\xi_3)\Big] \\ &\quad + \text{cyclical permutations of } \big\{(\xi_1,k_1),(\xi_2,k_2),(\xi_3,k_3)\big\}\,, \end{aligned} \tag{4.25e}$$

$$\begin{aligned} V^{3,(2)}_{\xi_1\xi_2\xi_3}(k_1,k_2,k_3) = \mathrm{i}\Big[& k_1\theta k_2 k_1\theta^{\xi_1}\Big((c^2_A - c^1_A)(k_{1,\xi_3}k_{3,\xi_2} - g_{\xi_2\xi_3}(k_1k_3)) \\ &+ c^2_A(k_{2,\xi_3}k_{3,\xi_2} - g_{\xi_2\xi_3}(k_2k_3))\Big) \\ &+ k_1\theta^{\xi_1}k_1\theta^{\xi_2}(c^2_A - c^1_A)(k_{2,\xi_3}(k_1k_3) - k_{1,\xi_3}(k_2k_3)) + (\xi_2,k_2)\leftrightarrow(\xi_3,k_3)\Big] \\ &+ \text{cyclical permutations of } \big\{(\xi_1,k_1),(\xi_2,k_2),(\xi_3,k_3)\big\}\,. \end{aligned} \tag{4.25f}$$

From the three second order ambiguities for the matter field only one, $c^2_{\psi,1}$, survives in the Feynman rules and thus has been denoted by c^2_{ψ}. The gauge field ambiguities at second order which appear in the Feynman rules are $c^2_{A,1}$ and $c^2_{A,2}$. As can be seen from (4.16) in the abelian case they coincide and hence we have introduced the notation $c^2_A = c^2_{A,1} + c^2_{A,2}$.
The TGB vertex has no ambiguities at $\mathcal{O}(\theta)$ stemming from the homogeneous part of the gauge equivalence equations.[1] At second order in θ only the ambiguities contribute, whereas there are no ambiguity-free contributions.
Since we are interested in $f\bar{f} \to \gamma\gamma$ scattering at tree level, we will need only the on-shell expression for the $\bar{f}\gamma\gamma f$ contact term. Therefore we have suppressed terms that vanish due to the equations of motion for the fermions in the vertex (4.25d). The complete expressions can be found in appendix B.

[1]The nonuniqueness of the coupling constant is of course also an ambiguity innate to the NCSM, but has different origin, as discussed in section 3.1.2.

4.3 The Rôle of the Ambiguities

It was stated in [70] that the ambiguities of the Seiberg-Witten maps correspond to field redefinitions and thus must have no physical consequences for observables. In this section this statement will be invalidated by showing that some ambiguities still survive in the cross section at second order in the noncommutative parameter θ. We will also give an explicit calculation of an NC-QED process up to $\mathcal{O}(\theta^2)$ and show the dependence on some ambiguities of the Seiberg-Witten map.
It has been shown that physical predictions of Quantum Field Theory, in particular on-shell S-matrix elements and scattering cross sections, do not depend on the choice of interpolating fields [73, 74, 75]. Any two theories which are related by non-singular local field redefinitions

$$\phi \leftrightarrow \phi'(\phi) \text{ with } \left.\frac{\partial \phi'}{\partial \phi}\right|_{\phi=0} = \mathbf{1} \tag{4.26a}$$

(up to finite renormalizations) and the corresponding change in the Lagrangian

$$\mathcal{L}(\phi) \leftrightarrow \mathcal{L}'(\phi') = \mathcal{L}(\phi(\phi')) \tag{4.26b}$$

will predict identical scattering cross sections. This is often referred to as reparametrization invariance and can be proven both in axiomatic quantum field theory [73] and in perturbation theory [75]. Therefore, we must carefully distinguish the ambiguities corresponding to field redefinitions from those which do not. The latter can have physical consequences and affect observable quantities.
In principle, the Seiberg-Witten maps

$$\begin{pmatrix} \lambda \\ A_\mu \\ \psi \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\lambda}(\lambda, A, \theta) \\ \hat{A}_\mu(A, \theta) \\ \hat{\psi}(\psi, A, \theta) \end{pmatrix} \tag{4.27}$$

appear to correspond to non-singular field redefinitions as described by (4.26). However, this is true only for some special cases, namely for $U(1)$ with unit charge or $U(N)$ gauge theories. We have shown previously that in this case the Seiberg-Witten maps are Lie algebra valued. Hence they are non-singular and correspond to field redefinitions. Their effect will cancel in observables, as we will immediately show for the $e^+e^- \rightarrow \gamma\gamma$ cross section. We are left only with the noncommutative effects stemming from the Moyal-Weyl $\star$-product. Nevertheless, we have seen that in order to allow for $SU(N)$ and $U(1)$ with arbitrary charges, we have to leave the Lie algebra and enter the universal enveloping algebra. The latter is strictly larger than the former and therefore, the maps relating the Lie algebra with its enveloping algebra must be singular. Hence, in general they do not correspond to field redefinitions any more. We have also seen, that the Seiberg-Witten maps are not unique. They differ by homogeneous solutions to the consistency and gauge equivalence equations, which in general are also enveloping algebra valued, such that we can not expect that they will cancel in observables. Only if the ambiguities are Lie algebra valued, they correspond to field redefinitions and must cancel in cross sections. We have seen in section 3.2.1 that in the case of fermion scattering into $Z\gamma$ at first order in θ, all ambiguities cancel in the cross section. Indeed, to $\mathcal{O}(\theta)$ the ambiguities c^1_λ, c^1_A and c^1_ψ in (2.51, 2.52) correspond to field redefinitions, since they are all Lie algebra valued:

$$A^1_{\xi,c^1_\lambda} = \mathrm{i}c^1_\lambda\,\theta^{\mu\nu}[D_\xi A_\mu, A_\nu]\,, \tag{4.28a}$$
$$A^1_{\xi,c^1_A} = -2\mathrm{i}\,c^1_A\,\theta^{\mu\nu}D_\sigma F_{\mu\nu}\,, \tag{4.28b}$$
$$\psi^1_{c^1_\lambda} = -\,c^1_\lambda\,\theta^{\mu\nu}A_\mu A_\nu\psi = -\frac{c^1_\lambda}{2}\,\theta^{\mu\nu}[A_\mu, A_\nu]\psi\,, \tag{4.28c}$$
$$\psi^1_{c^1_\psi} = \frac{c^1_\psi}{2}\,\theta^{\mu\nu}F_{\mu\nu}\psi\,. \tag{4.28d}$$

It was just shown in section 4.1.2 that the first order Seiberg-Witten maps enter the equations for the second order Seiberg-Witten maps. Thus, these equations and their solution space depend on the value of the triple $(c^1_\lambda, c^1_\psi, c^1_A)$. It is not guaranteed that the resulting dependence of the second order Seiberg-Witten maps on $(c^1_\lambda, c^1_\psi, c^1_A)$ again correspond to field redefinitions and must cancel in observables. In fact, we will see, that the first order ambiguity c^1_A will give nonvanishing contributions to observables in second order of θ. The ambiguities c^1_λ and c^1_ψ in section 4.1.2 will not contribute to the process since they enter the second order as coefficients for terms with more than two gauge fields. Only A^2_{ξ,c^1_A} in (4.11) contains a term with two gauge fields. Since this one is an anticommutator of gauge fields, it is enveloping algebra valued and is thus not expected to vanish in observables.
Contrary to [70], we also have found additional ambiguities (4.16a,4.16b) at second order for the gauge field. Exactly these are the crucial ones, since these will not vanish in observables, as we will show considering the $\mathcal{O}(\theta^2)$ scattering cross section for $e^+e^- \to \gamma\gamma$ in NCQED as a prototype of the neutral sector of the NCSM.[2] For a straightforward discrimination of the ambiguities which vanish in observables and those which might give physical contributions, it is helpful to perform a change of basis $\{c^2_{A,i}\}^6_{i=1} \to \{\tilde{c}^2_{A,i}\}^6_{i=1}$ in the six dimensional parameter space of the ambiguities and to consider the linear combinations:

$$\tilde{c}^2_{A,1} = \frac{1}{2}\big(c^2_{A,1} + c^2_{A,2}\big)\,, \tag{4.29a}$$
$$\tilde{c}^2_{A,2} = \frac{1}{2}\big(c^2_{A,3} + c^2_{A,4}\big)\,, \tag{4.29b}$$
$$\tilde{c}^2_{A,3} = \frac{1}{2}\big(c^2_{A,5} + c^2_{A,6}\big)\,, \tag{4.29c}$$
$$\tilde{c}^2_{A,4} = \frac{1}{2}\big(c^2_{A,1} - c^2_{A,2}\big)\,, \tag{4.29d}$$
$$\tilde{c}^2_{A,5} = \frac{1}{2}\big(c^2_{A,3} - c^2_{A,4}\big)\,, \tag{4.29e}$$

[2]The additional Z-boson couplings and its mass in the NCSM will not add to our conclusions about the ambiguities.

$$\tilde{c}^2_{A,6} = \frac{1}{2}\left(c^2_{A,5} - c^2_{A,6}\right). \tag{4.29f}$$

This way, the homogeneous solutions (4.16) can be rewritten:

$$A^2_{\xi,\tilde{c}^2_{A,1}} = \tilde{c}^2_{A,1}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\{D_\kappa F_{\mu\nu}, F_{\lambda\xi}\}\,, \tag{4.30a}$$

$$A^2_{\xi,\tilde{c}^2_{A,2}} = \tilde{c}^2_{A,2}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\{F_{\mu\kappa}, D_\xi F_{\nu\lambda}\}\,, \tag{4.30b}$$

$$A^2_{\xi,\tilde{c}^2_{A,3}} = \tilde{c}^2_{A,3}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\{F_{\kappa\lambda}, D_\xi F_{\mu\nu}\}\,, \tag{4.30c}$$

$$A^2_{\xi,\tilde{c}^2_{A,4}} = \tilde{c}^2_{A,4}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}[D_\kappa F_{\mu\nu}, F_{\lambda\xi}]\,, \tag{4.30d}$$

$$A^2_{\xi,\tilde{c}^2_{A,5}} = \tilde{c}^2_{A,5}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}[F_{\mu\kappa}, D_\xi F_{\nu\lambda}]\,, \tag{4.30e}$$

$$A^2_{\xi,\tilde{c}^2_{A,6}} = \tilde{c}^2_{A,6}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}[F_{\kappa\lambda}, D_\xi F_{\mu\nu}]\,. \tag{4.30f}$$

The ambiguities in the new basis $\{\tilde{c}^2_{A,i}\}^6_{i=1}$ are more transparent w.r.t. their physical contribution to observables. We can immediately exclude the last three from appearing in observables, since they are clearly in the Lie algebra. Thus, we have reduced the ambiguities with possible nonvanishing contributions to cross sections from six to three. Note that $\tilde{c}^2_{A,1} = c^2_A$. Furthermore, the ambiguities (4.30b) and (4.30c) vanish in the Feynman rules after contraction with the corresponding vertex momenta.

We now show exemplary by means of the second order ambiguity of the gauge field in a simple calculation how it is possible for the ambiguities not to vanish. In the case of NCQED, the two ambiguities $c^2_{A,1}$ and $c^2_{A,2}$ are identical and their sum was denoted by c^2_A. The only relevant term for our tree level process in (4.30a) is

$$A^2_{\xi,c^2_A} = \mathrm{i}c^2_A\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa A_\nu(\partial_\lambda A_\xi - \partial_\xi A_\lambda)\,. \tag{4.31}$$

In the Lagrangian, it can only contribute to the contact term and the three gauge boson vertex. The corresponding Feynman rule for the contact term containing c^2_A is:

$$g^2(\mathrm{i})^5 c^2_A\theta^{\mu\nu}\theta^{\kappa\lambda}\left[k_1\theta^{\mu_1}(k_1\theta k_2\gamma^{\mu_2} - k_1\theta^{\mu_2}\not{k}_2) + k_2\theta^{\mu_2}(k_2\theta k_1\gamma^{\mu_1} - k_2\theta^{\mu_1}\not{k}_1)\right]\,, \tag{4.32}$$

giving immediately the corresponding contribution to the contact amplitude

$$A^{(2)}_c = \ldots + \mathrm{i}g^2 c^2_A\left[k_1\theta\varepsilon_1(k_1\theta k_2\not{\varepsilon}_2 - k_1\theta\varepsilon_2\not{k}_2) + k_2\theta\varepsilon_2(k_2\theta k_1\not{\varepsilon}_1 - k_2\theta\varepsilon_1\not{k}_1)\right]\,. \tag{4.33}$$

This contribution will not vanish on its own, nor can it be cancelled by any term coming from the t- or u-channel diagram, since these cannot contain terms proportional to c^2_A. The only possible cancellation can be provided by the s-channel diagram. In fact, with the $\gamma\gamma\gamma$ interaction (4.25f), the s-channel diagram will yield exactly the terms needed to cancel (4.33). But, there is a caveat: while the normalization of the contact term (4.24b), including the terms involving c^2_A, is fixed, the normalization of the three boson vertex (4.24c) depends on the choice

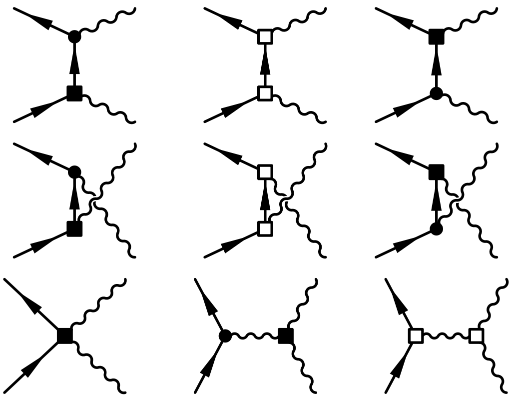


Figure 4.1: Feynman diagrams contributing to $e^+e^- \to \gamma\gamma$ in $\mathcal{O}(\theta^2)$. The black box stands for the full vertex (SM + $\mathcal{O}(\theta)$ + $\mathcal{O}(\theta^2)$), while the white box denotes the vertex up to first order (SM + $\mathcal{O}(\theta)$).

of the representation of the enveloping algebra. Therefore, the contributions from these vertices can not cancel in the general case, but only for the special choice $g_{[\rho]} = g$, and would need to vanish separately if the ambiguities were to drop out of observables.
For the first order ambiguity c_A^1, the same arguments come into play and it will not disappear from cross sections. Yet, the calculations are a little more intricate and not as straightforward as for c_A^2, therefore we will not present them explicitly.
We will now study the process $e^+e^- \to \gamma\gamma$ in more detail. The contributing Feynman diagrams up to second order in θ are given in figure 4.1. In the on-shell NCQED scattering amplitude for $e^+e^- \to \gamma\gamma$ up to second order in θ

$$A(e^+e^- \to \gamma\gamma) = g^2 A^{\mathrm{SM}} + g^2 A^{(1)} + g g_{[\rho]} A_s^{(1)} + g^2 A^{(2)} + g g_{[\rho]} A_s^{(2)}\,, \qquad (4.34)$$

we have split off the separately gauge invariant s-channel contributions, because their normalization depends on the representation of the enveloping algebra. For completeness, we restate the SM

$$A^{\mathrm{SM}} = -\frac{\mathrm{i}}{q_u^2}\bar{v}(p_2)\not{\epsilon}_1\not{q}_u\not{\epsilon}_2 u(p_1) - \frac{\mathrm{i}}{q_t^2}\bar{v}(p_2)\not{\epsilon}_2\not{q}_t\not{\epsilon}_1 u(p_1) \qquad (4.35)$$

(using $q_t = p_1 - k_1$ and $q_u = p_1 - k_2$) and first order contributions

$$\begin{aligned} A^{(1)} = &-\frac{\mathrm{i}}{q_u^2}\left(\frac{p_1\theta p_2 + k_1\theta k_2}{2\mathrm{i}}\right)\bar{v}(p_2)\not{\epsilon}_1\not{q}_u\not{\epsilon}_2 u(p_1) \\ &-\frac{\mathrm{i}}{q_t^2}\left(\frac{p_1\theta p_2 - k_1\theta k_2}{2\mathrm{i}}\right)\bar{v}(p_2)\not{\epsilon}_2\not{q}_t\not{\epsilon}_1 u(p_1) + A_c^{(1)}\,. \qquad (4.36) \end{aligned}$$

The $1/t$- and $1/u$-pole terms in (4.36) and (4.43) can be derived from the Moyal phase alone. The contributions coming from the Seiberg-Witten maps

are cancelled by corresponding terms in the contact diagram[3] after applying the Dirac equation to remove q_t^2 and q_u^2 in the denominators.
At each vertex we have

$$\mathrm{e}^{-\mathrm{i}p_1\theta p_2} = \mathrm{e}^{-\mathrm{i}p_2\theta p_3} = \mathrm{e}^{-\mathrm{i}p_3\theta p_1} \tag{4.37}$$

with all momenta incoming. Therefore in the t-channel we obtain:

$$\mathrm{e}^{-\mathrm{i}(-k_2)\theta(p_1-k_1)}\mathrm{e}^{-\mathrm{i}p_1\theta(p_2-k_2)} = \mathrm{e}^{-\mathrm{i}(p_1\theta p_2 - k_1\theta k_2)} \tag{4.38}$$

and in the u-channel, from $k_1 \leftrightarrow k_2$, we get:

$$\mathrm{e}^{-\mathrm{i}(p_1\theta p_2 + k_1\theta k_2)}\,. \tag{4.39}$$

Thus, in (4.36) the first terms in the Taylor expansions from above appear. The remaining terms yield:

$$A_c^{(1)} = \bar{v}(p_2)\left[\frac{1}{2}\varepsilon_1\theta\varepsilon_2(\not{k}_1 - \not{k}_2) - k_1\theta\varepsilon_2\not{\varepsilon}_1 - k_2\theta\varepsilon_1\not{\varepsilon}_2\right]u(p_1)\,, \tag{4.40}$$

where in the latter all ambiguities in the Seiberg-Witten maps have cancelled after application of the equations of motion. The s-channel contribution turns out to be proportional to the part of the amplitude without t- or u-channel poles and a $1/s$-pole contribution steming from the Moyal-Weyl $\star$-product alone:

$$A_s^{(1)} = -A_c^{(1)} + A_{s,\star}^{(1)}\,, \tag{4.41}$$

with

$$A_{s,\star}^{(1)} = \frac{1}{s}(k_1\theta k_2)\,\bar{v}(p_2)\left[\frac{1}{2}(\varepsilon_1\varepsilon_2)(\not{k}_2 - \not{k}_1) + (k_1\varepsilon_2)\not{\varepsilon}_1 - (k_2\varepsilon_1)\not{\varepsilon}_2\right]u(p_1)\,. \tag{4.42}$$

The cancellation of the $1/s$-poles in the s-channel amplitude occurs in the contributions from the Seiberg-Witten maps after applying $s+t+u=0$. Thus, if $g_{[\rho]} = g$ all contributions stemming from the Seiberg-Witten maps will cancel.
The contribution from $\mathcal{O}(\theta^2)$ reads:

$$\begin{aligned} A^{(2)} = &-\frac{\mathrm{i}}{q_u^2}\frac{1}{2}\left(\frac{p_1\theta p_2 + k_1\theta k_2}{2\mathrm{i}}\right)^2 \bar{v}(p_2)\not{\varepsilon}_1\not{q}_u\not{\varepsilon}_2 u(p_1) \\ &- \frac{\mathrm{i}}{q_t^2}\frac{1}{2}\left(\frac{p_1\theta p_2 - k_1\theta k_2}{2\mathrm{i}}\right)^2 \bar{v}(p_2)\not{\varepsilon}_2\not{q}_t\not{\varepsilon}_1 u(p_1) + A_c^{(2)}\,. \end{aligned} \tag{4.43}$$

Again, the $1/t$- and $1/u$-poles are accompanied by the second terms in the Taylor series expansion of the vertex factors induced by the Moyal-Weyl $\star$-product. The pole-free terms are summarized in

[3]We must make a remark regarding our notations: $A_c^{(1)}$ does not designate the amplitude for the contact diagram in figure 4.1, instead it is a notation for all terms in $A^{(1)}$ which do not contain $1/t$- and $1/u$-pole terms. Thus, $A_c^{(1)}$ is not the same as A_c^{NC} from section 3.2.1.

$$A_c^{(2)} = -\frac{\mathrm{i}}{2} p_1\theta p_2 A_c^{(1)} + \mathrm{i}(c_A^2 - c_A^1)\bar{v}(p_2)\Big[k_1\theta\varepsilon_1 k_1\theta\varepsilon_2 \not{k}_1 + k_2\theta\varepsilon_1 k_2\theta\varepsilon_2 \not{k}_2 + k_1\theta k_2\Big(k_1\theta\varepsilon_1 \not{\varepsilon}_2 - k_2\theta\varepsilon_2\not{\varepsilon}_1\Big)\Big]u(p_1)\,. \quad (4.44)$$

As in first order, we find that the s-channel term exactly cancels the contact term if an appropriate representation of the enveloping algebra is used:

$$A_s^{(2)}(c_A^1, c_A^2) = -A_c^{(2)}(c_A^1, c_A^2)\,. \quad (4.45)$$

Here, we explicitly write down the dependence on the ambiguities of the Seiberg-Witten maps. As qualitatively sketched at the beginning of this section, we observe that the only parts of the amplitude in which the ambiguities appear are $A_c^{(2)}$ and $A_s^{(2)}$. They enter the total amplitude (4.34) with different coefficients, g and $g_{[\rho]}$, and hence they can cancel only for the special case $g_{[\rho]} = g$. Thus, in the general case we are left with the ambiguities, enlarging thus the parameter space of the model by two dimensions.

This result is surprising, because the consensus was that theories described by different Seiberg-Witten maps should be equivalent. We have seen at the beginning of this section, that this is not true for theories formulated in the enveloping algebra. Observables are not identical for different ambiguities because the corresponding field transformations relating the different Lagrangians do not correspond to field redefinitions. Compare the two classes of transformations:

$$A_\mu = A_\mu^a T^a \to A'_\mu = A_\mu^a T^a + \alpha_\mu^{\nu\rho} A_\nu^a T^a A_\rho^b T^b \quad (4.46a)$$

and

$$A_\mu = A_\mu^a T^a \to A'_\mu = \big(A_\mu^a + \alpha_\mu^{\nu\rho} A_\nu^a A_\rho^a\big) T^a\,. \quad (4.46b)$$

The latter remains in the Lie algebra and scattering matrix elements are invariant under such reparametrizations. The first class, however, since T^aT^b leaves the Lie algebra generated by the T^a requires us to consider the enveloping algebra. Thus, additional degrees of freedom are introduced the transformation must be singular. Consequently the transformation

$$A_\mu \to (A'_{1,\mu}, A'_{2,\mu})\,, \quad (4.47)$$

with $A'_\mu = A'^a_{1,\mu} T^a + A'_{2,\mu}$ is singular and reparametrization invariance of matrix elements need not hold, which is exactly what we observe. Only in the fundamental representation of the $U(1)$ the fields remain in the Lie Algebra and the cancellation of the ambiguities will take place. In a $U(1)$ gauge theory we can choose an arbitrary Hermitian matrix $\rho(T)$ as a generator. The choice $\rho(T) = \sigma_3$ leads to $g_{[\rho]} = 0$, while choosing $\rho(T) = 1$ yields $g_{[\rho]} = g$. Only in the latter case, the anticommutator remains in the Lie algebra representation. This is in agreement with our statement at the beginning: for $U(1)$ with unit charge, the Seiberg-Witten maps correspond to field redefinitions and thus do not contribute to the cross section.

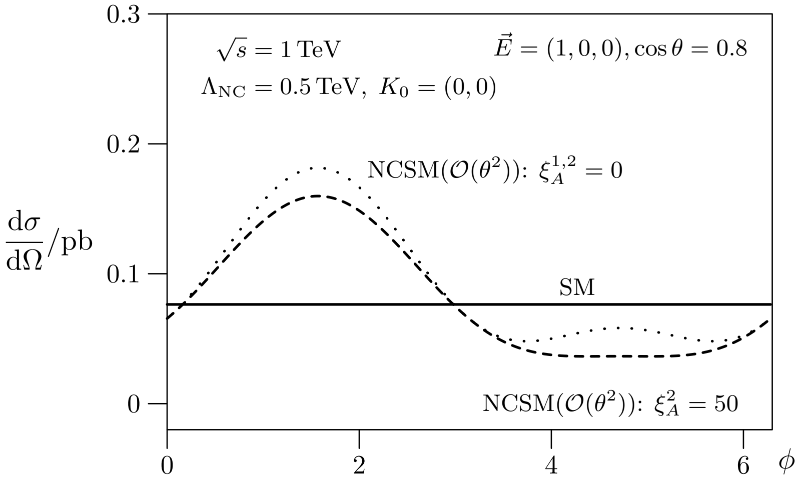


Figure 4.2: The azimuthal distribution of the differential cross section for $e^+e^- \to Z\gamma$ in the framework of the minimal NCSM at $\mathcal{O}(\theta^2)$ showing the dependence on the ambiguities of the Seiberg-Witten maps.

4.3.1 Numerical Dependence on Ambiguities in $e^+e^- \to Z\gamma$

After we have presented the dependence on the ambiguities in analitical calculations for the process $e^+e^- \to \gamma\gamma$ as an example, we finally show the numerical dependence on these ambiguities for the scattering process $e^+e^- \to Z\gamma$ at $\mathcal{O}(\theta^2)$ in figure 4.2. For illustration, we have chosen rather high values for c_A^1 and c_A^2. We observe that if both ambiguities have the same value, $c_A^1 = c_A^2$, they will cancel each other. We do not find a particular reason for this behaviour and hence it seems to be an accidental feature for this process.

It is particularly interesting to remark that for space-like noncommutativity, i.e. $E^i = 0$, all ambiguities cancel in the cross section. Space-like noncommutativity is exactly the type of noncommutativity favored by many authors since it preserves unitary as well as gauge invariance. In addition, in string theories only space-like noncommutativity is predicted.

4.4 Phenomenological Outlook

Now the mathematical apparatus describing the neutral current sector of the NCSM[4] up to $\mathcal{O}(\theta^2)$ stands and phenomenological studies within this framework can be performed. However, this goes beyond the purpose of this work. Nevertheless, we will close with one brief example of $\mathcal{O}(\theta^2)$ NCSM phenomenology.

One of the motivations for going to higher orders in θ given at the beginning of this section was the unphysical behavior (i.e. negative cross sections) at $\mathcal{O}(\theta)$ for high scattering energies compared to the noncommutative scale Λ_{NC}. We

[4] In principle, using the Seiberg-Witten maps found in section 4.1 and proceeding like in the case of the neutral current sector, the complete NCSM up to $\mathcal{O}(\theta^2)$ can be build.

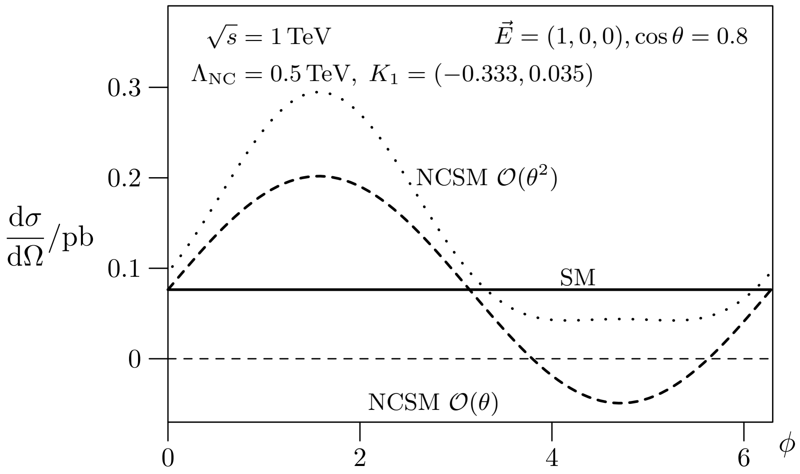


Figure 4.3: The unphysical behaviour of the differential cross section for $f\bar{f} \to Z\gamma$ at $\mathcal{O}(\theta)$ is cured by the $\mathcal{O}(\theta^2)$ contributions. All ambiguities are set to zero.

promised, that this was to be repaired by the second order in θ contributions, at least for a wider range of the scattering energy. And indeed, this happens. Figure 4.3 is similar to figure 3.5, but now the full differential cross section up to second order in θ is added. The $\mathcal{O}(\theta)$ contribution is such that it shifts the cross section from negative to positive values and thus our headaches regarding unphysical contributions for some regions in phase space when performing Monte Carlo simulation are soothed for a while. Of course, the cure is temporary since the theory is still an effective theory. We just shifted its region of validity to higher energies. Qualitatively, the form of the $\mathcal{O}(\theta^2)$ differential cross section in figure 4.3 justifies the brute force approach in section 3.2.3, where unphysical contributions were regularized by setting them to zero.

We also have stated that some processes which at $\mathcal{O}(\theta)$, depending on whether we choose the minimal or the nonminimal NCSM, show no or few noncommutative effects in the unpolarized cross section (e.g. $e^+e^- \to \gamma\gamma$), possess a richer noncommutative phenomenology at second order in θ. Thus, processes which are important at the LHC for other reasons, and will therefore be studied in detail, might be interesting from the point of view of noncommutativity.

As an example, we have calculated the process $pp \to \gamma\gamma$ up to $\mathcal{O}(\theta^2)$. This certainly will be analyzed in detail at the LHC due to the Higgs searches. In figure 4.4 we show the invariant di-photon mass distribution to $\mathcal{O}(\theta^2)$. Our expectations to see clear noncommutative signals are not disappointed. The Z-exchange s-channel diagram led at first order to minute observable noncommutative effects since only its interference with the SM amplitude entered the cross section. The second order accounts for the squared diagram which leads to a peak at the Z-mass in the $\gamma\gamma$ distribution. Its height depends on the value of the TGB coupling. The cross section in figure 4.4 was calculated using only the special solutions (4.6) and (4.7) to the gauge equivalence equations, that is with all ambiguities set to zero.

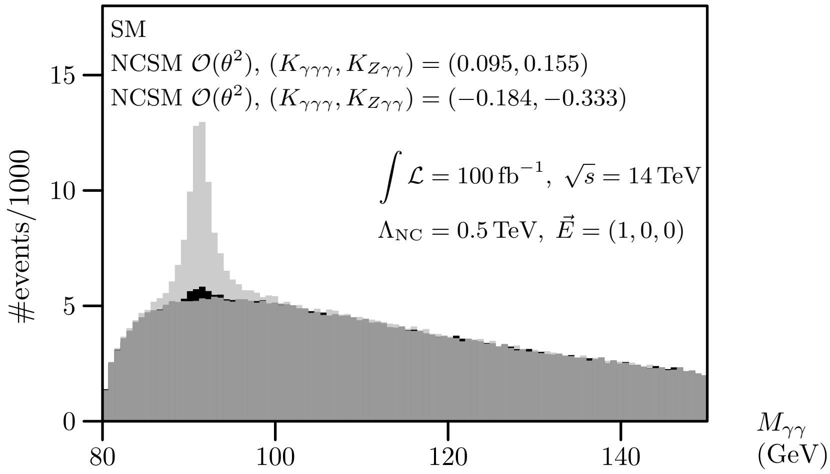


Figure 4.4: The invariant di-photon mass distribution for the process $pp \to \gamma\gamma$ to $\mathcal{O}(\theta^2)$.

We have used typical cuts as in studies regarding the light Higgs search.

$$E(\gamma) \geq 10\,\mathrm{GeV}\,, \tag{4.48a}$$
$$5^\circ \leq \theta(\gamma) \leq 175^\circ\,, \tag{4.48b}$$
$$p_T \geq 25\,\mathrm{GeV}, \tag{4.48c}$$
$$|\eta| \geq 2.5, \tag{4.48d}$$
$$80\,\mathrm{GeV} \leq M_{\gamma\gamma} \leq 150\,\mathrm{GeV}, \tag{4.48e}$$
$$DR_{\gamma\gamma} \geq 0.4\,. \tag{4.48f}$$

The last aspect justifying the $\mathcal{O}(\theta^2)$ effort is related to our ignorance about the noncommutative parameter θ. As already discussed in section 2.5.3 and section 3.2.8, due to the permanent variaton of the earth position w.r.t. a galactic reference-frame, in observables these effects are averaged over. Thus, at $\mathcal{O}(\theta)$ it might very well occur that all noncommutative signals drop out unless the orientation of the experiment is recorded for each event, since they enter the cross sections linearly as harmonic functions of the angles. In this case, only the second or higher order contribution will survive.

Chapter 5

Conclusions

Apart from the popular motivation due to string theory, quantum field theories on noncommutative space-time developed a life of their own giving rise to numerous theoretical and phenomenological efforts. In this thesis we view noncommutative QFT as intermediate theory between the SM and physics at the Planck scale, possibly occurring at a scale, which is broken down to the TeV scale and thus being accessible by the next generation of colliders. In the light of the upcoming LHC and assuming that noncommutative space-time might occur already at energy scales lying in the TeV range, we have performed in the first part of this thesis a phenomenological analysis of the process $pp \to Z\gamma$ with subsequent leptonic decay of the Z boson for the LHC and of e^+e^- annihilation into a Z and a photon at a future linear collider. During these studies the necessity of a further theoretical development of the considered model arose, which constituted the second part of this thesis.

We have presented a possible realization of the SM on noncommutative space-time. Its main ingredients are the Moyal-Weyl $\star$-product of functions on ordinary space-time which reproduces thus the noncommutativity inherent to the noncommutative operators $\hat{x}^\mu$ on an algebra of functions on the ordinary space-time, and the Seiberg-Witten maps. The latter map the ordinary fields to noncommutative fields in such a way that ordinary gauge transformations induce noncommutative transformations. This requirement was described mathematically by the so called gauge equivalence conditions for the gauge and matter field, and the consistency equation for the gauge parameter. These differential equations can be solved order by order in the noncommutative parameter θ and their solutions are the Seiberg-Witten maps, determined nonuniquely, since they differ by homogeneous solutions of the differential equations. The result is an effective theory as expansion in powers of θ, which preserves noncommutative gauge invariance, is anomaly free, does not modify the SM particle content and accommodates fractional charges. Being an effective theory, unitarity is ensured below a certain cut-off scale. The model was shown to be one-loop renormalizable within the gauge sector. Even if the inclusion of fermions in general still spoils multiplicative renormalizability, this is not relevant for the processes under consideration within this thesis.

The construction of this model is nevertheless not unique for two reasons. The first one is given by the freedom to choose between the minimal NCSM, a model which forbids triple gauge boson (TGB) couplings as in the SM, and the nonminimal NCSM allowing for TGB couplings. These again are not fixed but can vary within a finite range of allowed values. The second ambiguity in constructing the NCSM was discovered as a result of this work, and is due to the ambiguities in the Seiberg-Witten maps.

In the phenomenological part of this thesis we concentrated on possible signals of noncommutativity in hadronic scattering processes. The absence of such studies for the LHC motivated one of the goals of this work, the phenomenological study of pp scattering at the LHC within the θ expanded approach presented in chapter 3 up to $\mathcal{O}(\theta)$. We have settled on the production of a Z boson and a photon in the final state. Choosing the nonsymmetric final state was motivated by the fact that in unpolarized cross sections, as being studied at the LHC, only the axial coupling of the Z boson will lead to observable noncommutative effects. We have seen in section 3.2.2 that noncommutative space-time breaks rotational invariance with respect to the beam axis giving rise to an azimuthal dependence of the cross section. This provides an unmisleadingly clear and typical signal for noncommutativity of space-time, which discriminates these models against other new physics models. The study of the partonic cross section in section 3.2.2 also provided valuable information w.r.t. the cuts which had to be done for the Monte Carlo simulation of the hadronic process in the next section, since it was not trivial to make the noncommutative signal visible in the symmetric pp final state. The azimuthal dependence of the cross section was then used to perform likelihood fits and derive constraints on the noncommutative scale Λ_{NC}.

Thus, we have found that the LHC is in this channel only sensitive on $\vec{E}$-type noncommutativity, unless the values of the TGB coupling constants lie in an unfavourable range. In this case the bounds on Λ_{NC} set by the LHC are of the order of 1.2 TeV. Compared to this, the ILC has better cards. We have shown that ILC measurements are complementary to those at LHC being most sensitive on TGB couplings lying in the opposite corner of the region of their allowed values. Not only this, the ILC is more or less sensitive on all values of the TGB couplings, as well as on both time- and space-like noncommutativity. Thus, the bounds obtained for space-like ($\vec{E} \neq 0, \vec{B} = 0$) noncommutativity are between $\Lambda_{\mathrm{NC}} \gtrsim 400 - 900\,\mathrm{GeV}$ and for time-like ($\vec{E} \neq 0, \vec{B} = 0$) noncommutativity they range even up to $\Lambda_{\mathrm{NC}} \gtrsim 6\,\mathrm{TeV}$.

The upcoming LHC data will hopefully provide clear directions for further theoretical and phenomenological efforts. We have seen that the absence of noncommutative signals in the process studied in this work does not necessarily exclude the possibility of a noncommutative structure of space-time. It merely means that either the noncommutative scale is shifted beyond 1 TeV, or that nature has chosen exactly such values of the TGB coupling constants which lie exactly in the unfortunate part of their theoretical allowed range which is not accessible at the LHC, or that only space-like noncommutativity is realized in nature. For all these cases, the ILC would provide better answers.

We have also briefly studied similar processes at present experiments, like $p\bar{p} \to Z\gamma \to \ell^+\ell^-\gamma$ at the Tevatron and $e^+e^- \to Z\gamma$ at LEP. In principle our analysis confirms the current existing bounds on $\Lambda_{\rm NC}$ from high energy scattering experiments.

In the second part of this work we continued the construction of the NCSM to next order in θ. The motivation came from the phenomenological side: we could not prevent high partonic center of mass energies for some cases, so that the validity of the $\sqrt{s}/\Lambda_{\rm NC}$ expansion broke down. We have circumvented negative cross sections by simply setting them to zero, relying on the fact, that the $\mathcal{O}(\theta^2)$ contribution will repair the unphysical behaviour. On the other hand, processes which at $\mathcal{O}(\theta)$ showed little or no noncommutative effects show within the NCSM to $\mathcal{O}(\theta^2)$ clear noncommutative signals.

Thus, we have first derived the complete expressions for the Seiberg-Witten maps to $\mathcal{O}(\theta^2)$ finding additional ambiguities compared to those present in the literature. We could show that the ambiguities do not always correspond to field redefinitions and thus must not always cancel in physical observable, as it was commonly believed. We have exemplified the dependency on the ambiguities by means of a NCQED process in analytical and numerical calculations and proved thus that at higher order the NCSM obtains more free parameters. In the last section we proved the importance of going to $\mathcal{O}(\theta^2)$ by means of some examples. At $\mathcal{O}(\theta^2)$ a wide range of possibilities to study further noncommutative effects opens in various processes. On the theoretical side, the question of the ambiguities and their importance for the theory should be further elucidated. Recent work [34], studying the Seiberg-Witten maps to all orders in the noncommutative parameter θ, points to the fact, that each order in θ introduces new ambiguities.

The upcoming experimental input will hopefully point out the road elementary particle theory in general and noncommutative quantum field theory in particular must follow. Thus, soon we will know better on which phenomenological aspects of noncommutativity we should concentrate and which theoretical direction we should pursue.

Appendix A

Seiberg-Witten Maps to $\mathcal{O}(\theta^2)$

We provide the complete $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta^2)$ Seiberg-Witten maps for the gauge parameter, gauge field and matter field as solutions to the consistency and gauge equivalence equations to the corresponding order. For each field and the gauge parameter, at each order we will split the full solution into one special solution, additional contributions coming either due to the ambiguity of the gauge parameter or the ambiguities of the same field but at the precedent order in θ and homogeneous solutions of the differential equations at the corresponding order in θ.

A.1 Gauge Parameter

The Seiberg-Witten map for the gauge parameter to $\mathcal{O}(\theta)$ is given by the following (special and homogeneous) solution to the consistency equation (2.44):

$$\lambda^1(\lambda, A, \theta) = \frac{1}{4}\theta^{\mu\nu}\left\{\partial_\mu\lambda, A_\nu\right\}, \tag{A.1}$$

$$\lambda^1_{c^1_\lambda}(\lambda, A, \theta) = \mathrm{i}\, c^1_\lambda \theta^{\mu\nu}\left[\partial_\mu\lambda, A_\nu\right]. \tag{A.2}$$

At $\mathcal{O}(\theta^2)$, one special solution to the consistency equation at $\mathcal{O}(\theta^2)$ for $c^1_\lambda = 0$, reads:

$$\begin{aligned}
\lambda^2(\lambda, A, \theta) = \frac{\mathrm{i}}{32}\theta^{\kappa\lambda}\theta^{\mu\nu}\big(& - 4A_\kappa A_\nu \partial_\lambda\lambda A_\mu - 3A_\kappa\partial_\lambda\lambda A_\mu A_\nu \\
& - 2A_\lambda\partial_\nu\lambda A_\mu A_\kappa - 2A_\mu A_\kappa A_\lambda \partial_\nu\lambda - A_\mu A_\nu A_\kappa \partial_\lambda\lambda \\
& - 2A_\nu A_\kappa \partial_\lambda\lambda A_\mu - 4A_\nu\partial_\lambda\lambda A_\mu A_\kappa - 2\partial_\mu A_\kappa \partial_\lambda\partial_\nu\lambda \\
& - 2\partial_\lambda\lambda A_\mu A_\nu A_\kappa - \partial_\nu\lambda A_\mu A_\kappa A_\lambda + 2\partial_\lambda\partial_\nu\lambda\partial_\mu A_\kappa \\
& - 3A_\kappa A_\lambda \partial_\nu\lambda A_\mu\big) \\
+\frac{1}{16}\theta^{\kappa\lambda}\theta^{\mu\nu}\big(& 4A_\kappa\partial_\lambda\partial_\nu\lambda A_\mu + A_\lambda\partial_\nu\lambda\partial_\mu A_\kappa + 2A_\mu A_\kappa\partial_\lambda\partial_\nu\lambda \\
& - 2A_\mu\partial_\kappa A_\nu \partial_\lambda\lambda - \partial_\kappa A_\nu\partial_\lambda\lambda A_\mu + \partial_\mu A_\kappa A_\lambda\partial_\nu\lambda \\
& - \partial_\lambda\lambda A_\mu\partial_\kappa A_\nu + 2\partial_\nu\lambda\partial_\mu A_\kappa A_\lambda + 2\partial_\lambda\partial_\nu\lambda A_\mu A_\kappa\big).
\end{aligned} \tag{A.3}$$

The first order ambiguity c^1_λ leads to the additional terms for λ^2:

$$\begin{aligned}\lambda^2_{c^1_\lambda}(\lambda, A, \theta) = \frac{c^1_\lambda}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&+ A_\kappa A_\lambda \partial_\nu\lambda A_\mu - A_\kappa \partial_\lambda\lambda A_\mu A_\nu - \mathrm{i}(A_\mu \partial_\kappa A_\nu \partial_\lambda\lambda \\ &+ \partial_\mu A_\kappa A_\lambda \partial_\nu\lambda + \partial_\lambda\lambda A_\mu \partial_\kappa A_\nu + \partial_\nu\lambda \partial_\mu A_\kappa A_\lambda)\big) \\ + \frac{\mathrm{i}(c^1_\lambda)^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&+ A_\kappa A_\lambda \partial_\nu\lambda A_\mu + A_\kappa \partial_\lambda\lambda A_\mu A_\nu \\ &- A_\mu A_\nu A_\kappa \partial_\lambda\lambda - \partial_\nu\lambda A_\mu A_\kappa A_\lambda\big)\,.\end{aligned} \tag{A.4}$$

The purely second order ambiguity is parametrized by a 15 parameter family and is given by the solution to the homogeneous consistency equation at $\mathcal{O}(\theta^2)$:

$$\begin{aligned}\lambda^2_{c^2_{\lambda,1,\ldots,15}}(\lambda, A, \theta) = c^2_{\lambda,1}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\lambda \partial_\nu\lambda A_\mu A_\kappa + \mathrm{i}A_\mu A_\kappa \partial_\lambda\partial_\nu\lambda \\ &+ A_\nu A_\kappa \partial_\lambda\lambda A_\mu - A_\nu \partial_\lambda\lambda A_\mu A_\kappa + \mathrm{i}A_\nu \partial_\lambda\lambda \partial_\mu A_\kappa \\ &- \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\nu\lambda + \mathrm{i}\partial_\nu A_\kappa \partial_\lambda\lambda A_\mu - \mathrm{i}\partial_\lambda\lambda A_\mu \partial_\kappa A_\nu \\ &- \mathrm{i}\partial_\lambda\partial_\nu\lambda A_\mu A_\kappa + A_\kappa A_\nu \partial_\lambda\lambda A_\mu\big) \\ + c^2_{\lambda,2}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\kappa A_\lambda \partial_\nu\lambda A_\mu + A_\kappa \partial_\lambda\lambda A_\mu A_\nu + \mathrm{i}A_\kappa \partial_\lambda\lambda \partial_\mu A_\nu \\ &+ \mathrm{i}A_\mu \partial_\kappa A_\nu \partial_\lambda\lambda - \mathrm{i}\partial_\kappa A_\lambda \partial_\nu\lambda A_\mu + \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\nu\lambda \\ &+ \mathrm{i}\partial_\lambda\lambda A_\mu \partial_\kappa A_\nu - \partial_\lambda\lambda \partial_\mu\partial_\kappa A_\nu + \mathrm{i}\partial_\nu\lambda \partial_\mu A_\kappa A_\lambda \\ &- \partial_\mu\partial_\kappa A_\nu \partial_\lambda\lambda\big) \\ + c^2_{\lambda,3}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&+ A_\kappa A_\lambda \partial_\nu\lambda A_\mu - A_\kappa A_\nu \partial_\lambda\lambda A_\mu - A_\kappa \partial_\lambda\lambda A_\mu A_\nu \\ &- \mathrm{i}A_\mu \partial_\kappa A_\nu \partial_\lambda\lambda + \mathrm{i}A_\mu \partial_\nu A_\kappa \partial_\lambda\lambda + A_\nu \partial_\lambda\lambda A_\mu A_\kappa \\ &+ \mathrm{i}\partial_\lambda\lambda \partial_\mu A_\kappa A_\nu - \mathrm{i}\partial_\nu\lambda \partial_\mu A_\kappa A_\lambda\big) \\ + c^2_{\lambda,4}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\kappa A_\nu \partial_\lambda\lambda A_\mu + A_\lambda \partial_\nu\lambda A_\mu A_\kappa - \mathrm{i}A_\lambda \partial_\nu\lambda \partial_\mu A_\kappa \\ &- \mathrm{i}A_\mu A_\kappa \partial_\lambda\partial_\nu\lambda - A_\nu A_\kappa \partial_\lambda\lambda A_\mu + A_\nu \partial_\lambda\lambda A_\mu A_\kappa \\ &- \mathrm{i}\partial_\kappa A_\nu \partial_\lambda\lambda A_\mu + \mathrm{i}\partial_\mu A_\kappa A_\nu \partial_\lambda\lambda + \mathrm{i}\partial_\lambda\lambda A_\mu \partial_\nu A_\kappa \\ &+ \mathrm{i}\partial_\lambda\partial_\nu\lambda A_\mu A_\kappa\big) \\ + c^2_{\lambda,5}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&+ A_\lambda \partial_\nu\lambda A_\mu A_\kappa - \mathrm{i}A_\mu \partial_\kappa A_\lambda \partial_\nu\lambda - A_\nu A_\kappa \partial_\lambda\lambda A_\mu \\ &+ \mathrm{i}\partial_\lambda\lambda \partial_\mu A_\nu A_\kappa\big) \\ + c^2_{\lambda,6}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&+ A_\kappa A_\lambda \partial_\nu\lambda A_\mu - A_\kappa \partial_\lambda\lambda A_\mu A_\nu - \mathrm{i}A_\mu \partial_\kappa A_\nu \partial_\lambda\lambda \\ &- \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\nu\lambda + \mathrm{i}\partial_\mu A_\nu A_\kappa \partial_\lambda\lambda - \mathrm{i}\partial_\lambda\lambda A_\mu \partial_\kappa A_\nu \\ &+ \partial_\lambda\lambda \partial_\mu\partial_\kappa A_\nu - \mathrm{i}\partial_\nu\lambda A_\mu \partial_\kappa A_\lambda - \mathrm{i}\partial_\nu\lambda \partial_\mu A_\kappa A_\lambda \\ &+ \partial_\mu\partial_\kappa A_\nu \partial_\lambda\lambda\big) \\ + c^2_{\lambda,7}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\kappa A_\lambda \partial_\nu\lambda A_\mu + A_\kappa \partial_\lambda\lambda A_\mu A_\nu + A_\mu A_\nu A_\kappa \partial_\lambda\lambda \\ &- \partial_\nu\lambda A_\mu A_\kappa A_\lambda\big) \\ + c^2_{\lambda,8}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\lambda \partial_\nu\lambda A_\mu A_\kappa - A_\mu A_\kappa A_\lambda \partial_\nu\lambda + A_\nu A_\kappa \partial_\lambda\lambda A_\mu \\ &+ \partial_\lambda\lambda A_\mu A_\nu A_\kappa\big) \\ + c^2_{\lambda,9}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\kappa A_\nu \partial_\lambda\lambda A_\mu + A_\mu A_\kappa A_\nu \partial_\lambda\lambda \\ &+ A_\nu \partial_\lambda\lambda A_\mu A_\kappa - \partial_\lambda\lambda A_\mu A_\kappa A_\nu\big)\end{aligned} \tag{A.5}$$

$$
\begin{aligned}
&+ c^2_{\lambda,10}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\lambda\partial_\nu\lambda\partial_\mu A_\kappa - \mathrm{i}\partial_\kappa A_\nu\partial_\lambda\lambda A_\mu - \mathrm{i}\partial_\mu A_\kappa A_\lambda\partial_\nu\lambda \\
&\qquad + \partial_\mu A_\kappa\partial_\lambda\partial_\nu\lambda - \mathrm{i}\partial_\lambda\lambda A_\mu\partial_\kappa A_\nu + \partial_\lambda\partial_\nu\lambda\partial_\mu A_\kappa\big) \\
&+ c^2_{\lambda,11}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\kappa A_\nu\partial_\lambda\lambda A_\mu - \mathrm{i}A_\mu A_\kappa A_\lambda\partial_\nu\lambda + A_\mu A_\kappa\partial_\lambda\partial_\nu\lambda \\
&\qquad - \mathrm{i}A_\nu\partial_\lambda\lambda A_\mu A_\kappa + A_\nu\partial_\lambda\lambda\partial_\mu A_\kappa + \partial_\mu A_\kappa A_\lambda\partial_\nu\lambda \\
&\qquad - \partial_\nu A_\kappa\partial_\lambda\lambda A_\mu - \mathrm{i}\partial_\lambda\lambda A_\mu A_\nu A_\kappa - \partial_\lambda\lambda A_\mu\partial_\kappa A_\nu \\
&\qquad + \partial_\lambda\partial_\nu\lambda A_\mu A_\kappa\big) \\
&+ c^2_{\lambda,12}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\kappa A_\lambda\partial_\nu\lambda A_\mu - \mathrm{i}A_\kappa A_\nu\partial_\lambda\lambda A_\mu - \mathrm{i}A_\kappa\partial_\lambda\lambda A_\mu A_\nu \\
&\qquad + 2A_\kappa\partial_\lambda\partial_\nu\lambda A_\mu - A_\mu\partial_\kappa A_\nu\partial_\lambda\lambda - A_\mu\partial_\nu A_\kappa\partial_\lambda\lambda \\
&\qquad - \mathrm{i}A_\nu\partial_\lambda\lambda A_\mu A_\kappa + \partial_\lambda\lambda\partial_\mu A_\kappa A_\nu + \partial_\nu\lambda\partial_\mu A_\kappa A_\lambda\big) \\
&+ c^2_{\lambda,13}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\lambda\partial_\nu\lambda A_\mu A_\kappa + A_\lambda\partial_\nu\lambda\partial_\mu A_\kappa - \mathrm{i}A_\mu A_\kappa A_\nu\partial_\lambda\lambda \\
&\qquad + A_\mu A_\kappa\partial_\lambda\partial_\nu\lambda - \mathrm{i}A_\nu A_\kappa\partial_\lambda\lambda A_\mu - \partial_\kappa A_\nu\partial_\lambda\lambda A_\mu \\
&\qquad + \partial_\mu A_\kappa A_\nu\partial_\lambda\lambda - \mathrm{i}\partial_\lambda\lambda A_\mu A_\kappa A_\nu - \partial_\lambda\lambda A_\mu\partial_\nu A_\kappa \\
&\qquad + \partial_\lambda\partial_\nu\lambda A_\mu A_\kappa\big) \\
&+ c^2_{\lambda,14}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\kappa A_\lambda\partial_\nu\lambda A_\mu + \mathrm{i}A_\kappa\partial_\lambda\lambda A_\mu A_\nu - A_\kappa\partial_\lambda\lambda\partial_\mu A_\nu \\
&\qquad - \mathrm{i}A_\mu A_\kappa A_\lambda\partial_\nu\lambda + A_\mu\partial_\kappa A_\lambda\partial_\nu\lambda + A_\mu\partial_\kappa A_\nu\partial_\lambda\lambda \\
&\qquad - \partial_\kappa A_\lambda\partial_\nu\lambda A_\mu + \partial_\mu A_\kappa A_\lambda\partial_\nu\lambda - \mathrm{i}\partial_\lambda\lambda A_\mu A_\nu A_\kappa \\
&\qquad - \partial_\lambda\lambda A_\mu\partial_\kappa A_\nu + \partial_\lambda\lambda\partial_\mu A_\nu A_\kappa - \mathrm{i}\partial_\lambda\lambda\partial_\mu\partial_\kappa A_\nu \\
&\qquad - \partial_\nu\lambda\partial_\mu A_\kappa A_\lambda + \mathrm{i}\partial_\mu\partial_\kappa A_\nu\partial_\lambda\lambda\big) \\
&+ c^2_{\lambda,15}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\kappa A_\lambda\partial_\nu\lambda A_\mu + \mathrm{i}A_\kappa\partial_\lambda\lambda A_\mu A_\nu - A_\kappa\partial_\lambda\lambda\partial_\mu A_\nu \\
&\qquad - \mathrm{i}A_\mu A_\nu A_\kappa\partial_\lambda\lambda - \partial_\kappa A_\lambda\partial_\nu\lambda A_\mu + \partial_\mu A_\nu A_\kappa\partial_\lambda\lambda \\
&\qquad - \mathrm{i}\partial_\nu\lambda A_\mu A_\kappa A_\lambda + \partial_\nu\lambda A_\mu\partial_\kappa A_\lambda\big)\,.
\end{aligned}
$$

A.2 Gauge Field

The full Seiberg-Witten map to $\mathcal{O}(\theta)$ for the gauge field is given by:

$$
\begin{aligned}
A^1_\xi(A,\theta) &= \frac{1}{4}\theta^{\mu\nu}\big\{F_{\mu\xi} + \partial_\mu A_\xi, A_\nu\big\}\,, && \text{(A.6)}\\
A^1_{\xi,c^1_\lambda}(A,\theta) &= \mathrm{i}c^1_\lambda\,\theta^{\mu\nu}[D_\xi A_\mu, A_\nu]\,, && \text{(A.7)}\\
A^1_{\xi,c^1_A}(A,\theta) &= -2\mathrm{i}\,c^1_A\,\theta^{\mu\nu}D_\sigma F_{\mu\nu}\,. && \text{(A.8)}
\end{aligned}
$$

At $\mathcal{O}(\theta^2)$ we obtain one special solution to the gauge equivalence equation, with all first order ambiguities $c^1_{\lambda,A}$ and the $\mathcal{O}(\theta^2)$ ambiguities $c^2_{\lambda,i}$ ($i = 1,\ldots,15$) set to zero:

$$
\begin{aligned}
A^2_\xi(A,\theta) = &\frac{\mathrm{i}}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(2[\partial_\nu\partial_\lambda A_\xi, \partial_\mu A_\kappa] + [\partial_\mu A_\kappa, \partial_\xi\partial_\lambda A_\nu]\big) \qquad \text{(A.9)}\\
&+\frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ 2A_\mu A_\kappa\partial_\nu\partial_\lambda A_\xi - A_\mu A_\kappa\partial_\nu\partial_\xi A_\lambda + A_\mu A_\kappa\partial_\xi\partial_\lambda A_\nu \\
&\qquad + 4A_\mu\partial_\nu A_\kappa\partial_\lambda A_\xi - 4A_\mu\partial_\nu A_\kappa\partial_\xi A_\lambda - 2A_\mu\partial_\kappa A_\nu\partial_\lambda A_\xi \\
&\qquad - 2A_\nu\partial_\lambda A_\xi\partial_\mu A_\kappa + 3A_\nu\partial_\xi A_\lambda\partial_\mu A_\kappa + 4A_\kappa\partial_\nu\partial_\lambda A_\xi A_\mu
\end{aligned}
$$

$$
\begin{aligned}
&+ 2A_\lambda\partial_\nu A_\xi\partial_\mu A_\kappa - A_\lambda\partial_\xi A_\nu\partial_\mu A_\kappa - A_\xi\partial_\mu A_\kappa\partial_\lambda A_\nu \\
&- 4\partial_\mu A_\kappa A_\nu\partial_\lambda A_\xi + \partial_\mu A_\kappa A_\nu\partial_\xi A_\lambda + \partial_\mu A_\kappa A_\lambda\partial_\xi A_\nu \\
&- \partial_\mu A_\kappa\partial_\lambda A_\nu A_\xi + 2\partial_\nu A_\kappa\partial_\lambda A_\xi A_\mu - 3\partial_\nu A_\kappa\partial_\xi A_\lambda A_\mu \\
&+ 2\partial_\nu A_\xi\partial_\mu A_\kappa A_\lambda - 2\partial_\kappa A_\nu\partial_\lambda A_\xi A_\mu + \partial_\kappa A_\nu\partial_\xi A_\lambda A_\mu \\
&+ 2\partial_\lambda A_\nu A_\xi\partial_\mu A_\kappa + 2\partial_\lambda A_\xi A_\mu\partial_\nu A_\kappa - 4\partial_\lambda A_\xi\partial_\mu A_\kappa A_\nu \\
&- \partial_\xi A_\lambda A_\mu\partial_\nu A_\kappa - \partial_\xi A_\lambda A_\mu\partial_\kappa A_\nu + 4\partial_\xi A_\lambda\partial_\mu A_\kappa A_\nu \\
&+ 2\partial_\nu\partial_\lambda A_\xi A_\mu A_\kappa + \partial_\nu\partial_\xi A_\lambda A_\mu A_\kappa - \partial_\xi\partial_\lambda A_\nu A_\mu A_\kappa\big) \\
+\frac{\mathrm{i}}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- 4A_\mu A_\nu A_\kappa\partial_\lambda A_\xi + 3A_\mu A_\nu A_\kappa\partial_\xi A_\lambda - 2\partial_\xi A_\lambda A_\mu A_\kappa A_\nu \\
&+ 4A_\mu A_\kappa A_\nu\partial_\lambda A_\xi - 2A_\mu A_\kappa A_\nu\partial_\xi A_\lambda - 4A_\mu A_\kappa A_\lambda\partial_\nu A_\xi \\
&- 2A_\mu A_\kappa\partial_\nu A_\lambda A_\xi + 2A_\mu A_\kappa\partial_\lambda A_\nu A_\xi - 8A_\mu\partial_\nu A_\kappa A_\lambda A_\xi \\
&- 4A_\nu A_\kappa\partial_\lambda A_\xi A_\mu + 4A_\nu A_\kappa\partial_\xi A_\lambda A_\mu + 8A_\nu A_\lambda A_\xi\partial_\mu A_\kappa \\
&- 4A_\nu\partial_\lambda A_\xi A_\mu A_\kappa - 2A_\nu\partial_\xi A_\lambda A_\mu A_\kappa - 4A_\kappa A_\nu\partial_\lambda A_\xi A_\mu \\
&- 2A_\kappa A_\nu\partial_\xi A_\lambda A_\mu - 4A_\kappa A_\lambda\partial_\nu A_\xi A_\mu + A_\kappa A_\lambda\partial_\xi A_\nu A_\mu \\
&- 8A_\kappa\partial_\lambda A_\nu A_\xi A_\mu - 4A_\kappa\partial_\lambda A_\xi A_\mu A_\nu + A_\kappa\partial_\xi A_\lambda A_\mu A_\nu \\
&- 4A_\lambda A_\nu A_\xi\partial_\mu A_\kappa - 4A_\lambda A_\xi A_\mu\partial_\nu A_\kappa + 4A_\lambda A_\xi A_\mu\partial_\kappa A_\nu \\
&+ 8A_\lambda A_\xi\partial_\mu A_\kappa A_\nu - 4A_\lambda\partial_\nu A_\xi A_\mu A_\kappa + 4A_\lambda\partial_\xi A_\nu A_\mu A_\kappa \\
&- 2A_\xi A_\mu A_\kappa\partial_\nu A_\lambda - 2A_\xi A_\mu A_\kappa\partial_\lambda A_\nu + 8A_\xi A_\mu\partial_\kappa A_\nu A_\lambda \\
&- 2A_\xi\partial_\mu A_\kappa A_\nu A_\lambda + 2A_\xi\partial_\mu A_\kappa A_\lambda A_\nu + 2\partial_\mu A_\kappa A_\nu A_\lambda A_\xi \\
&+ 2\partial_\mu A_\kappa A_\lambda A_\nu A_\xi - 4\partial_\nu A_\kappa A_\lambda A_\xi A_\mu + 4\partial_\nu A_\lambda A_\xi A_\mu A_\kappa \\
&- 4\partial_\nu A_\xi A_\mu A_\kappa A_\lambda + 4\partial_\kappa A_\nu A_\lambda A_\xi A_\mu - 8\partial_\lambda A_\nu A_\xi A_\mu A_\kappa \\
&- 4\partial_\lambda A_\xi A_\mu A_\nu A_\kappa + 4\partial_\lambda A_\xi A_\mu A_\kappa A_\nu + 3\partial_\xi A_\nu A_\mu A_\kappa A_\lambda\big) \\
+\frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- 3A_\mu A_\nu A_\kappa A_\lambda A_\xi + 2A_\mu A_\kappa A_\nu A_\lambda A_\xi - 4A_\nu A_\kappa A_\lambda A_\xi A_\mu \\
&+ 4A_\nu A_\lambda A_\xi A_\mu A_\kappa - 4A_\nu A_\xi A_\mu A_\kappa A_\lambda + 4A_\kappa A_\nu A_\lambda A_\xi A_\mu \\
&- 4A_\kappa A_\lambda A_\nu A_\xi A_\mu - 2A_\kappa A_\lambda A_\xi A_\mu A_\nu - 8A_\lambda A_\nu A_\xi A_\mu A_\kappa \\
&- 4A_\lambda A_\xi A_\mu A_\nu A_\kappa + 4A_\lambda A_\xi A_\mu A_\kappa A_\nu \\
&- 3A_\xi A_\mu A_\nu A_\kappa A_\lambda + 2A_\xi A_\mu A_\kappa A_\nu A_\lambda\big)\,.
\end{aligned}
$$

Contributions from $\mathcal{O}(\theta)$-ambiguities are given by:

$$
\begin{aligned}
A^2_{\xi,c^1_\lambda}(A,\theta) = \frac{c^1_\lambda}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- \mathrm{i}\{A_\mu\partial_\kappa A_\nu,\partial_\lambda A_\xi\} - \mathrm{i}\{\partial_\mu A_\kappa A_\lambda,\partial_\nu A_\xi\}\big) \\
&[A_\mu A_\nu,\{A_\kappa,\partial_\lambda A_\xi\}] - A_\mu A_\nu A_\kappa\partial_\xi A_\lambda + \partial_\xi A_\nu A_\mu A_\kappa A_\lambda\big) \\
&+ \mathrm{i}[A_\nu A_\xi A_\mu, A_\kappa A_\lambda] + \mathrm{i}[A_\xi, A_\mu A_\nu A_\kappa A_\lambda]\big) \\
+\frac{(c^1_\lambda)^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&\mathrm{i}[A_\kappa\partial_\xi A_\lambda, A_\mu A_\nu] + \mathrm{i}[A_\kappa A_\lambda,\partial_\xi A_\nu A_\mu] \\
&\{A_\mu A_\nu A_\kappa A_\lambda, A_\xi\} - A_\kappa A_\lambda A_\xi A_\mu A_\nu\big)\,,
\end{aligned}
\tag{A.10}
$$

$$
A^2_{\xi,c^1_A}(A,\theta) = -\frac{\mathrm{i}c^1_A}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\{A_\lambda,\partial_\xi\partial_\mu\partial_\kappa A_\nu\}
\tag{A.11}
$$

$$
\begin{aligned}
&+\frac{c_A^1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu \partial_\xi\partial_\kappa A_\lambda + 2A_\mu A_\kappa \partial_\nu\partial_\xi A_\lambda + 2A_\mu \partial_\nu A_\kappa \partial_\xi A_\lambda \\
&\qquad - 2A_\mu \partial_\kappa A_\lambda \partial_\nu A_\xi - 2A_\mu \partial_\nu\partial_\kappa A_\lambda A_\xi + 2A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa \\
&\qquad + 2A_\nu \partial_\xi\partial_\kappa A_\lambda A_\mu + 2A_\kappa \partial_\lambda A_\xi \partial_\mu A_\nu + 2A_\kappa \partial_\nu\partial_\xi A_\lambda A_\mu \\
&\qquad - 2A_\kappa \partial_\xi\partial_\lambda A_\nu A_\mu - 2A_\lambda A_\xi \partial_\mu\partial_\kappa A_\nu - 2A_\xi \partial_\mu\partial_\kappa A_\nu A_\lambda \\
&\qquad + 2\partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu - 2\partial_\kappa A_\lambda \partial_\nu A_\xi A_\mu + 2\partial_\lambda A_\xi \partial_\mu A_\nu A_\kappa \\
&\qquad + 2\partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu - 2\partial_\nu\partial_\kappa A_\lambda A_\xi A_\mu + \partial_\xi\partial_\kappa A_\lambda A_\mu A_\nu \\
&\qquad - 2\partial_\xi\partial_\lambda A_\nu A_\mu A_\kappa\big) \\
&+\frac{\mathrm{i}c_A^1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu \partial_\kappa A_\lambda A_\xi - A_\mu A_\nu A_\kappa \partial_\xi A_\lambda + 2A_\mu A_\kappa A_\lambda \partial_\nu A_\xi \\
&\qquad + 2A_\mu A_\kappa \partial_\nu A_\lambda A_\xi + 2A_\mu \partial_\nu A_\kappa A_\lambda A_\xi - 2A_\nu A_\kappa \partial_\xi A_\lambda A_\mu \\
&\qquad + 2A_\nu A_\xi \partial_\mu A_\kappa A_\lambda + 2A_\nu \partial_\kappa A_\lambda A_\xi A_\mu - A_\kappa A_\lambda A_\xi \partial_\mu A_\nu \\
&\qquad + 2A_\kappa A_\lambda \partial_\nu A_\xi A_\mu + A_\kappa A_\lambda \partial_\xi A_\nu A_\mu + 2A_\kappa \partial_\nu A_\lambda A_\xi A_\mu \\
&\qquad - 2A_\kappa \partial_\lambda A_\xi A_\mu A_\nu - A_\kappa \partial_\xi A_\lambda A_\mu A_\nu + 2A_\lambda A_\xi A_\mu \partial_\kappa A_\nu \\
&\qquad - 2A_\lambda A_\xi \partial_\mu A_\nu A_\kappa + 2A_\lambda \partial_\xi A_\nu A_\mu A_\kappa + 2A_\xi A_\mu \partial_\kappa A_\nu A_\lambda \\
&\qquad - A_\xi \partial_\mu A_\nu A_\kappa A_\lambda + 2A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + 2\partial_\nu A_\kappa A_\lambda A_\xi A_\mu \\
&\qquad + \partial_\kappa A_\lambda A_\xi A_\mu A_\nu - 2\partial_\lambda A_\xi A_\mu A_\nu A_\kappa + \partial_\xi A_\nu A_\mu A_\kappa A_\lambda\big) \\
&+\frac{c_A^1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big([A_\mu A_\nu A_\kappa A_\lambda, A_\xi] + 2[A_\nu A_\kappa, A_\lambda A_\xi A_\mu]\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
A^2_{\xi,c^1_\lambda,c^1_A}(A,\theta) = c^1_\lambda\, c^1_A\, \theta^{\mu\nu}\theta^{\kappa\lambda}\big(&\mathrm{i}[A_\mu A_\nu, \partial_\xi\partial_\kappa A_\lambda] + [A_\mu A_\nu, A_\kappa \partial_\xi A_\lambda] \\
&+ \big[A_\kappa A_\lambda, [A_\xi, \partial_\mu A_\nu]\big] - [A_\kappa A_\lambda, \partial_\xi A_\nu A_\mu] \\
&\quad + \mathrm{i}[A_\mu A_\nu, A_\kappa A_\lambda A_\xi] + \mathrm{i}[A_\xi A_\mu A_\nu, A_\kappa A_\lambda]\big)\,. \qquad (A.12)
\end{aligned}
$$

The $\mathcal{O}(\theta^2)$-ambiguities of the gauge parameter also contribute to A^2_ξ and yield the additional terms:

$$
\begin{aligned}
A^2_{\xi,c^2_{\lambda,1,\dots,15}} = \; & c^2_{\lambda,1}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(-A_\mu A_\kappa A_\lambda \partial_\xi A_\nu - A_\mu A_\kappa \partial_\nu A_\lambda A_\xi \qquad (A.13) \\
&\qquad + \mathrm{i}A_\mu A_\kappa \partial_\nu\partial_\xi A_\lambda + A_\nu A_\kappa \partial_\xi A_\lambda A_\mu + \mathrm{i}A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa \\
&\qquad - A_\lambda \partial_\xi A_\nu A_\mu A_\kappa + \mathrm{i}A_\xi A_\mu A_\kappa A_\lambda A_\nu + A_\xi A_\mu A_\kappa \partial_\nu A_\lambda \\
&\qquad - A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + \partial_\mu A_\kappa A_\lambda A_\nu A_\xi - \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu \\
&\qquad + \mathrm{i}\partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu + \partial_\xi A_\lambda A_\mu A_\nu A_\kappa - \mathrm{i}\partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu \\
&\qquad - \mathrm{i}A_\mu A_\kappa A_\lambda A_\nu A_\xi - \mathrm{i}\partial_\xi\partial_\lambda A_\nu A_\mu A_\kappa\big) \\
&+ c^2_{\lambda,2}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+\mathrm{i}A_\mu A_\kappa A_\lambda A_\nu A_\xi + A_\mu A_\kappa A_\lambda \partial_\xi A_\nu + A_\mu A_\kappa \partial_\nu A_\lambda A_\xi \\
&\qquad - \mathrm{i}A_\mu A_\kappa \partial_\nu\partial_\xi A_\lambda + A_\mu \partial_\nu A_\kappa A_\lambda A_\xi - \mathrm{i}A_\mu \partial_\nu A_\kappa \partial_\xi A_\lambda \\
&\qquad - A_\mu \partial_\kappa A_\nu A_\lambda A_\xi + \mathrm{i}A_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda - A_\mu \partial_\kappa A_\lambda A_\nu A_\xi \\
&\qquad + \mathrm{i}A_\mu \partial_\kappa A_\lambda \partial_\xi A_\nu + \mathrm{i}A_\mu \partial_\nu\partial_\kappa A_\lambda A_\xi - A_\nu A_\kappa \partial_\xi A_\lambda A_\mu \\
&\qquad - \mathrm{i}A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa - \mathrm{i}A_\nu \partial_\xi\partial_\kappa A_\lambda A_\mu - \mathrm{i}A_\kappa \partial_\nu\partial_\xi A_\lambda A_\mu \\
&\qquad + \mathrm{i}A_\kappa \partial_\xi\partial_\lambda A_\nu A_\mu + A_\lambda \partial_\xi A_\nu A_\mu A_\kappa - A_\lambda \partial_\xi\partial_\mu\partial_\kappa A_\nu
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{i}A_\xi A_\mu A_\kappa A_\lambda A_\nu - A_\xi A_\mu A_\kappa \partial_\nu A_\lambda - A_\xi A_\mu \partial_\nu A_\kappa A_\lambda \\
&+ A_\xi A_\mu \partial_\kappa A_\nu A_\lambda + A_\xi A_\mu \partial_\kappa A_\lambda A_\nu - \mathrm{i}A_\xi A_\mu \partial_\nu \partial_\kappa A_\lambda \\
&+ A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + \mathrm{i}A_\xi \partial_\mu \partial_\kappa A_\nu A_\lambda - \partial_\mu A_\kappa A_\lambda A_\nu A_\xi \\
&+ \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu - \mathrm{i}\partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu + \mathrm{i}\partial_\xi A_\nu \partial_\mu A_\kappa A_\lambda \\
&- \partial_\xi A_\lambda A_\mu A_\nu A_\kappa + \mathrm{i}\partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu - \mathrm{i}\partial_\xi A_\lambda \partial_\mu A_\nu A_\kappa \\
&- \mathrm{i}\partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu - \partial_\xi A_\lambda \partial_\mu \partial_\kappa A_\nu - \mathrm{i}\partial_\mu \partial_\kappa A_\nu A_\lambda A_\xi \\
&- \partial_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda + \mathrm{i}\partial_\xi \partial_\lambda A_\nu A_\mu A_\kappa - \partial_\xi \partial_\mu \partial_\kappa A_\nu A_\lambda \big) \\
+ c^2_{\lambda,3}\, \theta^{\mu\nu}\theta^{\kappa\lambda} \big(&- A_\mu \partial_\nu A_\kappa A_\lambda A_\xi + \mathrm{i}A_\mu \partial_\nu A_\kappa \partial_\xi A_\lambda + A_\mu \partial_\kappa A_\nu A_\lambda A_\xi \\
&- \mathrm{i}A_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda + \mathrm{i}A_\kappa \partial_\nu \partial_\xi A_\lambda A_\mu - \mathrm{i}A_\kappa \partial_\xi \partial_\lambda A_\nu A_\mu \\
&+ A_\xi A_\mu \partial_\nu A_\kappa A_\lambda - A_\xi A_\mu \partial_\kappa A_\nu A_\lambda - \mathrm{i}\partial_\xi A_\nu \partial_\mu A_\kappa A_\lambda \\
&+ \mathrm{i}\partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu \big) \\
+ c^2_{\lambda,4}\, \theta^{\mu\nu}\theta^{\kappa\lambda} \big(&+ \mathrm{i}A_\mu A_\kappa A_\nu A_\lambda A_\xi + A_\mu A_\kappa A_\nu \partial_\xi A_\lambda + A_\mu A_\kappa \partial_\lambda A_\nu A_\xi \\
&- \mathrm{i}A_\mu A_\kappa \partial_\xi \partial_\lambda A_\nu + A_\nu \partial_\xi A_\lambda A_\mu A_\kappa - A_\kappa A_\nu \partial_\xi A_\lambda A_\mu \\
&- \mathrm{i}A_\lambda \partial_\xi A_\nu \partial_\mu A_\kappa - A_\xi A_\mu A_\kappa A_\nu A_\lambda - A_\xi A_\mu A_\kappa \partial_\lambda A_\nu \\
&+ A_\xi \partial_\mu A_\kappa A_\nu A_\lambda - \partial_\mu A_\kappa A_\nu A_\lambda A_\xi + \mathrm{i}\partial_\mu A_\kappa A_\nu \partial_\xi A_\lambda \\
&- \mathrm{i}\partial_\kappa A_\nu \partial_\xi A_\lambda A_\mu - \partial_\xi A_\lambda A_\mu A_\kappa A_\nu + \mathrm{i}\partial_\xi A_\lambda A_\mu \partial_\nu A_\kappa \\
&+ \mathrm{i}\partial_\nu \partial_\xi A_\lambda A_\mu A_\kappa \big) \\
+ c^2_{\lambda,5}\, \theta^{\mu\nu}\theta^{\kappa\lambda} \big(&+ A_\mu \partial_\kappa A_\lambda A_\nu A_\xi - \mathrm{i}A_\mu \partial_\kappa A_\lambda \partial_\xi A_\nu + \mathrm{i}A_\nu \partial_\xi \partial_\kappa A_\lambda A_\mu \\
&- A_\xi A_\mu \partial_\kappa A_\lambda A_\nu + \mathrm{i}\partial_\xi A_\lambda \partial_\mu A_\nu A_\kappa \big) \\
+ c^2_{\lambda,6}\, \theta^{\mu\nu}\theta^{\kappa\lambda} \big(&+ \mathrm{i}A_\mu A_\nu A_\kappa A_\lambda A_\xi + A_\mu A_\nu A_\kappa \partial_\xi A_\lambda - A_\mu A_\nu \partial_\kappa A_\lambda A_\xi \\
&+ \mathrm{i}A_\mu A_\nu \partial_\xi \partial_\kappa A_\lambda - \mathrm{i}A_\mu A_\kappa A_\lambda A_\nu A_\xi - A_\mu A_\kappa A_\lambda \partial_\xi A_\nu \\
&- A_\mu A_\kappa \partial_\nu A_\lambda A_\xi + \mathrm{i}A_\mu A_\kappa \partial_\nu \partial_\xi A_\lambda - A_\mu \partial_\nu A_\kappa A_\lambda A_\xi \\
&+ \mathrm{i}A_\mu \partial_\nu A_\kappa \partial_\xi A_\lambda + A_\mu \partial_\kappa A_\nu A_\lambda A_\xi - \mathrm{i}A_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda \\
&+ A_\mu \partial_\kappa A_\lambda A_\nu A_\xi - \mathrm{i}A_\mu \partial_\kappa A_\lambda \partial_\xi A_\nu - \mathrm{i}A_\mu \partial_\nu \partial_\kappa A_\lambda A_\xi \\
&+ A_\nu A_\kappa \partial_\xi A_\lambda A_\mu + \mathrm{i}A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa + \mathrm{i}A_\nu \partial_\xi \partial_\kappa A_\lambda A_\mu \\
&- A_\kappa A_\lambda \partial_\xi A_\nu A_\mu + A_\kappa \partial_\xi A_\lambda A_\mu A_\nu + \mathrm{i}A_\kappa \partial_\xi A_\lambda \partial_\mu A_\nu \\
&+ \mathrm{i}A_\kappa \partial_\nu \partial_\xi A_\lambda A_\mu - \mathrm{i}A_\kappa \partial_\xi \partial_\lambda A_\nu A_\mu - A_\lambda \partial_\xi A_\nu A_\mu A_\kappa \\
&+ A_\lambda \partial_\xi \partial_\mu \partial_\kappa A_\nu - \mathrm{i}A_\xi A_\mu A_\nu A_\kappa A_\lambda + A_\xi A_\mu A_\nu \partial_\kappa A_\lambda \\
&+ \mathrm{i}A_\xi A_\mu A_\kappa A_\lambda A_\nu + A_\xi A_\mu A_\kappa \partial_\nu A_\lambda + A_\xi A_\mu \partial_\nu A_\kappa A_\lambda \\
&- A_\xi A_\mu \partial_\kappa A_\nu A_\lambda - A_\xi A_\mu \partial_\kappa A_\lambda A_\nu + \mathrm{i}A_\xi A_\mu \partial_\nu \partial_\kappa A_\lambda \\
&+ A_\xi \partial_\mu A_\nu A_\kappa A_\lambda - A_\xi \partial_\mu A_\kappa A_\lambda A_\nu - \mathrm{i}A_\xi \partial_\mu \partial_\kappa A_\nu A_\lambda \\
&- \partial_\mu A_\nu A_\kappa A_\lambda A_\xi + \mathrm{i}\partial_\mu A_\nu A_\kappa \partial_\xi A_\lambda + \partial_\mu A_\kappa A_\lambda A_\nu A_\xi \\
&- \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu + \mathrm{i}\partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu - \mathrm{i}\partial_\kappa A_\lambda \partial_\xi A_\nu A_\mu \\
&- \partial_\xi A_\nu A_\mu A_\kappa A_\lambda - \mathrm{i}\partial_\xi A_\nu A_\mu \partial_\kappa A_\lambda - \mathrm{i}\partial_\xi A_\nu \partial_\mu A_\kappa A_\lambda \\
&+ \partial_\xi A_\lambda A_\mu A_\nu A_\kappa - \mathrm{i}\partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu + \mathrm{i}\partial_\xi A_\lambda \partial_\mu A_\nu A_\kappa \\
&+ \mathrm{i}\partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu + \partial_\xi A_\lambda \partial_\mu \partial_\kappa A_\nu + \mathrm{i}\partial_\mu \partial_\kappa A_\nu A_\lambda A_\xi \\
&+ \partial_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda + \mathrm{i}\partial_\xi \partial_\kappa A_\lambda A_\mu A_\nu - \mathrm{i}\partial_\xi \partial_\lambda A_\nu A_\mu A_\kappa \\
&+ \partial_\xi \partial_\mu \partial_\kappa A_\nu A_\lambda \big)
\end{aligned}
$$

$$
\begin{aligned}
&+ c^2_{\lambda,7}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\mu A_\nu A_\kappa A_\lambda A_\xi + A_\mu A_\nu A_\kappa \partial_\xi A_\lambda - A_\kappa A_\lambda \partial_\xi A_\nu A_\mu \\
&\qquad + A_\kappa \partial_\xi A_\lambda A_\mu A_\nu - \mathrm{i}A_\xi A_\mu A_\nu A_\kappa A_\lambda - \partial_\xi A_\nu A_\mu A_\kappa A_\lambda \big) \\
&+ c^2_{\lambda,8}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\mu A_\kappa A_\lambda A_\nu A_\xi - A_\mu A_\kappa A_\lambda \partial_\xi A_\nu + A_\nu A_\kappa \partial_\xi A_\lambda A_\mu \\
&\qquad - A_\lambda \partial_\xi A_\nu A_\mu A_\kappa + \mathrm{i}A_\xi A_\mu A_\kappa A_\lambda A_\nu + \partial_\xi A_\lambda A_\mu A_\nu A_\kappa \big) \\
&+ c^2_{\lambda,9}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\mu A_\kappa A_\nu A_\lambda A_\xi + A_\mu A_\kappa A_\nu \partial_\xi A_\lambda + A_\nu \partial_\xi A_\lambda A_\mu A_\kappa \\
&\qquad - A_\kappa A_\nu \partial_\xi A_\lambda A_\mu - \mathrm{i}A_\xi A_\mu A_\kappa A_\nu A_\lambda - \partial_\xi A_\lambda A_\mu A_\kappa A_\nu \big) \\
&+ c^2_{\lambda,10}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\mu A_\kappa A_\nu A_\lambda A_\xi + A_\mu A_\kappa A_\nu \partial_\xi A_\lambda - \mathrm{i}A_\mu A_\kappa A_\lambda A_\nu A_\xi \\
&\qquad - A_\mu A_\kappa A_\lambda \partial_\xi A_\nu - A_\mu A_\kappa \partial_\nu A_\lambda A_\xi + A_\mu A_\kappa \partial_\lambda A_\nu A_\xi \\
&\qquad + \mathrm{i}A_\mu A_\kappa \partial_\nu \partial_\xi A_\lambda - \mathrm{i}A_\mu A_\kappa \partial_\xi \partial_\lambda A_\nu + A_\nu A_\kappa \partial_\xi A_\lambda A_\mu \\
&\qquad + A_\nu \partial_\xi A_\lambda A_\mu A_\kappa + \mathrm{i}A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa - A_\kappa A_\nu \partial_\xi A_\lambda A_\mu \\
&\qquad - A_\lambda \partial_\xi A_\nu A_\mu A_\kappa - \mathrm{i}A_\lambda \partial_\xi A_\nu \partial_\mu A_\kappa - \mathrm{i}A_\xi A_\mu A_\kappa A_\nu A_\lambda \\
&\qquad + \mathrm{i}A_\xi A_\mu A_\kappa A_\lambda A_\nu + A_\xi A_\mu A_\kappa \partial_\nu A_\lambda - A_\xi A_\mu A_\kappa \partial_\lambda A_\nu \\
&\qquad + A_\xi \partial_\mu A_\kappa A_\nu A_\lambda - A_\xi \partial_\mu A_\kappa A_\lambda A_\nu - \mathrm{i}A_\xi \partial_\mu A_\kappa \partial_\lambda A_\nu \\
&\qquad - \partial_\mu A_\kappa A_\nu A_\lambda A_\xi + \mathrm{i}\partial_\mu A_\kappa A_\nu \partial_\xi A_\lambda + \partial_\mu A_\kappa A_\lambda A_\nu A_\xi \\
&\qquad - \mathrm{i}\partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu + \mathrm{i}\partial_\mu A_\kappa \partial_\lambda A_\nu A_\xi + \partial_\mu A_\kappa \partial_\xi \partial_\lambda A_\nu \\
&\qquad + \mathrm{i}\partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu - \mathrm{i}\partial_\kappa A_\nu \partial_\xi A_\lambda A_\mu + \partial_\xi A_\lambda A_\mu A_\nu A_\kappa \\
&\qquad - \partial_\xi A_\lambda A_\mu A_\kappa A_\nu + \mathrm{i}\partial_\xi A_\lambda A_\mu \partial_\nu A_\kappa - \mathrm{i}\partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu \\
&\qquad + \mathrm{i}\partial_\nu \partial_\xi A_\lambda A_\mu A_\kappa - \mathrm{i}\partial_\xi \partial_\lambda A_\nu A_\mu A_\kappa + \partial_\xi \partial_\lambda A_\nu \partial_\mu A_\kappa \big) \\
&+ c^2_{\lambda,11}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\mu A_\kappa \partial_\nu A_\lambda A_\xi + A_\mu A_\kappa \partial_\nu \partial_\xi A_\lambda + A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa \\
&\qquad - \mathrm{i}A_\xi A_\mu A_\kappa \partial_\nu A_\lambda - \mathrm{i}A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + \mathrm{i}\partial_\mu A_\kappa A_\lambda A_\nu A_\xi \\
&\qquad + \partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu - \partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu - \partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu \\
&\qquad + \partial_\xi \partial_\lambda A_\nu A_\mu A_\kappa \big) \\
&+ c^2_{\lambda,12}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\mu \partial_\nu A_\kappa A_\lambda A_\xi - A_\mu \partial_\nu A_\kappa \partial_\xi A_\lambda - \mathrm{i}A_\mu \partial_\kappa A_\nu A_\lambda A_\xi \\
&\qquad - A_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda + A_\kappa \partial_\nu \partial_\xi A_\lambda A_\mu + A_\kappa \partial_\xi \partial_\lambda A_\nu A_\mu \\
&\qquad + \mathrm{i}A_\xi A_\mu \partial_\nu A_\kappa A_\lambda + \mathrm{i}A_\xi A_\mu \partial_\kappa A_\nu A_\lambda + \partial_\xi A_\nu \partial_\mu A_\kappa A_\lambda \\
&\qquad + \partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu \big) \\
&+ c^2_{\lambda,13}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\mu A_\kappa \partial_\lambda A_\nu A_\xi + A_\mu A_\kappa \partial_\xi \partial_\lambda A_\nu + A_\lambda \partial_\xi A_\nu \partial_\mu A_\kappa \\
&\qquad - \mathrm{i}A_\xi A_\mu A_\kappa \partial_\lambda A_\nu - \mathrm{i}A_\xi \partial_\mu A_\kappa A_\nu A_\lambda + \mathrm{i}\partial_\mu A_\kappa A_\nu A_\lambda A_\xi \\
&\qquad + \partial_\mu A_\kappa A_\nu \partial_\xi A_\lambda - \partial_\kappa A_\nu \partial_\xi A_\lambda A_\mu - \partial_\xi A_\lambda A_\mu \partial_\nu A_\kappa \\
&\qquad + \partial_\nu \partial_\xi A_\lambda A_\mu A_\kappa \big) \\
&+ c^2_{\lambda,14}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+ \mathrm{i}A_\mu A_\kappa \partial_\nu A_\lambda A_\xi + A_\mu A_\kappa \partial_\nu \partial_\xi A_\lambda + \mathrm{i}A_\mu \partial_\nu A_\kappa A_\lambda A_\xi \\
&\qquad + A_\mu \partial_\nu A_\kappa \partial_\xi A_\lambda + \mathrm{i}A_\mu \partial_\kappa A_\nu A_\lambda A_\xi + A_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda \\
&\qquad - A_\mu \partial_\nu \partial_\kappa A_\lambda A_\xi + A_\nu \partial_\xi A_\lambda \partial_\mu A_\kappa - A_\kappa \partial_\nu \partial_\xi A_\lambda A_\mu \\
&\qquad - A_\kappa \partial_\xi \partial_\lambda A_\nu A_\mu - \mathrm{i}A_\lambda \partial_\xi \partial_\mu \partial_\kappa A_\nu - \mathrm{i}A_\xi A_\mu A_\kappa \partial_\nu A_\lambda \\
&\qquad - \mathrm{i}A_\xi A_\mu \partial_\nu A_\kappa A_\lambda - \mathrm{i}A_\xi A_\mu \partial_\kappa A_\nu A_\lambda + A_\xi A_\mu \partial_\nu \partial_\kappa A_\lambda \\
&\qquad - \mathrm{i}A_\xi \partial_\mu A_\kappa A_\lambda A_\nu + A_\xi \partial_\mu \partial_\kappa A_\nu A_\lambda + \mathrm{i}\partial_\mu A_\kappa A_\lambda A_\nu A_\xi \\
&\qquad + \partial_\mu A_\kappa A_\lambda \partial_\xi A_\nu - \partial_\nu A_\kappa \partial_\xi A_\lambda A_\mu - \partial_\xi A_\nu \partial_\mu A_\kappa A_\lambda
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \partial_\xi A_\lambda A_\mu \partial_\kappa A_\nu - \partial_\xi A_\lambda \partial_\mu A_\kappa A_\nu - \mathrm{i}\partial_\xi A_\lambda \partial_\mu \partial_\kappa A_\nu \\
&\quad - \partial_\mu \partial_\kappa A_\nu A_\lambda A_\xi + \mathrm{i}\partial_\mu \partial_\kappa A_\nu \partial_\xi A_\lambda + \partial_\xi \partial_\lambda A_\nu A_\mu A_\kappa \\
&\quad + \mathrm{i}\partial_\xi \partial_\mu \partial_\kappa A_\nu A_\lambda\big) \\
&+ c^2_{\lambda,15}\, \theta^{\mu\nu}\theta^{\kappa\lambda}\big(- \mathrm{i}A_\mu A_\nu \partial_\kappa A_\lambda A_\xi \\
&\quad - A_\mu A_\nu \partial_\xi \partial_\kappa A_\lambda - A_\kappa \partial_\xi A_\lambda \partial_\mu A_\nu + \mathrm{i}A_\xi A_\mu A_\nu \partial_\kappa A_\lambda \\
&\quad - \mathrm{i}A_\xi \partial_\mu A_\nu A_\kappa A_\lambda + \mathrm{i}\partial_\mu A_\nu A_\kappa A_\lambda A_\xi + \partial_\mu A_\nu A_\kappa \partial_\xi A_\lambda \\
&\quad - \partial_\kappa A_\lambda \partial_\xi A_\nu A_\mu + \partial_\xi A_\nu A_\mu \partial_\kappa A_\lambda + \partial_\xi \partial_\kappa A_\lambda A_\mu A_\nu\big)\,.
\end{aligned}
$$

The purely $\mathcal{O}(\theta^2)$-ambiguities from homogeneous solutions to the gauge equivalence equation at $\mathcal{O}(\theta^2)$ read:

$$A^2_{\xi,c^2_{A,1}}(A,\theta) = c^2_{A,1}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\kappa F_{\mu\nu})F_{\lambda\xi}\,, \tag{A.14a}$$

$$A^2_{\xi,c^2_{A,2}}(A,\theta) = c^2_{A,2}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\lambda\xi}(D_\kappa F_{\mu\nu})\,, \tag{A.14b}$$

$$A^2_{\xi,c^2_{A,3}}(A,\theta) = c^2_{A,3}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\mu\kappa}(D_\xi F_{\nu\lambda})\,, \tag{A.14c}$$

$$A^2_{\xi,c^2_{A,4}}(A,\theta) = c^2_{A,4}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\xi F_{\nu\lambda})F_{\mu\kappa}\,, \tag{A.14d}$$

$$A^2_{\xi,c^2_{A,5}}(A,\theta) = c^2_{A,5}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}F_{\kappa\lambda}(D_\xi F_{\mu\nu})\,, \tag{A.14e}$$

$$A^2_{\xi,c^2_{A,6}}(A,\theta) = c^2_{A,6}\frac{\mathrm{i}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(D_\xi F_{\mu\nu})F_{\kappa\lambda}\,. \tag{A.14f}$$

A.3 Matter Field

The Seiberg-Witten map for the matter field to $\mathcal{O}(\theta)$ including the ambiguity due to the gauge parameter ambiguity is given by

$$\psi^1(\psi,A,\theta) = \frac{1}{2}\theta^{\mu\nu}\left(A_\mu\partial_\nu\psi + \frac{\mathrm{i}}{2}A_\mu A_\nu\psi\right)\,, \tag{A.15}$$

$$\psi^1_{c^1_\lambda}(\psi,A,\theta) = -\,c^1_\lambda\,\theta^{\mu\nu}A_\mu A_\nu\psi\,, \tag{A.16}$$

$$\psi^1_{c^1_\psi}(\psi,A,\theta) = \frac{c^1_\psi}{2}\,\theta^{\mu\nu}F_{\mu\nu}\psi\,. \tag{A.17}$$

At $\mathcal{O}(\theta^2)$ we obtain the special solution for the matter field, when all ambiguities from $\mathcal{O}(\theta)$ and from the gauge parameter are set to zero:

$$
\begin{aligned}
\psi^2(\psi,A,\theta) = &-\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu A_\kappa \partial_\nu \partial_\lambda \psi \\
&+\frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- 2A_\mu \partial_\kappa A_\nu \partial_\lambda \psi - 2\partial_\mu A_\kappa A_\nu \partial_\lambda \psi \\
&\quad + 2A_\mu A_\kappa \partial_\nu \partial_\lambda \psi + 4A_\mu \partial_\nu A_\kappa \partial_\lambda \psi - \partial_\mu A_\kappa \partial_\nu A_\lambda \psi\big) \\
&+\frac{\mathrm{i}}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(- 2A_\mu \partial_\nu A_\kappa A_\lambda \psi + \partial_\mu A_\kappa A_\nu A_\lambda \psi \\
&\quad - A_\mu A_\kappa A_\lambda \partial_\nu \psi + A_\mu A_\kappa A_\nu \partial_\lambda \psi - A_\mu A_\nu A_\kappa \partial_\lambda \psi\big)
\end{aligned}
\tag{A.18}
$$

$$+\frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(-3A_\mu A_\nu A_\kappa A_\lambda\psi + 4A_\mu A_\kappa A_\nu A_\lambda\psi - 2A_\mu A_\kappa A_\lambda A_\nu\psi\big)\,.$$

Contributions from $\mathcal{O}(\theta)$-ambiguities read:

$$\begin{aligned}\psi^2_{c^1_\lambda}(\psi,A,\theta) =& \frac{c^1_\lambda}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu A_\kappa D_\lambda\psi - \mathrm{i}[A_\mu,\partial_\kappa A_\nu]\partial_\lambda\psi\big)\\ &+\frac{(c^1_\lambda)^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\nu A_\kappa A_\lambda\psi\big)\,,\end{aligned} \tag{A.19}$$

$$\begin{aligned}\psi^2_{c^1_\psi}(\psi,A,\theta) = \frac{c^1_\psi}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(&A_\mu A_\nu A_\kappa A_\lambda\psi + 2\mathrm{i}A_\mu A_\kappa A_\lambda\partial_\nu\psi\\ &+2\mathrm{i}A_\mu A_\kappa\partial_\nu A_\lambda\psi + 2\mathrm{i}A_\mu\partial_\nu A_\kappa A_\lambda\psi - 2A_\mu\partial_\kappa A_\lambda\partial_\nu\psi\\ &+\mathrm{i}A_\mu A_\nu\partial_\kappa A_\lambda\psi + 2A_\lambda\partial_\mu\partial_\kappa A_\nu\psi\big)\,,\end{aligned} \tag{A.20}$$

$$\psi^2_{c^1_\lambda,c^1_\psi}(\psi,A,\theta) = c^1_\lambda\,c^1_\psi\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathrm{i}A_\mu A_\nu A_\kappa A_\lambda\psi - A_\mu A_\nu\partial_\kappa A_\lambda\psi\big)\,. \tag{A.21}$$

Contributions from the $\mathcal{O}(\theta^1)$-ambiguities of the gauge parameter to the Seiberg-Witten map for the matter field at $\mathcal{O}(\theta^2)$ are

$$\begin{aligned}\psi^2_{c^1_{\lambda,1,\ldots,15}}(\psi,A,\theta) = \;& c^2_{\lambda,1}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(+\partial_\mu A_\kappa A_\nu A_\lambda\psi + \mathrm{i}\partial_\mu A_\kappa\partial_\nu A_\lambda\psi\\ &\quad -\mathrm{i}A_\mu A_\kappa A_\nu A_\lambda\psi - A_\mu A_\kappa\partial_\lambda A_\nu\psi - \mathrm{i}\partial_\mu A_\kappa\partial_\lambda A_\nu\psi\big)\\ &+c^2_{\lambda,2}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathrm{i}A_\mu A_\kappa A_\nu A_\lambda\psi + A_\mu A_\kappa\partial_\lambda A_\nu\psi + A_\mu\partial_\nu A_\kappa A_\lambda\psi\\ &\quad -A_\mu\partial_\kappa A_\nu A_\lambda\psi - A_\mu\partial_\kappa A_\lambda A_\nu\psi - \mathrm{i}A_\lambda\partial_\mu\partial_\kappa A_\nu\psi\\ &\quad -\partial_\mu A_\kappa A_\nu A_\lambda\psi - \mathrm{i}\partial_\mu A_\kappa\partial_\nu A_\lambda\psi + \mathrm{i}\partial_\mu A_\kappa\partial_\lambda A_\nu\psi\\ &\quad -\mathrm{i}\partial_\mu\partial_\kappa A_\nu A_\lambda\psi\big)\\ &+c^2_{\lambda,3}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(-A_\mu\partial_\nu A_\kappa A_\lambda\psi + A_\mu\partial_\kappa A_\nu A_\lambda\psi\big)\\ &+c^2_{\lambda,4}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathrm{i}A_\mu A_\kappa A_\nu A_\lambda\psi + A_\mu A_\kappa\partial_\lambda A_\nu\psi - \partial_\mu A_\kappa A_\nu A_\lambda\psi\big)\\ &+c^2_{\lambda,5}\,\theta^{\mu\nu}\theta^{\kappa\lambda}A_\mu\partial_\kappa A_\lambda A_\nu\psi\\ &+c^2_{\lambda,6}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(-\mathrm{i}A_\mu A_\kappa A_\nu A_\lambda\psi - A_\mu A_\kappa\partial_\lambda A_\nu\psi - A_\mu\partial_\nu A_\kappa A_\lambda\psi\\ &\quad +A_\mu\partial_\kappa A_\nu A_\lambda\psi + A_\mu\partial_\kappa A_\lambda A_\nu\psi + \mathrm{i}A_\lambda\partial_\mu\partial_\kappa A_\nu\psi\\ &\quad +\mathrm{i}\partial_\mu A_\nu\partial_\kappa A_\lambda\psi + \partial_\mu A_\kappa A_\nu A_\lambda\psi + \mathrm{i}\partial_\mu A_\kappa\partial_\nu A_\lambda\psi\\ &\quad -\mathrm{i}\partial_\mu A_\kappa\partial_\lambda A_\nu\psi + \mathrm{i}\partial_\mu\partial_\kappa A_\nu A_\lambda\psi\big)\\ &+\mathrm{i}\,c^2_{\lambda,7}\,\theta^{\mu\nu}\theta^{\kappa\lambda}A_\mu A_\nu A_\kappa A_\lambda\psi - \mathrm{i}\,c^2_{\lambda,8}\,\theta^{\mu\nu}\theta^{\kappa\lambda}A_\mu A_\kappa A_\lambda A_\nu\psi\\ &+\mathrm{i}\,c^2_{\lambda,9}\,\theta^{\mu\nu}\theta^{\kappa\lambda}A_\mu A_\kappa A_\nu A_\lambda\psi + \mathrm{i}\,c^2_{\lambda,10}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu A_\kappa\partial_\nu A_\lambda\psi\\ &+c^2_{\lambda,11}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\kappa A_\nu A_\lambda\psi - A_\mu A_\kappa A_\lambda A_\nu\psi + 2\mathrm{i}A_\mu A_\kappa\partial_\nu A_\lambda\psi\\ &\quad -\mathrm{i}A_\mu A_\kappa\partial_\lambda A_\nu\psi + \mathrm{i}\partial_\mu A_\kappa A_\nu A_\lambda\psi - \partial_\mu A_\kappa\partial_\nu A_\lambda\psi\\ &\quad +\partial_\mu A_\kappa\partial_\lambda A_\nu\psi\big)\\ &+\mathrm{i}\,c^2_{\lambda,12}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(-A_\mu\partial_\nu A_\kappa A_\lambda\psi - A_\mu\partial_\kappa A_\nu A_\lambda\psi\big)\\ &+\mathrm{i}\,c^2_{\lambda,13}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\kappa\partial_\lambda A_\nu\psi + \partial_\mu A_\kappa A_\nu A_\lambda\psi\big)\\ &+c^2_{\lambda,14}\,\theta^{\mu\nu}\theta^{\kappa\lambda}\big(A_\mu A_\kappa A_\nu A_\lambda\psi - A_\mu A_\kappa A_\lambda A_\nu\psi + 2\mathrm{i}A_\mu A_\kappa\partial_\nu A_\lambda\psi\end{aligned} \tag{A.22}$$

$$
\begin{aligned}
&- \mathrm{i}A_\mu A_\kappa \partial_\lambda A_\nu \psi + \mathrm{i}A_\mu \partial_\nu A_\kappa A_\lambda \psi + \mathrm{i}A_\mu \partial_\kappa A_\nu A_\lambda \psi \\
&+ A_\lambda \partial_\mu \partial_\kappa A_\nu \psi + \mathrm{i}\partial_\mu A_\kappa A_\nu A_\lambda \psi - \partial_\mu A_\kappa \partial_\nu A_\lambda \psi \\
&+ \partial_\mu A_\kappa \partial_\lambda A_\nu \psi - \partial_\mu \partial_\kappa A_\nu A_\lambda \psi\big) \\
+ c^2_{\lambda,15}\, \theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- A_\mu A_\nu A_\kappa A_\lambda \psi - 2\mathrm{i}A_\mu A_\nu \partial_\kappa A_\lambda \psi + \partial_\mu A_\nu \partial_\kappa A_\lambda \psi\big)\,.
\end{aligned}
$$

The purely $\mathcal{O}(\theta^2)$-ambiguities can be written compactly as

$$
\psi^2_{c^2_{\psi,1}}(\psi, A, \theta) = \mathrm{i}c^2_{\psi,1}\, \theta^{\mu\nu}\theta^{\kappa\lambda}(D_\mu F_{\nu\kappa}) D_\lambda \psi\,, \tag{A.23a}
$$

$$
\psi^2_{c^2_{\psi,2}}(\psi, A, \theta) = -\frac{c^2_{\psi,2}}{4}\theta^{\mu\nu}\theta^{\kappa\lambda} F_{\mu\nu} F_{\kappa\lambda}\psi\,, \tag{A.23b}
$$

$$
\psi^2_{c^2_{\psi,3}}(\psi, A, \theta) = \frac{c^2_{\psi,3}}{2}\theta^{\mu\nu}\theta^{\kappa\lambda} F_{\mu\kappa} F_{\nu\lambda}\psi\,. \tag{A.23c}
$$

Our original solution for the $\mathcal{O}(\theta^2)$ ambiguity parametrized by $c^2_{\psi,1}$ is

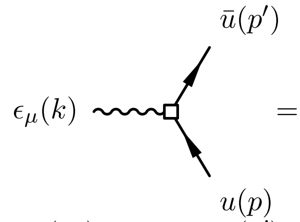
$$
\begin{aligned}
\psi^2_{c^2_{\psi,1}}(\psi, A, \theta) = c^2_{\psi,1}\, \theta^{\mu\nu}\theta^{\kappa\lambda}\big(&- \mathrm{i}A_\mu A_\nu \partial_\kappa A_\lambda \psi - A_\mu A_\kappa A_\nu A_\lambda \psi \\
&- \mathrm{i}A_\mu A_\kappa A_\lambda \partial_\nu \psi - \mathrm{i}A_\mu A_\kappa \partial_\nu A_\lambda \psi + \mathrm{i}A_\mu A_\kappa \partial_\lambda A_\nu \psi \\
&+ A_\mu \partial_\kappa A_\nu \partial_\lambda \psi - \mathrm{i}A_\mu \partial_\kappa A_\nu A_\lambda \psi + A_\mu \partial_\kappa A_\lambda \partial_\nu \psi \\
&- \mathrm{i}A_\mu \partial_\kappa A_\lambda A_\nu \psi - \partial_\mu A_\nu A_\kappa \partial_\lambda \psi + \partial_\mu A_\nu \partial_\kappa A_\lambda \psi \\
&- \mathrm{i}\partial_\mu A_\kappa A_\nu A_\lambda \psi + \partial_\mu A_\kappa A_\lambda \partial_\nu \psi + \partial_\mu A_\kappa \partial_\nu A_\lambda \psi \\
&- \partial_\mu A_\kappa \partial_\lambda A_\nu \psi + \mathrm{i}\partial_\mu \partial_\kappa A_\nu \partial_\lambda \psi + \partial_\mu \partial_\kappa A_\nu A_\lambda \psi \\
&+ \mathrm{i}A_\mu A_\nu A_\kappa \partial_\lambda \psi\big)\,.
\end{aligned} \tag{A.24}
$$

The difference between this solution and (A.23a) consists in homogeneous solutions to the gauge equivalence equation at $\mathcal{O}(\theta^2)$ (4.4b).

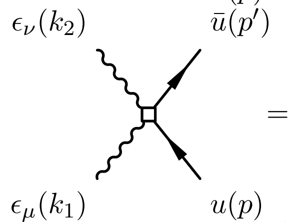
Appendix B

Feynman Rules to $\mathcal{O}(\theta^2)$

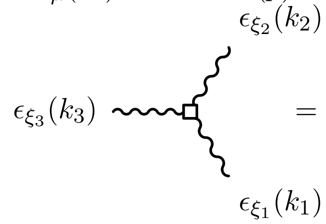
We give the Feynman rules for neutral currents and triple gauge boson interactions to $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta^2)$. Since the chiral structure of the fermionic currents remains unaffected by the SWMs, we have written the following vertex factors involving fermions as pure vector currents. The necessary substitutions $\gamma_\mu \to g_V\gamma_\mu - g_A\gamma_\mu\gamma_5$ depending on the fermion flavor and the vector boson contracted with to γ_μ can be copied directly from the SM Lagrangian. All momenta are incoming.



$$= \mathrm{i}g \cdot V_\mu(p', k, p)\,, \tag{B.1a}$$



$$= \mathrm{i}g^2 \cdot V^c_{\mu_2\mu_1}(p', k_2, k_1, p)\,, \tag{B.1b}$$



$$= \mathrm{i}g_{[\rho]} \cdot V^3_{\xi_1\xi_2\xi_3}(k_1, k_2, k_3)\,, \tag{B.1c}$$

where $g_{[\rho]}$ indicates the representation-dependence of the TGB coupling. The vertex functions $V^{(i)}_{\mu\nu\ldots}(p, k, \ldots)$ read:

$$V^{(1)}_\mu(p', k, p) = \frac{\mathrm{i}}{2}\Big[k\theta^\mu \not{p}(1 - 4c^1_\psi) + 2\,k\theta^\mu \not{k}(c^1_A - c^1_\psi) - p\theta^\mu \not{k} - (k\theta p)\gamma_\mu\Big]\,, \tag{B.2a}$$

$$V_{\mu}^{(2)}(p',k,p) = \frac{1}{8}(k\theta p)\Big[k\theta^{\mu}\not{p}(1-16c_{\psi}^{2}) + 4k\theta^{\mu}\not{k}(c_{A}^{1}-2c_{\psi}^{2}) - p\theta^{\mu}\not{k} - (k\theta p)\gamma_{\mu}\Big]\,, \tag{B.2b}$$

$$V_{\nu\mu}^{c,(1)}(p',k_2,k_1,p) = \frac{\mathrm{i}}{2}\Big[k_2\theta^{\mu}\gamma^{\nu} - k_1\theta^{\mu}\gamma^{\nu}(1-4c_{\psi}^{1}) - \theta^{\mu\nu}\not{k}_1 + (\mu\leftrightarrow\nu, k_1\leftrightarrow k_2)\Big]\,. \tag{B.2c}$$

We split the expression for the contact vertex at $\mathcal{O}(\theta^2)$ into the *on-shell* and the *off-shell* contributions, since for the processes calculated within this work only the first part is needed:

$$V_{\nu\mu}^{c,(2)}(p',k_2,k_1,p) \;=\; V_{\nu\mu}^{c,(2),\textit{on-shell}}(p',k_2,k_1,p) \,+\, V_{\nu\mu}^{c,(2),\textit{off-shell}}(p',k_2,k_1,p)\,. \tag{B.2d}$$

with

$$\begin{aligned} V_{\nu\mu}^{c,(2),\textit{on-shell}}(p',k_2,k_1,p) = \frac{1}{8}\Big[&k_1\theta k_2\,k_1\theta^{\mu}\gamma^{\nu}\,(8c_A^2 - 4c_\psi^1 + 8c_\psi^2 - 1)\\ &+ k_1\theta p\,k_1\theta^{\mu}\gamma^{\nu}\,(16c_\psi^2 - 1) + 2\,k_2\theta p\,k_1\theta^{\mu}\gamma^{\nu}\,(4c_\psi^1 - 1)\\ &- k_1\theta k_2\,k_2\theta^{\mu}\gamma^{\nu} + 3\,k_1\theta p\,k_2\theta^{\mu}\gamma^{\nu} + 2\,k_2\theta p\,k_2\theta^{\mu}\gamma^{\nu} - 3\,k_1\theta k_2\,p\theta^{\mu}\gamma^{\nu}\\ &+ 4\,k_1\theta^{\mu}k_1\theta^{\nu}\not{k}_1(2c_A^2 - c_A^1 - c_\psi^1) + 2\,k_1\theta^{\mu}p\theta^{\nu}\not{k}_1(1-4c_\psi^1) + 2\,k_2\theta^{\mu}p\theta^{\nu}\not{k}_1\\ &- 4\,\theta^{\mu\nu}k_1\theta p\not{k}_1 + (\mu\leftrightarrow\nu, k_1\leftrightarrow k_2)\Big] \end{aligned} \tag{B.2e}$$

and

$$\begin{aligned} V_{\nu\mu}^{c,(2),\textit{off-shell}}(p',k_2,k_1,p) = \frac{1}{8}\Big[&k_2\theta^{\mu}p\theta^{\nu}(\not{k}_1+\not{k}_2) + 3\,k_1\theta p\,\theta^{\mu\nu}(\not{k}_1+\not{k}_2)\\ &+ (\mu\leftrightarrow\nu, k_1\leftrightarrow k_2)\Big]\\ + k_1\theta^{\mu}k_2\theta^{\nu}\Big[C_1(\not{k}_1+\not{k}_2) + C_2\not{p}\Big] &+ k_1\theta^{\nu}k_2\theta^{\mu}\Big[C_3(\not{k}_1+\not{k}_2) + C_4\not{p}\Big]\\ &+ k_1\theta k_2\,\theta^{\mu\nu}\Big[(C_3 - \frac{1}{8})(\not{k}_2+\not{k}_1) + C_4\not{p}\Big]\,, \end{aligned} \tag{B.2f}$$

where the C_i are combinations of the ambiguity parameters:

$$\begin{aligned} C_1 &= \frac{c_A^1}{2} - \frac{c_\psi^1}{2} - 2c_A^1c_\psi^1 + (c_\psi^1)^2 + 2(c_{\lambda,15}^2 - c_{\lambda,6}^2)\\ &\quad +2(c_{\psi,2}^2 - c_{\psi,2}^2)\,,\\ C_2 &= \frac{1}{4} + 2((c_\psi^1)^2 - c_\psi^1) + 4(c_{\lambda,15}^2 + c_{\psi,1}^2 - c_{\psi,2}^2)\,,\\ C_3 &= 2(c_{\lambda,2}^2 - c_{\lambda,1}^2 - c_{\lambda,6}^2 - c_{\lambda,10}^2\\ &\quad - c_{\lambda,11}^2 - c_{\lambda,14}^2 + c_{\psi,1}^2 + c_{\psi,3}^2) + c_{A,3}^2 + c_{A,4}^2\,,\\ C_4 &= -\frac{1}{2} - 4(c_{\lambda,11}^2 + c_{\lambda,14}^2 - c_{\psi,1}^2 - c_{\psi,3}^2)\,. \end{aligned}$$

For the TGB interaction we have:

$$V_{\xi_1\xi_2\xi_3}^{3,(1)}(k_1,k_2,k_3) =$$

$$
\begin{aligned}
&\theta_{\xi_1\xi_2}\left[(k_1k_3)k_{2,\xi_3}-(k_2k_3)k_{1,\xi_3}\right]+(k_1\theta k_2)\left[k_{3,\xi_1}g_{\xi_2\xi_3}-g_{\xi_1\xi_3}k_{3,\xi_2}\right]\\
&\quad+\Big[(k_1\theta)_{\xi_1}\left[k_{2,\xi_3}k_{3,\xi_2}-(k_2k_3)g_{\xi_2\xi_3}\right]-(\xi_1\leftrightarrow\xi_2)-(\xi_1\leftrightarrow\xi_3)\Big]\\
&\qquad+\text{cyclical permutations of }\Big\{(\xi_1,k_1),(\xi_2,k_2),(\xi_3,k_3)\Big\}\,, \qquad \text{(B.2g)}
\end{aligned}
$$

$$
\begin{aligned}
V^{3,(2)}_{\xi_1\xi_2\xi_3}(k_1,k_2,k_3)={}&\mathrm{i}\Big[k_1\theta k_2k_1\theta^{\xi_1}\Big((c_A^2-c_A^1)(k_{1,\xi_3}k_{3,\xi_2}-g_{\xi_2\xi_3}(k_1k_3))\\
&+c_A^2(k_{2,\xi_3}k_{3,\xi_2}-g_{\xi_2\xi_3}(k_2k_3))\Big)\\
&+k_1\theta^{\xi_1}k_1\theta^{\xi_2}(c_A^2-c_A^1)(k_{2,\xi_3}(k_1k_3)-k_{1,\xi_3}(k_2k_3))+(\xi_2,k_2)\leftrightarrow(\xi_3,k_3)\Big]\\
&+\text{cyclical permutations of }\Big\{(\xi_1,k_1),(\xi_2,k_2),(\xi_3,k_3)\Big\}\,. \qquad \text{(B.2h)}
\end{aligned}
$$

Bibliography

[1] HARTLAND S. SNYDER: *Quantized Space-Time*. Phys. Rev., 71(1):38–41, 1947.

[2] HARTLAND S. SNYDER: *The Electromagnetic Field in Quantized Space-Time*. Phys. Rev., 72(1):68–71, 1947.

[3] *Letter of Heisenberg to Pauli (1930)*. Wolfgang Pauli, Scientific Correspondence, II(15), 1985.

[4] *Letter of Pauli to Oppenheimer (1946)*. Wolfgang Pauli, Scientific Correspondence, III(380), 1993.

[5] N. SEIBERG und E. WITTEN: *String Theory and Noncommutative Geometry*. JHEP, 09:032, 1999. hep-th/9908142.

[6] SERGIO DOPLICHER, KLAUS FREDENHAGEN und JOHN E. ROBERTS: *The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields*. Commun. Math. Phys., 172:187–220, 1995. hep-th/0303037.

[7] ANA ALBOTEANU, THORSTEN OHL und REINHOLD RÜCKL: *Probing the Noncommutative Standard Model at Hadron Colliders*. Phys. Rev., D74:096004, 2006.

ANA ALBOTEANU, THORSTEN OHL und REINHOLD RÜCKL: *Collider Tests of the Non-Commutative Standard Model*. PoS, HEP2005:322, 2006.

[8] CHRISTIAN SPECKNER: *Die Produktion geladener Eichbosonen im nichtkommutativen Standardmodell*. Diploma Thesis, 2006.

[9] ANA ALBOTEANU, THORSTEN OHL und REINHOLD RÜCKL: *The Noncommutative Standard Model at $\mathcal{O}(\theta^2)$*. arXiv:0707.3595 [hep-ph].

[10] H. WEYL: *Quantenmechanik und Gruppentheorie*. Z. Physik, 46:1, 1927.

[11] M. KONTSEVICH: *Deformtion Quantization of Poisson Manifolds, I*. 1997. q-alg/9709040.

[12] M. HAYAKAWA: *Perturbative Analysis on Infrared Aspects of Noncommutative QED on R**4*. Phys. Lett., B478:394–400, 2000. hep-th/9912094.

[13] M. HAYAKAWA: *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4*. 1999. hep-th/9912167.

[14] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari und A. Tureanu: *Noncommutative Standard Model: Model Building.* 2001. hep-th/0107055.

[15] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari und A. Tureanu: *Noncommutative Gauge Field Theories: A No-Go Theorem.* Phys. Lett., B526:132–136, 2002. hep-th/0107037.

[16] X. Calmet, B. Jurco, P. Schupp, J. Wess und M. Wohlgenannt: *The Standard Model on Non-Commutative Space-Time.* Eur. Phys. J., C23:363–376, 2002. hep-ph/0111115.

[17] T. Filk: *Divergencies in a Field Theory on Quantum Space.* Phys. Lett., B376:53–58, 1996.

[18] Shiraz Minwalla, Mark Van Raamsdonk und Nathan Seiberg: *Noncommutative Perturbative Dynamics.* JHEP, 02:020, 2000. hep-th/9912072.

[19] Harald Grosse und Raimar Wulkenhaar: *Renormalisation of ϕ^4 Theory on Noncommutative R^4 in the Matrix Base.* Commun. Math. Phys., 256:305–374, 2005. hep-th/0401128.

[20] Vincent Rivasseau: *Non-Commutative Renormalization.* 2007. arXiv:0705.0705 [hep-th].

Vincent Rivasseau und Fabien Vignes-Tourneret: *Renormalisation of Non-Commutative Field Theories.* 2007. hep-th/0702068.

[21] Axel de Goursac, Jean-Christophe Wallet und Raimar Wulkenhaar: *Noncommutative Induced Gauge Theory.* 2007. hep-th/0703075.

[22] Harald Grosse und Michael Wohlgenannt: *Noncommutative QFT and Renormalization.* J. Phys. Conf. Ser., 53:764–792, 2006. hep-th/0607208.

[23] Daniel N. Blaschke, Harald Grosse und Manfred Schweda: *Noncommutative U(1) Gauge Theory on R^4 with Oscillator Term.* 2007. arXiv:0705.4205 [hep-th].

[24] Raimar Wulkenhaar: *Private communication.*

[25] Andeas Bichl und others: *Renormalization of the noncommutative photon self-energy to all orders via Seiberg-Witten map.* JHEP, 06:013, 2001. hep-th/0104097.

[26] Dusko Latas, Voja Radovanovic und Josip Trampetic: *Noncommutative SU(N) gauge theories and asymptotic freedom.* 2007. hep-th/0703018.

[27] Maja Buric, Voja Radovanovic und Josip Trampetic: *The one-loop renormalization of the gauge sector in the noncommutative standard model.* 2006. hep-th/0609073.

[28] Maja Buric, Dusko Latas und Voja Radovanovic: *Renormalizability of noncommutative SU(N) gauge theory.* JHEP, 02:046, 2006. hep-th/0510133.

[29] Raimar Wulkenhaar: *Non-renormalizability of θ-expanded noncommutative QED.* JHEP, 03:024, 2002. hep-th/0112248.

[30] Jesper M. Grimstrup und Raimar Wulkenhaar: *Quantisation of θ-expanded non-commutative QED.* Eur. Phys. J., C26:139–151, 2002. hep-th/0205153.

[31] Jaume Gomis und Thomas Mehen: *Space-time noncommutative field theories and unitarity.* Nucl. Phys., B591:265–276, 2000.

[32] Y. Liao und K. Sibold: *Time-ordered perturbation theory on noncommutative spacetime: basic rules.* Eur. Phys. J., C25:469–477, 2002. hep-th/0205269.

Yi Liao und Klaus Sibold: *Time-ordered perturbation theory on noncommutative spacetime. II. Unitarity.* Eur. Phys. J., C25:479–486, 2002.

[33] Th. Ohl, R. Rückl und J. Zeiner: *Unitarity of time-like noncommutative gauge theories: The violation of Ward identities in time-ordered perturbation theory.* Nucl. Phys., B676:229–242, 2004 hep-th/0309021

[34] Jörg Zeiner: *Noncommutative Quantumelectrodynamics from Seiberg-Witten Maps to all orders in $\theta^{\mu\nu}$*, 2007. Ph.D. Thesis, 2007

[35] JoAnne L. Hewett, Frank J. Petriello und Thomas G. Rizzo: *Signals for Non-Commutative Interactions at Linear Colliders.* Phys. Rev., D64:075012, 2001. hep-ph/0010354.

[36] Carl E. Carlson, Christopher D. Carone und Nahum Zobin: *Noncommutative Gauge Theory Without Lorentz Violation.* Phys. Rev., D66:075001, 2002. hep-th/0206035.

[37] M. Chaichian, P. P. Kulish, K. Nishijima und A. Tureanu: *On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT.* Phys. Lett., B604:98–102, 2004. hep-th/0408069.

[38] M. M. Sheikh-Jabbari: *Discrete Symmetries (C,P,T) in Noncommutative Field Theories.* Phys. Rev. Lett., 84:5265–5268, 2000. hep-th/0001167.

[39] Paolo Aschieri, Branislav Jurco, Peter Schupp und Julius Wess: *Non-commutative GUTs, Standard Model and C, P, T.* Nucl. Phys., B651:45–70, 2003. hep-th/0205214.

[40] I. Hinchliffe und N. Kersting: *Review of the Phenomenology of Noncommutative Geometry.* 2002. hep-ph/0205040.

[41] G. Abbiendi et. al: *Test of Non-Commutative QED in the Process $e^+e^- \to \gamma\gamma$ at LEP.* Phys. Lett., B568:181–190, 2003. hep-ex/0303035.

[42] Harald Grosse und Yi Liao: *Anomalous C-violating Three Photon Decay of the Neutral Pion in Noncommutative Quantum Electrodynamics.* Phys. Lett., B520:63–68, 2001. hep-ph/0104260.

[43] Peter Schupp, Josip Trampetic, Julius Wess und Georg Raffelt: *The photon neutrino interaction in non-commutative gauge field theory and astrophysical bounds.* Eur. Phys. J., C36:405–410, 2004. hep-ph/0212292.

[44] M. Buric, D. Latas, V. Radovanovic und J. Trampetic: *Nonzero $Z \to \gamma\gamma$ decays in the renormalizable gauge sector of the noncommutative standard model.* Phys. Rev., D75:097701, 2007.

[45] Ihab. F. Riad und M. M. Sheikh-Jabbari: *Noncommutative QED and anomalous dipole moments.* JHEP, 08:045, 2000. hep-th/0008132.

[46] N. Kersting: *$(g-2)_\mu$ from Noncommutative Geometry.* Phys. Lett., B527:115–118, 2002. hep-ph/0109224.

[47] I. Hinchliffe und N. Kersting: *CP violation from noncommutative geometry.* Phys. Rev., D64:116007, 2001. hep-ph/0104137.

[48] E. O. Iltan: *The inclusive $b \to s\gamma$ decay in the noncommutative standard model.* New J. Phys., 4:54, 2002. hep-ph/0204129.

[49] M. Chaichian, M. M. Sheikh-Jabbari und A. Tureanu: *Hydrogen atom spectrum and the Lamb shift in noncommutative QED.* Phys. Rev. Lett., 86:2716, 2001. hep-th/0010175.

M. Chaichian, M. M. Sheikh-Jabbari und A. Tureanu: *Comments on the hydrogen atom spectrum in the noncommutative space.* Eur. Phys. J., C36:251–252, 2004. hep-th/0212259.

[50] Irina Mocioiu, Maxim Pospelov und Radu Roiban: *Limits on the non-commutativity scale.* 2001. hep-ph/0110011.

[51] Ana Alboteanu: *Phänomenologische Untersuchungen nichtkommutativer Quantenfeldtheorien an Hadron-Collidern.* Diploma Thesis, 2003

[52] Thorsten Ohl und Jurgen Reuter: *Testing the noncommutative standard model at a future photon collider.* Phys. Rev., D70:076007, 2004. hep-ph/0406098.

[53] Blazenka Melic, Kornelija Passek-Kumericki, Josip Trampetic, Peter Schupp und Michael Wohlgenannt: *The standard model on non-commutative space-time: Electroweak currents and Higgs sector.* Eur. Phys. J., C42:483–497, 2005. hep-ph/0502249.

[54] Blazenka Melic, Kornelija Passek-Kumericki, Josip Trampetic, Peter Schupp und Michael Wohlgenannt: *The standard model on non-commutative space-time: Strong interactions included.* Eur. Phys. J., C42:499–504, 2005. hep-ph/0503064.

[55] W. Behr, N.G. Deshpande, G. Duplančić, P. Schupp, J. Trampetić und J. Wess: *The $Z \to \gamma\gamma, gg$ Decays in the Noncommutative Standard Modell*. 2002. hep-ph/0202121.

[56] G. Duplancic, P. Schupp und J. Trampetic: *Comment on triple gauge boson interactions in the non- commutative electroweak sector*. 2003. hep-ph/0309138.

[57] H. L. Lai und others: *Improved parton distributions from global analysis of recent deep inelastic scattering and inclusive jet data*. Phys. Rev., D55:1280–1296, 1997. hep-ph/9606399.

[58] T. Ohl: *O'Mega: An Optimizing Matrix Element Generator*. Proceedings of 7th International Workshop on Advanced Computing and Analysis Techniques in Physics Research (ACAT 2000).

[59] J. Reuter, M. Moretti, T. Ohl: *O'Mega: An optimizing matrix element generator*. hep-ph/0102195.

[60] C. Schwinn, T. Ohl, J. Reuter: *O'Mega*. long write-up, in preparation.

[61] W. Kilian: *WHIZARD 1.0: A generic Monte-Carlo integration and event generation package for multi-particle processes. Manual.* `http://www-ttp.physik.uni-karlsruhe.de/whizard/`.

[62] W. Kilian: *WHIZARD: Complete simulations for electroweak multi-particle processes*. Proceedings of ICHEP, Amsterdam 2002.

[63] D. S. Sivia: *Data Analysis*. Oxford University Press, Oxford, 1996.

[64] F. Abe *et al.* [CDF Collaboration]: *Limits on Z - Photon Couplings from $p\bar{p}$ Interactions at $\sqrt{s}$ = 1.8 TeV*. Phys. Rev. Lett., 74:1941, 1995.

D. Acosta *et al.* [CDF II Collaboration]: *Measurement of $W\gamma$ and $Z\gamma$ Production in $p\bar{p}$ Collisions at $\sqrt{s}$ = 1.96 TeV*. Phys. Rev. Lett., 94:041803, 2005. hep-ex/0410008.

[65] S. Abachi *et al.* [D0 Collaboration]: *Measurement of the $ZZ\gamma$ and $Z\gamma\gamma$ Couplings in $p\bar{p}$ Collisions at $\sqrt{s}$ = 1.8 TeV*. Phys. Rev. Lett., 75:1028, 1995.

S. Abachi *et al.* [D0 Collaboration]: *Study of the $ZZ\gamma$ and $Z\gamma\gamma$ Couplings in $Z(\nu\nu)\gamma$ Production*. Phys. Rev. Lett., 78:3640, 1997. hep-ex/9702011].

B. Abbott *et al.* [D0 Collaboration]: *$Z\gamma$ production in $p\bar{p}$ Collisions at $\sqrt{s}$ = 1.8 TeV and Limits on Anomalous $ZZ\gamma$ and $Z\gamma\gamma$ Couplings*. Phys. Rev., D57:3817, 1998. hep-ex/9710031.

V. M. Abazov *et al.* [D0 Collaboration]: *Study of $Z\gamma$ Events and Limits on Anomalous $ZZ\gamma$ and $Z\gamma\gamma$ Couplings in $p\bar{p}$ Collisions at $\sqrt{s}$ = 1.96 TeV*. Phys. Rev. Lett., 95:051802, 2005.

[66] F. James und M. Roos: *Minuit: A System for Function Minimization and Analysis of the Parameter Errors and Correlations.* Comput. Phys. Commun., 10:343–367, 1975.

[67] Thorsten Ohl: *CIRCE version 1.0: Beam spectra for simulating linear collider physics.* Comput. Phys. Commun., 101:269–288, 1997. hep-ph/9607454.

[68] M. Buric, D. Latas, V. Radovanovic und J. Trampetic: *Improved $Z \to \gamma\gamma$ decay in the renormalizable gauge sector of the noncommutative standard model.* 2006. hep-ph/0611299.

Josip Trampetic: *Signal for space-time noncommutativity: The $Z \to \gamma\gamma$ decay in the renormalizable gauge sector of the θ-expanded NCSM.* 2007. arXiv:0704.0559 [hep-ph].

[69] J. A. M. Vermaseren: *Symbolic Manipulation with* `FORM` *version 2: Tutorial and Reference Manual.* CAN (Amsterdam), 1992. ISBN 90-74116-01-9.

[70] Lutz Moller: *Second order of the expansions of action functionals of the noncommutative standard model.* JHEP, 10:063, 2004. hep-th/0409085.

[71] B. Jurco, L. Moller, S. Schraml, P. Schupp und J. Wess: *Construction of non-Abelian gauge theories on noncommutative spaces.* Eur. Phys. J., C21:383–388, 2001. hep-th/0104153.

[72] B. L. Cerchiai, A. F. Pasqua und B. Zumino: *The Seiberg-Witten map for noncommutative gauge theories.* 2002. hep-th/0206231.

[73] R. Haag: *Quantum field theories with composite particles and asymptotic conditions.* Phys. Rev., 112:669, 1958.

[74] S. R. Coleman, J. Wess und B. Zumino: *Structure of phenomenological Lagrangians. 1.* Phys. Rev., 177:2239, 1969.

C. G. Callan, S. R. Coleman, J. Wess und B. Zumino: *Structure of phenomenological Lagrangians. 2.* Phys. Rev., 177:2247, 1969.

[75] Steven Weinberg: *Phenomenological Lagrangians.* Physica A, 96:327, 1979.

Danksagungen

An dieser Stelle möchte ich mich bei all jenen bedanken, die zum Gelingen dieser Arbeit beigetragen haben. Insbesondere gilt mein Dank:

- PD Dr. Thorsten Ohl, auf dessen fachliche Unterstützung ich immer zurückgreifen konnte, für die vielen Diskussionen, in denen wir der Nichtkommutativität der Raumzeit zumindest im Verständnis einen Schritt näher gekommen sind und der mancherlei Computer- bzw. Software-Tücke beigekommen ist,
- Prof. Dr. Reinhold Rückl, der mir die Auseinandersetzung mit diesem spannenden Promotionsthema ermöglicht hat und der mich durch Gespräche sowie durch die Möglichkeit, an Workshops, Konferenzen und Schulen teilzunehmen, die eine fachliche Verbreiterung und neue Denkanstöße mit sich brachten, ermutigt und unterstützt hat,
- für die Unterstützung während meiner Promotion
  - dem Evangelischen Studienwerk e. V. Villigst,
  - der Deutschen Forschungsgemeinschaft, Förderkennzeichen RU 311/1-1,
  - und dem Graduiertenkolleg 1147 *Theoretische Astrophysik und Teilchenphysik* ,
- Brigitte Wehner, für ihre Hilfe in den ganzen Verwaltungsangelegenheiten und für ihre Unterstützung weit darüber hinaus,
- Alexander Wagner und Prof. Dr. Hans Fraas für ihre Hilfe in manch einer Lebenslage,
- allen Mitarbeitern des Lehrstuhls und der gesamten Fakultät für die gute Arbeitsatmosphäre und ihre Hilfsbereitschaft,
- meinen Eltern, die mich immer unterstützt haben.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Würzburg, den 27. Juli 2007

Ana Alboteanu