One-Loop Calculations in a Network of Non-Linear Sigma-Models

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Zusammenfassung

Diese Arbeit untersucht ein bestimmtes Modell im Rahmen des Little Higgs Mechanismus. Dieser Mechanismus bietet eine realistische Lösung des Hierarchieproblems im Standardmodell, alternativ zur Supersymmetrie. Ausgehend von einem 5D Lagrangian mit einer diskretisierten Dimension wird ein effektives 4D Modell mit zwei nicht-linear realisierten Linkfeldern mit der Symmetriegruppe $\mathrm{SU}(n) \times \mathrm{SU}(n)$ betrachtet. Die Invarianz des Lagrangian unter BRS Transformation wird explizit gezeigt und dessen Feynmanregeln werden abgeleitet. Die quadratisch und logarithmisch divergenten Anteile sämtlicher Selbstenergien werden in R_ξ-Eichung in dimensionaler Regularisierung berechnet. Zur Überprüfung der Ergebnisse wird die Gültigkeit von Slavnov-Taylor Identitäten gezeigt. Mittels der Hintergrund-Feld Methode wird die β-Funktion der Eichkopplungskonstante des Modells berechnet.

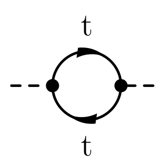
Contents

1 Introduction

The Standard Model (SM) has shown a remarkable success in describing physics at length scales ranging from atomic scales all the way down to the shortest currently probed scales of about 10^{-18}m. Experimental data is in very good agreement at least up to an energy scale of a few hundred GeV. Therefore, it may appear strange that so much work is devoted to discovering physics beyond the SM. However, there are questions, which are not answered by the SM, e.g. can its 19 free parameters (couplings, masses and mixing angles) be reduced and why is the gauge group of the SM the direct product of SU(3) $\times$ SU(2) $\times$ U(1)? This leads to GUT's which unify the weak, electro-magnetic and strong force at energies of 10^{15} GeV. Gravity is also not included which introduces another energy scale, the Planck scale at 10^{19} GeV.

Due to the great success of the SM, one might be tempted to postulate a minimal scenario: the LHC will discover the last missing particle in the SM, the Higgs boson with a mass somewhere between the current lower bound of 114 GeV and the upper bound of approximately 500 GeV, and there will be no additional new physics discovered at the LHC. This implies a very delicate and unnatural fine tuning of parameters, as we will see below. The LHC will probe the SM in the energy scale of 1-10 TeV. Let us assume that the SM is valid up to a cut-off scale of $\Lambda = 10$ TeV. At higher energies new physics may take over, which implies that we don't know how to calculate loop diagrams with momenta larger than Λ. Thus we cut off such loops at this scale. The hierarchy problem arises from the fact that quadratically divergent loop contributions drive the Higgs mass up to unacceptably large values unless the tree level mass parameter is finely tuned to cancel the large quantum corrections. The most significant of these divergences come from three sources. They are one-loop diagrams involving the top quark, the SU(2) $\times$ U(1) gauge bosons $W^\pm, Z, \gamma$ and the Higgs itself.

$\Delta m_H^2 \quad = $



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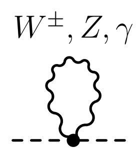

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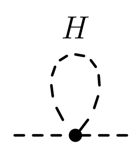


(1.1)

All other quadratically divergent diagrams involve small coupling constants and do not contribute significantly at 10 TeV. The contributions are

- $\frac{3}{8\pi^2}\lambda_t\Lambda^2 \sim -(2\text{TeV})^2$ from the top loop,

- $\frac{1}{16\pi^2}g^2\Lambda^2 \sim (700\text{GeV})^2$ from the gauge loops, and
- $\frac{1}{16\pi^2}\lambda^2\Lambda^2 \sim (500\text{GeV})^2$ from the Higgs loop,

where $\lambda_t \propto m_t/v$ is the top quark Yukawa coupling, g the coupling constants of the group SU(2) $\times$ U(1) and λ the unknown self-coupling of the Higgs boson, which is assumed to be of $\mathcal{O}(1)$. In order to add up to a Higgs mass between 100 and 500 GeV as required in the SM, a fine tuning on the level of one percent is necessary (see figure 1.1). This is called the hierarchy problem. It arises already at a cut-off scale at 10 TeV which can be probed in the near future (If we assume the GUT scale as cut-off, the fine tuning has to be done to about 13 orders of magnitude). If we set

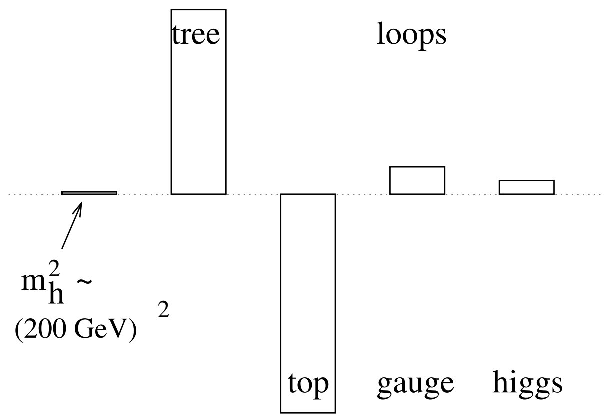


Figure 1.1: required fine tuning in the SM with a cut-off of 10 TeV, from [Sch02]

Λ = 1 TeV, which is probed with current accelerators no fine-tuning is necessary. The biggest contribution from the top quark then is about $(200\text{GeV})^2$, so the SM is perfectly natural at the current energy scale.

We can turn the argument around and use the hierarchy problem to predict new physics. If we want a natural cancellation of the divergences, we predict new particles in the mass range of 2-10 TeV which are related to the particles that produce the quadratic divergences. These particles must be related by symmetry.

One approach for solving the hierarchy problem is supersymmetry (SUSY). There, every particle has a superpartner with *opposite* statistics. These form loop diagrams with quadratic divergences which cancel the divergences of the SM particles exactly, due to a relative minus sign. If SUSY were exact, the diagrams would cancel completely. Since we do not see superparticles, it must be (at least) softly broken. The cancellation takes place only above the mass scale of the superpartners, below only the SM particles exist. Thus, the cutoff Λ is replaced by M_{SUSY}.

For long, it was offen stated in the literature that quadratic divergences in realistic theories only cancel between fermion and boson loops. However, this is wrong. The cancellation between particles with the same spin already occurs in the Higgs sector in the supersymmetric SM (MSSM), but this was offen ignored.

Aside from SUSY, there are two ways how one can get a naturally light boson: spontaneously broken global symmetries that can produce massless Goldstone bosons and gauge symmetries that protect vector boson masses. Both of them seem not to be relevant, since the Higgs doesn't look like a boson with spin. Neither does it look like a Goldstone boson, since the Goldstone mechanism only allows derivative couplings of the Goldstone boson. Non-derivative quartic couplings, Yukawa and gauge couplings are not allowed[1]. However, there is a way how to combine these two ideas. These models are inspired by extra dimensions and revive the idea that the Higgs is a pseudo-Goldstone boson resulting from a spontaneously broken approximate symmetry.

Consider a five dimensional SU(n) gauge theory where the fifth dimension is put on a lattice with N sites. This can be illustrated by the following moose diagram (cf. [Geo86]):

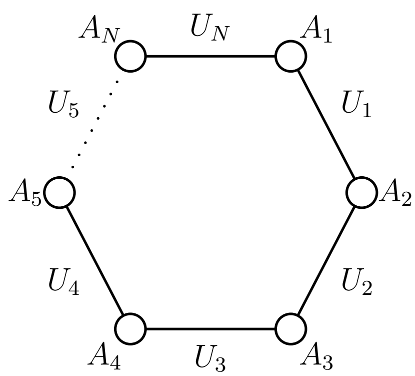


Figure 1.2: moose diagram

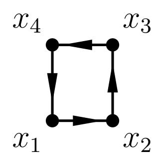
To derive the corresponding Lagrangian it is useful to briefly introduce the geometrical interpretation of gauge invariance, see [Pes] or [Böh] for further details. Field equations connect the fields ψ (e.g. a Dirac field) at different space time points x and $x + \mathrm{d}x$ (if derivatives of ψ are involved in the Lagrangian). The corresponding

[1]This can be seen by considering a complex scalar field $\phi(x)$ with a potential that preserves a global symmetry, e.g. U(1) in the simplest case. If symmetry is spontaneously broken (the potential induces a vev for ϕ), we have to expand the scalar field about one of the ground states $\phi_0(x) = v\mathrm{e}^{\mathrm{i}\Theta}$. We use the parametrization $\phi(x) = (v + \eta(x))\mathrm{e}^{i(\Theta+\xi(x))}$, where η and ξ are real fields and insert this in the Lagrangian. We find that ξ is massless and couples to η only with derivative couplings.

infinitesimal parallel displacement $\mathcal{U}(x, x + \mathrm{d}x)$ of the field $\psi(x)$ is described by a connection, in physics it is called a gauge field $A_\mu(x)$. The Wilson link is defined by

$$\mathcal{U}(x_a, x_b) = \exp\left(\mathrm{i}\int_{x_a}^{x_b} \mathrm{d}x^\mu \, A_\mu(x)\right) = \mathcal{U}(x_b, x_a)^\dagger = 1 + iA_\mu(x)\mathrm{d}x^\mu. \tag{1.2}$$

where the last equality holds for infinitesimal displacements. Imposing local gauge invariance, the covariant derivative of ψ can be defined which transforms as the field ψ itself and the transformation law for the connections can be found. For a locally invariant Lagrangian, we also have to find kinetic terms for A_μ, which involve terms depending on A_μ and its derivatives, but not on ψ. One way to find these is by linking four Wilson links around a small square in four-dimensional spacetime:

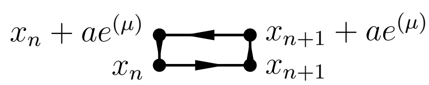


(1.3)

The plaquette action is defined by

$$L(x_1, x_2, x_3, x_4) = \mathrm{tr}\left(\mathcal{U}(x_1, x_2)\mathcal{U}(x_2, x_3)\mathcal{U}(x_3, x_4)\mathcal{U}(x_4, x_1)\right). \tag{1.4}$$

Now we assume that the points x_1, x_2, x_3, x_4 are infinitesimally separated by $\mathrm{d}x$ and $\mathrm{d}y$. Then we expand the Wilson links and keep terms up to $\mathcal{O}(\mathrm{d}x\mathrm{d}y)$. Integrating over the surface results in the well-known field strength tensor $(F_{\mu\nu})^2$. Now we apply this method to a 5-D Lagrangian where we carry out the continuum limit only in four directions. The displacement in the fifth direction is kept finite.



(1.5)

Using the notation

$$U(x_n) = \mathcal{U}(x_n, x_{n+1}), \tag{1.6a}$$

$$U^\dagger(x_n) = \mathcal{U}(x_{n+1}, x_n), \tag{1.6b}$$

the Wilson loop around the fifth dimension is given by

$$\begin{aligned} &L(x_n, x_{n+1}, x_{n+1} + ae^{(\mu)}, x_n + ae^{(\mu)}) \\ &\quad = \mathrm{tr}\left(U(x_n)\mathrm{e}^{\mathrm{i}aA_\mu(x_{n+1})}U^\dagger(x_n + ae^{(\mu)})\mathrm{e}^{-\mathrm{i}aA_\mu(x_n)}\right) = \mathrm{tr}\left(U(x_n)\mathrm{e}^{aD_\mu}U^\dagger(x_n)\right) \end{aligned} \tag{1.7}$$

with the covariant derivatives

$$D_\mu U(x_n) = \partial_\mu U(x_n) + \mathrm{i}A_\mu(x_n)U(x_n) - \mathrm{i}U(x_n)A_\mu(x_{n+1}), \tag{1.8a}$$

$$D_\mu U^\dagger(x_n) = \partial_\mu U^\dagger(x_n) + \mathrm{i}A_\mu(x_{n+1})U^\dagger(x_n) - \mathrm{i}U^\dagger(x_n)A_\mu(x_n). \tag{1.8b}$$

We expand the exponentiated derivative up to $\mathcal{O}(a^2)$. A careful consideration shows that only $DU(DU)^\dagger$ remains; the other terms are surface terms and cancel. Thus, we arrive at the low-energy Lagrangian for a generalized N-sided polygon,

$$\mathcal{L} = \sum_{i=1}^{N} \frac{v^2}{4} \operatorname{tr}\left((D^{i,\mu}U_i)^\dagger D^i_\mu U_i\right) - \frac{1}{8}\operatorname{tr}\left(F^{\mu\nu,i}F_{\mu\nu,i}\right) + \ldots \quad . \tag{1.9}$$

On each site there is an $\mathrm{SU}(n)$ gauge group and on the link pointing from the i'th to the $(i+1)$'th site, we have the field U_i which is represented by a nonlinear sigma model field. They transform as $U_i \to G^i U_i (G^{i-1})^\dagger$ under the $\mathrm{SU}(n)^N$ gauge symmetry. Let's consider the symmetries. If the gauge couplings are turned off, there is no coupling between the U's at different sites and the theory has a large $\mathrm{SU}(n)^{2N}$ accidental *global* 'chiral' symmetry

$$U_i \to L_i U_i R^\dagger_{i+1} \tag{1.10}$$

where L_i, R_i are independent $\mathrm{SU}(n)$ matrices. By the gauge interactions this is spontaneously broken down at the scale $\Lambda = 4\pi v$ to $\mathrm{SU}(n)^N$, which results in N Goldstone bosons. Now, the gauge couplings preserve only the $\mathrm{SU}(n)^N$ gauge group where $L_i = R_i$. Using the gauge freedom, we can go to unitarity gauge where we can gauge $N-1$ U_i's to one. Thus, $N-1$ Goldstone bosons are eaten by gauge bosons. The remaining Goldstone boson is classically massless. It is associated with the product $U_1 U_2 \cdots U_N$ which transforms homogeneously under the diagonal sum of all G_i's and cannot be transformed to unity. This operator is the discretization of the Wilson line in the continuum case. The linear combination $\phi = (\pi_1 + \pi_2 \ldots \pi_N)/\sqrt{N}$ ('little Higgs') corresponds to the zero mode of A_5 and transforms under the surviving diagonal subgroup $\mathrm{SU}(n)$. It is essential that no one operator alone breaks the global symmetry protecting the mass of the remaining Goldstone. The light scalar is a 'chain' of nonlinear sigma models, a 'non-local' object in the fifth dimension. Above the symmetry breaking scale Λ, the description with nonlinear sigma models is no longer valid and a UV completion is needed[2]. However, the mass of ϕ is insensitive to the physical details at Λ, so we don't need to care about this. As a consequence of this special symmetry breaking mechanism, no quadratically divergent contributions to the little Higgs exist at one loop. The necessary cancellations come from loops with the *same* spin.

[2] In the original paper [ArH014] this is done in the same way as QCD completes the theory of pions.

This work studies a special Little Higgs Lagrangian and is organized as follows:
Chapter 2 introduces the concept of spontaneously broken theories. Nonlinear sigma models are shown to be very useful tools to describe effective field theories where only the pattern of symmetry breaking is known.
The specific Little Higgs Lagrangian with $N = 2$ sites and two link fields U_1 and U_2 is considered in chapter 3. A low-energy expansion is made and the Lagrangian is rewritten in terms of fields in the mass eigenbasis. Feynman rules and transformation properties of the involved fields are derived. The Lagrangian is quantized with the BRS method in chapter 4. The nilpotence of the Lagrangian under BRS transformation is explicitly shown, and the Feynman rules for the ghosts are obtained.
Chapter 5 discusses the background field method, where the gauge field A is split into a quantum field A and a classical background field $\hat{A}$. This method allows one to fix a gauge without loosing explicit gauge invariance with respect to the background field $\hat{A}$. Relevant Feynman rules are derived in this gauge.
The quadratically and logarithmically divergent parts of the self-energies for all fields are calculated in a general R_ξ-gauge in chapter 6. The cancellation of quadratic divergences in the self-energy of the Little Higgs can be seen in detail.
In chapter 7, the correctness of the obtained results is checked by verifying a Slavnov-Taylor identity at one loop for Green functions involving the massive gauge boson. The validity of the Goldstone boson equivalence theorem is also shown for a tree level process.
Some remarks on the renormalization program and renormalization group equations are found in chapter 8. The β-function is obtained with the background field method by the calculation of the self-energy of the background field $\hat{A}$. The thesis is concluded in chapter 9.
All Feynman rules and important relations for dimensional regularization can be found in the appendices. Additionally, scalar contributions to the self-energy of a massive gauge boson are calculated in a linear sigma model and can be found in appendix B.

2 Spontaneously Broken Symmetries

One of the most important principles in building models for quantum field theories is that the action S is invariant under local gauge transformations. These symmetries generate dynamics, called gauge interactions. The prototype of such a gauge theory is QED and it is now believed that all fundamental interactions are described by gauge theories. However, this implies that all vector particles have to be massless, since mass terms are not gauge invariant. But massive vector bosons, like the Z and $W^\pm$ bosons in the Standard Model are observed. If we want to avoid this contradiction between theory and experiment, we have to break the gauge invariance of the theory somehow. Introducing explicit breaking terms in the form of gauge-boson masses leads to nonrenormalizable theories. However, if only the ground state is not invariant under gauge transformations we can have masses *and* constrain the interactions by the underlying gauge symmetry. This situation is called *spontaneous symmetry breaking*.

2.1 Linear Sigma Model

As an example, consider a complex scalar field coupled to itself and to an electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi) \tag{2.1}$$

with $D_\mu = \partial_\mu + ieA_\mu$ and $V(\phi) = -\frac{\mu^2}{2}\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2$. This Lagrangian is invariant under local U(1) transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \qquad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \tag{2.2}$$

For $\mu^2 > 0$, the minimum of the potential occurs at

$$\langle\phi\rangle = \phi_0 = \left(\frac{\mu^2}{\lambda}\right)^{1/2} := \frac{v}{\sqrt{2}}. \tag{2.3}$$

So, the field ϕ gets a vacuum expectation value (vev).
Plotting the potential as a function of Reϕ and Imϕ we see that the potential is minimal on the whole circle with radius $v/\sqrt{2}$. In (2.3) we made an explicit choice for

a vacuum state. While the Lagrangian is invariant under U(1) gauge transformations, the vacuum state is not. We have a spontaneously broken or 'hidden' symmetry.

Now we expand the Lagrangian about the vacuum state, parametrize the field ϕ as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \sigma(x) + i\pi(x)) \tag{2.4}$$

and rewrite the Lagrangian as

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - \mu^2\sigma^2) + \frac{1}{2}(\partial_\mu\pi\partial^\mu\pi) - \frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}e^2v^2A_\mu A^\nu + evA_\mu\partial^\mu\pi \\ + \text{cubic and quartic terms.}\end{aligned} \tag{2.5}$$

The π is the massless scalar Goldstone boson for a spontaneously broken global symmetry, predicted by the Goldstone theorem. However, there is a problem when interpreting the mixing term between A_μ and π and also when we count the number of degrees of freedom: massless vector bosons have two degrees of freedom (two physical polarization states), while massive vector bosons have three. So we have four degrees of freedom before symmetry breaking and five afterwards, which doesn't make much sense. However with a special gauge transformation, leading to the so-called 'unitarity gauge', we can show that the massless scalar field is unphysical. We chose $\alpha(x)$ such that $\phi(x)$ is real-valued at every point x. Thus, the π-field is removed from the theory, and so one degree of freedom and the mixing between π and A_μ. The acquiring of one extra degree of freedom for the gauge boson by 'eating' the unphysical Goldstone boson (would-be Goldstone boson) is called the 'Higgs mechanism'. So, the particle content is one scalar particle with mass μ, the σ, and one massive vector boson with mass ev, the A_μ, although there were a massless vector boson and a complex scalar before symmetry breaking. The advantage of this gauge is that only physical particles appear and thus the theory is manifestly unitary. However, the propagator of the massive gauge boson has a bad high energy behaviour and the theory is not manifestly renormalizable. To show renormalizability, it is better to use the so-called R_ξ-gauges.

2.2 Nonlinear Parametrization

Let's consider a model in which the mass of the σ-particle is much bigger than the energy where the theory is probed. Then we can take the limit $\mu \to \infty$, $\lambda \to \infty$ while keeping the vev unchanged. The potential becomes infinitely steep, so the dynamics happens only on the circle

$$|\phi(x)| = \frac{v}{\sqrt{2}}. \tag{2.6}$$

The σ boson can therefore be removed from the theory if we maintain the constraint (2.6). A suitable representation is the exponential representation

$$\phi(x) = \frac{v}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\zeta(x)/v} := \frac{v}{\sqrt{2}} U \tag{2.7}$$

where (2.6) is fulfilled automatically. The Lagrangian in this parametrization reads

$$\mathcal{L} = \frac{v^2}{2} (D_\mu U)^\dagger (D^\mu U) - \frac{1}{4} (F_{\mu\nu})^2. \tag{2.8}$$

All other terms are constants which do not depend on the fields and thus can be ignored.

2.3 Perturbation Theory

The nonlinear Lagrangian is no longer a polynomial in the fields. After a series expansion we find an infinite number of interacting terms and thus an infinite number of Feynman rules. How can we perform meaningful calculations in perturbation theory? Well, we also couldn't do that *before* the reparametrization, since we had sent the coupling constants to infinity. After the reparametrization the couplings μ and λ are no longer present in the Lagrangian and only derivative coupling appear. So, all vertices are proportional to powers of p/v, where p is the momentum. In this way we can make a perturbation theory in powers of the momentum. As long as $|p| < v$[1] this yields meaningful results.

Renormalizability

The Lagrangian has dimension 4, since the action S is dimensionless (in natural units) and the four-volume element has dimension -4. Since scalar and gauge bosons have dimension 1 (this can be seen from the kinetic part in the Lagrangian), a term with n Goldstone fields and p derivatives has dimension $p+n$ and its coupling constant therefore has dimension $4-p-n$. Power counting arguments show that only theories with non-negative coupling constants are renormalizable. That is all divergences occurring in higher order perturbation theory can be absorbed by a redefinition of a finite number of parameters of the theory. Additionally, we also need boson propagators which behave as k^{-2} and fermion propagators with k^{-1} for large momenta k.

An example for an interaction term in the nonlinear sigma model (NLSM) is $(\pi\partial\pi)^2$, and its coupling constant has dimension -2. All other terms are even worse. So, the NLSM is *not* renormalizable. But we do not care much about this: we do not claim

[1] Or more exactly $|p| < 4\pi v$, since in loop calculations always characteristic denominator factors of $16\pi^2 v^2$ appear.

that our theory is valid up to the highest energies; it will breakdown at energies near the symmetry-breaking scale $4\pi v$.

Nonlinearly realized symmetries are excellent tools to describe unknown theories where the pattern of symmetry breaking is known, since the (unknown) details of the interactions do not enter the Lagrangian. All interactions depend only on the pattern of symmetry breaking and appear as derivative couplings, where we can make a low energy expansion. To derive the effective Lagrangian, we have to write down all terms consistent with the symmetry. Terms with $2n$ derivatives, like

$$\Big((D_\mu U)^\dagger (D^\mu U)\Big)^{2n} \tag{2.9}$$

have to be added. The general Lagrangian can be organized by the dimensionality of the operators,

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots \tag{2.10}$$

The important point is that almost all terms are small at low energies, since each derivative comes with a factor of the momentum p. Therefore terms involving more than two derivatives are suppressed. For the lowest energies, only the Lagrangian with two derivatives has to be considered. We call this an '$\mathcal{O}(E^2)$' contribution. When including loops it would appear that the perturbation theory in powers of p/v would break down. This might happen when two of the momentum factors of an $\mathcal{O}(E^4)$ Lagrangian are involved in the loop and thus are proportional to the loop momentum. Integrating over the loop momentum apparently leaves only two factors of the 'low' energy variable p. It would therefore seem that for certain loop diagrams, an $\mathcal{O}(E^4)$ Lagrangian could behave as if it were $\mathcal{O}(E^2)$. This would be disastrous, because arbitrarily high order terms in the Lagrangian would contribute at $\mathcal{O}(E^2)$ when loops were calculated. But the reverse happens. This can be shown in general with Weinberg's power counting theorem, cf. [Wei79]. The idea is that higher order loop diagrams need more vertices and each vertex in a diagram contributes powers of $1/v$. Thus, the overall momentum power of an amplitude will *increase* rather than decrease, when loops are formed. The end result is very simple for counting the order of the energy expansion. The lowest order (E^2) is given by the two-derivative Lagrangian at tree-level. There are two sources at the next order (E^4): (i) amplitudes with two insertions of $\mathcal{O}(E^2)$ and (ii) the tree-level $\mathcal{O}(E^4)$ amplitudes. Finite predictions result after renormalizing the coefficients of the E^4 Langrangians.

3 Gauged Sigma-Models

3.1 Lagrangian

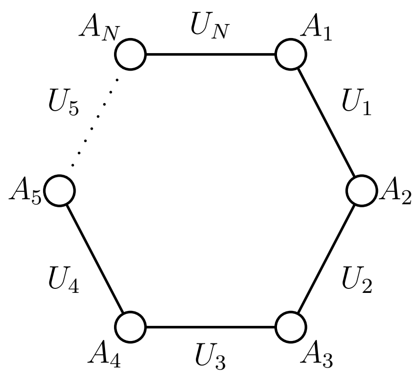


Figure 3.1: moose diagram

As already explained in chapter 1 the above figure is a pictorial representation of a compactified fifth dimension, called a moose diagram. On each site there is an $\mathrm{SU}(n)$ gauge group and on the link pointing from the i'th to the $(i+1)$'th site, there is a nonlinear sigma model field U_i. The low-energy effective Lagrangian for the Little-Higgs model is

$$\mathcal{L} = \sum_{i=1}^{N} \frac{v^2}{4} \operatorname{tr}\left((D^{i,\mu}U_i)^\dagger D^i_\mu U_i\right) - \frac{1}{8}\operatorname{tr}\left(F^{\mu\nu,i}F_{\mu\nu,i}\right) + \dots \tag{3.1}$$

with the nonlinear sigma model fields

$$U_i(x) = \mathrm{e}^{\mathrm{i}\pi_i(x)/v}, \qquad \pi(x) = \lambda^a \pi_i^a(x) \tag{3.2}$$

and the covariant derivative

$$D^i_\mu U_i = \partial_\mu U_i - \mathrm{i}g_i A^i_\mu U_i + \mathrm{i}g_{i-1} U_i A^{i-1}_\mu . \tag{3.3}$$

U_i and $D_\mu U_i$ transform covariantly under the gauge groups $\mathbf{L}^{\times N}$

$$U_i \to L^i U_i (L^{i-1})^\dagger \tag{3.4a}$$

$$D^i_\mu U_i \to L^i (D^i_\mu U_i)(L^{i-1})^\dagger \tag{3.4b}$$

provided that the gauge fields transform as

$$A^i_\mu \to L^i A^i_\mu (L^i)^\dagger + \mathrm{i}\frac{1}{g_i} L^i(\partial_\mu (L^i)^\dagger). \tag{3.5}$$

The dots in (3.1) represent higher dimensional operators ($D \geq 6$) which are irrelevant at low energies. The λ's are twice the generators of the gauge groups $\mathrm{SU}(n)$ and satisfy

$$\left[\lambda^a, \lambda^b\right] = 2\mathrm{i} f^{abc}\lambda^c, \qquad tr(\lambda^a \lambda^b) = 2\delta^{ab}. \tag{3.6}$$

This is also the reason for the factor $1/8$ of $(F_{\mu\nu})^2$. In order to make a low-energy expansion, we first concentrate on the simplest case, $N = 1$, one scalar field with two neighboring sites, A^μ_L and A^μ_R. The corresponding Lagrangian is given by

$$\mathcal{L} = \frac{v^2}{4} \operatorname{tr}\left((D^\mu U)^\dagger D_\mu U\right) - \frac{1}{8}\left(\operatorname{tr}\left(F^{\mu\nu,L} F_{\mu\nu,L}\right) + \operatorname{tr}\left(F^{\mu\nu,R} F_{\mu\nu,R}\right)\right) \tag{3.7}$$

with

$$D_\mu U = \partial_\mu U - \mathrm{i} g_L A^L_\mu U + \mathrm{i} g_R U A^R_\mu, \tag{3.8}$$

where U and $D_\mu U$ both transform covariantly under $\mathbf{L} \times \mathbf{R}$

$$U \to L U R^\dagger, \tag{3.9a}$$

$$D_\mu U \to L (D_\mu U) R^\dagger, \tag{3.9b}$$

provided the gauge fields transform as

$$A^L_\mu \to L A^L_\mu L^\dagger + \mathrm{i}\frac{1}{g_L} L(\partial_\mu L^\dagger), \tag{3.10a}$$

$$A^R_\mu \to R A^R_\mu R^\dagger + \mathrm{i}\frac{1}{g_R} R(\partial_\mu R^\dagger). \tag{3.10b}$$

Expanding the covariant derivatives in the first term of (3.7), we find

$$\begin{aligned}\mathcal{L} = {} & \frac{v^2}{4} \operatorname{tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) + \frac{(g_L v)^2}{4} \operatorname{tr}\left(A^L_\mu A^{L,\mu}\right) + \frac{(g_R v)^2}{4} \operatorname{tr}\left(A^R_\mu A^{R,\mu}\right) \\ & + \mathrm{i}\frac{g_L v^2}{2} \operatorname{tr}\left(A^L_\mu (\partial_\mu U) U^\dagger\right) - \mathrm{i}\frac{g_R v^2}{2} \operatorname{tr}\left(A^R_\mu U^\dagger (\partial_\mu U)\right) - \frac{g_L g_R v^2}{2} \operatorname{tr}\left(A^R_\mu U^\dagger A^{L,\mu} U\right).\end{aligned} \tag{3.11}$$

With the explicit realization of U in (3.2), the first term in

$$\operatorname{tr}\left(A^R_\mu U^\dagger A^{L,\mu} U\right) = \operatorname{tr}\left(A^R_\mu A^{L,\mu}\right) + \mathrm{i}\frac{1}{v}\operatorname{tr}\left([A^R_\mu, A^{L,\mu}]\pi\right) + \frac{1}{2v^2}\operatorname{tr}\left([\pi, A^R_\mu][\pi, A^{L,\mu}]\right) + \mathcal{O}(v^{-3}) \tag{3.12}$$

contributes an off-diagonal mass term for the gauge bosons and we have to diagonalize the mass matrix

$$\begin{pmatrix} (g_R v)^2 & -g_L g_R v^2 \\ -g_L g_R v^2 & (g_L v)^2 \end{pmatrix} \tag{3.13}$$

by

$$A_\mu = \frac{g_L A^R_\mu + g_R A^L_\mu}{\sqrt{g_L^2 + g_R^2}}, \tag{3.14a}$$

$$Z_\mu = \frac{g_R A^R_\mu - g_L A^L_\mu}{\sqrt{g_L^2 + g_R^2}}, \tag{3.14b}$$

where A is massless and Z receives a mass, ($m^2 = g_L g_R v^2/2$). This is hardly surprising, because the $L = R$ subgroup

$$U \to L U L^\dagger \tag{3.15}$$

is realized linearly on $\pi \to L\pi L^\dagger$, while the orthogonal $L = R^\dagger$ subgroup is realized nonlinearly (cf. section 3.4). If we choose for simplicity $g_L = g_R = g$, we obtain

$$A_\mu = \frac{1}{\sqrt{2}}\left(A^R_\mu + A^L_\mu\right), \tag{3.16a}$$

$$Z_\mu = \frac{1}{\sqrt{2}}\left(A^R_\mu - A^L_\mu\right), \tag{3.16b}$$

and

$$A^L_\mu = \frac{1}{\sqrt{2}}\left(A_\mu - Z_\mu\right), \tag{3.17a}$$

$$A^R_\mu = \frac{1}{\sqrt{2}}\left(A_\mu + Z_\mu\right). \tag{3.17b}$$

The Lagrangian reads

$$\begin{aligned}\mathcal{L} = {} & \frac{v^2}{4}\operatorname{tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) + \frac{(gv)^2}{2}\operatorname{tr}\left(Z_\mu Z^\mu\right) \\ & + \mathrm{i}\frac{gv^2}{2\sqrt{2}}\operatorname{tr}\left(A_\mu\left[(\partial_\mu U)U^\dagger - U^\dagger(\partial_\mu U)\right]\right) - \mathrm{i}\frac{gv^2}{2\sqrt{2}}\operatorname{tr}\left(Z_\mu\left[(\partial_\mu U)U^\dagger + U^\dagger(\partial_\mu U)\right]\right)\end{aligned}$$

$$+\,\mathrm{i}\frac{g^2v}{2}\,\mathrm{tr}\left([A_\mu, Z^\mu]\pi\right)$$
$$-\frac{1}{8}g^2\,\mathrm{tr}\left([\pi, A_\mu][\pi, A^\mu]\right) + \frac{1}{8}g^2\,\mathrm{tr}\left([\pi, Z_\mu][\pi, Z^\mu]\right) + \mathcal{O}(v^{-1}). \quad (3.18)$$

To read off the mass of the Z boson, we have to compare the kinetic term with the mass term, after evaluating the traces and commutators by using (3.6).

3.2 Derivatives

The treatment of derivatives in exponentials is not trivial, but with the Baker-Hausdorff-formula we can find useful expressions for the above terms involving U's and their derivatives. The Hausdorff-formula

$$\mathrm{e}^A(D\mathrm{e}^{-A}) = \mathrm{e}^{[A,\cdot]}D - D = [A, D] + \frac{1}{2!}[A,[A,D]] + \ldots = -DA - \frac{1}{2!}[A, DA] + \ldots \quad (3.19)$$

for derivations D[1] follows from

$$\mathrm{e}^A B \mathrm{e}^{-A} = \mathrm{e}^{[A,\cdot]}B = B + [A, B] + \frac{1}{2}[A,[A,B]] + \ldots \quad . \quad (3.20)$$

Note that derivations not only act on operators to their right, but also on the function which follows the operator. For

$$U(x) = \mathrm{e}^{\mathrm{i}\pi(x)/v} \quad (3.21)$$

we find from

$$\begin{aligned} U^\dagger(\partial_\mu U) &= \mathrm{e}^{-\mathrm{i}/v[\pi,\cdot]}\partial_\mu - \partial_\mu \\ &= -\mathrm{i}\frac{1}{v}[\pi, \partial_\mu] - \frac{1}{2v^2}[\pi,[\pi,\partial_\mu]] + \mathrm{i}\frac{1}{6v^3}[\pi,[\pi,[\pi,\partial_\mu]]] + \mathcal{O}(v^{-4}) \end{aligned} \quad (3.22)$$

the expansions

$$U^\dagger(\partial_\mu U) = +\mathrm{i}\frac{1}{v}\partial_\mu\pi + \frac{1}{2v^2}\left[\pi, \partial_\mu\pi\right] - \mathrm{i}\frac{1}{6v^3}\left[\pi,[\pi,\partial_\mu\pi]\right] + \mathcal{O}(v^{-4}), \quad (3.23a)$$
$$U(\partial_\mu U^\dagger) = -\mathrm{i}\frac{1}{v}\partial_\mu\pi + \frac{1}{2v^2}\left[\pi, \partial_\mu\pi\right] + \mathrm{i}\frac{1}{6v^3}\left[\pi,[\pi,\partial_\mu\pi]\right] + \mathcal{O}(v^{-4}), \quad (3.23b)$$
$$(\partial_\mu U)U^\dagger = +\mathrm{i}\frac{1}{v}\partial_\mu\pi - \frac{1}{2v^2}\left[\pi, \partial_\mu\pi\right] - \mathrm{i}\frac{1}{6v^3}\left[\pi,[\pi,\partial_\mu\pi]\right] + \mathcal{O}(v^{-4}), \quad (3.23c)$$

[1] A derivation maps an algebra into itself, $D: A \to A$. It is linear: $D(\alpha v + \beta w) = \alpha(Dv) + \beta(Dw)$ and obeys the product rule, $D(vw) = (Dv)w + v(Dw)$.

$$(\partial_\mu U^\dagger)U = -\mathrm{i}\frac{1}{v}\partial_\mu\pi - \frac{1}{2v^2}\left[\pi,\partial_\mu\pi\right] + \mathrm{i}\frac{1}{6v^3}\left[\pi,\left[\pi,\partial_\mu\pi\right]\right] + \mathcal{O}(v^{-4}). \tag{3.23d}$$

The latter three expressions can be derived from the first by conjugation $\pi \to -\pi$

$$U(\partial_\mu U^\dagger) = U^\dagger(\partial_\mu U)\Big|_{\pi\to-\pi} \tag{3.24a}$$

$$(\partial_\mu U)U^\dagger = (\partial_\mu U^\dagger)U\Big|_{\pi\to-\pi} \tag{3.24b}$$

and by using $\partial_\mu(U^\dagger U) = \partial_\mu(UU^\dagger) = 0$, i.e.

$$U(\partial_\mu U^\dagger) = -(\partial_\mu U)U^\dagger, \tag{3.25a}$$

$$(\partial_\mu U^\dagger)U = -U^\dagger(\partial_\mu U). \tag{3.25b}$$

3.3 Low-Energy-Expansion

Now, we are able to expand the field U in terms of commutators of π's and their derivatives. Let's evaluate the first term in (3.11):

$$\mathcal{L}_0 = \frac{v^2}{4}\,\mathrm{tr}\left(\partial^\mu U^\dagger\partial_\mu U\right) = \frac{v^2}{4}\,\mathrm{tr}\left((U^\dagger\partial^\mu U)^\dagger(U^\dagger\partial_\mu U)\right) \tag{3.26}$$

Using (3.23) and $\mathrm{tr}([\pi,\partial^\mu\pi]\partial_\mu\pi) = 0$ from the cyclic invariance of the trace, we find

$$\begin{aligned}\mathcal{L}_0 = {}& \frac{1}{4}\,\mathrm{tr}\left(\partial^\mu\pi\partial_\mu\pi\right) - \frac{1}{24v^2}\,\mathrm{tr}\left(\partial^\mu\pi\left[\pi,\left[\pi,\partial_\mu\pi\right]\right]\right) \\ & - \frac{1}{16v^2}\,\mathrm{tr}\left(\left[\pi,\partial^\mu\pi\right]\left[\pi,\partial_\mu\pi\right]\right) - \frac{1}{24v^2}\,\mathrm{tr}\left(\left[\pi,\left[\pi,\partial^\mu\pi\right]\right]\partial_\mu\pi\right) + \mathcal{O}(v^{-4}).\end{aligned} \tag{3.27}$$

Again using the cyclic invariance, (3.27) simplifies to

$$\mathcal{L}_0 = \frac{1}{4}\,\mathrm{tr}\left(\partial^\mu\pi\partial_\mu\pi\right) + \frac{1}{48v^2}\,\mathrm{tr}\left(\left[\pi,\partial_\mu\pi\right]\left[\pi,\partial^\mu\pi\right]\right) + \mathcal{O}(v^{-4}). \tag{3.28}$$

With (3.23), the Lagrangian (3.18) results in

$$\begin{aligned}\mathcal{L} = \frac{1}{4}\,\mathrm{tr}\left(\partial^\mu\pi\partial_\mu\pi\right) + \frac{1}{48v^2}\,\mathrm{tr}\left(\left[\pi,\partial_\mu\pi\right]\left[\pi,\partial^\mu\pi\right]\right) + \frac{(gv)^2}{2}\,\mathrm{tr}\left(Z_\mu Z^\mu\right) \\ + \frac{gv}{\sqrt{2}}\,\mathrm{tr}\left(Z^\mu\partial_\mu\pi\right) - \frac{g}{6\sqrt{2}v}\,\mathrm{tr}\left(Z^\mu[\pi,[\pi,\partial_\mu\pi]]\right) \\ - \mathrm{i}\frac{g}{2\sqrt{2}}\,\mathrm{tr}\left(A^\mu\left[\pi,\partial_\mu\pi\right]\right) + \mathrm{i}\frac{g^2v}{2}\,\mathrm{tr}\left([A_\mu,Z^\mu]\pi\right)\end{aligned}$$

$$-\frac{1}{8}g^2\,\mathrm{tr}\left([\pi, A_\mu][\pi, A^\mu]\right) + \frac{1}{8}g^2\,\mathrm{tr}\left([\pi, Z_\mu][\pi, Z^\mu]\right) + \mathcal{L}_{5,6} + \mathcal{O}(v^{-3}). \quad (3.29)$$

Note the cancellation of two (one) terms when subtracting (adding) two terms in (3.23) which are coupled to A_μ and Z_μ. $\mathcal{L}_{5,6}$ contains all remaining terms of $\mathcal{O}(v^{-1})$ and $\mathcal{O}(v^{-2})$ which originate from (3.12) and (3.23). They contain no derivatives and involve five or six fields (two gauge bosons and two or three pions, respectively). The term $\mathrm{tr}(A^\mu[\pi,[\pi,[\pi,\partial_\mu\pi]]])$ involves one derivative but also consists of five fields. These vertices are not required for the calculation of two-point functions to one-loop order and are thus ignored.

One also has to add the kinetic term for A^L_μ and A^R_μ:

$$\mathcal{L}_{\mathrm{kin}} = -\frac{1}{8}\Big(\mathrm{tr}\left(F^{\mu\nu,L}F_{\mu\nu,L}\right) + \mathrm{tr}\left(F^{\mu\nu,R}F_{\mu\nu,R}\right)\Big) \quad (3.30)$$

with

$$F^{\mu\nu}_i = \partial_\mu A^\nu_i - \partial_\nu A^\mu_i - \mathrm{i}g[A^\mu_i, A^\nu_i], \qquad i = L, R \quad (3.31)$$

expressed in terms of A^μ and Z^μ. We also have a ‘parity’ symmetry $L \leftrightarrow R$:

$$\begin{pmatrix} A_\mu \\ \pi \\ Z_\mu \end{pmatrix} \longrightarrow \begin{pmatrix} +A_\mu \\ -\pi \\ -Z_\mu \end{pmatrix} \quad (3.32)$$

All equations must be consistent with this parity transformation.

N = 2

From now on we set $g_1 = g_2$ for simplicity. Extending our model to two sites is trivial, since for $N = 2$ the second site is the mirror image of the first, that is

$$D^2_\mu = D^1_\mu\Big|_{A^L_\mu \leftrightarrow A^R_\mu} \quad (3.33)$$

which amounts to an overall replacement $Z_\mu = \frac{1}{\sqrt{2}}(A^R_\mu - A^L_\mu) \leftrightarrow -Z_\mu$ and $\pi_1 \leftrightarrow \pi_2$. The resulting mixing term is $(gv/\sqrt{2})\,\mathrm{tr}(Z^\mu\partial_\mu(\pi_1 - \pi_2))$. This indicates that the field

$$\pi_- = \frac{1}{\sqrt{2}}\left(\pi_1 - \pi_2\right) \quad (3.34a)$$

is the unphysical would-be Goldstone boson and is eaten by the Z boson, while the field

$$\pi_+ = \frac{1}{\sqrt{2}}\left(\pi_1 + \pi_2\right) \quad (3.34b)$$

is a physical degree of freedom, since there is no mixing term for π_+ (the A boson remains strictly massless). This field is the Little Higgs particle. Note that it is a linear combination of the fields π_1 and π_2, a non-local object in figure 3.1. This guaranties that it is free from quadratically divergent mass contributions as we will see in chapter 6. The full Lagrangian for $N = 2$ is (without the $(F_{\mu\nu})^2$-terms)

$$\begin{aligned}
\mathcal{L}_{\text{scalar}} = & \frac{1}{4}\operatorname{tr}\left(\partial^\mu \pi_+ \partial_\mu \pi_+\right) + \frac{1}{4}\operatorname{tr}\left(\partial^\mu \pi_- \partial_\mu \pi_-\right) \\
& + \frac{1}{2\cdot 48 v^2}\operatorname{tr}\Big([\pi_+, \partial_\mu \pi_+][\pi_+, \partial^\mu \pi_+] + [\pi_-, \partial_\mu \pi_-][\pi_-, \partial^\mu \pi_-] \\
& \quad + [\pi_+, \partial_\mu \pi_-][\pi_+, \partial^\mu \pi_-] + [\pi_-, \partial_\mu \pi_+][\pi_-, \partial^\mu \pi_+] \\
& \quad + 2\,[\pi_+, \partial_\mu \pi_+][\pi_-, \partial^\mu \pi_-] + 2\,[\pi_+, \partial_\mu \pi_-][\pi_-, \partial^\mu \pi_+] \Big) \\
& + (gv)^2 \operatorname{tr}\left(Z_\mu Z^\mu\right) + gv \operatorname{tr}\left(Z^\mu \partial_\mu \pi_-\right) \\
& - \frac{g}{12v}\operatorname{tr}\Big(Z^\mu[\pi_-, [\pi_-, \partial_\mu \pi_-]] + Z^\mu[\pi_+, [\pi_+, \partial_\mu \pi_-]] \\
& \quad + Z^\mu[\pi_+, [\pi_-, \partial_\mu \pi_+]] + Z^\mu[\pi_-, [\pi_+, \partial_\mu \pi_+]] \Big) \\
& - \mathrm{i}\frac{g}{2\sqrt{2}}\operatorname{tr}\left(A^\mu [\pi_+, \partial_\mu \pi_+] + A^\mu [\pi_-, \partial_\mu \pi_-]\right) + \mathrm{i}\frac{g^2 v}{\sqrt{2}}\operatorname{tr}\left([A_\mu, Z^\mu]\pi_-\right) \\
& - \frac{1}{8} g^2 \operatorname{tr}\left([\pi_+, A_\mu][\pi_+, A^\mu] + [\pi_-, A_\mu][\pi_-, A^\mu]\right) \\
& + \frac{1}{8} g^2 \operatorname{tr}\left([\pi_+, Z_\mu][\pi_+, Z^\mu] + [\pi_-, Z_\mu][\pi_-, Z^\mu]\right) + \mathcal{O}(v^{-3})
\end{aligned} \tag{3.35}$$

Together with another Lagrangian derived in chapter 4 this Lagrangian will be used to find all relevant vertices to compute all self-energies to one-loop approximation. Since the Lagrangian doesn't change under the relabelling of the two sites, we have the following parity symmetry,

$$\begin{pmatrix} A_\mu \\ \pi_+ \\ \pi_- \\ Z_\mu \end{pmatrix} \longrightarrow \begin{pmatrix} +A_\mu \\ +\pi_+ \\ -\pi_- \\ -Z_\mu \end{pmatrix} \tag{3.36}$$

to be respected by all equations.

3.4 Gauge Transformation of the Fields

For the BRS transformation (see chapter 4) we need the transformation properties of the scalars π_+, π_- and the vector bosons A_μ, Z_μ. These can be obtained from (3.9) and (3.10). Since we have two gauge groups **L** and **R**, we can parametrize them with ζ and α in the following way:

$$L = \mathrm{e}^{\mathrm{i}g\zeta}\mathrm{e}^{\mathrm{i}g\alpha}, \tag{3.37a}$$
$$R = \mathrm{e}^{-\mathrm{i}g\zeta}\mathrm{e}^{\mathrm{i}g\alpha}. \tag{3.37b}$$

Varying these parameters independently, we obtain

$$\mathrm{e}^{\mathrm{i}\pi/v} \xrightarrow{\alpha} \mathrm{e}^{\mathrm{i}g\alpha}\mathrm{e}^{\mathrm{i}\pi/v}\mathrm{e}^{-\mathrm{i}g\alpha}, \tag{3.38a}$$
$$\mathrm{e}^{\mathrm{i}\pi/v} \xrightarrow{\zeta} \mathrm{e}^{\mathrm{i}g\zeta}\mathrm{e}^{\mathrm{i}\pi/v}\mathrm{e}^{\mathrm{i}g\zeta}. \tag{3.38b}$$

Using the Hausdorff-formulae

$$\mathrm{e}^A\mathrm{e}^B\mathrm{e}^{-A} = \exp(\mathrm{e}^{[A,\cdot]}B) = \mathrm{e}^{B+[A,B]+\frac{1}{2}[A,[A,B]]+\mathcal{O}(A^3)} \tag{3.39}$$

and

$$\mathrm{e}^A\mathrm{e}^B\mathrm{e}^A = \mathrm{e}^{B+2A+\frac{1}{6}[B,[B,A]]+\mathcal{O}(A^2,B^3)} \tag{3.40}$$

we obtain as infinitesimal transformations of the π-field ($N = 1$)

$$\delta\pi = \delta_\zeta\pi + \delta_\alpha\pi = 2gv\zeta - \frac{g}{6v}[\pi,[\pi,\zeta]] + \mathrm{i}g[\alpha,\pi] + \mathcal{O}(\pi^4\zeta) \tag{3.41}$$

where we can see that α parametrizes the linearly realized symmetry and ζ parametrizes the nonlinearly realized one. Note that a term of $\mathcal{O}(\pi^3\zeta)$ has parity $+1$ with respect to (3.32) and therefore cannot occur in the transformation of π. From this we can easily derive the transformations of π_+ and π_-. The nonlinear sigma model fields transform as $U_1 \to LU_1R^\dagger$ and $U_2 \to RU_2L^\dagger$. From (3.37) and (3.38) we find that the linear part of the transformation has the same sign both for π_1 and π_2, but an opposite sign for the nonlinear part,

$$\delta_\zeta\pi_1 = 2gv\zeta - \frac{g}{6v}[\pi_1,[\pi_1,\zeta]] + \mathcal{O}(\pi_1^4), \tag{3.42a}$$
$$\delta_\zeta\pi_2 = -2gv\zeta + \frac{g}{6v}[\pi_2,[\pi_2,\zeta]] + \mathcal{O}(\pi_2^4), \tag{3.42b}$$
$$\delta_\alpha\pi_1 = \mathrm{i}g[\alpha,\pi_1], \tag{3.42c}$$
$$\delta_\alpha\pi_2 = \mathrm{i}g[\alpha,\pi_2]. \tag{3.42d}$$

With

$$\delta_{\zeta,\alpha}\pi_\pm = \frac{1}{\sqrt{2}}\left(\delta_{\zeta,\alpha}\pi_1 \pm \delta_{\zeta,\alpha}\pi_2\right) \tag{3.43}$$

we find the transformation laws

$$\delta\pi_+ = \delta_\zeta\pi_+ + \delta_\alpha\pi_+ = -\frac{g}{6\sqrt{2}v}\Big([\pi_-,[\pi_+,\zeta]] + [\pi_+,[\pi_-,\zeta]]\Big) + ig[\alpha,\pi_+] + \mathcal{O}(\zeta\pi^4), \tag{3.44a}$$

$$\delta\pi_- = \delta_\zeta\pi_- + \delta_\alpha\pi_- = 2\sqrt{2}gv\zeta - \frac{g}{6\sqrt{2}v}\Big([\pi_+,[\pi_+,\zeta]] + [\pi_-,[\pi_-,\zeta]]\Big) + ig[\alpha,\pi_-] + \mathcal{O}(\zeta\pi^4). \tag{3.44b}$$

for the physical scalar π_+ and the would-be Goldstone boson π_- in the $N = 2$ Lagrangian. While (3.44a) is less interesting because it will not be used in a gauge fixing functional, (3.44b) shows that *both* bosons $\pi_\pm$ will couple quartically to the Faddeev-Popov ghosts (and with the same strength), see section 4.2.

Let's turn to the transformation laws of the gauge bosons in the mass-eigenbasis. With (3.10), (3.16) and (3.17) we find

$$A_\mu \to \frac{1}{2}\left(RA_\mu R^\dagger + LA_\mu L^\dagger + RZ_\mu R^\dagger - LZ_\mu L^\dagger\right) + \mathrm{i}\frac{1}{\sqrt{2}g}\left(R\partial_\mu R^\dagger + L\partial_\mu L^\dagger\right), \tag{3.45a}$$

$$Z_\mu \to \frac{1}{2}\left(RA_\mu R^\dagger - LA_\mu L^\dagger + RZ_\mu R^\dagger + LZ_\mu L^\dagger\right) + \mathrm{i}\frac{1}{\sqrt{2}g}\left(R\partial_\mu R^\dagger - L\partial_\mu L^\dagger\right). \tag{3.45b}$$

In particular, varying α and ζ independently, one has

$$A_\mu \xrightarrow{\alpha} LA_\mu L^\dagger + \mathrm{i}\frac{\sqrt{2}}{g}L\partial_\mu L^\dagger, \tag{3.46a}$$

$$Z_\mu \xrightarrow{\alpha} LZ_\mu L^\dagger, \tag{3.46b}$$

$$A_\mu \xrightarrow{\zeta} \frac{1}{2}\left(L^\dagger A_\mu L + LA_\mu L^\dagger + L^\dagger Z_\mu L - LZ_\mu L^\dagger\right) + \mathrm{i}\frac{1}{\sqrt{2}g}\left(L^\dagger\partial_\mu L + L\partial_\mu L^\dagger\right), \tag{3.46c}$$

$$Z_\mu \xrightarrow{\zeta} \frac{1}{2}\left(L^\dagger A_\mu L - LA_\mu L^\dagger + L^\dagger Z_\mu L + LZ_\mu L^\dagger\right) + \mathrm{i}\frac{1}{\sqrt{2}g}\left(L^\dagger\partial_\mu L - L\partial_\mu L^\dagger\right). \tag{3.46d}$$

The infinitesimal transformations are

$$\delta A_\mu = \mathrm{i}g[\alpha, A_\mu] + \sqrt{2}\partial_\mu\alpha - \mathrm{i}g[\zeta, Z_\mu], \tag{3.47a}$$

$$\delta Z_\mu = \mathrm{i}g[\alpha, Z_\mu] - \mathrm{i}g[\zeta, A_\mu] - \sqrt{2}\partial_\mu\zeta. \tag{3.47b}$$

3.5 Feynman Rules

In the preceding sections we have derived an effective low-energy Lagrangian up to $\mathcal{O}(v^{-2})$ from the original Lagrangian (3.1). At this point we can translate the Lagrangian into Feynman rules. In the path integral representation of quantum field theory, Feynman rules are obtained by applying functional derivatives on generating functionals of Green and vertex functions. After that we go from coordinate space to momentum space via Fourier transformation. This thesis is not the place to give a detailed description, which can be found in textbooks. Only the 'recipe' shall be sketched here:

1. Take all terms in $\mathrm{i}\mathcal{L}$ with a certain combination of fields; these are the outer lines of the vertex.

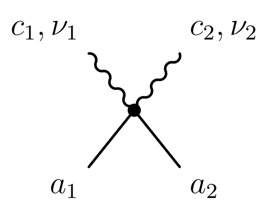
2. Replace all derivatives by $(-\mathrm{i})$ times the incoming momenta of the fields on which they act. This corresponds to Fourier transformation.

3. Symmetrize all indices and momenta of *identical* fields and add the corresponding symmetry factor. Then discard all outer fields. This corresponds to the functional derivative.

Propagators are obtained by inverting the quadratic vertices (and multiplying by -1). Now we apply this to our Lagrangian. First, we have to evaluate the traces over the generators of the group, cf. (3.6) (remember that $\phi = \phi_a \lambda^a$).
As an example, consider the last term in (3.35),

$$\begin{aligned} \mathrm{i}\mathcal{L}_{4,Z} &= \frac{1}{8}\mathrm{i}g^2 \cdot 2 \cdot (2\mathrm{i})^2 g^{\nu_1\nu_2} f^{a_1c_1e} f^{a_2c_2e} \pi_+^{a_1} Z_{\nu_1}^{c_1} \pi_+^{a_2} Z_{\nu_2}^{c_2} \\ &= \frac{1}{2}\mathrm{i}g^2 g^{\nu_1\nu_2} (f^{a_1c_1e} f^{a_2c_2e} + f^{a_1c_2e} f^{a_2c_1e}) \pi_+^{a_1} Z_{\nu_1}^{c_1} \pi_+^{a_2} Z_{\nu_2}^{c_2}. \end{aligned} \tag{3.48}$$

Note that the second line is symmetric with respect to the interchange of $a_1 \leftrightarrow a_2$ and/or $c_1 \leftrightarrow c_2$. Discarding the fields and multiplying by a symmetry factor of $(2!)^2$ (since we have two identical π_+ and Z's) yields the vertex



$$= -2\mathrm{i}g^2 \left[f^{a_1c_1e} f^{a_2c_2e} + f^{a_1c_2e} f^{a_2c_1e} \right] g^{\nu_1\nu_2}. \tag{3.49}$$

The four Goldstone vertex with four identical scalars is more interesting:

$$\begin{aligned} \frac{\mathrm{i}}{96v^2} \operatorname{tr}\left([\pi, \partial_\mu \pi] \, [\pi, \partial^\mu \pi] \right) &= -\frac{8\mathrm{i}}{96v^2} f^{a_1a_2b} f^{a_3a_4b} (\pi^{a_1} \partial_\mu \pi^{a_2})(\pi^{a_3} \partial^\mu \pi^{a_4}) \\ &= -\frac{\mathrm{i}}{48} \cdot \frac{1}{v^2} f^{a_1a_2b} f^{a_3a_4b} (\pi^{a_1} \overleftrightarrow{\partial_\mu} \pi^{a_2})(\pi^{a_3} \overleftrightarrow{\partial^\mu} \pi^{a_4}) \end{aligned}$$

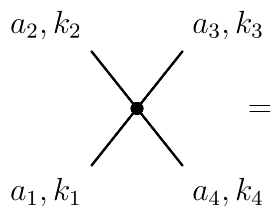
$$
\xrightarrow{\partial\to-\mathrm{i}k} \frac{\mathrm{i}}{48}\cdot\frac{1}{v^2} f^{a_1a_2b}f^{a_3a_4b}(k_1-k_2)(k_3-k_4)\pi^{a_1}\pi^{a_2}\pi^{a_3}\pi^{a_4}
$$
$$
= \frac{\mathrm{i}}{48}\cdot\frac{1}{3v^2}\Big[(k_1-k_2)(k_3-k_4)f^{a_1a_2b}f^{a_3a_4b} + (k_1-k_3)(k_4-k_2)f^{a_1a_3b}f^{a_4a_2b} + (k_1-k_4)(k_2-k_3)f^{a_1a_4b}f^{a_2a_3b}\Big]\pi^{a_1}\pi^{a_2}\pi^{a_3}\pi^{a_4}. \quad (3.50)
$$

In the second line the symmetrization of the derivatives is obtained with the help of

$$
f^{a_1a_2b}\pi^{a_1}\partial_\mu\pi^{a_2} = \frac{1}{2}f^{a_1a_2b}\pi^{a_1}\partial_\mu\pi^{a_2} + \frac{1}{2}f^{a_2a_1b}\pi^{a_2}\partial_\mu\pi^{a_1} = \frac{1}{2}f^{a_1a_2b}(\pi^{a_1}\overleftrightarrow{\partial_\mu}\pi^{a_2}) \quad (3.51)
$$

where in the last step we have used the antisymmetry of the structure constants and $\pi^{a_1}\overleftrightarrow{\partial_\mu}\pi^{a_2} = \pi^{a_1}\partial_\mu\pi^{a_2} - \pi^{a_2}\partial_\mu\pi^{a_1}$.

In the fourth line we symmetrized the indices a_1, a_2, a_3, a_4, since they belong to four identical fields. The symmetry factor is therefore 4! and we find



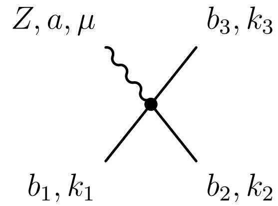
$$
= \frac{\mathrm{i}}{6v^2}\begin{bmatrix} (k_1-k_2)(k_3-k_4)f^{a_1a_2b}f^{a_3a_4b} \\ +(k_1-k_3)(k_4-k_2)f^{a_1a_3b}f^{a_4a_2b} \\ +(k_1-k_4)(k_2-k_3)f^{a_1a_4b}f^{a_2a_3b}\end{bmatrix}. \quad (3.52)
$$

We also have a mixed four vertex with two π_+ and two π_- fields, to which four terms in (3.35) contribute. The symmetrization in these terms is slightly different (since we only have two pairs of identical fields) and the symmetry factor is $(2!)^2$. However, we obtain the same Lorentz and gauge structure as for the pure vertex in (3.52). This is true for *all* other mixed vertices.

Another four vertex is found from

$$
\mathrm{i}\mathcal{L}_4 = -\frac{\mathrm{i}g}{12v}\,\mathrm{tr}(Z^\mu[\pi_-,[\pi_-,\partial_\mu\pi_-]]) = -\frac{8\mathrm{i}g}{12v}f^{a_2a_3b}f^{a_1b_a}Z^{\mu,a}\pi_-^{a_1}\pi_-^{a_2}\partial_\mu\pi_-^{a_3}
$$
$$
= \frac{\mathrm{i}g}{3v}\cdot\frac{1}{3}\Big[f^{aa_1b}f^{a_2a_3b}Z^a\pi_-^{a_1}(\pi_-^{a_2}\overleftrightarrow{\partial}\pi_-^{a_3}) + f^{aa_2b}f^{a_3a_1b}Z^a\pi_-^{a_2}(\pi_-^{a_3}\overleftrightarrow{\partial}\pi_-^{a_1}) + f^{aa_3b}f^{a_1a_2b}Z^a\pi_-^{a_3}(\pi_-^{a_1}\overleftrightarrow{\partial}\pi_-^{a_2})\Big]. \quad (3.53)
$$

After replacing the derivatives with the incoming momenta of the π_- and multiplying with 3!, since we have three identical π_-, we get



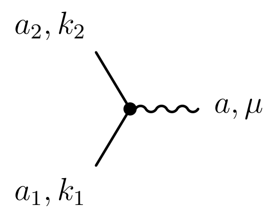
$$= \frac{2g}{3v}\begin{bmatrix} (k_1 - k_2)^\mu f^{a_1 a_2 c} f^{a_3 a c} \\ +(k_2 - k_3)^\mu f^{a_2 a_3 c} f^{a_1 a c} \\ +(k_3 - k_1)^\mu f^{a_3 a_1 c} f^{a_2 a c} \end{bmatrix}. \tag{3.54}$$

We also find an analogous vertex where two of the π_- are replaced by π_+.

Here is an example for a three vertex,

$$\begin{aligned} \mathrm{i}\mathcal{L}_3 &= \frac{g}{2\sqrt{2}} \operatorname{tr}(A^\mu[\pi_+, \partial_\mu \pi_+]) = \sqrt{2} g f^{a_1 a_2 a} A^{\mu,a} \pi_+^{a_1} \partial_\mu \pi_+^{a_2} \\ &= \frac{g}{\sqrt{2}} f^{a a_1 a_2} A^{\mu,a} (\pi_+^{a_1} \overleftrightarrow{\partial_\mu} \pi_+^{a_2}). \end{aligned} \tag{3.55}$$

Again, replacing ∂_μ with $-\mathrm{i}k_\mu$ and multiplying by 2!, we find

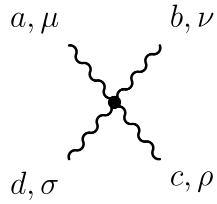


$$= -\sqrt{2} g (k_1 - k_2)^\mu f^{a a_1 a_2}. \tag{3.56}$$

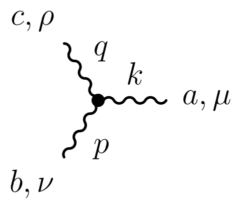
Let's turn to the pure gauge part of the Lagrangian, (3.30). Since the derivation of the Feynman rules for non-abelian gauge bosons can be found in standard text books on quantum field theory this will not be done in this work. We only have to think about many gauge-boson vertices and their corresponding prefactors we have. After rewriting the Lagrangian in terms of physical A and Z fields we find for the three vertices a pure AAA-vertex and an AZZ-vertex, both with parity $+1$, consistent with the $L \leftrightarrow R$ symmetry of the Lagrangian. An explicit calculation shows that the gauge and Lorentz structure of the mixed AZZ-vertex is the same as for the AAA-vertex. We also have to take care of the coupling constant g, since there are various factors of 2 and $1/\sqrt{2}$ due to commutators and traces of the generators (factor 4, see (3.6)) and normalization constants (factor $(1/\sqrt{2})^3$, see (3.17)). Another factor of 2 arises because both A_L and A_R contribute to the vertices and a factor of 4 comes from squaring $F_{\mu\nu}$. Finally, we have $-1/8g \cdot 4 \cdot (1/\sqrt{2})^3 \cdot 2 \cdot 4 = -\sqrt{2}g$ as coupling constant for the three vertices.

In analogy we find an $AAAA$-, $ZZZZ$- and a $AAZZ$-vertex, with coupling constant $2g^2$.

We obtain



$$= -2\mathrm{i}g^2 \begin{bmatrix} (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})f^{abe}f^{cde} \\ +(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})f^{ace}f^{bde} \\ +(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})f^{ade}f^{bce} \end{bmatrix}, \tag{3.57}$$



$$= \sqrt{2}gf^{abc} \begin{bmatrix} g^{\mu\nu}(k-p)^{\rho} \\ +g^{\nu\rho}(p-q)^{\mu} \\ +g^{\rho\mu}(q-k)^{\nu} \end{bmatrix} \tag{3.58}$$

for the gauge-boson vertices. A complete list of all vertices can be found in the appendix.

4 BRS Invariance

Until now, we haven't determined the propagators of the gauge bosons. Following the recipe for calculating propagators given in section (3.5), we obtain for the massless gauge-boson propagator

$$\begin{aligned} \mathrm{i}\mathcal{L}_2 &= -\frac{\mathrm{i}}{4}(\partial_\mu A^{\nu,a})(\partial^\mu A^b_\nu)\delta^{ab} \\ &= -\frac{\mathrm{i}}{2}(\partial_\nu A^a_\mu)(\partial^\mu A^{\nu,b})\delta^{ab} + \frac{\mathrm{i}}{2}(\partial_\mu A^a_\nu)(\partial^\nu A^{\mu,b})\delta^{ab} \\ &= \frac{\mathrm{i}}{2}A^a_\nu g^{\mu\nu}\partial^2 A^b_\mu \delta^{ab} - \frac{i}{2}A^a_\nu \partial^\mu\partial^\nu A^b_\mu \delta^{ab} + \text{surface terms} \\ &\longrightarrow -\mathrm{i}\delta^{ab}\big[k^2 g^{\mu\nu} - k^\mu k^\nu\big] \\ &= -(\text{propagator})^{-1} \end{aligned} \tag{4.1}$$

Here, we can see the problem: the operator $K^{\mu\nu} = g^{\mu\nu}k^2 - k^\mu k^\nu$ has an eigenvector k_μ with eigenvalue zero, so its inverse is not defined. This problem is due to gauge invariance of the Lagrangian $\mathcal{L}$. In the generating functionals we integrate over all possible field configurations at every space-time point x, including those that are connected by a gauge transformation. Recall that the Lagrangian is invariant under general gauge transformations of the form

$$A_\mu(x) \to A_\mu(x) + \frac{1}{g}\partial_\mu \alpha(x). \tag{4.2}$$

The troublesome modes are those for which $A_\mu(x) = \frac{1}{g}\partial_\mu\alpha(x)$ which are equivalent to $A_\mu(x) = 0$. In the functional integral we integrate over a continuous infinity of physically equivalent field configurations, thus it is badly defined. To fix this problem, we have to choose a local functional $G[A;x]$ in the gauge fields that fixes the gauge in the sense that the condition $G[A;x] = 0$ selects one solution out of the set of gauge-transformed, physically equivalent potentials $A(x)$.

Following [Böh], one way to accomplish this is the Faddeev-Popov procedure where one inserts

$$1 = \Delta\{\mathbf{A}\}\int \mathcal{D}[\mu(g)]\delta\{G[A]\} \tag{4.3}$$

consisting of a delta-functional and a Jacobi determinant into the functional integral.

The generating functional for Green functions is

$$\begin{aligned} T\{\mathbf{J}\} &= Z\{\mathbf{J}\}/Z\{0\}, \\ Z\{\mathbf{J}\} &= \int \mathcal{D}[\mathbf{A}] e^{\mathrm{i}S\{\mathbf{A}\}+\mathrm{i}\int d^4x J\cdot A} \end{aligned} \tag{4.4}$$

The measure $\mathcal{D}[\mathbf{A}]$ involves at each space-time point a product over all group and vector components of the field $A_\mu^a(x)$. What remains is a divergent multiplicative factor resulting from the integration over the gauge group ($\int \mathcal{D}[\mu(g)]$) that cancels out when forming the ratio $Z\{\mathbf{J}\}/Z\{0\}$. At the end this results in two new terms in the Lagrangian. The first term is a gauge-fixing part of the form $\mathcal{L}_{\text{gf}} = -1/(2\xi)G^2$, e.g. $G = \partial_\mu A^\mu$ in Lorentz gauge for massless gauge bosons. The second term is the ghost Lagrangian $\mathcal{L}_{\text{ghost}}$ containing unphysical anticommuting Grassmann-valued scalar fields, called Faddeev-Popov ghost fields.
By introducing the gauge-fixing term $G[A;x]$ the manifest gauge-invariance is lost. However, it can be shown that this new effective Lagrangian leads to physical results such as S-matrix elements which are gauge independent. Unphysical contributions contained in the gauge-fixing Lagrangian (which is necessary to define the propagator) are cancelled by contributions of the ghost propagators and vertices.

4.1 BRS Transformation

Although the gauge invariance of the Lagrangian has been destroyed by gauge fixing, a new symmetry of the effective action appears. It implies all the consequences of gauge invariance for physical results by an extension of the gauge transformation to the ghost fields. This extended gauge transformation is the *Becchi-Rouet-Stora* or *BRS transformation*. The canonical BRS formalism is equivalent to the path integral method (via Faddeev-Popov) but somewhat more elegant. Furthermore, it reveals us more insight into quantum field theories.
For gauge-boson and scalar fields the BRS transformation is a gauge transformation (cf. (3.44) and (3.47)) with $\alpha(x) = \delta\lambda\eta_\alpha(x)$ and $\zeta(x) = \delta\lambda\eta_\zeta(x)$, where $\delta\lambda$ is an infinitesimal, Grassmann-valued constant which anticommutes with the ghost fields η_α and η_ζ[1]. The constant $\delta\lambda$ has been introduced so that the transformations do not change the statistics of the fields. The BRS operator s is defined as the left derivative with respect to $\delta\lambda$ of the BRS transformed fields. Thus, the product rule reads

$$s(FG) = (sF)G \pm FsG, \tag{4.5}$$

[1] η_α and η_ζ are linear combinations of left- and right-ghosts, in analogy to the gauge bosons A and Z, see (3.17) and (3.16). Their BRS transformations are defined by $\delta\eta_L \propto [\eta_L, \eta_L]$ and $\delta\eta_R \propto [\eta_R, \eta_R]$

where the minus sign occurs for fermionic F (odd number of Grassmann variables). The transformations of arbitrary fields Ψ are written as

$$s\Psi = \frac{\delta\Psi}{\delta\lambda}. \tag{4.6}$$

They are given by

$$\delta\pi_+ = \delta\lambda(\mathrm{i}g[\eta_\alpha, \pi_+] - \frac{g}{6\sqrt{2}v}([\pi_-, [\pi_+, \eta_\zeta]] + [\pi_+, [\pi_-, \eta_\zeta]])), \tag{4.7a}$$

$$\delta\pi_- = \delta\lambda(\mathrm{i}g[\eta_\alpha, \pi_-] + 2\sqrt{2}gv\eta_\zeta - \frac{g}{6\sqrt{2}v}([\pi_+[\pi_+, \eta_\zeta]] + [\pi_-[\pi_-, \eta_\zeta]])), \tag{4.7b}$$

$$\delta Z_\mu = \delta\lambda(ig[\eta_\alpha, Z_\mu] - ig[\eta_\zeta, A_\mu] - \sqrt{2}\partial\eta_\zeta), \tag{4.7c}$$

$$\delta A_\mu = \delta\lambda(ig[\eta_\alpha, A_\mu] - ig[\eta_\zeta, Z_\mu] + \sqrt{2}\partial\eta_\alpha), \tag{4.7d}$$

$$\delta\eta_\alpha = \delta\lambda(\frac{\mathrm{i}}{2\sqrt{2}}g \cdot ([\eta_\alpha^b, \eta_\alpha^c] + [\eta_\zeta^b, \eta_\zeta^c])), \tag{4.7e}$$

$$\delta\eta_\zeta = \delta\lambda(\frac{\mathrm{i}}{\sqrt{2}}g \cdot [\eta_\alpha^b, \eta_\zeta^c]), \tag{4.7f}$$

$$\delta\overline{\eta}_\zeta = \delta\lambda \cdot B_\zeta, \tag{4.7g}$$

$$\delta\overline{\eta}_\alpha = \delta\lambda \cdot B_\alpha, \tag{4.7h}$$

$$\delta B_\alpha = 0, \tag{4.7i}$$

$$\delta B_\zeta = 0. \tag{4.7j}$$

Note that the ghost transformations are consistent with our parity symmetry (η_α has parity +1, η_ζ has parity -1, similar to A and Z). The B-fields are introduced to obtain off-shell BRS invariance. They are (commuting) auxiliary scalar fields. In the Lagrangian they will appear only in quadratic terms without derivatives and can be removed using their equations of motion.

The scalar and kinetic Lagrangian, (3.35) and (3.30) are evidently BRS-invariant, $\delta\mathcal{L}_{\text{scalar}} = \delta\mathcal{L}_{\text{kin}} = 0$.

In addition, we can construct another BRS-invariant Lagrangian,

$$\mathcal{L}_{\text{BRS}} = \frac{1}{2}\,\mathrm{tr}\left[s\left(\overline{\eta}_\zeta\left(G_\zeta + \frac{1}{2}\xi B_Z\right)\right)\right] + \frac{1}{2}\,\mathrm{tr}\left[s\left(\overline{\eta}_\alpha\left(G_\alpha + \frac{1}{2}\xi B_A\right)\right)\right] \tag{4.8}$$

written as a pure BRS transformation (since the BRS transformation is *nilpotent*, see below) with the gauge-fixing functionals

$$G_\zeta = \partial Z - \xi m\pi_-, \tag{4.9a}$$

$$G_\alpha = \partial A. \tag{4.9b}$$

The BRS transformation is nilpotent, that is

$$s(s\Psi) = 0, \tag{4.10}$$

which will be proven in section (4.3). This property is *essential* for a general proof of renormalizability. It also allows us to divide the Hilbert space into a physical and unphysical part what will be sketched here.
Because our Lagrangian is invariant under this (continuous) transformation, there is a conserved current, and the integral over the time component of this current will be a conserved charge Q that commutes with the Hamiltonian H and thus with the S-matrix. It acts on anticommuting/commuting fields as $[Q, \Psi]_\pm = s\Psi$, where the plus sign stands for the anticommutator. The relation (4.10) is equivalent to the operator identity

$$Q^2 = 0. \tag{4.11}$$

The charge Q is used to define physical states[2] (see [Ku79])

$$Q|\Psi\rangle_{\text{phys}} = 0. \tag{4.12}$$

A nilpotent operator that commutes with H divides the eigenstates of H into three subspaces.

1. The subspace $\mathcal{V}_1$ of states that are not annihilated by Q.
2. the subspace $\mathcal{V}_2$ of states of the form $|\Psi_2\rangle = Q|\Psi_1\rangle$ where $|\Psi_1\rangle$ is in $\mathcal{V}_1$.
3. the subspace $\mathcal{V}_0$ of states that are annihilated by Q but are not in $\mathcal{V}_2$.

The states in $\mathcal{V}_1$ are characterized as unphysical, by (4.12). It can be easily seen that all states $|\Psi_2\rangle$ in $\mathcal{V}_2$ have zero norm and are orthogonal to all states $|\Psi_0\rangle$ in $\mathcal{V}_0$.

$$\begin{aligned} \langle\Psi_2|\Psi_2\rangle &= \langle\Psi_1|Q|\Psi_2\rangle = 0, \\ \langle\Psi_2|\Psi_0\rangle &= \langle\Psi_1|Q|\Psi_0\rangle = 0. \end{aligned} \tag{4.13}$$

As a consequence these states decouple, i.e. they disappear from any physical matrix element. Physical states that differ only by zero-norm states are physically equivalent. Thus, the physical Hilbert space $\mathcal{H}_{\text{phys}}$ can be defined as quotient space of $\mathcal{V}_0$ with respect to the subspace $\mathcal{V}_2$,

$$\mathcal{H}_{\text{phys}} \equiv \mathcal{V}_0/\mathcal{V}_2. \tag{4.14}$$

Distinguishable physical states are defined as equivalence classes of states with strictly positive norm.

[2]This can be seen as a generalization of the Gupta-Bleuler method in QED, where one needs the additional identity for the *physical* states $\partial^\mu A^+_\mu(x)|\psi\rangle_{\text{phys}} = 0$.

In general, asymptotic states (which are one-particle states) containing ghosts, antighosts or gauge bosons with unphysical polarization always belong to $\mathcal{V}_1$ or $\mathcal{V}_2$. There are exactly four unphysical modes, the ghost, the antighost, the would-be Goldstone boson and the B mode. This is an example of the so-called *quartet mechanism*. According to this, unphysical states always appear as quartets and only combinations of the quartet states with zero norm can appear in the physical space.

Further information to this subject can be found in [Ku79], [Wei], [Böh] and [Pes]. The BRS transformations as defined in (4.7) are not fully nilpotent. In order to get a nilpotent transformation, we have to renormalize the ghosts with a factor of $\sqrt{2}$. This compensates the $\sqrt{2}$ in the transformation of the vector bosons. The Z antighost is multiplied by $-1/2$ and the A antighost by $1/2$. This leads to canonical normalized ghost propagators. Thus, the transformation laws for the nilpotent BRS transformation reads

$$\delta\pi_-^a = \sqrt{2}\delta\lambda(2gf^{abc}\pi_-^b\eta_\alpha^c + 2\sqrt{2}gv\eta_\zeta^a + \frac{4g}{6\sqrt{2}v}f^{abe}f^{cde}(\pi_+^b\pi_+^c\eta_\zeta^d + \pi_-^b\pi_-^c\eta_\zeta^d)), \tag{4.15a}$$

$$\delta\pi_+^a = \sqrt{2}\delta\lambda(2gf^{abc}\pi_+^b\eta_\alpha^c + \frac{4g}{6\sqrt{2}v}(f^{abe}f^{cde} + f^{ace}f^{bde})\pi_+^b\pi_-^c\eta_\zeta^d), \tag{4.15b}$$

$$\delta A_\mu^a = \sqrt{2}\delta\lambda(\sqrt{2}\partial_\mu\eta_\alpha^a + 2gf^{abc}A_\mu^b\eta_\alpha^c - 2gf^{abc}Z_\mu^b\eta_\zeta^c), \tag{4.15c}$$

$$\delta Z_\mu^a = \sqrt{2}\delta\lambda(-\sqrt{2}\partial_\mu\eta_\zeta^a + 2gf^{abc}Z_\mu^b\eta_\alpha^c - 2gf^{abc}A_\mu^b\eta_\zeta^c), \tag{4.15d}$$

$$\delta\eta_\alpha^a = \delta\lambda(-\frac{1}{2}g\cdot 2f^{abc}(\eta_\alpha^b\eta_\alpha^c + \eta_\zeta^b\eta_\zeta^c)), \tag{4.15e}$$

$$\delta\eta_\zeta^a = \delta\lambda(-g\cdot 2f^{abc}\eta_\alpha^b\eta_\zeta^c). \tag{4.15f}$$

4.2 Feynman Rules

Now we have considered all parts of the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{BRS}}. \tag{4.16}$$

This Lagrangian is BRS invariant and all Feynman rules can be derived from it. The vertices from $\mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{kin}}$ were already derived in the last chapter, here we write down the propagators and ghost vertices.

It is useful to decompose the BRS Lagrangian into two parts,

$$\begin{aligned}\mathcal{L}_{\text{BRS}} &= \frac{1}{2}\sum_{i=\alpha,\zeta}\operatorname{tr}\left[(s\overline{\eta}_i)(G_i + \frac{1}{2}\xi B_i) - \overline{\eta}_i s(G_i + \frac{1}{2}\xi B_i)\right] \\ &=: \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}.\end{aligned} \tag{4.17}$$

$\mathcal{L}_{\mathrm{gf}}$ corresponds to the gauge-fixing Lagrangian and enables us to define proper gauge-boson propagators, whereas $\mathcal{L}_{\mathrm{ghost}}$ contains all vertices with ghost fields.
Replacing $s\overline{\eta}_i$ with B_i we obtain:

$$\mathcal{L}_{\mathrm{gf}} = \frac{1}{2}\,\mathrm{tr}\left[\frac{1}{2}\xi B_A^2 + B_A G_\alpha\right] + \frac{1}{2}\,\mathrm{tr}\left[\frac{1}{2}\xi B_Z^2 + B_Z G_\zeta\right]. \tag{4.18}$$

The B-field is an auxiliary field without derivatives. We can replace it via the equation of motion ($\partial\mathcal{L}/\partial B = 0$) according to

$$B = -\frac{1}{\xi}G. \tag{4.19}$$

Thus, we find

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\xi}(\partial_\mu A^{\mu,a})^2 - \frac{1}{2\xi}(\partial_\mu Z^{\mu,a} - \xi m \pi_-^a)^2. \tag{4.20}$$

We can determine the gauge-boson propagators $G^{\mu\nu}$ from the terms $\propto (\partial A)^2, (\partial Z)^2$ in analogy to (4.1):

$$-(G_A^{\mu\nu})^{-1} = -\mathrm{i}\delta^{ab}[k^2 g^{\mu\nu} - (1-\frac{1}{\xi})k^\mu k^\nu]. \tag{4.21}$$

In order to invert this, we define the transverse and longitudinal projection operators

$$g_T^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \qquad g_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}. \tag{4.22}$$

With the ansatz

$$G_A^{\mu\nu} = a g_T^{\mu\nu} + b g_L^{\mu\nu}, \qquad G_A^{\mu\nu}(G_{A,\nu\rho})^{-1} = \delta^\mu_\rho, \tag{4.23}$$

we find

$$a \overset{A}{\sim\!\sim\!\sim\!\sim} b \; = \frac{-\mathrm{i}}{k^2}\left(g^{\mu\nu} - \frac{(1-\xi)k^\mu k^\nu}{k^2}\right)\delta^{ab}. \tag{4.24}$$

Let's turn to the Z boson. In (3.35) we found

$$\mathcal{L}_{\mathrm{scalar}} = (gv)^2\,\mathrm{tr}(Z_\mu Z^\mu) + gv\,\mathrm{tr}(Z^\mu \partial_\mu \pi_-) + \ldots \qquad . \tag{4.25}$$

The first term is a mass term for the Z boson, $m = 2gv$ (remember that taking the trace yields a factor of 2). The second is a mixing term, which is cancelled by the term $m\pi_-^a\partial_\mu Z^{\mu,a}$, coming from $1/(2\xi)(\partial_\mu Z^{\mu,a} - \xi m \pi_-^a)^2$ in $\mathcal{L}_{\mathrm{gf}}$. Our BRS Lagrangian corresponds to the R_ξ-gauges[3], where all bilinear terms involving two different fields

[3] *Arbitrary* other gauges can be obtained by using $\mathcal{L}_{\mathrm{BRS}} = s\Psi$, where Ψ is an arbitrary functional with ghost number -1. The ghost number is defined as $+1$ for η, -1 for $\overline{\eta}$ and 0 for all gauge and matter fields.

cancel in the Lagrangian.

With the additional mass term we get the Z boson propagator

$$a \overset{Z}{\sim\!\sim\!\sim\!\sim} b = \frac{-\mathrm{i}}{k^2 - m^2}\left(g^{\mu\nu} - \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m^2}\right)\delta^{ab}. \tag{4.26}$$

$\mathcal{L}_{\text{gf}}$ also contains a mass term for the field π_- ($m_{\pi_-} = \sqrt{\xi}m$). Thus, the bilinear Lagrangian for π_- reads

$$\mathcal{L}_{2,\text{scalar}} = \frac{1}{4}\operatorname{tr}(\partial\pi_-^2) + \frac{1}{4}\xi m^2 \operatorname{tr}(\pi_-^2), \tag{4.27}$$

and the corresponding scalar propagators are given by

$$a \overset{\pi_+}{\overline{\qquad\quad}} b = \frac{\mathrm{i}}{k^2}\delta^{ab}, \tag{4.28}$$

$$a \overset{\pi_-}{\overline{\qquad\quad}} b = \frac{\mathrm{i}}{k^2 - \xi m^2}\delta^{ab}. \tag{4.29}$$

Remember that π_- with its ξ-dependent mass is the would-be Goldstone boson which is eaten by the massive Z boson, while π_+ is the Little Higgs and a physical degree of freedom.

Unitarity gauge, where there are no unphysical states, is obtained for $\xi \to \infty$. The would-be Goldstone boson gets infinitely heavy and decouples from the S matrix. For $k \to \infty$, the Z boson propagator is of $\mathcal{O}(1)$, while for finite ξ the propagator behaves as $1/k^2$.

Ghosts

Now we are ready to derive the Feynman rules from the part of the BRS Lagrangian

$$\mathcal{L}_{\text{ghost}} = -\frac{1}{2}\operatorname{tr}[\overline{\eta}_\alpha s G_\alpha + \overline{\eta}_\zeta s G_\zeta], \tag{4.30}$$

where the BRS operator s acts on the gauge-fixing functional. We obtain the ghost propagators by considering only terms with two fields in the BRS transformed gauge-fixing functionals G.

$$\begin{aligned}
\mathrm{i}\mathcal{L}_{\zeta,2} &= -\frac{\mathrm{i}}{2}\operatorname{tr}\left[-\frac{1}{2}\overline{\eta}_\zeta\partial^\mu(-\sqrt{2}\partial_\mu(\sqrt{2}\eta_\zeta)) - \xi m(-\frac{1}{2}\overline{\eta}_\zeta(\sqrt{2}m\sqrt{2}\eta_\zeta))\right] \\
&= -\frac{\mathrm{i}}{2}\operatorname{tr}\left[\overline{\eta}_\zeta\partial^\mu\partial_\mu\eta_\zeta + \xi m^2\overline{\eta}_\zeta\eta_\zeta\right] \\
&= \mathrm{i}\overline{\eta}_\zeta^a(-\Box - \xi m^2)\eta_\zeta^a.
\end{aligned} \tag{4.31}$$

The propagator term for the α ghost is analogous, but with zero mass. Hence, we find

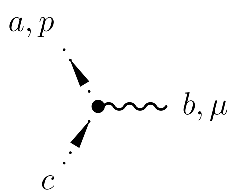
$$a \cdots\overset{\alpha}{\blacktriangleright}\cdots b \;=\frac{\mathrm{i}}{k^2}\delta^{ab}, \tag{4.32}$$

$$a \cdots\overset{\zeta}{\blacktriangleright}\cdots b \;=\frac{\mathrm{i}}{k^2-\xi m^2}\delta^{ab}. \tag{4.33}$$

Gauge-boson ghost three vertices result from

$$\begin{aligned}\mathcal{L}_{\zeta,3} &= -\frac{1}{2}\operatorname{tr}\Big(-\frac{1}{2}\overline{\eta}_\zeta\partial^\mu(+ig[\sqrt{2}\eta_\alpha, Z_\mu]-ig[\sqrt{2}\eta_\zeta, A_\mu])\Big)\\ &= -\frac{\sqrt{2}}{4}\mathrm{i}g2\Big(\overline{\eta}^a_\zeta(-2\mathrm{i}f^{abc}\partial^\mu(A^b_\mu\eta^c_\zeta)+2\mathrm{i}f^{abc}\partial^\mu(Z^b_\mu\eta^c_\alpha))\Big)\\ &= \sqrt{2}gf^{abc}\left((\partial^\mu\overline{\eta}^a_\zeta)(A^b_\mu\eta^c_\zeta)-(\partial^\mu\overline{\eta}^a_\zeta)(Z^b_\mu\eta^c_\alpha)\right).\end{aligned} \tag{4.34}$$

Replacing $\partial\to\mathrm{i}p$ (*outgoing* momentum of the antighost) and multiplying with i, we obtain the $A\overline{\eta}_\zeta\eta_\zeta$-and the $Z\overline{\eta}_\zeta\eta_\alpha$-vertices



$$= \pm\sqrt{2}gp^\mu f^{abc}. \tag{4.35}$$

The positive sign is for the vertices with an A boson.The $A\overline{\eta}_\alpha\eta_\alpha$- and the $Z\overline{\eta}_\alpha\eta_\zeta$-vertices from $\mathcal{L}_\alpha$ are analogous.

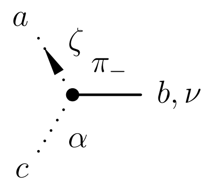
Scalar three and four vertices result from $\delta\pi_-$, i.e.

$$\mathcal{L}_3 = \frac{\xi m}{2}\operatorname{tr}\left(-\frac{1}{2}\overline{\eta}_\zeta[\pi_-,\sqrt{2}\eta_\alpha]\right) = -2\sqrt{2}g^2v\xi f^{abc}\,\overline{\eta}^a_\zeta\pi^b_-\eta^c_\alpha, \tag{4.36}$$

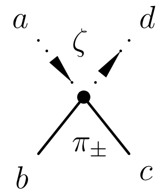
and

$$\begin{aligned}\mathcal{L}_4 &= \frac{\sqrt{2}}{4}\xi m\operatorname{tr}\Big(\overline{\eta}_\zeta(-\frac{g}{6\sqrt{2}}\delta\lambda([\pi_+,[\pi_+,\eta_\zeta]]+[\pi_-,[\pi_-,\eta_\zeta]]))\Big)\\ &= \frac{2g^2v}{4\cdot 6v}\delta\lambda\operatorname{tr}(\lambda^a[\lambda^b,[\lambda^c.\lambda^d]])\cdot(\overline{\eta}^a_\zeta\pi^b_+\pi^c_+\eta^d_\zeta+\overline{\eta}^a_\zeta\pi^b_-\pi^c_-\eta^d_\zeta)\\ &= -\frac{1}{3}\xi g^2 2f^{abe}f^{cde}\cdot(\overline{\eta}^a_\zeta\pi^b_+\pi^c_+\eta^d_\zeta+\overline{\eta}^a_\zeta\pi^b_-\pi^c_-\eta^d_\zeta),\end{aligned} \tag{4.37}$$

which leads to the vertices



$$= -2\sqrt{2}\mathrm{i}g^3 v f^{abc}, \tag{4.38}$$



$$= -\frac{2\mathrm{i}}{3}\xi g^2 \left[f^{abe} f^{cde} + f^{ace} f^{bde} \right]. \tag{4.39}$$

4.3 Nilpotence

In this section we show that the BRS transformation of any product of fields Φ is nilpotent, that is[4]

$$\delta_{BRS}(s\Phi) = 0, \tag{4.40}$$

or equivalently

$$s(s\Phi) = 0. \tag{4.41}$$

4.3.1 Nilpotence of the BRS Transformation of the Lagrangian

First, we prove that it is sufficient to prove the nilpotence for a single field, $s(s\Psi) = 0$. Applying the transformation on a prduct of two fields reads

$$\delta_{BRS}(\Psi_1\Psi_2) = \delta\lambda(s\Psi_1)\Psi_2 + \Psi_1(\delta\lambda s\Psi_2) = \delta\lambda[(s\Psi_1)\Psi_2 \pm \Psi_1 s\Psi_2], \tag{4.42}$$

where the sign $\pm$ is plus for bosonic Ψ_1 and minus for fermionic Ψ_1. For $\delta_{BRS}(s\Psi) = 0$, the BRS transformation on $s(\Psi_1\Psi_2)$ is

$$\delta_{BRS}s(\Psi_1\Psi_2) = (s\Psi_1)\delta\lambda(s\Psi_2) \pm \delta\lambda(s\Psi_1)(s\Psi_2). \tag{4.43}$$

Since $s\Psi$ has statistics opposite to Ψ, moving $\delta\lambda$ to the left introduces a sign factor $\mp$ and one finds

$$\delta_{BRS}(\Psi_1\Psi_2) = \delta\lambda[\mp s(\Psi_1)(s\Psi_2) \pm s(\Psi_1)(s\Psi_2)] = 0. \tag{4.44}$$

[4] We do note write $\delta_{BRS}(\delta_{BRS}\Phi)$ since this would involve $\delta\lambda^2$ which is trivially zero.

Continuing this way, we see that BRS transformations are nilpotent on any products of fields,

$$\delta_{BRS} s(\Psi_1 \Psi_2 \ldots) = 0. \tag{4.45}$$

Any functional $F[\Psi]$ can be written as a sum of integrals of such products with c-number coefficients, so this completes the proof of the nilpotence of $\mathcal{L}_{BRS}$ under the assumption of the nilpotence of a single field, which will be proven in the following sections.

In the following, we need the BRS transformations (4.15a) and use the Jacobi-identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0. \tag{4.46}$$

With $[T^a, T^b] = \mathrm{i} f^{abc} T^c$ (4.46) leads to

$$f^{12e} f^{34e} + f^{13e} f^{42e} + f^{14e} f^{23e} = 0, \tag{4.47}$$

where we replaced a, b, c, d with the numbers $1, 2, 3, 4$ to emphasize the cyclic structure.

4.3.2 Nilpotence of the BRS Transformation of the Ghosts

The transformation of the α ghost is given by

$$\begin{aligned}
&\delta_{BRS}(s\eta^a_\alpha) \propto f^{abc}[(\delta\eta^b)\eta^c_\alpha] + \eta^b_\alpha(\delta\eta^c_\alpha) + (\delta\eta^b_\zeta)\eta^c_\zeta + \eta^b_\zeta(\delta\eta^c_\zeta)] \\
&= f^{abc}[f^{bde}\delta\lambda(\eta^d_\alpha\eta^e_\alpha + \eta^d_\zeta\eta^e_\zeta)\eta^c_\alpha + \eta^b_\alpha\delta\lambda f^{cde}(\eta^d_\alpha\eta^e_\alpha + \eta^d_\zeta\eta^e_\zeta) + 2\delta\lambda f^{bde}\eta^d_\alpha\eta^e_\zeta\eta^c_\zeta + 2f^{cde}\eta^b_\zeta\delta\lambda\eta^d_\alpha\eta^e_\zeta] \\
&= \delta\lambda f^{abc}[f^{bde}(\eta^d_\alpha\eta^e_\alpha\eta^c_\alpha + \eta^d_\zeta\eta^e_\zeta\eta^c_\alpha) - f^{cde}(\eta^b_\alpha\eta^d_\alpha\eta^e_\alpha + \eta^b_\alpha\eta^d_\zeta\eta^e_\alpha) + 2f^{bde}\eta^d_\alpha\eta^e_\zeta\eta^c_\zeta - 2f^{cde}\eta^b_\zeta\eta^d_\alpha\eta^e_\zeta].
\end{aligned} \tag{4.48}$$

The product $\eta^b_\alpha\eta^d_\alpha\eta^e_\alpha$ is antisymmetric ($\eta^2_\alpha\eta^3_\alpha\eta^4_\alpha = -\eta^3_\alpha\eta^2_\alpha\eta^4_\alpha = \eta^3_\alpha\eta^4_\alpha\eta^2_\alpha = \eta^4_\alpha\eta^2_\alpha\eta^3_\alpha$), so terms with three equal ghosts vanish due to the Jacobi-identity. The first and last term in the mixed ghost terms also cancel for the same reason,

$$\begin{aligned}
f^{abc} f^{bde}\eta^d_\zeta\eta^e_\zeta\eta^c_\alpha - 2f^{abc} f^{cde}\eta^e_\zeta\eta^b_\zeta\eta^d_\zeta &= (f^{abc} f^{bde} - 2f^{aeb} f^{bcd})\eta^d_\zeta\eta^e_\zeta\eta^c_\alpha \\
&= (f^{abc} f^{bde} - f^{aeb} f^{bcd} + f^{adb} f^{bce})\eta^d_\zeta\eta^e_\zeta\eta^c_\alpha = 0,
\end{aligned} \tag{4.49}$$

where we relabelled the indices, used the anticommutativity of the ghosts and the antisymmetry of f^{abc}. The calculation for the second and third term is analogous.

The nilpotence of the transformation acting on the antighost (and the auxiliary B field) is obvious, cf. (4.7).

4.3.3 Nilpotence of the BRS Transformation of the Vector Bosons

We use the same manipulations as above, only the number of terms is larger. Again, only one vector boson is shown, the other is analogous.

$$\begin{aligned}
&s(sZ^a_\mu) = 2gf^{abc}\partial_\mu(\eta^b_\alpha\eta^c_\zeta) \\
&\quad - 2gf^{abc}\big(\sqrt{2}\partial_\mu\eta^b_\alpha + 2gf^{bde}A^d_\mu\eta^e_\alpha - 2gf^{bde}Z^d_\mu\eta^e_\zeta\big)\eta^c_\zeta - 2gf^{abc}A^b_\mu\big(-g\cdot 2f^{cde}\eta^d_\alpha\eta^e_\zeta\big) \\
&+2gf^{abc}\big(-\sqrt{2}\partial_\mu\eta^b_\zeta+2gf^{bde}Z^d_\mu\eta^e_\alpha-2gf^{bde}A^d_\mu\eta^e_\zeta\big)\eta^c_\alpha+2gf^{abc}Z^b_\mu\big(-\frac{1}{2}g\cdot 2f^{cde}(\eta^d_\alpha\eta^e_\alpha+\eta^d_\zeta\eta^e_\zeta)\big).
\end{aligned} \tag{4.50}$$

The terms with derivatives add up to zero (note that $\partial_\mu(\eta^b_\alpha\eta^c_\zeta) = (\partial_\mu\eta^b_\alpha)\eta^c_\zeta - (\partial_\mu\eta^c_\zeta)\eta^b_\alpha$). For the A_μ and Z_μ terms the Jacobi identity applies, i.e.

$$\begin{aligned}
&-4g^2f^{abc}f^{bde}A^d_\mu\eta^e_\alpha\eta^c_\zeta + 4g^2f^{abc}f^{cde}A^b_\mu\eta^d_\alpha\eta^e_\zeta - 4g^2f^{abc}f^{bde}A^d_\mu\eta^e_\zeta\eta^c_\alpha \\
&\qquad\qquad = 4g^2A^d_\mu\eta^e_\alpha\eta^c_\zeta(-f^{abc}f^{bde}+f^{adb}f^{bec}+f^{abe}f^{bdc}) = 0, \\
&4g^2f^{abc}f^{bde}Z^d_\mu\eta^e_\zeta\eta^c_\zeta - 2g^2f^{abc}f^{cde}Z^b_\mu\eta^d_\zeta\eta^e_\zeta \\
&\qquad\qquad = 2g^2[(f^{abc}f^{bde}-f^{abe}f^{bdc})Z^d_\mu\eta^e_\zeta\eta^c_\zeta - f^{adb}f^{bec}Z^d_\mu\eta^e_\zeta\eta^c_\zeta] = 0.
\end{aligned} \tag{4.51}$$

4.3.4 Nilpotence of the BRS Transformation of the Scalars

This is the hard part, since we have to check the nilpotence involving terms containing up to four fields (we cannot do more, since in the transformation (3.44) of the scalars, we have ignored terms of $\mathcal{O}(\pi^4\zeta)$ which contribute to terms with five fields). The calculation involves terms with three structure constants f^{abc}, so we would need nested Jacobi-identities. We will avoid this by rewriting these terms as commutators of four generators T. There are two possibilities to arrange four generators in commutators,

$$\begin{aligned}
[T^b[T^c[T^d,T^e]]] &= -\mathrm{i}f^{deg}f^{cgf}f^{bfa}T^a = -\mathrm{i}f^{bfa}f^{cgf}f^{deg}T^a, \\
[[T^b,T^c],[T^d,T^e]] &= -f^{bcf}f^{deg}[T^f,T^g] = -\mathrm{i}f^{bcf}f^{deg}f^{fga}T^a = -\mathrm{i}f^{bcf}f^{fga}f^{deg}T^a.
\end{aligned} \tag{4.52}$$

Note the different contractions in (4.52). Applying s^2 on π_-, we obtain

$$\begin{aligned}
&s(s\pi^a_-) = -2\sqrt{2}gv2gf^{abc}\eta^b_\alpha\eta^c_\zeta \\
&+2gf^{abc}\Big[2gf^{bde}\pi^d_-\eta^e_\alpha + 2\sqrt{2}gv\eta^b_\zeta + \frac{2g}{6\sqrt{2}v}(f^{bdi}f^{fgi}+f^{bfi}f^{dgi})(\pi^d_+\pi^f_+\eta^g_\zeta+\pi^d_-\pi^f_-\eta^g_\zeta)\Big]\eta^c_\alpha \\
&\qquad\qquad - gf^{cde}\pi^b_-\Big[\eta^d_\alpha\eta^e_\alpha+\eta^d_\zeta\eta^e_\zeta\Big] + \frac{2g}{6\sqrt{2}v}(f^{abe}f^{cde}+f^{ace}f^{bde}).
\end{aligned}$$

$$\Big[(\delta\pi_+^b)\pi_+^c\eta_\zeta^d + (\delta\pi_+^c)\pi_+^b\eta_\zeta^d + \pi_+^b\pi_+^c(\delta\eta_\zeta^d) + \underbrace{(\delta\pi_-^b)}_{2\sqrt{2}gv\eta_\zeta^b + 2gf^{bde}\pi_-^d\eta_\alpha^e + \dots}\pi_-^c\eta_\zeta^d + (\delta\pi_-^c)\pi_-^b\eta_\zeta^d + \pi_-^b\pi_-^c(\delta\eta_\zeta^d)\Big]. \tag{4.53}$$

We have only two terms involving two (ghost)fields, these terms cancel each other. There are also two terms containing $\pi_-\eta_\alpha\eta_\alpha$ which cancel due to the Jacobi-identity. For $\pi_-\eta_\zeta\eta_\zeta$ we have three terms, two of them result from $\delta\pi_-$ in the last line of (4.53). Since these are symmetric in b and c this results in a factor of two. Thus, we obtain

$$\begin{aligned} &-2g^2 f^{abc}f^{cde}\pi_-^b\eta_\zeta^d\eta_\zeta^e + \frac{4}{3}g^2(f^{abe}f^{cde} + f^{ace}f^{bde})\eta_\zeta^b\pi_-^c\eta_\zeta^d \\ &\qquad = (-2g^2 f^{ace}f^{ebd} + \frac{4}{3}g^2(f^{abe}f^{cde} + f^{ace}f^{bde}))\eta_\zeta^b\pi_-^c\eta_\zeta^d \\ &\qquad\qquad = \frac{2}{3}g^2\eta_\zeta^b\pi_-^c\eta_\zeta^d(f^{ace}f^{bde} + 2f^{abe}f^{cde}). \end{aligned} \tag{4.54}$$

which also cancels due to the Jacobi-identity. Now, consider the terms with four fields. It is sufficient to treat $\pi_-\pi_-\eta_\zeta\eta_\alpha$, since the calculation is analogous for π_+. We also suppress an overall factor of $4g^2/6\sqrt{2}$ and obtain

$$\begin{aligned} s(s\pi_-) = \Big[&2f^{abc}f^{bdi}f^{fgi}\pi_-^d\pi_-^f\eta_\zeta^g\eta_\alpha^c + (f^{abe}f^{cde} + f^{ace}f^{bde}) \\ &[\underbrace{f^{bfg}\pi_-^f\pi_-^c\eta_\alpha^g\eta_\zeta^d + f^{cfg}\pi_-^f\pi_-^b\eta_\alpha^g\eta_\zeta^d}_{2f^{bfg}\pi_-^f\pi_-^c\eta_\alpha^g\eta_\zeta^d} - \underbrace{f^{dfg}\pi_-^b\pi_-^c\eta_\alpha^f\eta_\zeta^g}_{\to 2f^{abe}f^{cde}f^{dfg}\dots}]\Big]T^a. \end{aligned} \tag{4.55}$$

We relabel the indices to factor out the fields $\pi_-^f\pi_-^c\eta_\alpha^g\eta_\zeta^d$ and use the antisymmetry of f^{abc} to get the indices into the order of (4.52). Expression (4.55) reads

$$\begin{aligned} &-\frac{2}{\mathrm{i}}\pi_-^f\pi_-^c\eta_\alpha^g\eta_\zeta^d T^a\Big[f^{gba}f^{ceb}f^{fde} + f^{fgb}f^{bea}f^{cde} - f^{cea}f^{dbe}f^{fgb} - f^{fea}f^{cbe}f^{gdb}\Big] \\ &\quad \to [T^g[T^c[T^f,T^d]]] + [[T^f,T^g],[T^c,T^d]] - [T^c[T^d[T^f,T^g]]] - [[T^f,T^c],[T^g,Td^d]]. \end{aligned} \tag{4.56}$$

Now, we have to expand all these commutators. The terms cancel completely, when using the symmetry in the indices f and c, since we have two identical particles. For example, $T^gT^cT^fT^d$ from the first commutator cancels $-T^gT^fT^cT^d$ from the second commutator.

5 Background Field Method

In the above, the final Lagrangian has not been gauge invariant due to the gauge fixing and the inclusion of ghosts. It is only invariant under the nonlinear BRS transformations. As a consequence, Green functions do not directly reflect the underlying gauge invariance, but rather satisfy complicated Slavnov-Taylor identities resulting from BRS invariance. They also depend on the particular gauge fixing chosen, and only physical quantities such as cross-sections are gauge-independent.

The background field method is a technique which allows for fixing a gauge without destroying explicit gauge invariance. This makes calculations in gauge theories easier both technically and conceptually. For example the β-function of non-abelian gauge theories can be calculated from the background field two-point function alone. No vertex functions have to be considered, which is a considerable simplification compared to the conventional method.

Fixing the gauge is necessary for defining propagators. For external fields gauge fixing is not mandatory. So, the basic idea of the background field method is to split the gauge field appearing in the classical action according to

$$A^a_\mu \to A^a_\mu + \hat{A}^a_\mu, \tag{5.1}$$

where $\hat{A}$ is an arbitrary classical background field and plays the role of an external field. Thus no gauge fixing is necessary for $\hat{A}$. A is the fluctuating quantum field with properly defined propagators, requiring gauge-fixing. The background field is treated as an external source while the quantum field is the variable of integration in the functional integral. Then, a gauge is chosen (the background field gauge) which breaks the gauge invariance only of the A field, but retains gauge invariance in terms of the $\hat{A}$ field.

A complete treatment of the BFM can be found in [Ab81] and the method will be only sketched here. The classical Yang-Mills Lagrangian for a gauge field reads

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} \tag{5.2}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^v_\nu. \tag{5.3}$$

After splitting the gauge field according to (5.1), the Lagrangian can be written as

$$\mathcal{L}_{\text{BFM}} = -\frac{1}{4}(\hat{F}^a_{\mu\nu} + \hat{D}^{ac}_\mu A^c_\nu - \hat{D}^{ac}_\nu A^c_\mu + gf^{abc}A^b_\mu A^c_\nu)^2, \tag{5.4}$$

where

$$\begin{aligned} \hat{F}^a_{\mu\nu} &= \partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu + gf^{abc}\hat{A}^b_\mu \hat{A}^v_\nu, \\ \hat{D}^{ac}_\mu &= \partial_\mu \delta^{ac} + gf^{abc}\hat{A}^b_\mu \end{aligned} \tag{5.5}$$

are the field strength and the covariant derivative with respect to the background field. For fixed background field, the Lagrangian (5.4) is invariant under the infinitesimal gauge transformations

$$\begin{aligned} A^a_\mu \to A'^a_\mu &= A^a_\mu + \frac{1}{g}\delta^{ac}\partial_\mu \alpha + f^{abc}(A^b_\mu + \hat{A}^b_\mu)\alpha^c \\ &= A^a_\mu + \frac{1}{g}\hat{D}^{ac}_\mu \alpha^c + f^{abc}A^b_\mu \alpha^c. \end{aligned} \tag{5.6}$$

In order to define the functional integral, we have to fix the gauge for A^a_μ. We choose the *background field gauge* condition

$$G^a_{\text{BFM}} = \partial_\mu A^a_\mu + gf^{abc}\hat{A}^b_\mu A^c_\mu = \hat{D}^{ab,\mu}A^b_\mu. \tag{5.7}$$

The BRS-Lagrangian is obtained from this gauge-fixing functional and the gauge transformation (5.6).

The complete Lagrangian

$$\mathcal{L}_{\text{BFM}} = \mathcal{L}_{\text{YM,BFM}} - \frac{1}{2\xi}(G^a_{\text{BFM}})^2 + \mathcal{L}_{\text{FP,BFM}} \tag{5.8}$$

is gauge-fixed, but remains invariant under the local transformations

$$\begin{aligned} \hat{A} \to \hat{A}'^a_\mu &= \hat{A}^a_\mu + \frac{1}{g}\hat{D}^{ab}\hat{\alpha}^b, \\ A^a_\mu \to A'^a_\mu &= A^a_\mu + f^{abc}A^b_\mu \hat{\alpha}^c, \\ \eta^a \to \eta'^a &= \eta^a + f^{abc}\eta^b \hat{\alpha}^c, \\ \overline{\eta}^a \to \overline{\eta}'^a &= \overline{\eta}^a + f^{abc}\overline{\eta}^b \hat{\alpha}^c. \end{aligned} \tag{5.9}$$

The background field $\hat{A}$ transforms inhomogeneously as a gauge field, while the quantum field A and the ghost fields $\eta, \overline{\eta}$ transform (homogeneously) as matter field. This invariance in the background field gauge follows directly from the fact that a background field $\hat{A}^a_\mu$ appears in (5.8) only within covariant derivatives and the field strength. To construct an explicitly gauge invariant effective action[1] $\hat{\Gamma}$, we simply set the source for the quantum field A^a_μ in the generating functional equal to zero,

$$\hat{\Gamma}\{\hat{A}\} = \Gamma_{\mathrm{BFM}}\{A = 0, \hat{A}\}. \tag{5.10}$$

One can ask for the relation of this gauge invariant effective action $\hat{\Gamma}$ and the effective action Γ_{conv} in the conventional method. Performing a shift $A^a_\mu \to A^a_\mu + \hat{A}^a_\mu$, it turns out that the BFM effective action and the conventional action are related by

$$\Gamma_{\mathrm{BFM}}\{A, \hat{A}\} = \Gamma_{\mathrm{conv}}\{\tilde{A}\}|_{\tilde{A}=A+\hat{A}}, \tag{5.11}$$

evaluated with the unconventional gauge-fixing term $\tilde{G}^a = \partial_\mu A^{\mu,a} - \partial_\mu \hat{A}^{\mu,a} + g f^{abc} \hat{A}^b_\mu A^{\mu,c}$. Thus, the gauge invariant effective action is given by

$$\hat{\Gamma}\{\hat{A}\} = \Gamma_{\mathrm{conv}}\{\tilde{A}\}|_{\tilde{A}=\hat{A}}. \tag{5.12}$$

From this effective action one computes the Feynman rules. In one-particle irreducible diagrams, quantum fields only appear in loops (since the source of A has been set to zero) and background fields only appear in external lines (since the functional integral is only over A).

In the model under study with two gauge groups and the mixing term $\mathcal{L}_M = gv\,\mathrm{tr}[Z\partial\pi_-]$ we choose a background field gauge which also cancels the mixing terms in the Lagrangian

$$G_{\mathrm{BFM},A_L} = \partial A_L - ig[\hat{A}_L, A_L] + \xi \frac{gv}{\sqrt{2}}\pi_-, \tag{5.13a}$$

$$G_{\mathrm{BFM},A_R} = \partial A_R - ig[\hat{A}_R, A_R] + \xi \frac{gv}{\sqrt{2}}\pi_- \ . \tag{5.13b}$$

Rewriting (5.13) in terms of physical fields

$$Z = \frac{1}{\sqrt{2}}(A_R - A_L), \tag{5.14}$$

$$A = \frac{1}{\sqrt{2}}(A_R + A_L), \tag{5.15}$$

we obtain

$$\mathcal{L}^{\mathrm{BRS}}_{\mathrm{BFM}} = \frac{1}{2}\,\mathrm{tr}\left[s\left(\overline{\eta}_\alpha G_A + \frac{1}{2}\xi\overline{\eta}_\alpha B_A\right)\right] + \frac{1}{2}\,\mathrm{tr}\left[s\left(\overline{\eta}_\zeta G_Z + \frac{1}{2}\xi\overline{\eta}_\zeta B_Z\right)\right] \tag{5.16}$$

[1] In the functional path integral method, the effective action is defined as a Legendre transformation of the generating functional of connected Green functions.

with

$$G_A(A,Z) = \partial A - \frac{ig}{\sqrt{2}}([\hat{A},A]+[\hat{Z},Z]), \tag{5.17a}$$

$$G_Z(A,Z,\pi_-) = \partial Z - \frac{ig}{\sqrt{2}}([\hat{Z},A]+[\hat{A},Z]) - 2\xi g v \pi_- \ . \tag{5.17b}$$

Feynman rules

The Feynman rules can be obtained from the scalar and BRS Lagrangian in the usual way. All vertices that result from the scalar part are equal to those in the conventional formalism, no matter whether they contain background or quantum fields. Different vertices result for fields which are involved in the gauge-fixing term. For calculating the vertices resulting from the BRS Lagrangian, we need the transformation properties of the fields involved:

$$\delta Z^a_\mu = -\sqrt{2}\partial_\mu \zeta^a - 2g f^{abc}\alpha^b (Z+\hat{Z})^c_\mu + 2g f^{abc}\zeta^b (A+\hat{A})^c_\mu, \tag{5.18}$$

$$\delta A^a_\mu = \sqrt{2}\partial_\mu \alpha^a - 2g f^{abc}\alpha^b (A+\hat{A})^c_\mu + 2g f^{abc}\zeta^b (Z+\hat{Z})^c_\mu. \tag{5.19}$$

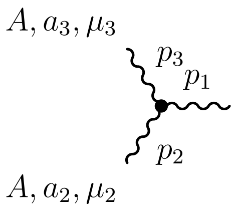
We only derive vertices with one or two $\hat{A}$'s from $\mathcal{L}_{\mathrm{BRS}}$, since in chapter 8 we only calculate the two-point function of the massless background field $\hat{A}$. The gauge-fixing Lagrangian in the background field gauge reads

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{4\xi}\,\mathrm{tr}\left[G_A^2 + G_Z^2\right] \tag{5.20}$$

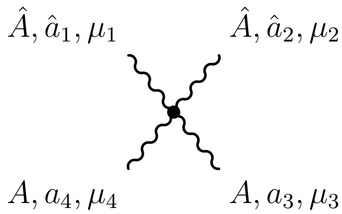
and results in

$$\mathcal{L} \propto \left(-\frac{\sqrt{2}g}{\xi} f^{a_1 b a_2}\, \partial_\mu A^{a_1,\mu} \hat{A}^b_\nu A^{a_2,\nu} - \frac{2g^2}{\xi} f^{a_1 b_1 e} f^{a_2 b_2 e}\, \hat{A}^{a_1}_\mu \hat{A}^{a_2}_\nu A^{b_1,\mu} A^{b_2,\nu} \right) \tag{5.21}$$

which imply the modified three and four gauge-boson vertices



$$\hat{A}, \hat{a}_1, \mu_1 = \sqrt{2} g f^{\hat{a}_1 a_2 a_3} \begin{bmatrix} g^{\mu_1\mu_2}(2p_1 + (1-\frac{1}{\xi}p_3))^{\mu_3} \\ +g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} \\ +g^{\mu_3\mu_1}(2p_1+(1-\frac{1}{\xi})p_2)^{\mu_2} \end{bmatrix}, \tag{5.22}$$



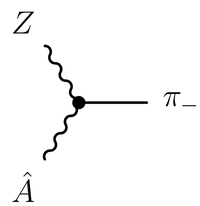
$$= -2\mathrm{i}g^2 \begin{bmatrix} (g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) & f^{\hat{a}_1\hat{a}_2 b} f^{a_3 a_4 b} \\ +(g^{\mu_1\mu_4}g^{\mu_3\mu_2} - g^{\mu_1\mu_2}g^{\mu_3\mu_4} - \frac{1}{\xi}g^{\mu_1\mu_3}g^{\mu_2\mu_4}) & f^{\hat{a}_1 a_3 b} f^{a_4\hat{a}_2 b} \\ +(g^{\mu_1\mu_2}g^{\mu_4\mu_3} - g^{\mu_1\mu_3}g^{\mu_4\mu_2} + \frac{1}{\xi}g^{\mu_1\mu_4}g^{\mu_3\mu_2}) & f^{\hat{a}_1 a_4 b} f^{\hat{a}_2 a_3 b} \end{bmatrix}. \tag{5.23}$$

Analog vertices can be found for the massive quantum fields Z.

From the mixed term in G_Z^2 results a contribution to a vertex with a π_--leg,

$$\mathrm{i}\mathcal{L}_3' = -\frac{i}{4\xi}\,\mathrm{tr}\left(\sqrt{2}\mathrm{i}g\xi m[\hat{A}, Z]\pi_-\right) = 2\sqrt{2}\mathrm{i}g^2 v g^{\mu\nu} f^{abc}\hat{A}^a Z^b \pi_-^c. \tag{5.24}$$

The same contribution to the vertex appears in $\mathcal{L}_{\text{scalar}}$, but with a relative minus-sign. The $\hat{A}Z\pi_-$-vertex vanishes

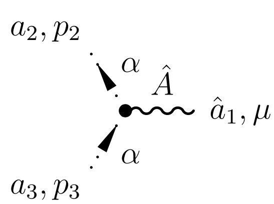


$$= 0. \tag{5.25}$$

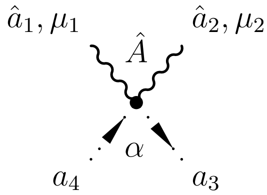
Let's turn to the ghost vertices. We explicitly derive vertices with one or two $\hat{A}$ coming from G_A, vertices from G_Z can be found analogously. The relevant terms in the Lagrangian are

$$\begin{aligned} \frac{1}{2}\,\mathrm{tr}\,(-\overline{\eta}_\alpha(sG_A)) &\to -\frac{1}{2}\,\mathrm{tr}\left(\overline{\eta}_\alpha s(\partial A - \frac{\mathrm{i}g}{\sqrt{2}}[\hat{A}, A])\right) = -\overline{\eta}^a_\alpha \partial(sA^a) - \sqrt{2}gf^{abc}\overline{\eta}^a_\alpha)\hat{A}^b(sA^c) \\ &\to \sqrt{2}gf^{abc}\overline{\eta}^a_\alpha\partial(\eta^b_\alpha\hat{A}^c) - \sqrt{2}gf^{abc}\overline{\eta}^a_\alpha\hat{A}^b\partial\eta^c_\alpha + 2g^2 f^{abc}f^{cde}\overline{\eta}^a_\alpha\hat{A}^b\eta^d_\alpha\hat{A}^e \\ &= \underbrace{-\sqrt{2}gf^{abc}\big((\partial\overline{\eta}^a_\alpha)\eta^b_\alpha\hat{A}^c - \overline{\eta}^a_\alpha\hat{A}^c(\partial\eta^b_\alpha)\big)}_{\to -\sqrt{2}gf^{abc}(p_2 - p_3)^\mu} + \underbrace{2g^2 f^{abc}f^{cde}\overline{\eta}^a_\alpha\hat{A}^b\eta^d_\alpha\hat{A}^e}_{\to 2\mathrm{i}g^2 g^{\mu\nu}(f^{abc}f^{cde} + f^{aec}f^{cbe})} \end{aligned} \tag{5.26}$$

In the intermediate steps, we evaluated the traces, renormalized the (anti)ghosts with factors of $1/2$ and $\sqrt{2}$, integrated by parts and replaced $\partial \to -\mathrm{i}p$, where p_2 is the (incoming) momentum of the antighost. The vertices read



$$= -\sqrt{2}g(p_2 - p_3)^\mu f^{\hat{a}_1 a_2 a_3}, \tag{5.27}$$



$$= 2\mathrm{i}g^2 \left[f^{\hat{a}_1 a_3 b} f^{\hat{a}_2 a_4 b} + f^{\hat{a}_1 a_4 b} f^{\hat{a}_2 a_3 b} \right] g^{\mu_1 \mu_2}. \tag{5.28}$$

$$\tag{5.29}$$

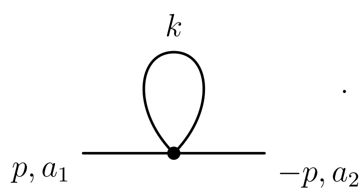
The background field vertices can be also found in appendix D.

6 Self-energies

Now we are able to calculate the self-energies for all fields. All relevant vertices and propagators have been derived in chapter 3 and chapter 4. We use dimensional regularization to regularize the divergent integrals and express the results in terms of scalar N-point integrals $A_0, B_0, \ldots$ and tensor coefficients (e.g. $B_1, B_{00}, B_{11}, C_{11}, C_{12}, \ldots$) which are functions of the scalar N-point integrals. We use the notation and conventions as defined in [Kil02]. All necessary formulas for this work can be found in appendix A. Especially note the treatment of quadratic divergences as a limit of $D \to 2$ in the general D-dimensional integrals.

6.1 Scalar Self-energies

The vertices with four scalars contribute in the loop



. (6.1)

Using $k_1 = -k_2 = p$ and $k_3 = -k_4 = k$, the numerator gets

$$\frac{\mathrm{i}}{6v^2}\left((p-k)^2 f^{a_1cb}f^{ca_2b} - (p+k)^2 f^{a_1cb}f^{a_2cb}\right) = -\frac{2\mathrm{i}}{6v^2}(p^2+k^2)C_A\delta^{a_1a_2}. \tag{6.2}$$

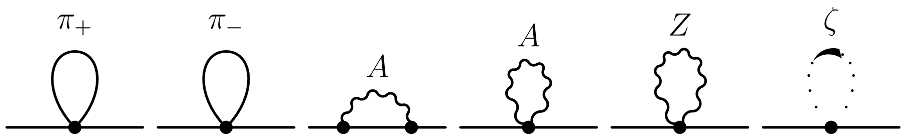
The product of the structure constants $f^{a_1cb}f^{a_2cb}$ with two of their indices contracted yields

$$f^{a_1cb}f^{a_2cb} = C_A\delta^{a_1a_2}, \tag{6.3}$$

where C_A the quadratic Casimir operator of the group. It depends on the representation of the group, e.g. $C_A = N$ in the adjoint representation of SU(N).

6.1.1 Self-energy of π_+

Since π_+ plays the role of our 'Little Higgs' particle, its mass should be protected from quadratic divergences. The following six graphs



(6.4)

contribute to the self-energy.

The following notation is used: Π stands for a self-energy diagram, its first index indicates if it results from a four vertex or from three vertices. The next index denotes the outer legs of the diagram and the last indices list all other involved particles.

The first graph yields

$$\Pi_{4\pi_+} = \frac{1}{2}\frac{-\mathrm{i}}{3v^2}C_A\delta^{ab}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{i}(p^2+k^2)}{k^2} = \frac{1}{6v^2}\frac{\mathrm{i}}{16\pi^2}C_A\delta^{ab}p^2A_0(0), \tag{6.5}$$

where the first factor of $1/2$ is a symmetry factor. Note that the integral is quadratically divergent, since the A_0 function has a (mass independent) pole when we send the dimension $d \to 2$. For $d \to 4$ we have no pole, since the mass is zero and thus not logarithmically divergent. The global factor of $\mathrm{i}/(16\pi^2)C_A\delta^{ab}$ appears in all other diagrams and will be consequently omitted in the following.

For the second graph we obtain

$$\Pi_{4\pi_+\pi_-} = \frac{1}{2}\frac{-\mathrm{i}}{3v^2}\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{i}(p^2+k^2-\xi m^2+\xi m^2)}{k^2-\xi m^2} = \frac{1}{6v^2}(p^2+\xi m^2)A_0(\xi m^2). \tag{6.6}$$

The third graph requires a bit more calculation,

$$\begin{aligned}
\Pi_{3\pi_+A} &= \int\frac{d^dk}{(2\pi)^d}2g^2(2p+k)^\mu f^{hal}(-2p-k)^\nu f^{hbl}\frac{\mathrm{i}}{(k+p)^2}\cdot\frac{-\mathrm{i}}{k^2}\left(g_{\mu\nu}-\frac{(1-\xi)k_\mu k_\nu}{k^2}\right)\\
&= -2g^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{(2p+k)^2}{k^2(k+p)^2}-(1-\xi)\frac{k_\mu(2p+k)^\mu k_\nu(2p+k)^\nu}{k^4(k+p)^2}\right)\\
&= -2g^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{4p^2+4pk+k^2}{k^2(k+p)^2}-(1-\xi)\frac{4p_\mu p_\nu k^\mu k^\nu+4p_\mu k^\mu k^2+k^4}{k^4(k+p)^2}\right)\\
&= -2g^2\left[4p^2B_0(p,0,0)+4p^2B_1(p,0,0)+A_0(0)-(1-\xi)(4p^2C_{00}+4p^2B_1+A_0(0))\right],
\end{aligned} \tag{6.7}$$

and the fourth graph results in

$$\Pi_{4\pi_+A} = 2g^2 g^{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{g_{\mu\nu}k^2 - (1-\xi)k_\mu k_\nu}{k^4} = 2g^2(d-1+\xi)A_0(0). \tag{6.8}$$

The fifth graph yields

$$\begin{aligned}
\Pi_{4\pi_+Z} &= -2g^2 g^{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-m^2}\left(g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2-\xi m^2}\right) \\
&= -2g^2 \int \frac{d^dk}{(2\pi)^d}\left(\frac{d}{k^2-m^2} - \frac{(1-\xi)k^2}{(k^2-m^2)(k^2-\xi m^2)}\right) \\
&= -2g^2\left[dA_0(m^2) - (1-\xi)dB_{00}(0,m^2,\xi m^2)\right],
\end{aligned} \tag{6.9}$$

and the last one is

$$\Pi_{4\pi_+\zeta} = (-1)(\frac{-4\mathrm{i}}{3}\xi g^2)f^{abe}f^{cae}\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2-\xi m^2} = \frac{4}{3}\xi g^2 A_0(\xi m^2). \tag{6.10}$$

In order to see the quadratic divergent part, we send $d \to 2$. Only the scalar one-point function A_0 and the tensor coefficient $B_{00} = \frac{1}{2}A_0 + \ldots$ contributes. In this limit, A_0 is given by $A_0 = -4\pi\mu^2 \cdot \Gamma\left(\frac{2-d}{2}\right) +$ finite terms. With $m = 2gv$, we find

$$\begin{aligned}
\Pi^{\pi_+(\text{quadr.})} &= A_0\left(\frac{p^2}{3v^2} + \frac{2}{3}\xi g^2 - 2\xi g^2 + 2g^2(1+\xi) - 2g^2(1+\xi) + \frac{4}{3}\xi g^2\right) \\
&= \frac{p^2}{3v^2}A_0.
\end{aligned} \tag{6.11}$$

Note the intricate cancellation of the mass terms in the scalar, gauge-boson and ghost loops. Expression (6.11) shows that we don't need a mass renormalization with a quadratic dependence on the cut-off scale, as expected. π_+ is naturally light. The quadratic divergence is only $\propto (p/v)^2$ and can be remedied by a wave-function renormalization.

The logarithmic divergence is

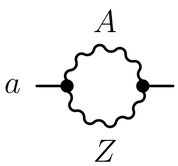
$$\Pi^{\pi_+(log.)} = \left[\frac{m^2p^2}{v^2}\left(\frac{2}{3}\xi - \frac{3}{2}\right) - \frac{3}{2}\frac{m^4}{v^2}\right]\Delta. \tag{6.12}$$

6.1.2 Self-energy of π_-

This calculation ist pretty much the same as for the π_+-self-energy. We have the same six graphs which contribute to quadratic divergencies. There is also a seventh graph, where an A and Z boson build the loop, but this is only logarithmically divergent what can be seen by simple power counting. The graphs 1,2,4,5 and 6 are identical. The third graph can be obtained by replacing $B_{\dots}(p,0,0)$ with $B_{\dots}(p,0,\xi m^2)$ and $A_0(0)$ with $A_0(\xi m^2)$, which is due to the massive π_--propagator. Therefore, the quadratic divergence is same as for π_+,

$$\Pi^{\pi_-(\text{quadr.})} = \frac{p^2}{3v^2} A_0. \tag{6.13}$$

The remaining seventh diagram is given by



$$= (\frac{-4\mathrm{i}}{\sqrt{2}} g^2 v)^2 (-\mathrm{i})^2 g^{\mu\nu} g^{\rho\sigma} \int \frac{d^dk}{(2\pi)^d} \left(\frac{g_{\mu\rho}}{k^2} - (1-\xi)\frac{k_\mu k_\rho}{k^4} \right)$$

$$\cdot \left(\frac{g_{\sigma\nu}}{(k-p)^2 - m^2} - \frac{(1-\xi)(k-p)_\sigma (k-p)_\nu}{[(k-p)^2 - m^2][(k-p)^2 - \xi m^2]} \right) f^{acd} f^{bcd}$$

$$= 8g^4 v^2 \int \frac{d^dk}{(2\pi)^d} \Big[\frac{g^{\mu\nu} g^{\rho\sigma} g_{\mu\rho} g_{\sigma\nu}}{k^2[(k-p)^2 - m^2]} - \frac{(1-\xi)k^2}{k^4[(k-p)^2 - m^2]}$$

$$- \frac{(1-\xi)[(k-p)^2 - \xi m^2 + \xi m^2]}{k^2[(k-p)^2 - m^2][(k-p)^2 - \xi m^2]} + \frac{(1-\xi)^2 k_\mu (k-p)^\mu k_\nu (k-p)^\nu}{k^4[(k-p)^2 - m^2][(k-p)^2 - \xi m^2]} \Big]$$

$$= 2m^2 g^2 \Big[(d-1+\xi) B_0(p,0,m^2) - (1-\xi)\big(B_0(p,0,m^2) + \xi m^2 C_0(\dots)\big)$$

$$+ (1-\xi)^2 \big(B_0(0,0,m^2) - 2p_\mu C^\mu(\dots) + p_\mu p_\nu D^{\mu\nu}(\dots) \big) \Big]$$

$$= 2m^2 g^2 \Big[(d-1+\xi^2) B_0(p,0,m^2) + \dots \Big]. \tag{6.14}$$

Note that C_0, C^μ and $D^{\mu\nu}$ are already convergent.

We obtain

$$\Pi^{\pi_-(log.)} = \Big(\frac{p^2 + \xi m^2}{6v^2} \xi m^2 - 2g^2(3p^2 - \xi p^2 + \xi^2 m^2) - 8g^2\big(m^2 - (1-\xi)\frac{3}{12}(m^2 + \xi m^2)\big)$$

$$+ \frac{4}{3}\xi g^2 \xi m^2 + 2m^2 g^2 (3 + \xi^2 m^2) \Big) \Delta$$

$$= \frac{m^2 p^2}{v^2} \left(\frac{2}{3}\xi - \frac{3}{2} \right) \Delta \tag{6.15}$$

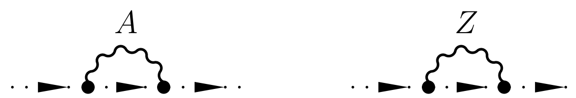
for the logarithmic divergence of the π_- self-energy.

6.2 Self-energies of the Ghosts

We can build loops out of three vertices with one gauge boson leg and from scalar four vertices, where only the ζ ghost couples to the scalars. We have also a scalar three vertex with an α ghost and a ζ antighost, but we cannot close this vertex to form a loop.

6.2.1 Self-energy of the α Ghost

There are two diagrams,



, (6.16)

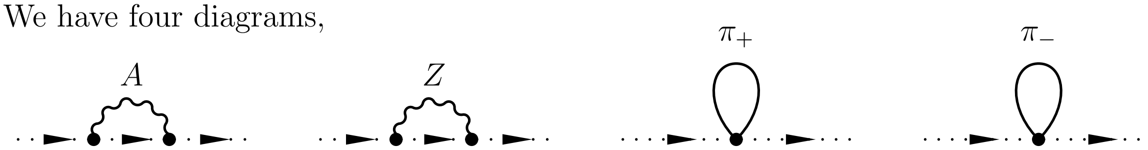
which contribute to the self-energy. The first diagram is given by

$$\begin{aligned} 2g^2 \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p+q)^2} \Big(\frac{g^{\mu\nu}}{p^2} - (1-\xi) \frac{p^\mu p^\nu}{p^4} \Big) f^{dca} (q+p)^\mu f^{bcd} q^\nu \\ = -2g^2 \int \frac{d^dp}{(2\pi)^d} \left(\frac{q(p+q)}{p^2(p+q)^2} - (1-\xi) \frac{p_\mu q^\mu p_\nu (q+p)^\nu}{p^4 (p+q)^2} \right) \\ = -2g^2 q^2 \big(B_0 + B_1 - (1-\xi)(C_{00} + q^2 C_{11} + B_1(q,0,0)) \big). \end{aligned} \tag{6.17}$$

The loop in the second diagram is built by a Z boson and a ζ ghost. Only the arguments of the above result change which only affects the finite part. Note that there are no quadratic divergences here.

6.2.2 Self-energy of the ζ Ghost

We have four diagrams,



(6.18)

The first two diagrams are calculated in analogy to (6.17). The third graph yields

$$\Pi_{4\zeta\pi_+} = \frac{1}{2} \left(\frac{-2\mathrm{i}}{3} \xi g^2 \right) \cdot 2 f^{ade} f^{dbe} \int \frac{d^dp}{(2\pi)^d} \frac{\mathrm{i}}{p^2} = -\frac{2}{3} \xi g^2 A_0(0). \tag{6.19}$$

The fourth graph with its massive π_- propagator results in

$$\Pi_{4\zeta\pi_-} = -\frac{2}{3}\xi g^2 A_0(\xi m^2). \tag{6.20}$$

Note that the second and third diagram are quadratically divergent and add up to

$$\Pi^{\zeta(\text{quadr.})} = -\frac{4}{3}\xi g^2 C_A \delta^{ab} \frac{\mathrm{i}}{16\pi^2} A_0. \tag{6.21}$$

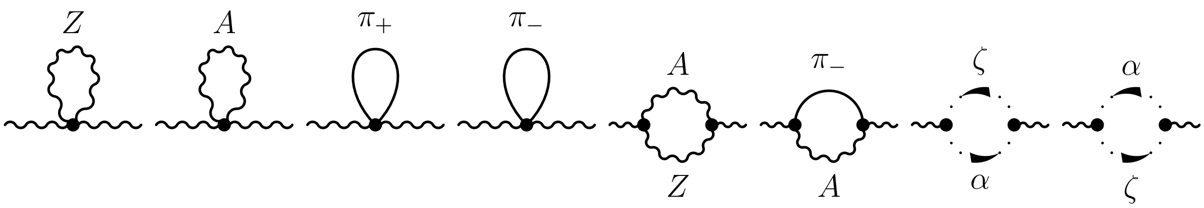
The logarithmic divergent part of the self-energy of the ζ ghost is

$$\Pi^{\zeta(\text{log.})} = -g^2 \Delta \left[\frac{3}{2}q^2 - \frac{1}{2}\xi q^2 + \frac{2}{3}\xi^2 m^2\right]. \tag{6.22}$$

6.3 Gauge-Boson Self-energies

6.3.1 Self-energy of the Z Boson

We can construct the following eight loop diagrams:



(6.23)

The first four graphs have a symmetry factor of 1/2 and the vertex contributes to a factor $-2\mathrm{i}g^2 \cdot 2C_A\delta^{ab}$ times metric tensors. The outer legs are labelled a, μ and b, ν. The first diagram yields

$$\begin{aligned}
\Pi^{\mu\nu}_{4Z} &= -2g^2 \int \frac{d^dp}{(2\pi)^d}\Big[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\Big]\left(\frac{g_{\rho\sigma}}{p^2-m^2} - \frac{(1-\xi)p_\rho p_\sigma}{(p^2-m^2)(p^2-\xi m^2)}\right) \\
&= -2g^2 \int \frac{d^dp}{(2\pi)^d}\left(\frac{g^{\mu\nu}(d-1)}{p^2-m^2} - \frac{(1-\xi)(p^2 g^{\mu\nu} - p^\mu p^\nu)}{(p^2-m^2)(p^2-\xi m^2)}\right) \\
&= -2g^2 g^{\mu\nu}(d-1)\Big(A_0(m^2) - (1-\xi)B_{00}(0, m^2, \xi m^2)\Big).
\end{aligned} \tag{6.24}$$

The second one can be obtained from the first by replacing the masses in the arguments with zero. Thus, this diagram has no logarithmic but only quadratic divergences. The third and fourth are simple, they yield

$$\Pi^{\mu\nu}_{4Z\pi_+} = 2g^2 g^{\mu\nu} A_0(0), \tag{6.25}$$

$$\Pi^{\mu\nu}_{4Z\pi_-} = 2g^2 g^{\mu\nu} A_0(\xi m^2). \tag{6.26}$$

The next one is harder. We denote A with group index d, momentum $q+p$ and the propagator with indices σ, α. Z is denoted by c, p, β, ρ.

$$\begin{aligned}\Pi^{\mu\nu}_{3ZA} = 2g^2(-\mathrm{i})^2 \int \frac{d^dp}{(2\pi)^d}\left(\frac{g_{\sigma\alpha}}{(q+p)^2} - (1-\xi)\frac{(q+p)_\sigma(q+p)_\alpha}{(q+p)^4}\right)\\ \cdot\left(\frac{g_{\beta\rho}}{p^2-m^2} - (1-\xi)\frac{p_\beta p_\rho}{(p^2-m^2)(p^2-\xi m^2)}\right) f^{acd} f^{bcd}\\ \cdot\left[g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-p-2q)^\rho\right]\\ \cdot\left[g^{\nu\beta}(-q+p)^\alpha + g^{\beta\alpha}(-2p-q)^\nu + g^{\alpha\nu}(2q+p)^\beta\right].\end{aligned} \tag{6.27}$$

To make things easier, let us use Feynman-gauge, $\xi = 1$. After multiplying and sorting the expressions, we obtain

$$-2g^{\mu\nu}p^2 + (6-4d)p^\mu p^\nu - 2g^{\mu\nu}pq + (6-4d)p^\mu q^\nu - 5g^{\mu\nu}q^2 + (6-d)q^\mu q^\nu \tag{6.28}$$

for the numerator structure. Inserting (6.28) in (6.27) yields

$$\begin{aligned}\Pi^{\mu\nu}_{3ZA} = -2g^2\Big(g^{\mu\nu}\left[-2A_0(0) - 2m^2B_0 + (6-4d)B_{00} - 2q^2B_1 - 5q^2B_0\right]\\ + q^\mu q^\nu\left[(6-4d)(B_{11}+B_1) + (6-d)B_0(q,m,0)\right]\Big).\end{aligned} \tag{6.29}$$

For a general ξ the calculation by hand is very tedious, but it can be implemented with the computer program FORM, [Ver01].

$$\Pi^{\mu\nu}_{3ZA} = 2g^2\Delta\left(g^{\mu\nu}\left[q^2\left(\frac{25}{6}-\xi\right) + \frac{3}{4}\xi m^2 + \frac{3}{4}\xi^2 m^2 + 3m^2\right] + q^\mu q^\nu\left[-\frac{14}{3}+\xi\right]\right) \tag{6.30}$$

is the result for the logarithmically divergent part. The sixth diagram is only logarithmically divergent, we obtain

$$\begin{aligned}\Pi^{\mu\nu}_{3ZA\pi_-} &= -8g^4v^2 g^{\mu\rho} f^{dac} g^{\nu\sigma} f^{dbc}\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2-\xi m^2}\left(\frac{g_{\rho\sigma}}{(p+q)^2} - (1-\xi)\frac{(p+q)_\sigma(p+q)_\rho}{(p+q)^4}\right)\\ &= -2g^2m^2g^{\mu\nu}\big(B_0(q,\xi m^2,0) - (1-\xi)C_{00}\big).\end{aligned} \tag{6.31}$$

The final two ghost loops are identical and yield

$$\begin{aligned}\Pi^{\mu\nu}_{3Z\zeta,\alpha} &= (-1)4g^2 \int \frac{d^dp}{(2\pi)^d}\left(\frac{\mathrm{i}}{(p^2-\xi m^2)}\frac{\mathrm{i}}{(q+p)^2} f^{dac}(q+p)^\mu f^{cbd}p^\nu\right)\\ &= -4g^2(B^{\mu\nu}+q^\mu B^\nu)\\ &= -4g^2(g^{\mu\nu}B_{00}+q^\mu q^\nu(B_{11}+B_1)).\end{aligned} \tag{6.32}$$

where the arguments of B are $(q, \xi m^2, 0)$.
We extract the quadratic divergent part by sending $d \to 2$ and obtain

$$\Pi^{\mu\nu}_{Z(\text{quadr.})} = g^2 g^{\mu\nu}A_0(-2-2+2+2-2[-2-2\cdot\frac{1}{2}]-1-1) = 4g^2g^{\mu\nu}A_0. \tag{6.33}$$

This result also holds for a general ξ since the ξ-dependent contributions of diagram 1,2 and 5 cancel each other (see next section). The logarithmic divergence is

$$\Pi^{\mu\nu}_{Z(\log.)} = g^2\Delta\Big(g^{\mu\nu}(\frac{26}{3}-2\xi+2\xi m^2)-q^\mu q^\nu(\frac{26}{3}-2\xi)\Big). \tag{6.34}$$

We decompose the self-energy in a transverse and a longitudinal part by

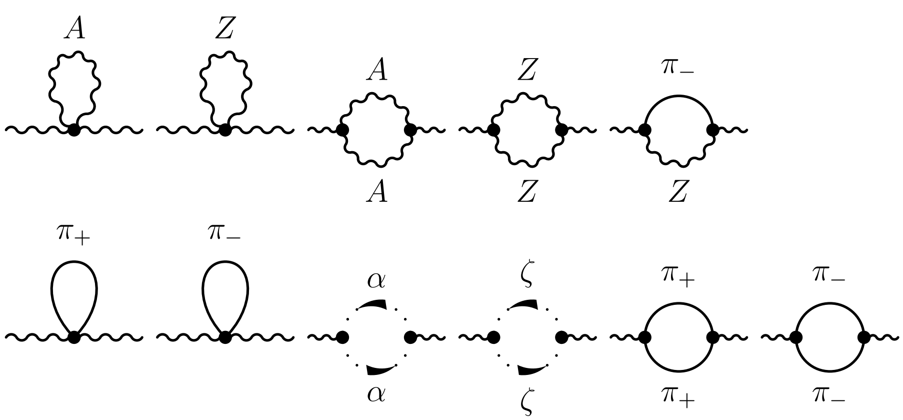
$$\Pi^{\mu\nu} = \left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\Pi_T(k^2)+\frac{q^\mu q^\nu}{q^2}\Pi_L(k^2) \tag{6.35}$$

and obtain

$$\Pi^Z_L = 2\xi m^2 g^2\Delta \tag{6.36}$$

for the longitudinal part.

6.3.2 Self-energy of the A Boson



(6.37)

The diagrams involving gauge boson loops can be obtained from the previous section by changing the arguments of the N-point functions, which are shown once in each equation. Note also the symmetry-factor of 1/2 for the third and fourth diagram and an opposite sign for diagram six and seven, due to the vertex. For the massless A boson we expect a transverse Lorentz structure of the self-energy (a longitudinal part would induce a radiative mass term, but a massive A boson is forbidden by the gauge symmetry) and a cancelling of the quadratic divergencies. Gauge-boson and ghost loops have to cancel independently from the scalar loops. This is shown in the following calculation.

$$\Pi^{\mu\nu}_{4A} = -2g^2 g^{\mu\nu}(d-1)\Big(A_0(0) - (1-\xi)B_{00}(0,0,0)\Big) \tag{6.38a}$$

$$\Pi^{\mu\nu}_{4AZ} = -2g^2 g^{\mu\nu}(d-1)\Big(A_0(m^2) - (1-\xi)B_{00}(0,m^2,\xi m^2)\Big) \tag{6.38b}$$

$$\begin{aligned}\Pi^{\mu\nu}_{3A} = g^2\Big(&g^{\mu\nu}\big[2A_0(0) - (6-4d)B_{00} + q^2(2B_1 - 5B_0)\big]\\ &- q^\mu q^\nu\big[(6-4d)(B_{11}+B_1) + (6-d)B_0(q,0,0)\big]\Big) \qquad (\xi = 1)\end{aligned} \tag{6.38c}$$

$$\begin{aligned}\Pi^{\mu\nu}_{3AZ} = g^2\Big(&g^{\mu\nu}\big[2A_0(m^2) + 2m^2 B_0 - (6-4d)B_{00} + q^2(2B_1 - 5B_0)\big]\\ &- q^\mu q^\nu\big[(6-4d)(B_{11}+B_1) + (6-d)B_0(q,m^2,m^2)\big]\Big) \qquad (\xi = 1)\end{aligned} \tag{6.38d}$$

$$\Pi^{\mu\nu}_{3AZ\pi_-} = -2g^2 m^2 g^{\mu\nu} B_0(q,m^2,m^2) \qquad (\xi = 1) \tag{6.38e}$$

Lets calculate the remaining scalar and ghost loops. The scalar loops are given by

$$\begin{aligned}\Pi^{\mu\nu}_{3A\pi_\pm} &= -g^2 \int \frac{d^dp}{(2\pi)^d} \frac{(2p+q)^\mu(-2p-q)^\nu}{(p^2-\xi m^2)[(p+q)^2 - \xi m^2]}\\ &= g^2[4B^{\mu\nu} + 4q^\mu B^\nu + 4k^\mu q^\nu B_0]\\ &= g^2(4g^{\mu\nu}B_{00} + q^\mu q^\nu(4B_{11} + 4B_1 + B_0)),\end{aligned} \tag{6.39}$$

where the argument of B is $(q,0,0)$ for π_+ and $(q,\xi m^2,\xi m^2)$ for π_-.

The ghost loops are given by

$$\begin{aligned}\Pi^{\mu\nu}_{3A\zeta,\alpha} &= (-1)2g^2 \int \frac{d^dp}{(2\pi)^d} \frac{\mathrm{i}}{(p^2-\xi m^2)} \frac{\mathrm{i}}{(q+p)^2 - \xi m^2} f^{dac}(q+p)^\mu f^{cbd} p^\nu\\ &= -2g^2(B^{\mu\nu} + q^\mu B^\nu)\\ &= -2g^2(g^{\mu\nu}B_{00} + q^\mu q^\nu(B_{11} + B_1)).\end{aligned} \tag{6.40}$$

Let's summarize the ghost and scalar loops:

$$\Pi^{\mu\nu}_{4A\pi_+} = -2g^2 g^{\mu\nu} A_0(0), \tag{6.41a}$$

$$\Pi^{\mu\nu}_{4A\pi_-} = -2g^2 g^{\mu\nu} A_0(\xi m^2), \tag{6.41b}$$

$$\Pi^{\mu\nu}_{3A\pi_+} = g^2\Big(4g^{\mu\nu}B_{00} + q^\mu q^\nu\big(4B_{11} + 4B_1 + B_0(q,0,0)\big)\Big), \tag{6.41c}$$

$$\Pi^{\mu\nu}_{3A\pi_-} = g^2\Big(4g^{\mu\nu}B_{00} + q^\mu q^\nu\big(4B_{11} + 4B_1 + B_0(q,\xi m^2,\xi m^2)\big)\Big), \tag{6.41d}$$

$$\Pi^{\mu\nu}_{3A\alpha} = -2g^2\Big(g^{\mu\nu}B_{00} + q^\mu q^\nu\big(B_{11} + B_1(q,0,0)\big)\Big), \tag{6.41e}$$

$$\Pi^{\mu\nu}_{3A\zeta} = -2g^2\Big(g^{\mu\nu}B_{00} + q^\mu q^\nu\big(B_{11} + B_1(q,\xi m^2,\xi m^2)\big)\Big). \tag{6.41f}$$

We obtain

$$\Pi^{\mu\nu}_{A(\text{quadr.})} = g^2 g^{\mu\nu} A_0(-2-2+3+3-2-2+2+2-1-1) = 0 \tag{6.42}$$

for the quadratically divergent part ($\xi = 1$). Note that the scalar contributions cancel among themselves and the gauge boson part is cancelled by the ghost part. If one wants to calculate in a general gauge, Π_{3A} and Π_{AZ} become rather lengthy, since higher C and D functions appear. One way around is to use the formulae (A.44) to (A.48) in [Pes]. In (6.24) only the p^4-terms in the $(1-\xi)$ part are quadratically divergent. Then one can bring the integral into the form of (A.47) and extract the pole ($\Gamma(1-\frac{d}{2}) \propto A_0$). The result is that $3A_0$ gets replaced by $(3+(\xi-1))A_0$ in Π_{3A} and Π_{3AZ}. This also leads to a vanishing quadratic divergence, but for a general ξ (additionally, this was also checked with FORM).

Being sure that we have no quadratic divergences, we can now safely send $d \to 4$ and verify the transverse Lorentz structure. Note that $\Pi^{\mu\nu}_{4A\pi_+}$ and $\Pi^{\mu\nu}_{4A}$ vanish for $d \to 4$.

With $\tilde{B}_{\dots} = B_{\dots}(k,0,0) + B_{\dots}(k,m,m)$ and $A_0(m^2) = A_0$ the scalar part can be written for ($\xi = 1$) as

$$\begin{aligned}\Pi^{\mu\nu,\text{scalar}} &= g^2 g^{\mu\nu}[4\tilde{B}_{00} - 2A_0] + q^\mu q^\nu[4\tilde{B}_{11} + 4\tilde{B}_1 + \tilde{B}_0] \\ &= \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi^{\text{scalar}}_T(k^2) + \frac{q^\mu q^\nu}{q^2}\Pi^{\text{scalar}}_L(k^2),\end{aligned} \tag{6.43}$$

with

$$\Pi^{\text{scalar}}_T(k^2) = g^2[4\tilde{B}_{00} - 2A_0], \tag{6.44a}$$

$$\Pi^{\text{scalar}}_L(k^2) = g^2[4\tilde{B}_{00} - 2A_0 + 4k^2\tilde{B}_{11} + k^2\tilde{B}_1 + k^2\tilde{B}_0]. \tag{6.44b}$$

Using the expressions for $B_{\dots}$ as defined in the [Kil02], one can easily show the vanishing of Π^{scalar}_L. The logarithmically divergent part is

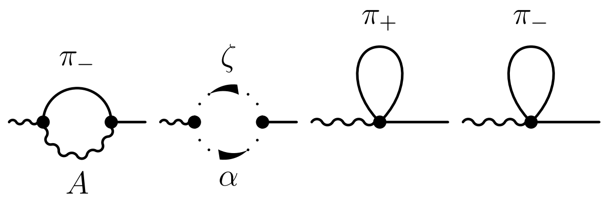
$$\Pi^{\text{scalar}}_T(q^2) = -\frac{2}{3}g^2 q^2 \cdot \Delta. \tag{6.45}$$

An analogous calculation can be done for the gauge sector (including ghosts), the longitudinal part of the self-energy also vanishes. Note that the sixth graph, where the loop is formed by a π_- and a Z is taken to the gauge sector because it only contributes mass terms which exactly cancel the mass contributions of diagrams with massive gauge-boson or ghost propagators. The transverse part of the A boson self-energy reads

$$\Pi_T^{\text{gauge}} = 2(2-d)A_0 - 4(2-d)\tilde{B}_{00} + 4q^2\tilde{B}_0 = \frac{20}{3}g^2 \cdot \Delta + \ldots \qquad . \tag{6.46}$$

6.4 Self-energy for the $Z\pi_-$-mixing

At one loop, we have four contributions to the $Z\pi$-mixing.



(6.47)

The outer legs are denoted by the group indices a and b and the momentum of the Z boson by p.
For the first graph, we obtain

$$\begin{aligned}
&\frac{-4\mathrm{i}}{\sqrt{2}}g^3 v(-\sqrt{2}) f^{dca} f^{cdb} g^{\mu\nu} \int \frac{d^dk}{(2\pi)^d}\Big(\frac{g_{\nu\rho}}{k^2} - (1-\xi)\frac{k_\nu k_\rho}{k^4}\Big)\frac{(k+p)^\rho + p^\rho}{[(k+p)^2 - \xi m^2]} \\
&\qquad = -4\mathrm{i}g^3 v \int \frac{d^dk}{(2\pi)^d}\left(\frac{(k+2p)^\mu}{[(k+p)^2-\xi m^2]k^2} - (1-\xi)\frac{k^\mu k_\nu (k+2p)^\nu}{[(k+p)^2-\xi m^2]k^4}\right) \\
&\qquad\quad = -4\mathrm{i}g^3 v\big(B^\mu(p,0,\xi m^2) + 2p^\mu B_0 - (1-\xi)(p^\mu B_1 + 2p_\mu C^{\mu\nu})\big) \\
&\qquad\qquad = -4\mathrm{i}g^3 v(2B_0 + B_1 - (1-\xi)(B_1 + 2C_{00}))p^\mu = -6\mathrm{i}g^3 v\Delta \cdot p^\mu,
\end{aligned} \tag{6.48}$$

and the second diagram results in

$$4\mathrm{i}\xi g^3 v p^\mu\big(B_0(p,\xi m^2,0) + B_1(p,\xi m^2,0)\big) = 2\mathrm{i}\xi g^3 v\Delta \cdot p^\mu. \tag{6.49}$$

The third graph yields a purely quadratic divergence,

$$\frac{1}{2}\frac{2\mathrm{i}g}{3v} f^{aa_1c} f^{a_1bc} \int \frac{d^dk}{(2\pi)^d}\frac{(-p+k)^\mu + (-p-k)^\mu}{k^2} = \frac{2\mathrm{i}g}{3v}p^\mu A_0(0). \tag{6.50}$$

Note the minus sign of p. This is due to the fact that in the corresponding vertex

only (incoming) momenta from the scalars appear. Due to momentum conservation the incoming momentum of the outer scalar leg is (-1) times the momentum of the gauge boson whose momentum is going into the loop.
The fourth graph is given by

$$\frac{2\mathrm{i}g}{3v}p^{\mu}A_0(\xi m^2). \tag{6.51}$$

We obtain

$$\Pi_{\mu}^{Z\pi,\text{quadr.}} = \frac{4\mathrm{i}g}{3v}p_{\mu}A_0 \tag{6.52}$$

for the quadratically divergent and

$$\begin{aligned}\Pi_{\mu}^{Z\pi,\text{log.}} &= 2\mathrm{i}\xi g^3 v\Delta \cdot p_{\mu} - 6\mathrm{i}g^3 v\Delta \cdot p_{\mu} + \frac{2\mathrm{i}g}{3v}\xi m^2\Delta \cdot p_{\mu} \\ &= -\frac{\mathrm{i}m^3}{v^2}\Delta\left(\frac{3}{4} - \frac{7}{12}\xi\right) \cdot p_{\mu}\end{aligned} \tag{6.53}$$

for the logarithmically divergent part.

6.5 Quadratic Divergences

We could explicitly show that the scalar π_+ boson, the little Higgs, is free of quadratic divergences, unlike the Higgs particle in the Standard Model.
But there is a non-vanishing quadratic divergence for the self-energy of the Z boson, $\Pi^{\mu\nu}_{Z(\text{quadr.})} = 4g^2 g^{\mu\nu}A_0$. This term contributes to a quadratically divergent mass correction. It results from the scalar loops of the third and fourth diagram in section 6.3.1 and can be traced back to the nonlinear parametrization of the sigma model. On a technical level, we expect an independent cancelling of the quadratic divergent parts from gauge and scalar loops, as it is the case for the massless A boson. When computing the divergent terms in the self-energies, the only difference between the two bosons is the mass, but this doesn't affect quadratic divergences (see the argument after (A.10). Here, only the terms in the gauge sector of the Z boson cancel, the contributions from the scalar loops remain.
In appendix B the scalar part in the self-energy of the Z boson is calculated also in an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *linear* sigma model, which is a possible UV-completion of the nonlinear sigma model. There, it turns out that there is a contribution from the massive σ boson, and the sum of all scalar contribution vanishes. Maybe we should not be too surprised about this. In the nonlinear sigma model we argued that the mass of the σ boson is much bigger than the energy where the theory is measured and therefore decouples. However, quadratic divergences are independent of masses. Therefore, even infinitely heavy particles yield a contribution to quadratic divergences,

which is missing in the nonlinear sigma model (more discussion about the limit of a large m_σ, linear and nonlinear sigma models can be found in [Vel]).
Do we have to worry about the quadratically divergent mass of the Z boson? No, the theory is only valid up to the cut-off scale $4\pi v$ and loop integrations are only carried out for momenta smaller than $4\pi v$. After that, a UV-completed theory takes over which is free of quadratic divergences. Thus, the mass of the Z boson would be of $\mathcal{O}(v^2)$. We do not have a fine tuning problem, since we have no bounds for the mass of this field, as it is the case for the Higgs boson.

7 Consistency Checks

Continuous symmetries in a classical Lagrangian lead to conserved currents and charges, due to Noether's theorem. These considerations can be extended to the path integral representation in quantum field theory. The results are Ward identities for Green functions, i.e. relations between Green functions resulting from a symmetry of the action.

The equations of motion for the classical fields can be generalized to the equations of motion for Green functions. Their derivation is similar to the derivation of the Ward identities. The starting point is the invariance of the path integral under field transformations, which simply corresponds to the invariance of the action S.

Details to the following discussion can be found in [Böh]. The *generating functional* is

$$Z\{\mathbf{J}\} = \int \mathcal{D}[\boldsymbol{\psi}] e^{\mathrm{i}S\{\psi\}+\mathrm{i}\int \mathbf{J}(x)\boldsymbol{\psi}(x)}, \tag{7.1}$$

where the vector $\boldsymbol{\psi}$ collectively denotes all fields ψ_i and $\mathbf{J}$ denotes all sources of the fields.

Now, let us perform an infinitesimal transformation of the fields,

$$\psi_i \to \psi_i + \delta\psi_i, \qquad \delta\psi_i = \epsilon f_i(x), \tag{7.2}$$

where ϵ is an infinitesimal parameter and f_i is an ordinary function of x. Under this transformation the action also varies. We obtain

$$\frac{\delta S}{\delta\psi_k(x)} = \frac{\partial\mathcal{L}}{\partial\psi_k(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_k(x))}. \tag{7.3}$$

A change of integration variables cannot change the integral $Z\{\mathbf{J}\}$. Thus, the variation of $\delta Z\{\mathbf{J}\}$ vanishes and we obtain

$$0 = \int \mathcal{D}[\boldsymbol{\psi}] e^{\mathrm{i}S\{\psi\}+\mathrm{i}\int d^4y \mathbf{J}(x)\boldsymbol{\psi}(x)} \left(\frac{\delta S}{\delta\psi_i(x)} + J_i(x)\right). \tag{7.4}$$

Green functions are obtained by differentiating the generating functional with respect to the sources and putting these equal to zero,

$$G(x_1,\dots,x_n)=\frac{\delta^n}{\mathrm{i}\delta J(x_1)\cdots\mathrm{i}\delta J(x_n)}Z\{J\}|_{J(x)\equiv 0}. \tag{7.5}$$

Generically, we write Green functions as $\langle T\prod_l \psi_{i_l}\rangle$. Applying the functional derivative to (7.4), we obtain the equations of motion for Green functions,

$$-\mathrm{i}\langle T\prod_l \psi_{i_l}(x_l)\frac{\delta S}{\delta\psi_i(x)}\rangle=\langle T\frac{\delta}{\delta\psi_i(x)}\prod_l \psi_{i_l}(x_l)\rangle. \tag{7.6}$$

The δ-functions in (7.6) result from the differentiation of the explicit sources $J_i(x)$ in (7.4). Later, we need the equation of motion when we have only one type of fields. Then (7.6) simplifies to

$$-\mathrm{i}\langle T\psi(y)\frac{\delta S}{\delta\psi(x)}\rangle=\delta^4(x-y). \tag{7.7}$$

From (7.6) we also obtain Ward identities for Green functions,

$$\delta\langle T\prod_l \psi_{i_l}(x_l)\rangle=0, \tag{7.8}$$

if the action S is invariant of the action under (global) transformations, $\delta S/\delta\psi_i=0$. The BRS transformation is an example for a global transformation ($\delta\lambda$ is a global variable), the resulting Ward identities are called Slavnov-Taylor identities[1] and read

$$s\langle T\prod_l \psi_{i_l}(x_l)\rangle=0. \tag{7.9}$$

7.1 Slavnov-Taylor Identities for the Z Boson

We are interested in identities which result from the Green function $\langle T\overline{\eta}^a(x)G^b\{Z;y\}\rangle$. (7.9) yields

$$\begin{aligned}0&=\langle T(s\overline{\eta}^a(x))G^b\{Z;y\}\rangle-\langle T\overline{\eta}^a(x)sG^b\{Z;y\}\rangle\\&=-\frac{1}{\xi}\langle TG^a\{Z;y\}G^b\{Z;y\}\rangle+\langle T(sG^b\{Z;y\})\overline{\eta}^a(x)\rangle.\end{aligned} \tag{7.10}$$

[1] This general Slavnov-Taylor identity can also be derived in the canonical BRS method by sandwiching the commutator (or anticommutator) ($s\Psi=[Q,\Psi]_\pm$) of an arbitrary product of fields with the BRS charge between physical fields: $\langle\phi_{\mathrm{phys}}|T[Q,\Psi_1\Psi_2\dots\Psi_n]_\pm|\psi_{\mathrm{phys}}\rangle=0$. Then the BRS charge can be taken out of the time-ordering and we arrive at (7.9).

We can replace the second term in (7.10) via the equation of motion for the antighost (cf. (7.7) with $\mathcal{L}_{\text{ghost}} = -\overline{\eta}^a sG^b$). We obtain

$$-\frac{1}{\xi}\langle TG^a\{Z;y\}G^b\{Z;y\}\rangle = \mathrm{i}\delta^{ab}\delta^4(x-y). \tag{7.11}$$

Using the gauge-fixing functional for the massive Z boson in (4.9), we find the following non-trivial relation

$$\partial^x_\mu\partial^y_\nu\langle TZ^{\mu,a}(x)Z^{\nu,b}(y)\rangle - 2\xi m\partial^x_\mu\langle TZ^{\mu,a}(x)\pi^b_-(y)\rangle + \xi^2m^2\langle T\pi^a_-(x)\pi^b_-(y) = -\mathrm{i}\xi\delta^{ab}\delta^4(x-y). \tag{7.12}$$

Going to momentum space, (7.12) reads

$$k^\mu k^\nu G^{ZZ}_{\mu\nu} - 2\mathrm{i}\xi m k^\mu G^{Z\pi}_\mu + \xi^2m^2G^{\pi\pi} = -\mathrm{i}\xi, \tag{7.13}$$

where the momentum k^μ in $G^{Z\pi}_\mu$ denotes the *outgoing* momentum of the gauge boson[2].

As already mentioned in section (3.5), the propagators (two-point Green functions) are obtained by inverting the two-point vertices. Here we have to invert the full (five dimensional) propagator matrix

$$\begin{pmatrix} \Gamma^{ZZ}_{\mu\nu} & \Gamma^{Z\pi}_\mu \\ \Gamma^{Z\pi}_\nu & \Gamma^{\pi\pi} \end{pmatrix} = -\begin{pmatrix} G^{ZZ}_{\mu\nu} & G^{Z\pi}_\mu \\ G^{Z\pi}_\nu & G^{\pi\pi} \end{pmatrix}^{-1}. \tag{7.14}$$

With the ansatz

$$\mathbf{G} = \begin{pmatrix} G^{ZZ}_{\nu\rho} & G^{Z\pi}_\nu \\ G^{Z\pi}_\rho & G^{\pi\pi} \end{pmatrix} = \begin{pmatrix} G^{ZZ}_T g^T_{\nu\rho} + G^{ZZ}_L g^L_{\nu\rho} & G^{Z\pi}_L k_\nu \\ G^{Z\pi}_L k_\rho & G^{\pi\pi} \end{pmatrix} \tag{7.15}$$

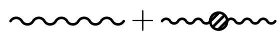
and

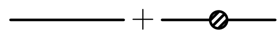
$$g^T_{\nu\rho} = g_{\nu\rho} - \frac{k_\nu k_\rho}{k^2}, \qquad g^L_{\nu\rho} = \frac{k_\nu k_\rho}{k^2}, \tag{7.16}$$

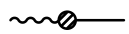
we get the propagators by comparing coefficients of $\mathbf{G}\mathbf{G}^{-1} = \mathbf{1}_{5\times5}$.

$$\begin{aligned} G^{ZZ}_T &= \frac{1}{\Gamma^{ZZ}_T}, & G^{Z\pi}_L &= \frac{-\Gamma^{Z\pi}_L}{\Gamma^{\pi\pi}\Gamma^{ZZ}_L - (\Gamma^{Z\pi}_L)^2k^2}, \\ G^{ZZ}_L &= \frac{\Gamma^{\pi\pi}}{\Gamma^{\pi\pi}\Gamma^{ZZ}_L - (\Gamma^{Z\pi}_L)^2k^2}, & G^{\pi\pi} &= \frac{1}{\Gamma^{\pi\pi}} + \frac{k^2(\Gamma^{Z\pi}_L)^2}{\Gamma^{\pi\pi}(\Gamma^{\pi\pi}\Gamma^{ZZ}_L - (\Gamma^{Z\pi}_L)^2k^2)}. \end{aligned} \tag{7.17}$$

[2] See the comment after (6.50): the momentum of Z going *into* the loop is the *outgoing* momentum at the space-time point x. Thus, ∂^x_μ gets replaced by $+ik_\mu$ and the signs are correct.

$\Gamma^{ZZ}_{\mu\nu} =$  $= g^{T}_{\mu\nu}\Gamma^{ZZ}_{T} + g^{L}_{\mu\nu}\Gamma^{ZZ}_{L}$

$\Gamma^{\pi\pi} =$  (7.18)

$\Gamma^{Z\pi}_{\mu} =$  $= \Gamma^{Z\pi}_{L} k_\mu$

The above two-point-vertices from $\mathrm{i}\mathcal{L}$ in momentum space are given by (including both the quadratically and logarithmically divergent part)

$$\Gamma^{ZZ}_{L} = -\mathrm{i}\left(\frac{k^2}{\xi} - m^2\right) + \frac{m^2}{v^2}A_0 + \frac{\xi m^4}{2v^2}\Delta, \tag{7.19a}$$

$$\Gamma^{\pi\pi} = \mathrm{i}\big(k^2 - \xi m^2\big) + \frac{k^2}{3v^2}A_0 + \frac{m^2k^2}{v^2}\left(\frac{2}{3}\xi - \frac{3}{2}\right)\Delta, \tag{7.19b}$$

$$\Gamma^{Z\pi}_{L} = \frac{2\mathrm{i}m}{3v^2}A_0 + \frac{\mathrm{i}m^3}{v^2}\left(-\frac{3}{4} + \frac{7}{12}\xi\right)\Delta. \tag{7.19c}$$

Since we only calculate vertex corrections to one loop, we can ignore second order contributions. Using $1/(1+x) = 1 - x$, we find ($m = 2gv$)

$$\begin{aligned} k^\mu k^\nu G^{ZZ}_{\mu\nu} &= -\frac{k^2\Gamma^{\pi\pi}}{\Gamma^{\pi\pi}\Gamma^{ZZ}_{L} - (\Gamma^{Z\pi}_{L})^2k^2} = -\frac{k^2}{\Gamma^{ZZ}_{L}} = \frac{-\mathrm{i}\xi k^2}{(k^2-\xi m^2)[1 + \frac{\mathrm{i}}{2v^2}(\frac{2\xi m^2 A_0 + \xi^2 m^4\Delta}{k^2-\xi m^2})]} \\ &= \frac{-\mathrm{i}\xi k^2(1 - \frac{\mathrm{i}}{2v^2}\frac{2\xi m^2 A_0 + \xi^2 m^4 \Delta}{k^2 - \xi m^2})}{k^2 - \xi m^2}, \end{aligned} \tag{7.20}$$

$$G^{\pi\pi} = -\frac{1}{\Gamma^{\pi\pi}} = \frac{i(1 + \frac{\mathrm{i}k^2}{3v^2}\frac{A_0 + m^2(2\xi - \frac{9}{2})\Delta}{k^2-\xi m^2})}{k^2 - \xi m^2}, \tag{7.21}$$

$$k^\mu G^{Z\pi}_{\mu} = \frac{\Gamma^{Z\pi}_{L}k^2}{\Gamma^{\pi\pi}\Gamma^{ZZ}_{L}} = \frac{\xi k^2\left[\frac{\mathrm{i}m}{v^2}(\frac{2}{3}A_0 + (-\frac{3}{4} + \frac{7}{12}\xi)m^2\Delta)\right]}{(k^2-\xi m^2)^2}. \tag{7.22}$$

Now, we can check if the Ward-identity (7.13) holds. The tree-level part is obviously fulfilled, which indicates that our signs are right. The one-loop contributions are

$$-\frac{\frac{m^2}{v^2}\xi^2k^2A_0}{(k^2-\xi m^2)^2}\left(1 - \frac{4}{3} + \frac{1}{3}\right) - \frac{\xi^2\frac{m^4}{v^2}k^2\Delta}{(k^2-\xi m^2)^2}\left(\frac{1}{2}\xi + \frac{3}{2} - \frac{7}{6}\xi + \frac{2}{3}\xi - \frac{3}{2}\right) \tag{7.23}$$

The prefactors of the divergent parts cancel. Thus, the Ward-identity holds.

7.2 Goldstone-Boson Equivalence Theorem

The Goldstone-boson equivalence theorem (ET) is an important consequence of the Slavnov-Taylor identities in spontaneously broken theories. It states that S-matrix elements for the emission of scalar gauge bosons can be obtained from S-matrix elements for the emission of Goldstone bosons. The scalar gauge boson is unphysical, for high energies however the scalar polarization vector goes over into the physical longitudinal polarization vector, up to $\mathcal{O}(\frac{M_Z}{E})$. Thus, the theorem can be used to facilitate the calculation of cross sections for reactions with longitudinal gauge bosons at high energies, as the amplitudes for external scalars are much easier to evaluate. On the other side it might allow to derive information on the mechanism of spontaneous symmetry breaking from the experimental study of longitudinal gauge bosons. In this work, it will be used as second check of the correctness of our Feynman rules. Crucial for the ET is again the fact that the BRS transformation on Green functions vanish,

$$s\langle T\prod_l \psi_{i_l}(x_l)\rangle = 0. \tag{7.24}$$

We consider a Green function with one antighost field and arbitrary other fields. We obtain

$$\begin{aligned} 0 &= s\langle T\overline{\eta}^a \prod_l \psi_{i_l}(x_l)\rangle \\ &= -\langle T\frac{1}{\xi}G^a \prod_l \psi_{i_l}(x_l)\rangle \\ &\qquad + \text{all other terms where } s \text{ acts on } \psi_{i_l}. \end{aligned} \tag{7.25}$$

Now, if all the fields are physical and on shell, their BRS variations vanish [cf. (4.12)] and we find

$$\langle TG^a \prod_l \psi_{i_l}^{phys}(x_l)\rangle = 0. \tag{7.26}$$

Inserting the gauge-fixing functional for the massive gauge boson and going into momentum space, (7.26) reads

$$\langle T(k^\mu Z_\mu(k) - \xi m\pi_-(k))\prod_l \psi_{i_l}^{phys}(k_l)\rangle = 0. \tag{7.27}$$

S-matrix elements can be obtained via the LSZ-reduction formula from truncated Green functions, that is removing the poles of internal lines. In the derivation one has to use the tree level relations

$$k_\mu G^{\mu\nu}_{ZZ} = -\xi G_{\overline{\eta}\eta}k^\nu, \qquad G_{\overline{\eta}\eta} = G_{\pi\pi}, \tag{7.28}$$

which can be derived from the identities

$$s\langle T Z^a_\mu(x) \overline{\eta}^b_\zeta(y)\rangle = 0, \qquad s\langle T \pi^a_-(x) \overline{\eta}^b_\zeta(y)\rangle = 0, \tag{7.29}$$

respectively, evaluated at tree level. Relation (7.28) can also be obtained from the explicit expressions, this is done in [Hor96]. Relation (7.27) for the amplitudes reads

$$\mathrm{i}\frac{k^\mu}{m}\mathcal{M}_\mu(A \to B + Z)) = \mathcal{M}(A \to B + \pi_-). \tag{7.30}$$

A and B are arbitrary other *physical* particles, including other gauge bosons as well. Note that k^μ/m is not a physical polarization vector[3], which must satisfy the condition

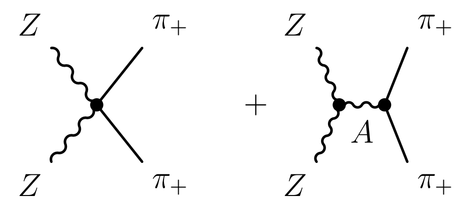
$$\epsilon^\mu(k) \cdot k_\mu = 0. \tag{7.31}$$

For our theory, we cannot construct a physical matrix element where only one Z boson is involved, due to the negative parity of Z. But a process with two Z bosons and two π_+ bosons (total parity +1) is possible. The relation

$$-\mathrm{i}k_3^\nu \mathcal{M}_{\mu\nu}(ZZ \to \pi_+\pi_+) \cdot \epsilon^\mu(k_4) = m\mathcal{M}_\mu(Z\pi_- \to \pi_+\pi_+) \cdot \epsilon^\mu(k_4) \tag{7.32}$$

with

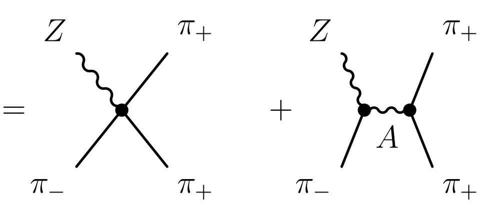
$$\mathcal{M}_{\mu\nu}(Z^{\mu,d}(k_4)Z^{\nu,c}(k_3) \to \pi^a_+(k_1)\pi^b_+(k_2)) =$$



(7.33)

and

$$\mathcal{M}_\mu(Z^{\mu,d}(k_4)\pi^c_-(k_3) \to \pi^a_+(k_1)\pi^b_+(k_2)) =$$



(7.34)

should be valid. The first diagram of $\mathcal{M}_{\mu\nu}$ is (using momentum conservation to replace k_3 with $-(k_1 + k_2 + k_4)$, (7.31) and putting external particles on-shell)

$$2g^2(k_1 + k_2)^\mu[f^{ace}f^{bde} + f^{ade}f^{bce}]\epsilon_\mu. \tag{7.35}$$

[3] For high energies however, the longitudinal polarization vector of a massive gauge boson, $\epsilon^\mu_L = (\frac{k}{m}, 0, 0, \frac{E_k}{m})$ gets parallel to k^μ

The second diagram of $\mathcal{M}_{\mu\nu}$ reads (after some manipulations)

$$\begin{aligned}2g^2 f^{ecd} f^{eab}(g^{\mu\nu}(k_4-k_3)^\rho + g^{\nu\rho}(k_3-k)^\mu + g^{\mu\rho}(k-k_4)^\nu)\frac{-\mathrm{i}g_{\rho\sigma}}{k^2}(k_1-k_2)^\sigma \cdot (-\mathrm{i}k_{3,\nu})\epsilon_\mu(k_4) \\ = 2g^2 f^{abe} f^{cde}(k_1-k_2)^\mu - 2m^2 g^2 f^{abe} f^{cde}\frac{(k_1-k_2)^\mu}{2k_1k_2}. \quad (7.36)\end{aligned}$$

The first and second diagram of $\mathcal{M}_\mu$ are given by $(m = 2gv)$

$$\frac{4}{3}g^2[(k_1-k_2)^\mu f^{abe} f^{cde} + (2k_1+k_2)^\mu f^{ace} f^{bde} + (k_1+2k_2)^\mu f^{ade} f^{bce}] \cdot \epsilon_\mu, \quad (7.37)$$

$$-8g^4 v^2 f^{abe} f^{cde}\frac{(k_1-k_2)^\mu}{2k_1k_2} \cdot \epsilon_\mu, \quad (7.38)$$

respectively. We observe that (7.38) cancels the last term in (7.36), however the other cancellations are not obvious. Naively, one would compare the coefficients of the $f^{\cdots}f^{\cdots}$ terms, but this is not correct, since we can use the Jacobi-identity to replace this term by two other $f^{\cdots}f^{\cdots}$ terms. We rewrite the $8/3g^2$-terms in (7.37) and find

$$\begin{aligned}&\frac{8}{3}g^2(k_1^\mu f^{abe} f^{cde} + k_2^\mu f^{ade} f^{bce} = \\ &2g^2(k_1^\mu f^{abe} f^{cde} + k_2^\mu f^{ade} f^{bce}) + \frac{2}{3}g^2 k_1^\mu[f^{ade} f^{bce} + f^{abe} f^{cde}] + \frac{2}{3}g^2 k_2^\mu[f^{ace} f^{bde} - f^{abe} f^{cde}].\end{aligned} \quad (7.39)$$

Now, the cancellation can be seen.
Another process involving one π_- can be constructed,

$$-\mathrm{i}k_1^\mu \mathcal{M}_{\mu\nu\rho}(ZZ \to A) \cdot \epsilon^\nu(k_2)\epsilon^\rho(k_3) = m\mathcal{M}_\mu(Z\pi_- \to A) \cdot \epsilon^\nu(k_2)\epsilon^\rho(k_3). \quad (7.40)$$

The calculation of $\mathcal{M}_{\mu\nu\rho}(ZZ \to A)$ is analogous to (7.36) and the above relation can be verified.
Having done these two non-trivial checks we can be confident that our Feynman rules and expressions for our self-energies are correct.

8 Renormalization

Quantum effects, resulting from Feynman graphs containing loops, lead to corrections to Green functions and S-matrix elements. These corrections change the relations among the parameters of the Lagrangian. As a result, the bare parameters are no longer directly related to physical quantities. Moreover, the bare parameters can even become divergent. These divergent quantities have to be regularized, e.g. by dimensional regularization. This amounts to a modification of the theory so that the possibly divergent expressions become well-defined, and that in a suitable limit the original (divergent) theory is recovered. Consequently, a redefinition of the original (*bare*) parameters $(m, g, \ldots)$, a *renormalization* of the theory is needed. In this process, also the fields have to be renormalized.

The requirement that divergences are compensated does not determine the finite parts of the renormalization constants. As a consequence, calculations in finite orders of perturbation theory performed in different renormalization schemes may differ by higher-order contributions. In an all-order calculation all different schemes would lead to equivalent relations between physical quantities. The dependence of the choice of the renormalization scheme and consequences are studied with the help of renormalization-group equations.

In our Lagrangian we absorb the divergences by a simple rescaling of the original (bare) quantities, denoted by the subscript 0. The renormalized quantities are defined as

$$\pi_\pm = Z_{\pi_\pm}^{1/2} \pi_{\pm,0}, \tag{8.1a}$$

$$A^\mu = Z_A^{1/2} A_0^\mu, \tag{8.1b}$$

$$Z^\mu = Z_Z^{1/2} Z_0^\mu, \tag{8.1c}$$

$$m_0^2 = Z_m m^2, \tag{8.1d}$$

$$g_0 = Z_g g, \tag{8.1e}$$

$$\xi_0 = Z_A \xi, \tag{8.1f}$$

where $Z_{\pi_\pm}$, Z_A and Z_Z are wave-function or field-strength renormalization constants; Z_g and Z_m are known as the coupling and the mass renormalization constants respectively. In perturbation theory we write

$$Z_i = 1 + \delta Z_i. \tag{8.2}$$

This *multiplicative renormalization* does not change the functional dependence of $\mathcal{L}(\psi_i, g, m)$ on ψ_i, g and m. $\mathcal{L}(\psi_i, g, m)$ yields the same Feynman rules for the renormalized fields and parameters as does $\mathcal{L}(\psi_{i,0}, g_0, m_0)$ does for the bare ones. The counterterm Lagrangian $\mathcal{L}_{\text{ct}}$ summarizes all terms containing the renormalization constants and generates counterterm Feynman rules[1]. The renormalization constants will absorb the divergences, up to finite parts. In the mass-independent *modified minimal-substraction scheme* $\overline{\text{MS}}$ only the divergences of the form $\Delta = 2/(4-D) - \gamma_E + \log 4\pi$ get subtracted, which is especially convenient for higher-order calculations and best suited for dimensional regularization. In the *on-shell scheme* one determines the renormalization constants by imposing renormalization conditions so that m and g are the physical masses and coupling constants. The difference of the various schemes is in the finite part of the renormalization constants. The transformations which link the coupling constants in different schemes leave the β-function invariant up to second order.

Renormalization group

The reparametrization (8.1) is not unique. Physical results are independent of the choice of renormalized parameters. Only the explicit expressions for physical quantities change, not the relation between them. This fact is the basis of the *renormalization group* (RG).

The renormalization group equation (RGE) follows from the fact that S-matrix elements (or n-point proper vertex functions Γ_n which are related to the generating functional of Green functions via a Legendre transformation) do not change (calculated in all orders of perturbation theory) under the shift of $\mu \to \mu + \delta\mu$, (where μ is the arbitrary mass parameter in dimensional regularization)

$$\mu \frac{\delta}{\delta\mu} \langle \text{out}|S|\text{in}\rangle = 0. \tag{8.3}$$

The proper vertex functions are renormalized by

$$\Gamma_n^{(r)} = Z_\phi^{n/2} \Gamma_n^{(0)}, \tag{8.4}$$

where the unrenormalized n-point vertex function $\Gamma_n^{(0)}$ only depends on the bare quantities $g_0, m_0, \ldots$ and *not* on the mass scale μ. The renormalization constants $Z_\phi^{1/2}$ for each field in the n-point vertex and the renormalized vertex function $\Gamma_n^{(r)}$ itself *do* depend on μ and the regularized masses m and coupling constants g. The RGE exploits

[1]For a massless gauge boson it is an easy exercise to show that $-\mathrm{i}\delta Z_A[k^2 g^{\mu\nu} - (1 - 1/\xi)k^\mu k^\nu] + \mathrm{i}\delta Z_\xi (1/\xi) k^\mu k^\nu$ is the counterterm vertex.

the μ-independence of $\Gamma_n^{(0)}$ to determine how $\Gamma_n^{(r)}$ must depend on μ.

$$\begin{aligned} \mu\frac{\partial}{\partial\mu}\Gamma_n^{(0)} = 0 &= \mu\frac{\partial}{\partial\mu}(Z_\phi^{-n/2}\Gamma_n^{(r)}(g,m,\mu)) \\ &= Z_\phi^{-n/2}\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_m\frac{\partial}{\partial m} - n\gamma(g)\right)\Gamma_n^{(r)}(g,m,\mu), \end{aligned} \tag{8.5}$$

where dimensionless functions are defined as

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}, \tag{8.6a}$$

$$\gamma_m(g) = \mu\frac{\partial}{\partial\mu}\ln m = m^{-1}\mu\frac{\partial}{\partial\mu}m, \tag{8.6b}$$

$$\gamma(g) = \mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi} = \frac{1}{2}Z_\phi^{-1}\mu\frac{\partial}{\partial\mu}Z_\phi. \tag{8.6c}$$

$\gamma(g)$ is the called the anomalous dimension, $\gamma_m(g)$ is the renormalization-group coefficient for the mass term and $\beta(g)$ is the renormalization-group function, or β-function[2]. These functions are the same for all vertex functions and thus a characteristic of the theory. They are related to the shifts in the coupling constant, mass and field strength that compensate for the shift in the renormalization scale μ. The behaviour of the coupling constant as a function of μ is of particular interest, since it determines the strength of the interaction and the conditions under which perturbation theory is valid. We can compute these functions by choosing convenient Green functions[3], where we insist that the expressions satisfy the RGE. Because the μ-dependence of a renormalized Green function originates in the counterterms that cancel its logarithmic divergences, we find that the β, γ_m and γ functions are simply related to these counterterms, or equivalently to the coefficients of the divergent logarithms. In order to determine the β-function we also need the counterterms for the three point functions, the vertex corrections. We only have calculated the two point functions (self-energies) which is not enough. But there is a way how to circumvent this problem: the gauge invariance of the effective action in the background field gauge relates the renormalization constant of the gauge coupling to the renormalization constant of the background field. Because explicit gauge invariance is retained in the background field method, the infinities appearing in the effective action must take the gauge invariant form of a divergent constant times $(F_{\mu\nu}^a)^2$. According to (8.1), $F_{\mu\nu}^a$ is renormalized by

$$(F_{\mu\nu}^a)_0 = Z_A^{1/2}\Big[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gZ_gZ_A^{1/2}f^{abc}A_\mu^bA_\nu^c\Big]. \tag{8.7}$$

[2]In principle, these functions can also depend on the dimensionless quantity m/μ. This problem is circumvented in the mass-independent $\overline{MS}$-scheme.

[3]The RGE holds also for n-point Green functions.

This is only invariant if

$$Z_g = Z_{\hat{A}}^{-1/2} \tag{8.8}$$

is satisfied. So, we can extract the β-function from the self-energy of the background field $\hat{A}$.

With (8.1), (8.6) and the fact that g_0 and $Z_{\hat{A}}$ are independent of μ, the β-function and the anomalous dimension γ are related to the coupling constant renormalization Z_g and the field renormalization $Z_{\hat{A}}$ by

$$\beta(g) = -g\mu \frac{\partial}{\partial \mu} \ln Z_g, \qquad \gamma(g) = \frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_{\hat{A}}. \tag{8.9}$$

Thus, β and γ are related by

$$\beta = g\gamma. \tag{8.10}$$

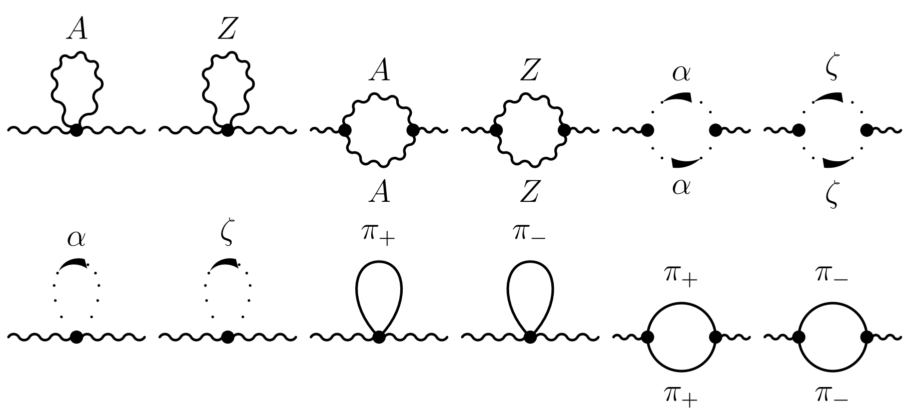
The β-function can be written as an expansion in g of the form

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0 - \frac{g^5}{(16\pi^2)^2}\beta_1 + \mathcal{O}(g^7) \tag{8.11}$$

where the first coefficient is obtained from a one loop calculation.

Self-energy for the background field gauge boson

The self-energy for $\hat{A}$ can be calculated similarly as in the conventional formalism (cf. section 6.3.2). The Feynman graphs with the external background fields are the same with one exception: the fifth graph in (6.37) where the loop contains π_- and Z doesn't exist, since the $\hat{A}Z\pi_-$-vertex is zero in the background field gauge. We also have two ghost diagrams coming from the four vertex with two ghosts. The graphs with scalar loops give the same result as in the conventional mechanism, while the graphs involving gauge bosons and ghosts yield different results owing to the modified Feynman rules (cf. appendix D).



(8.12)

The calculation was implemented in FORM. It turns out that the mass terms for graphs with massive propagators have a complicated dependence on ξ, but the sum of all contributions is zero. This is not surprising, we expect the cancellation of all mass terms, since the gauge boson is strictly massless. We also expect that the final result is ξ-independent and transverse, but already each graph has a transverse nature, when ignoring the mass terms. It is given by

$$\Pi_T^{\text{total}} = 14g^2k^2\Delta. \tag{8.13}$$

As always, the factor $\mathrm{i}/(16\pi^2)C_A\delta^{ab}$ has been suppressed. The gauge boson and ghost loops contribute a factor of $11/3 \cdot (2g)^2k^2\Delta$ (in agreement with textbook results for a pure Yang-Mills theory) and the scalar loops a factor of $-1/6 \cdot (2g)^2k^2\Delta$ to the final result. The factor of 2 results form the fact that we use twice the generators of the group (cf. (3.30)). Note that for each diagram with massless propagators exists an analogous diagram with massive propagators which equally contributes to the β function.

From this we obtain the background field renormalization constant

$$Z_{\hat{A}} = 1 + 14C_A\left(\frac{g^2}{16\pi^2}\right)\Delta \tag{8.14}$$

and the β-function

$$\beta(g) = -\frac{g^3}{16\pi^2}\beta_0, \qquad \beta_0 = 14C_A. \tag{8.15}$$

We can easily solve this differential equation by integrating

$$\int_g^{\overline{g}} \frac{1}{g^3}\mathrm{d}g = -\int_{m^2}^{Q^2} \frac{\beta_0}{16\pi^2}\frac{\mathrm{d}\mu}{\mu} \tag{8.16}$$

and we obtain

$$\overline{g}^2(Q^2) = \frac{g^2}{1 + \frac{g^2}{16\pi^2}2\beta_0 \ln\frac{Q^2}{m^2}} \tag{8.17}$$

for the *running coupling constant*. Additional matter multiplets would give a positive contribution to the β-function. The coupling constant tends to zero at a logarithmic rate as the energy scale increases. This is called *asymptotic freedom*.

9 Conclusion

The last missing particle in the Standard Model, the Higgs boson is not protected by a symmetry from radiative corrections to its mass. Quadratically divergent contributions from gauge boson, top and Higgs loops arise. It was long believed that only supersymmetry provides a realistic mechanism to cancel the quadratic dependence of the Higgs mass on the cut-off scale. Recently, a new way of cancelling of quadratic divergences was found where loops with the same spin cancel. These models are called Little Higgs models, and the Higgs boson is naturally light. This is also due to a new symmetry. The Higgs boson is a Goldstone boson, resulting from the spontaneous breaking of a global symmetry in such a way that no single operator alone breaks the symmetry. The idea is based on the deconstruction of extra dimensions, but it can be shown that the mechanism can be generalized and doesn't depend on this concrete realization (cf. [ArH02]).

This work investigates a special Little Higgs Model. Starting from the Lagrangian with $N = 2$ sites, motivated from the deconstruction of a fifth dimension, we made an expansion in the inverse of the symmetry breaking scale $4\pi v$ and derived the relevant Feynman rules for a one loop calculation. We also discussed the importance of the BRS formalism, which yields a deeper insight into the structure of quantum field theory. We quantized the theory using the BRS method and explicitly showed the nilpotence of the transformation. This completed our Feynman rules. Then we calculated the quadratically and logarithmically divergent contributions to all self-energies in a general R_ξ-gauge, using dimensional regularization. We showed that π_+, the Little Higgs, is free of quadratically divergent mass contributions. We discovered a quadratic divergence for the mass terms of the massive gauge boson. We have attributed this to the nonlinear realization of the sigma model. We derived Slavnov-Taylor identities and the Goldstone boson equivalence theorem and successfully applied these to the results. Finally, we could compute the β-function of the model using the background field method.

For a future work, an extension of the loop calculations to a model with N sites and the calculation of vertex functions to determine all renormalization constants can be carried out. But for a realistic model which completely includes the Standard Model, further modifications have to applied. This is done in [ArH02], for example. Thus, Little Higgs models are serious competitors for an alternative solution of the hierarchy problem. Currently, we cannot say whether SUSY or the Little Higgs models describe nature best. Forthcoming experiments at the LHC or at a future linear collider have to decide.

A Dimensional Regularization

Higher-order corrections to Green functions and S-matrix elements result from Feynman graphs containing loops. This is a conceptual and technical complication because *divergences* occur in the evaluation of loop diagrams. The simplest example is the scalar tadpole diagram (Fig. A.1).

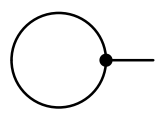


Figure A.1: tadpole graph

It involves the integral

$$A_0(m) = \frac{1}{\mathrm{i}\pi^2} \int d^4k \frac{1}{k^2 - m^2}. \tag{A.1}$$

The factor $(\mathrm{i}\pi^2)^{-1}$is added for convenience. According to power counting, the integral is quadratically divergent in four dimensions. For large momenta, $A_0 \propto \int_0^\Lambda \frac{k^3 \mathrm{d}k}{k^2} = \int_0^\Lambda k \mathrm{d}k = \Lambda^2$, where we send the cut-off parameter Λ to infinity.

In the dimensional regularization scheme, calculations are performed in D instead of four dimensions. Since loop integrals converge for small enough D ($D < 2$ for A_0) the usual calculational rules for integrals, such as linearity, translational and rotational invariance, and the usual scaling can be used. The analytic structure of these integrals allows for an analytic continuation to arbitrary complex D. The UV divergences manifest themselves as poles at integer values of D. Changing the dimension of the integral changes also the dimension of $A_0(m)$. We compensate this by multiplying with μ^{4-D}, where μ has the dimension of a mass. So, we replace

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \mu^{(4-D)} \int \frac{d^Dk}{(2\pi)^D}. \tag{A.2}$$

A precise definition of dimensional regularization implies that integrals vanish if they do not depend on any scale

$$\int d^D k (k^2)^\alpha = 0, \tag{A.3}$$

although being formally infinite for all α and integer D. The following is taken from [Kil02] which is an excellent introduction for dimensional regularization.

Standardized One-Loop Integrals

General N-point tensor integral:

$$T^N_{\mu_1,\cdots\mu_M}(p_1,\dots p_{N-1}, m_0,\dots,m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int d^D q \frac{q_{\mu_1}\cdots q_{\mu_M}}{N_0 N_1\cdots N_{N-1}} \tag{A.4}$$

with

$$N_0 = (q^2 - m^2), \qquad N_i = [(q+p_i)^2 - m_i^2], \qquad i = 1,\dots,N-1 \tag{A.5}$$

The p_i are external momenta. Note that the integral is only over $d^D q$. In calculations this leads to factors of $\mathrm{i}/(16\pi^2)$.

In the following, we will write T^1 as A, T^2 as B, T^3 as C, and so forth. Scalar integrals with no momenta in the numerator are denoted with the subscript 0.

Tensor integrals with loop momenta in the numerator can be built up from a complete set of Lorentz tensors and invariant scalar coefficient functions. The Lorentz tensors are constructed from the external momenta and the d-dimensional metric tensor $g^{\mu\nu}$.

$$B^\mu = p_1^\mu B_1 \tag{A.6a}$$
$$B^{\mu\nu} = g^{\mu\nu} B_{00} + p_1^\mu p_1^\nu B_{11} \tag{A.6b}$$
$$C^\mu = p_1^\mu B_1 + p_2^\mu B_2 \tag{A.6c}$$
$$C^{\mu\nu} = g^{\mu\nu} C_{00} + p_1^\mu p_1^\nu C_{11} + (p_1^\mu p_2^\nu + p_1^\nu p_1^\mu) C_{12} + p_2^\mu p_2^\nu C_{22} \tag{A.6d}$$
$$\dots$$

Now we have to evaluate the scalar integrals. We define the auxiliary integral

$$I_n(A) = \int d^D q \frac{1}{(q^2 - A + \mathrm{i}\epsilon)^n}, \quad D < 2n, \quad A > 0. \tag{A.7}$$

After performing a Wick-rotation (euclidian coordinates, time component of the four vector multiplied by i), using Cauchy's theorem and integrating in polar coordinates (the integral over the surface of a D-dimensional unit sphere is $2\pi^{D/2}/\Gamma(D/2)$) we obtain

$$I_n(A) = \mathrm{i}(-1)^n \pi^{D/2} \frac{\Gamma(n - D/2)}{\Gamma(n)} (A - \mathrm{i}\epsilon)^{\frac{D}{2}-n}. \tag{A.8}$$

The properties of $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ are:

- poles at $z = 0, -1, -2, -3, \ldots,$
- $\frac{1}{\Gamma(z)}$ is analytical,
- $\Gamma(z+1) = z\Gamma(z),$
- $\Gamma(n+1) = n! \quad \text{for} \quad n = 0, 1, 2, 3, \ldots; \Gamma(\frac{1}{2}) = \sqrt{\pi},$
- $\lim\limits_{z \to 0} \Gamma(z) = \frac{1}{z} - \gamma_E + \mathcal{O}(z) \quad \text{with} \quad \gamma_E = 0.5772\ldots$

one-point function $A_0(m)$:

$$\begin{aligned} A_0(m) &= \frac{(2\pi)^{4-D}}{\mathrm{i}\pi^2} \int d^D q (q^2 - m^2 + \mathrm{i}\epsilon)^{-1} \\ &= \frac{(2\pi)^{4-D}}{\mathrm{i}\pi^2} I_1(m^2) \end{aligned} \tag{A.9}$$

In chapter 6 we are interested in the quadratic and logarithmic divergences of two-point functions. Quadratic divergences appear as a pole in $(D-2)$ whereas the logarithmic divergences appear as poles in $(D-4)$.
In the limit of $D \to 2$ we obtain

$$A_0 = -4\pi\mu^2 \left(\frac{2}{2-D} - \gamma_E + \log 4\pi - \log \frac{m^2}{\mu^2} \right). \tag{A.10}$$

Note that the coefficient of the pole is independent of the mass. Since we are only interested in the coefficients of the poles we can ignore finite terms. Thus we can drop the argument of A_0. This implies that we do *not* set $\int d^D q (q^{-2}) = 0$ for $D \to 2$. A quadratic divergence vanishes if its coefficients can be written in the form $D-2$, that is if the coefficient is zero for $D = 2$.
In the limit of $D \to 4$ we obtain after an Taylor expansion

$$A_0(m) = m^2 \left[\Delta - \log \left(\frac{m^2}{\mu^2} \right) + 1 \right] + \mathcal{O}(D-4) \tag{A.11}$$

with

$$\Delta = \frac{2}{4-D} - \gamma_E + \log 4\pi \tag{A.12}$$

Note that A_0 vanishes for zero mass. Thus, $A_0(0)$ is purely quadratically divergent.

N-point functions:
With the Feynman parametrization

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x[a(1-x)+bx]^{-2} \tag{A.13}$$

and its generalizations we can also find expressions for higher n-point functions. We also need expressions for the tensor coefficients B_1, B_{11}, $B_{00}, \ldots$ in terms of scalar integrals A_0, B_0, C_0 and D_0. Note that only A_0 and B_0 are logarithmically divergent. The calculational details can be found in [Kil02]. All logarithmically divergent ($D \to 4$) parts of the tensor integrals (including higher C and D functions are listed in [Den03].

$$A_0(m_0) = m_0^2\Delta \tag{A.14a}$$

$$A_{00}(m_0) = \frac{1}{4}m_0^2\Delta \tag{A.14b}$$

$$B_0(p_1, m_0, m_1) = \Delta \tag{A.14c}$$

$$B_1(p_1, m_0, m_1) = -\frac{1}{2}\Delta \tag{A.14d}$$

$$B_{00}(p_1, m_0, m_1) = -\frac{1}{12}[p^2 - 3(m_0^2 + m_1^2)]\Delta \tag{A.14e}$$

$$B_{11}(p_1, m_0, m_1) = \frac{1}{3}\Delta \tag{A.14f}$$

$$C_{00}(p_1, p_2, m_0, m_1, m_2) = \frac{1}{4}\Delta \tag{A.14g}$$

All other tensor integrals are finite. In the limit of $D \to 2$ only $A_0, B_{00} = 1/2A_0$ and $C_{0000} = 1/8A_0$ are divergent.

B Linear Sigma-Model

In this appendix a gauged *linear* sigma model is written down and Feynman rules are derived for it. We show that quadratic divergences for the massive gauge boson resulting from *scalar* loops cancel, in contrast to the nonlinear realization.

The $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$-invariant Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial\vec{\pi})^2 + \frac{1}{2}(\partial\sigma)^2 - \frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 \tag{B.1}$$

where $(\sigma, \vec{\pi})$ forms a vector in $O(4)$ which locally isomorphic to $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$. In a more compact matrix notation

$$\Sigma = \sigma + \mathrm{i}\vec{\tau}\vec{\pi} \tag{B.2}$$

we rewrite the Lagrangian

$$\mathcal{L}_0 = \frac{1}{4}\,\mathrm{tr}(\partial_\mu\Sigma)(\partial^\mu\Sigma)^\dagger - \frac{\mu^2}{4}\,\mathrm{tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left[\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\right]^2 \tag{B.3}$$

and the invariance of $\mathcal{L}_0$ under $\Sigma \to L\Sigma R^\dagger$ becomes obvious. The τ_i are the Pauli matrices which satisfy the following relations

$$\tau^a\tau^b = \delta^{ab} + i\epsilon^{abc}\tau^c, \tag{B.4a}$$

$$\mathrm{tr}\,\tau^a = 0, \tag{B.4b}$$

$$\mathrm{tr}\,\tau^a\tau^b = 2, \tag{B.4c}$$

$$\mathrm{tr}(\tau^a\tau^b\tau^c) = 2\mathrm{i}\epsilon^{abc}. \tag{B.4d}$$

For $\mu < 0$ the minimum of the potential $V(\sigma, \vec{\pi}) = \frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$ occurs at

$$\langle\sigma\rangle_0 = \sqrt{\frac{-\mu^2}{\lambda}} \equiv v \tag{B.5}$$

We introduce a shifted field $\tilde{\sigma}$ by subtracting the vev from σ

$$\tilde{\sigma} = \sigma - v \tag{B.6}$$

The Lagrangian in terms of $\tilde{\sigma}$ and π reads

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\tilde{\sigma}\partial^\mu\tilde{\sigma} - \frac{1}{2}(-2\mu^2)\tilde{\sigma}^2 + \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} - \lambda v\tilde{\sigma}(\tilde{\sigma}+\vec{\pi})^2 - \frac{\lambda}{4}[(\tilde{\sigma}+\vec{\pi})^2 - v^4] \quad \text{(B.7)}$$

Now we gauge the model by introducing covariant derivatives (in complete analogy to the nonlinear sigma model)

$$D_\mu\Sigma = \partial_\mu\Sigma - \mathrm{i}gA^L_\mu\Sigma + \mathrm{i}g\Sigma A^R_\mu, \quad \text{(B.8)}$$

and the locally gauge invariant Lagrangian reads

$$\mathcal{L} = \frac{1}{4}(D_\mu\Sigma)(D^\mu\Sigma)^\dagger - \frac{\mu^2}{4}\operatorname{tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left[\operatorname{tr}\left(\Sigma^\dagger\Sigma\right)\right]^2. \quad \text{(B.9)}$$

Σ and $D_\mu\Sigma$ both transform covariantly under $\mathbf{L}\times\mathbf{R}$

$$\Sigma \to L\Sigma R^\dagger, \quad \text{(B.10a)}$$

$$D_\mu\Sigma \to L(D_\mu\Sigma)R^\dagger \quad \text{(B.10b)}$$

with

$$L = \mathrm{e}^{-\mathrm{i}\vec{\alpha}_L\cdot\vec{\tau}/2}, \qquad R = \mathrm{e}^{-\mathrm{i}\vec{\alpha}_R\cdot\vec{\tau}/2}, \quad \text{(B.11)}$$

provided the gauge fields transform as

$$A^L_\mu \to LA^L_\mu L^\dagger + \mathrm{i}\frac{1}{g_L}L(\partial_\mu L^\dagger) \quad \text{(B.12a)}$$

$$A^R_\mu \to RA^R_\mu R^\dagger + \mathrm{i}\frac{1}{g_R}R(\partial_\mu R^\dagger). \quad \text{(B.12b)}$$

We obtain the transformation properties for σ by taking the trace of (B.10). For π^a we first multiply with τ^a and then take the trace. A simple calculation yields

$$\delta\sigma = \frac{1}{2}(\alpha_L - \alpha_R)^a\pi^a, \quad \text{(B.13a)}$$

$$\delta\pi^a = -\frac{1}{2}(\alpha_L - \alpha_R)^a\sigma - \frac{1}{2}\epsilon^{abc}\pi^b(\alpha_L + \alpha_R)^c. \quad \text{(B.13b)}$$

Next, we want to rewrite the Lagrangian in terms of the fields $\tilde{\sigma}$ and π. In order to do this, we have to find expressions for the covariant derivatives for $\tilde{\sigma}$ and π. For $D_\mu\sigma$ this can be found by taking the trace of (B.8), for $D_\mu\pi$ we first multiply with τ^a.

$$D_\mu\sigma = \partial_\mu\sigma - g(A^R - A^L)^a_\mu\pi^a, \quad \text{(B.14a)}$$

$$D_\mu\pi^a = \partial_\mu\pi^a + g(A^R - A^L)^a_\mu\sigma + g\epsilon^{abc}(A^R + A^L)^b_\mu\pi^c. \quad \text{(B.14b)}$$

As in the nonlinear model, we define A and Z bosons as linear combinations of A^L and A^R,

$$A_\mu^a = A_\mu^{R,a} + A_\mu^{L,a}, \qquad Z_\mu^a = A_\mu^{R,a} - A_\mu^{L,a}. \tag{B.15}$$

The squares of the covariant derivatives are

$$\frac{1}{2}(D_\mu\sigma)^2 = \frac{1}{2}(\partial_\mu\tilde{\sigma} - gZ_\mu^a\pi^a)^2 = \frac{1}{2}(\partial\tilde{\sigma})^2 + \frac{1}{2}g^2(Z_\mu^a\pi^a)^2 - g\partial_\mu\tilde{\sigma}(Z^{\mu,a}\pi^a) \tag{B.16}$$

and

$$\begin{aligned}\frac{1}{2}(D_\mu\pi)^2 = \frac{1}{2}(\partial_\mu\pi^a + gZ_\mu^a(\tilde{\sigma}+v) + g\epsilon^{abc}A_\mu^b\pi^c)^2& \\ = \frac{1}{2}(\partial\pi)^2 + \frac{1}{2}g^2Z_\mu^aZ^{\mu,a}(\tilde{\sigma}+v)^2 + \frac{1}{2}g^2\epsilon^{abc}\epsilon^{ade}A_\mu^b\pi^cA^{\mu,d}\pi^e& \\ + g\partial_\mu\pi Z^{\mu,a}(\tilde{\sigma}+v) + g\epsilon^{abc}\partial_\mu\pi^aA^{\mu,b}\pi^c + g^2\epsilon^{abc}Z_\mu^aA^{\mu,b}(\tilde{\sigma}+v)\pi^c&.\end{aligned} \tag{B.17}$$

Together with the potential terms in (B.7) we obtain (partially suppressed indices)

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial\tilde{\sigma})^2 - \frac{1}{2}(-2\mu^2)\tilde{\sigma}^2 + \frac{1}{2}(\partial\pi)^2 + \frac{1}{2}(gv)^2Z^2 + gv\partial\pi Z& \\ + \frac{1}{2}g^2(Z\pi)^2 - g\partial\tilde{\sigma}(Z\pi) + \frac{1}{2}g^2Z^2\tilde{\sigma}^2 + g^2vZ^2\tilde{\sigma}& \\ + \frac{1}{2}g^2\epsilon^{abc}\epsilon^{ade}A^b\pi^cA^d\pi^e + g\partial\pi Z\tilde{\sigma} + g\epsilon^{abc}\partial\pi^aA^b\pi^c + g^2\epsilon^{abc}Z^aA^b\tilde{\sigma}\pi^c& \\ + g^2v\epsilon^{abc}Z^aA^b\pi^c - \lambda v\tilde{\sigma}^3 - \lambda v\tilde{\sigma}\pi^2 - \frac{\lambda}{4}\tilde{\sigma}^2\pi^2 - \frac{\lambda}{4}\tilde{\sigma}^4 - \frac{\lambda}{4}\pi^4&.\end{aligned} \tag{B.18}$$

Note the mass term for the Z boson, $m = gv$. There is also a mixing term, $gv\partial\pi Z$. To this Lagrangian we add the kinetic terms of Z and A and also the BRS invariant Lagrangian

$$\mathcal{L}_{\mathrm{BRS}} = s\Big(\overline{\eta}_\zeta(G_\zeta + \frac{1}{2}\xi B_Z)\Big) + s\Big(\overline{\eta}_\alpha(G_\alpha + \frac{1}{2}\xi B_A)\Big) \tag{B.19}$$

with the gauge-fixing functionals for Z and A

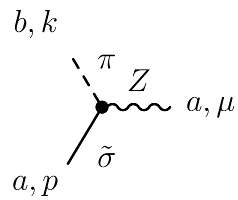
$$G_\zeta = \partial Z - \xi m\pi, \qquad G_\alpha = \partial A. \tag{B.20}$$

This cancels the mixing term and yields properly defined gauge-boson propagators and ghost vertices. The particle content of our model is one massive gauge boson Z which acquires its extra degree of freedom by eating the unphysical would-be Goldstone boson π, one massless gauge boson A and one massive scalar boson $\tilde{\sigma}$ (heavy higgs). Extending our model to two link fields ($N = 2$) would be straightforward, the argumentation is analogous to the nonlinear model. This would lead to a second massive $\tilde{\sigma}$-boson and an extra *physical* scalar π (but with zero mass, the little higgs) which is not eaten by a gauge boson.

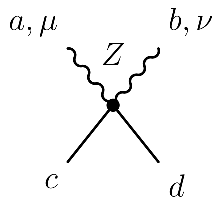
We are only interested in scalar loops which contribute in the self-energy of the Z boson. For this it is sufficient to consider only the $N = 1$ case. To calculate these terms, we only need three and four vertices with two Z boson legs. The ghost terms do not contribute. The relevant terms are

$$\frac{1}{2}g^2 g^{\mu\nu}\delta^{ab} Z^a_\mu Z^b_\nu \tilde{\sigma}^2, \quad \frac{1}{2}g^2 g^{\mu\nu}\delta^{ab}\delta^{cd} Z^a_\mu \pi^b Z^c_\nu \pi^d, \quad g\partial_\mu \pi^a Z^{a,\mu}\tilde{\sigma} - g\partial_\mu\tilde{\sigma}(Z^{\mu,a}\pi^a). \tag{B.21}$$

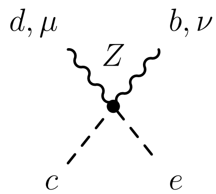
This leads to the following vertices:



$$= g(k-p)^\mu \delta^{ab}, \tag{B.22}$$



$$= \mathrm{i}g^2 g^{\mu\nu}\left[\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right], \tag{B.23}$$

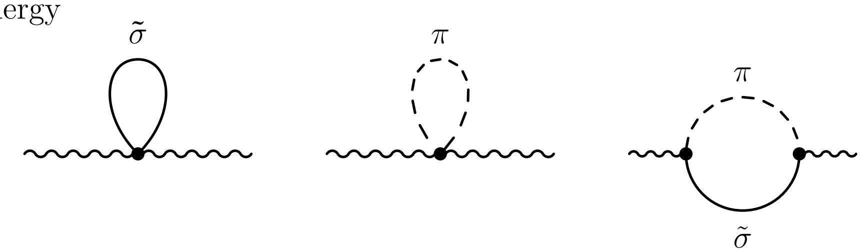


$$= \mathrm{i}g^2 g^{\mu\nu}\left[\epsilon^{abc}\epsilon^{ade} + \epsilon^{abe}\epsilon^{adc}\right]. \tag{B.24}$$

(B.25)

(for $N = 2$ the number of vertices doubles, the signs do not change)

From this we can build the following scalar loops which contribute to the Z boson self-energy



. (B.26)

The first diagram yields (symmetry factor 1/2 and suppressing the factor

$i/(16\pi^2)C_A\delta^{ab})$

$$ig^2 g^{\mu\nu}\int \frac{d^dp}{(2\pi)^d}\frac{i}{p^2+2\mu^2} = -g^2 g^{\mu\nu}A_0(-2\mu^2). \tag{B.27}$$

The second is analogous

$$-g^2 g^{\mu\nu}A_0(\xi m^2), \tag{B.28}$$

the last diagram is

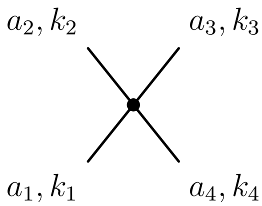
$$g^2\int \frac{d^dp}{(2\pi)^d}\frac{(k+2p)^\mu(k+2p)^\nu}{(p^2+2\mu^2)[(p+k)^2-\xi m^2]} = g^2(4B^{\mu\nu}+4k^\mu B^\nu + k^\mu k^\nu B_0). \tag{B.29}$$

Only A_0 and $B_{00} = 1/2A_0 + \dots$ contribute to quadratic divergences and the masses are irrelevant. So the quadratic divergences coming from the first and second diagram are exactly cancelled by the third diagram which is absent in the nonlinear sigma model.

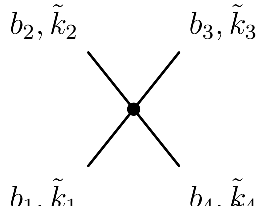
C Feynman Rules in R_ξ-gauge

C.1 Vertices

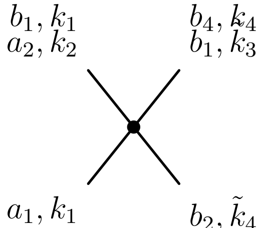
Unless otherwise denoted, the massless vector boson A is denoted by the indices a_i, μ_i, the massive vector boson $Z^{c_i}_{\nu_i}$, the real scalar $\pi^{a_i}_+(k_i)$ and the would-be Goldstone boson $\pi^{b_i}_-(\tilde{k}_i)$.



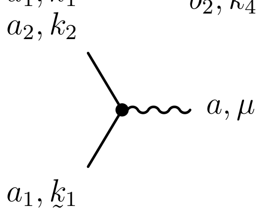
$$= \mathrm{i}\frac{1}{6v^2}\begin{bmatrix} (k_1-k_2)(k_3-k_4)f^{a_1a_2c}f^{a_3a_4c} \\ +(k_1-k_3)(k_4-k_2)f^{a_1a_3c}f^{a_4a_2c} \\ +(k_1-k_4)(k_2-k_3)f^{a_1a_4c}f^{a_2a_3c} \end{bmatrix} = S_{a_1a_2a_3a_4}(k_1,k_2,k_3,k_4) \tag{C.1}$$



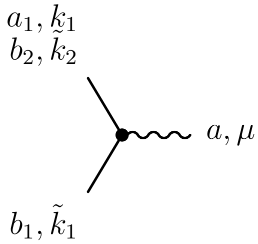
$$= S_{b_1b_2b_3b_4}(\tilde{k}_1,\tilde{k}_2,\tilde{k}_3,\tilde{k}_4) \tag{C.2}$$



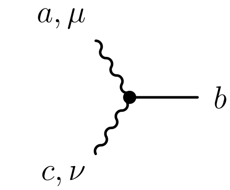
$$= S_{a_1a_2b_1b_2}(k_1,k_2,\tilde{k}_3,\tilde{k}_4) \tag{C.3}$$



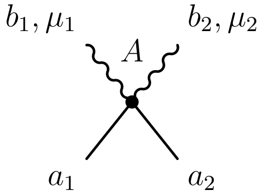
$$= -\sqrt{2}g(k_1-k_2)^\mu f^{aa_1a_2} \tag{C.4}$$



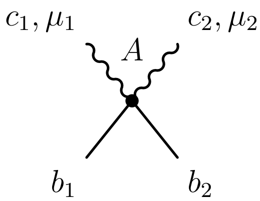
$$= -\sqrt{2}g(\tilde{k}_1-\tilde{k}_2)^\mu f^{ab_1b_2} \tag{C.5}$$



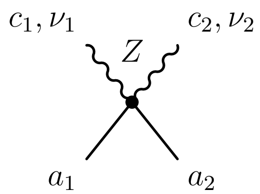
$$= 2\sqrt{2}\mathrm{i}g^2 v g^{\mu\nu} f^{abc} \tag{C.6}$$



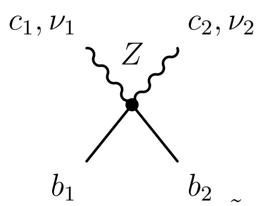
$$= 2\mathrm{i}g^2 \left[f^{a_1 b_1 e} f^{a_2 b_2 e} + f^{a_1 b_2 e} f^{a_2 b_1 e} \right] g^{\mu_1 \mu_2} = F^{\mu_1 \mu_2}_{a_1 a_2 b_1 b_2} \tag{C.7}$$



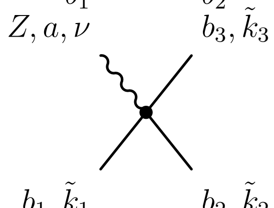
$$= F^{\mu_1 \mu_2}_{b_1 b_2 c_1 c_2} \tag{C.8}$$



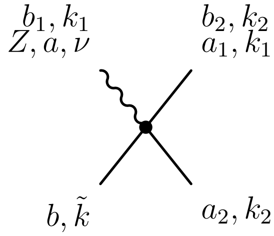
$$= F^{\nu_1 \nu_2}_{a_1 a_2 c_1 c_2} \tag{C.9}$$



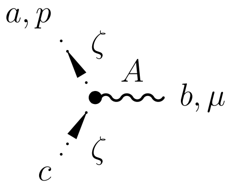
$$= F^{\nu_1 \nu_2}_{b_1 b_2 c_1 c_2} \tag{C.10}$$



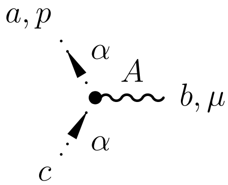
$$= \frac{2g}{3v} \begin{bmatrix} (\tilde{k}_1 - \tilde{k}_2)^\nu f^{a_1 a_2 c} f^{a_3 a c} \\ +(\tilde{k}_2 - \tilde{k}_3)^\nu f^{a_2 a_3 c} f^{a_1 a c} \\ +(\tilde{k}_3 - \tilde{k}_1)^\nu f^{a_3 a_1 c} f^{a_2 a c} \end{bmatrix} = M^\nu_{a a_1 a_2 a_3}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \tag{C.11}$$



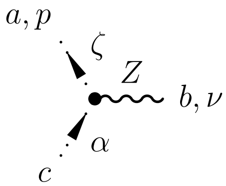
$$= M^\nu_{a a_1 a_2 b}(k_1, k_2, \tilde{k}) \tag{C.12}$$



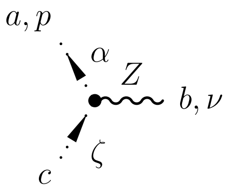
$$= -\sqrt{2} g p^\mu f^{abc} = T^\mu_{abc}(p) \tag{C.13}$$



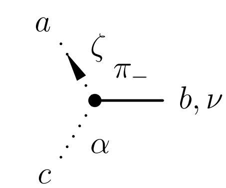
$$= T^\mu_{abc}(p) \tag{C.14}$$



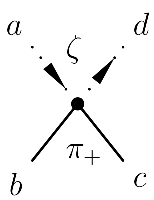
$$= -T^\nu_{abc}(p) \tag{C.15}$$



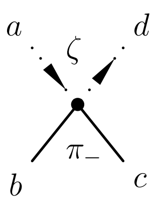
$$= -T^\nu_{abc}(p) \tag{C.16}$$



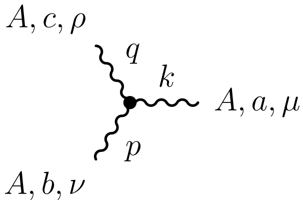
$$= -2\sqrt{2} \mathrm{i} g^2 v f^{abc} \tag{C.17}$$



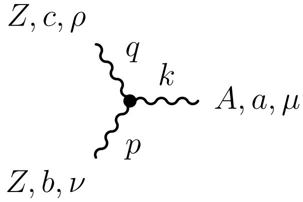
$$= -\frac{2\mathrm{i}}{3} \xi g^2 \left[f^{abe} f^{cde} + f^{ace} f^{bde} \right] \tag{C.18}$$



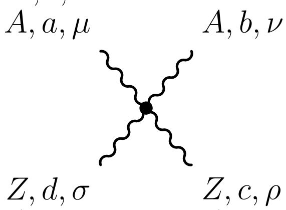
$$= -\frac{2\mathrm{i}}{3} \xi g^2 \left[f^{abe} f^{cde} + f^{ace} f^{bde} \right] \tag{C.19}$$



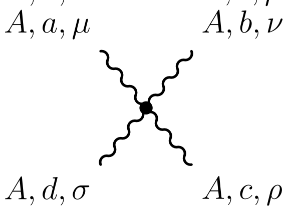
$$= \sqrt{2} g f^{abc} \begin{bmatrix} g^{\mu\nu}(k-p)^{\rho} \\ +g^{\nu\rho}(p-q)^{\mu} \\ +g^{\rho\mu}(q-k)^{\nu} \end{bmatrix} = G^{\mu\nu\rho}_{abc}(k,p,q) \tag{C.20}$$



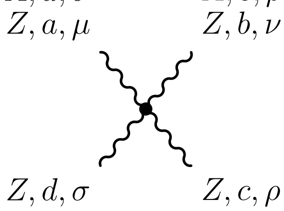
$$= G^{\mu\nu\rho}_{abc}(k,p,q) \tag{C.21}$$



$$= -2\mathrm{i}g^2 \begin{bmatrix} (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) f^{abe} f^{cde} \\ +(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) f^{ace} f^{bde} \\ +(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) f^{ade} f^{bce} \end{bmatrix} = H^{\mu\nu\rho\sigma}_{abcd} \tag{C.22}$$



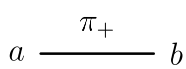
$$= H^{\mu\nu\rho\sigma}_{abcd} \tag{C.23}$$



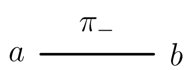
$$= H^{\mu\nu\rho\sigma}_{abcd} \tag{C.24}$$

C.2 Propagators

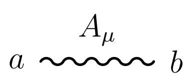
We have the usual propagators for massless/massive scalars, bosons and ghosts. The ghost fields were renormalized in the BRS transformation by a factor of $\sqrt{2}$ and the antighost fields by a factor of $1/2$ for the α ghost and a factor of $-1/2$ for the ζ ghost to get the canonical normalized propagators.



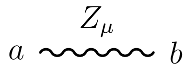
$$= \frac{\mathrm{i}}{k^2}\delta^{ab} \tag{C.25}$$



$$= \frac{\mathrm{i}}{\tilde{k}^2 - \xi m^2}\delta^{ab} \tag{C.26}$$

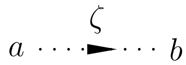


$$= \frac{-\mathrm{i}}{k^2}\left(g^{\mu\nu} - \frac{(1-\xi)k^\mu k^\nu}{k^2}\right)\delta^{ab} \tag{C.27}$$



$$= \frac{-\mathrm{i}}{k^2 - m^2} \left(g^{\mu\nu} - \frac{(1-\xi)k^\mu k^\nu}{k^2 - \xi m^2} \right) \delta^{ab} \tag{C.28}$$

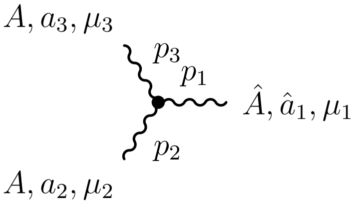
$$= \frac{\mathrm{i}}{k^2} \delta^{ab} \tag{C.29}$$



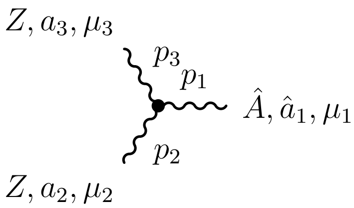
$$= \frac{\mathrm{i}}{k^2 - \xi m^2} \delta^{ab} \tag{C.30}$$

D Feynman Rules in Background Field Gauge

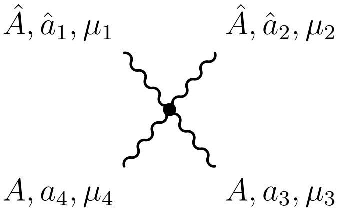
Only vertices which are relevant for the two-point function of the massless background field $\hat{A}$ are listed. All momenta (including ghosts) go into the vertex.



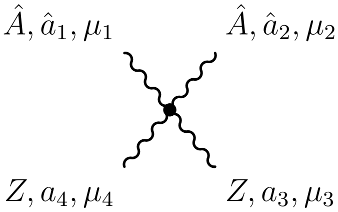
$$= \sqrt{2} g f^{\hat{a}_1 a_2 a_3} \begin{bmatrix} g^{\mu_1\mu_2}(2p_1 + (1 - \frac{1}{\xi} p_3))^{\mu_3} \\ +g^{\mu_2\mu_3}(p_2 - p_3)^{\mu_1} \\ +g^{\mu_3\mu_1}(2p_1 + (1 - \frac{1}{\xi})p_2)^{\mu_2} \end{bmatrix} = G^{\mu_1\mu_2\mu_3}_{\hat{a}_1 a_2 a_3}(p_1, p_2, p_3) \tag{D.1}$$



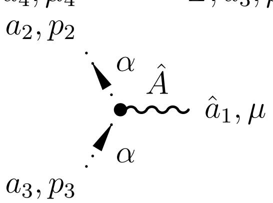
$$= G^{\mu_1\mu_2\mu_3}_{\hat{a}_1 a_2 a_3}(p_1, p_2, p_3) \tag{D.2}$$



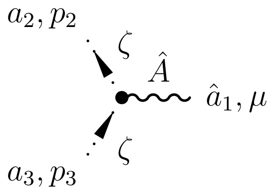
$$= -2\mathrm{i}g^2 \begin{bmatrix} (g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) & f^{\hat{a}_1\hat{a}_2 b} f^{a_3 a_4 b} \\ +(g^{\mu_1\mu_4}g^{\mu_3\mu_2} - g^{\mu_1\mu_2}g^{\mu_3\mu_4} - \frac{1}{\xi}g^{\mu_1\mu_3}g^{\mu_2\mu_4}) & f^{\hat{a}_1 a_3 b} f^{a_4 \hat{a}_2 b} \\ +(g^{\mu_1\mu_2}g^{\mu_4\mu_3} - g^{\mu_1\mu_3}g^{\mu_4\mu_2} + \frac{1}{\xi}g^{\mu_1\mu_4}g^{\mu_3\mu_2}) & f^{\hat{a}_1 a_4 b} f^{\hat{a}_2 a_3 b} \end{bmatrix} = K^{\mu_1\mu_2\mu_3\mu_4}_{\hat{a}_1\hat{a}_2 a_3 a_4} \tag{D.3}$$



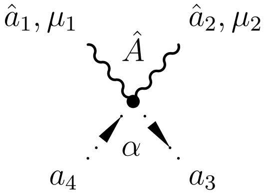
$$= K^{\mu_1\mu_2\mu_3\mu_4}_{\hat{a}_1\hat{a}_2 a_3 a_4} \tag{D.4}$$



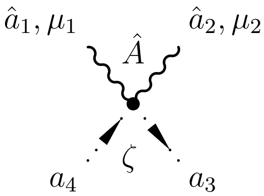
$$= -\sqrt{2} g (p_2 - p_3)^{\mu} f^{\hat{a}_1 a_2 a_3} \tag{D.5}$$



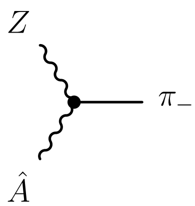
$$= -\sqrt{2}g(p_2 - p_3)^\mu f^{\hat{a}_1 a_2 a_3} \tag{D.6}$$



$$= 2\mathrm{i}g^2 \left[f^{\hat{a}_1 a_3 b} f^{\hat{a}_2 a_4 b} + f^{\hat{a}_1 a_4 b} f^{\hat{a}_2 a_3 b} \right] g^{\mu_1 \mu_2} \tag{D.7}$$



$$= 2\mathrm{i}g^2 \left[f^{\hat{a}_1 a_3 b} f^{\hat{a}_2 a_4 b} + f^{\hat{a}_1 a_4 b} f^{\hat{a}_2 a_3 b} \right] g^{\mu_1 \mu_2} \tag{D.8}$$



$$= 0 \tag{D.9}$$

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Würzburg, den 26.07.04

Stefan Karg