12. Problemset “Theoretical Particle Physics”
July 2, 2017

Jets

12.1 $e^+e^- \rightarrow q\bar{q}g$

Compute the cross section for the process $e^+e^- \rightarrow q\bar{q}g$ differential in the energies of the outgoing quarks

$$\frac{d^2\sigma}{dx_1dx_2}(x_1,x_2,s) = N_c \frac{4\pi\alpha^2 Q^2}{3s} \frac{\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$ (1)

with $Q$ the electric charge of the quark in units of $e$. You may assume $s \ll M_Z^2$ and ignore the $Z^0$-exchange.

1. Compute the total cross section for the process $e^+e^- \rightarrow q\bar{q}$.

2. Write the matrix element $T$ for the process $e^+e^- \rightarrow q\bar{q}g$, using $p_1$ and $p_2$ for the incoming $e^-$ and $e^+$ and $q_1$, $q_2$ and $k$ for the outgoing $q$, $\bar{q}$ and gluon.

3. Show that no unphysical gluons with polarization vector $\epsilon_\mu(k) = k_\mu$ are produced, i.e. verify the Ward identity $T|_{\epsilon_\mu(k) \rightarrow k_\mu} = 0$. Hint: use the Dirac equation for the external quarks and antiquarks, e.g.

$$\frac{1}{-\not{p} - \not{k} - m + i\epsilon} \not{k} v(p) = \frac{1}{-\not{\not{p}} - \not{k} - m + i\epsilon} (\not{k} + \not{p} + m) v(p) = -v(p)$$ (2)

4. Show that the squared matrix element summed over color, spins and polarizations can be written

$$\sum_{\text{colors, spins, pol.}} |T|^2 = \frac{e^4 g_\sigma^2 Q^2}{s^2} L_{\mu\nu}(p_1, p_2, 0) H^{\mu\nu}(q_1, q_2, k)$$ (3)

with the leptonic tensor

$$L_{\mu\nu}(p, q, m) = \text{tr} \left[ (\not{p} + m) \gamma_\mu (\not{q} - m) \gamma_\nu \right]$$ (4)

the hadronic tensor

$$H^{\mu\nu}(q_1, q_2, k) = \sum_{\text{colors, spins, } \epsilon} J^\mu(q_1, q_2, k, \epsilon) J^{\nu,*}(q_1, q_2, k, \epsilon)$$ (5)
and current matrix elements

\[ J^\mu(q_1, q_2, k, \epsilon) = J_1^\mu(q_1, q_2, k, \epsilon) + J_2^\mu(q_1, q_2, k, \epsilon) \]  

(6a)

\[ J_1^\mu(q_1, q_2, k, \epsilon) = \bar{u}(q_1) T_a \delta^{\mu}(k)(\frac{q_1 + k + m}{(q_1 + k)^2 - m^2 + i\epsilon}) \gamma^\mu v(q_2) \]  

(6b)

\[ J_2^\mu(q_1, q_2, k, \epsilon) = \bar{u}(q_1) \gamma^\mu(\frac{-\mathbf{q}_2 - \mathbf{k} + m}{(q_2 + k)^2 - m^2 + i\epsilon}) \]  

(6c)

5. Use the same argument as for the Ward identity to show that

\[ (q_1^\mu + q_2^\mu + k^\mu) J_\mu(q_1, q_2, k, \epsilon) = 0 \]  

(7)

and using the center of mass momentum \( p = p_1 + p_2 = q_1 + q_2 + k \)

\[ p_\mu H^{\mu\nu}(q_1, q_2, k) = p_\nu H^{\mu\nu}(q_1, q_2, k) = 0 \]  

(8)

6. The angular dependence of \( H^{\mu\nu}(q_1, q_2, k) \) is too complicated. Use the variables

\[ x_1 = 2q_1 p/p^2, \quad x_2 = 2q_2 p/p^2, \quad x_3 = 2k p/p^2 \]  

(9)

and the formula

\[ \int \tilde{d}q_1 \tilde{d}q_2 \tilde{d}k (2\pi)^4 \delta^4(q_1 + q_2 + k - p) f(x_1, x_2, x_3) = \frac{s}{128\pi^3} \int dx_1 dx_2 f(x_1, x_2, 2 - x_1 - x_2) \]  

(10)

to integrate over the angles.

7. The integral over the angles

\[ \int d\tilde{\Omega} H^{\mu\nu}(q_1, q_2, k) = \tilde{H}^{\mu\nu}(p, x_1, x_2) \]  

(11)

can only depend on the center of mass momentum \( p \). Argue that

\[ \tilde{H}^{\mu\nu}(p, x_1, x_2) = \left( \frac{p^\mu p^\nu}{p^2} - g^{\mu\nu} \right) \tilde{H}(p, x_1, x_2) \]  

(12)

with

\[ \tilde{H}(p, x_1, x_2) = -\frac{1}{3} \int d\tilde{\Omega} H_{\mu\nu}(q_1, q_2, k) \]  

(13)

NB: this is useful because \( H^{\mu\nu}(q_1, q_2, k) \) is much easier to compute than \( H^{\mu\nu}(q_1, q_2, k) \).
8. Compute $\tilde{H}(p, x_1, x_2)$ and the differential cross section.

Formulae:

- Sums of color matrices

$$\sum_a T_a T_a = C_F \cdot 1 = \frac{N_C^2 - 1}{2N_C} \cdot 1$$

(14)