2. Problemset "Theoretical Particle Physics"
May 3, 2017

Poincaré Algebra

2.1 Poincaré Algebra

In the lecture, we have introduced the commutation relations of the Poincaré algebra

$$[P_\mu, P_\nu] = 0 \tag{1a}$$
$$[J_{\mu\nu}, P_\rho] = \mathrm{i}\,(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \tag{1b}$$
$$[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}\,(g_{\mu\rho}J_{\nu\sigma} - g_{\mu\sigma}J_{\nu\rho} - g_{\nu\rho}J_{\mu\sigma} + g_{\nu\sigma}J_{\mu\rho})\ . \tag{1c}$$

and the Pauli-Lubanski vector

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma\ . \tag{2}$$

- Show that

$$[P_\mu, P^\rho P_\rho] = 0 \tag{3a}$$
$$[J_{\mu\nu}, P^\rho P_\rho] = 0 \tag{3b}$$
$$[P_\mu, W^\rho W_\rho] = 0 \tag{3c}$$
$$[J_{\mu\nu}, W^\rho W_\rho] = 0 \tag{3d}$$

2.2 Poincaré Algebra Spin 0

- Show that the differential operators

$$P_\mu = \mathrm{i}\partial_\mu \tag{4a}$$
$$J_{\mu\nu} = \mathrm{i}\,(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{4b}$$

  realize the Poincaré algebra (1).

- Compute the Pauli-Lubanski vector W_μ in the realization (4).

- Compute

$$W^2 = W^\mu W_\mu = -m^2 s(s+1) \tag{5}$$

  and deduce s.

2.3 Poincaré Algebra Spin 1/2

The constant Dirac matrices $\{\gamma_\mu\}_{\mu=0,1,2,3,4}$ satisfy the *anti*-commutation relations

$$[\gamma_\mu, \gamma_\nu]_+ = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \,. \tag{6}$$

- Compute the commutation relations

  $$[\sigma_{\mu\nu}, \sigma_{\kappa\lambda}]_- \tag{7}$$

  of the matrices

  $$\sigma_{\mu\nu} = \frac{\mathrm{i}}{2} [\gamma_\mu, \gamma_\nu]_- \,. \tag{8}$$

- Show that the sum of differential operators and matrices

  $$P_\mu = \mathrm{i}\partial_\mu \tag{9a}$$

  $$J_{\mu\nu} = \mathrm{i}\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + \frac{1}{2}\sigma_{\mu\nu} \tag{9b}$$

  also realize the Poincaré algebra (1).

- Compute the Pauli-Lubanski vector again in the realization (9).

- Compute

  $$W^2 = W^\mu W_\mu = -m^2 s(s+1) \tag{10}$$

  and deduce s. Hint:

  $$\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\mu}{}_{\alpha\beta\gamma} = -g_{\alpha\alpha'} g_{\beta\beta'} g_{\gamma\gamma'} + \text{signed permutations of } \alpha',\ \beta',\ \gamma'. \tag{11}$$