Theoretical (Elementary) Particle Physics

(Summer 2015)

Thorsten Ohl
Institut für Theoretische Physik und Astrophysik
Universität Würzburg
Am Hubland
97070 Würzburg
Germany

Personal Manuscript!
Use at your own peril!

July 1, 2015

Abstract

- Fundamentale Teilchen und Kräfte
- Symmetrien und Gruppen
- Quarkmodell
- Grundlagen der Quantenfeldtheorie
- Eichtheorien
- Spontane Symmetriebrechung
- Elektroschwaches Standardmodell
- Quantenchromodynamik
- Erweiterungen des Standardmodells

Contents

Vorbemerkung

Dieses Manuskript ist mein persönliches Vorlesungsmanuskript, an vielen Stellen nicht ausformuliert und kann jede Menge Fehler enthalten. Es handelt sich hoffentlich um weniger Denk- als Tippfehler, trotzdem kann ich deshalb ich keine Verantwortung für Fehler übernehmen. Zeittranslationsinvarianz ist natürlich auch nicht gegeben ...

Dennoch, oder gerade deshalb, bin ich für alle Korrekturen und Vorschläge dankbar!

Organisatorisches

Kontakt

- Büro: 22.02.009 (Hubland Nord, Emil-Hilb-Weg 22, 2. Stock)
- Sprechstunde: nach Vereinbarung
- Mail: `ohl@physik.uni-wuerzburg.de`
- URL: `http://physik.uni-wuerzburg.de/ohl/`

Aktuelle Informationen

Vorlesungs-URL zu lang, einfach auf

- `http://physik.uni-wuerzburg.de/ohl/` gehen und den Links folgen,

dort auch Inhaltsangabe.

Übungsgruppen

- #01: Mittwoch, 8:30 Uhr

Übungszettel

- ...

—1—
Introduction

1.1 Literature

Lecture 01: Tue, 14. 04. 2015

1.1.1 Elementary Particle Physics and Standard Model

Advanced

- Howard Georgi: *Weak Interactions and Modern Particle Theory*, Dover, 2009. NB: the author makes a PDF file of an updated version available at his home page: http://www.people.fas.harvard.edu/~hgeorgi/weak.pdf
- John F. Donoghue, Eugene Golowich, Barry R. Holstein: *Dynamics of the Standard Model*, Cambridge University Press, 1992.

1.1.2 Quantum Field Theory

Introductory

- Micheal E. Peskin, Daniel V. Schroeder: *An Introduction to Quantum Field Theory*, Addison-Wesley Publishing Company, 1995.
- Claude Itzykson, Jean-Bernard Zuber: *Quantum Field Theory*, McGraw-Hill, 1990.

Advanced

- Steven Weinberg: *The Quantum Theory of Fields. Volume I: Foundations*, Cambridge University Press, 1995.

- Steven Weinberg: *The Quantum Theory of Fields. Volume II: Modern Applications*, Cambridge University Press, 1996.

1.1.3 Group Theory

Introductory

- Howard Georgi: *Lie Algebras in Particle Physics, 2nd ed.*, Perseus Books, 1999.

Unorthodox

- Predrag Cvitanović: *Group Theory: Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, 2008. NB: the author makes a PDF file of the book available at http://birdtracks.eu/

1.2 The Setting

1.2.1 Dramatis Personae

Stable Particles

These have never been observed to decay if left alone

- electrons (e^-) and positrons (e^+)
- photons (γ)
- protons (p) and anti-protons ($\bar{p}$)

and γ, e^- and p, together with neutrons, make up all "normal matter".

Almost Stable Particles

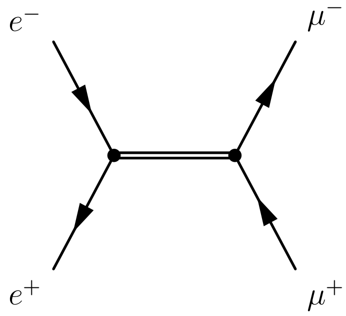
These live long enough to leave macroscopic $O(1\,\mathrm{m})$ tracks in detectors:

- neutrons (n) and anti-neutrons ($\bar{n}$)
- muons (μ^-) and antimuons (μ^-)

Unstable Particles (a. k. a. Resonances)

Everything else decays too quickly to be seen as a track in detectors.

The exchange of a particle with mass M corresponds to an amplitude



$$\propto \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\epsilon} \tag{1.1}$$

and a cross section

$$\sigma(s) \propto \left| \frac{\mathrm{i}}{s - M^2} \right|^2 = \frac{1}{(s - M^2)^2} \tag{1.2}$$

with an unphysical singularity at $s = M^2$. A more careful computation reveals a finite width with

$$\frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}\epsilon} \to \frac{\mathrm{i}}{p^2 - M^2 + \mathrm{i}M\Gamma} \tag{1.3}$$

and a Breit-Wigner resonance shape

$$\sigma(s) \propto \left| \frac{\mathrm{i}}{s - M^2 + \mathrm{i}M\Gamma} \right|^2 = \frac{1}{(s - M^2)^2 + M^2\Gamma^2} \,. \tag{1.4}$$

The mass M of the particle can then be measured as the location of the peak of the cross section and the lifetime τ of the particle as the inverse of the width $\Gamma = 1/\tau$ of the resonance. A typical example is the spectrum of resonances decaying into muon pairs at LHC, shown in figure 1.1. Obviously, these resonances must correspond to uncharged particles and we can expect more resonances in other channels, corresponding to charged particles.

1.2.2 Place

Typical energies in nuclear reactions are $O(10\,\mathrm{MeV})$ corresponding to length scales $O(10\,\mathrm{fm}) = O(10^{-14}\,\mathrm{m})$, using the conversion factor

$$\hbar c = 197\,\mathrm{MeV}\,\mathrm{fm}\,. \tag{1.5}$$

"Interesting" elementary particle starts with $O(1\,\mathrm{GeV})$ and we are now testing the "terascale" $O(1\,\mathrm{TeV})$ for the first time.

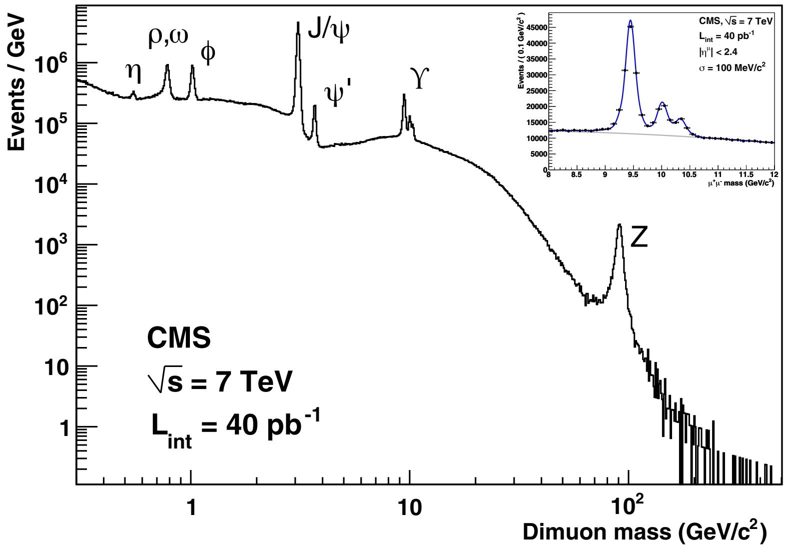


Figure 1.1: *Resonances in* $pp \to \mu^+\mu^- + X$ *measured by the CMS experiment at LHC in 2010 [1].*

1.2.3 Tools

Experiment

Accelerators and colliders

- natural (cosmic)
- man made (LHC etc.)

Theory

- Quantum Field Theory (QFT) for computing cross sections and decay rates
- group theory for organizing particles and interactions

1.2.4 Approaches

There are two complementary approaches that are both required for progress in our understanding of the microcosmos

Bottom-Up

1. write down the most general *mathematically consistent* interaction of the observed particles, consistent with the observed symmetries and conservation laws (cf. Noether theorem)
2. fit the free parameters to observations
3. compute cross section and decay rates for the observed particles
4. compare with experiment
5. if necessary, add new particles and *repeat*

Top-Down

1. propose an improved microscopic model of elementary particles and their interactions
2. compute cross section and decay rates for the observed particles, which might be bound states of the elementary particles
3. compare with experiment
4. *repeat*

1.3 The Frontier (as of today): LHC

In figure 1.1, the Standard Model (SM) predictions are tested by a single experiment over more than two orders of magnitude in energy and five orders of magnitude in cross section.

As we are speaking, the LHC is restarting for “Run 2”, which will probe the SM predictions well into the terascale.

—2—
Fundamental Particles and Forces

2.1 What Is *an Elementary Particle?*

The answer to this natural question is both trivial and subtle:

> *a particle is considered* elementary*, if and only if (iff) there is no evidence that it is composite, i. e.*
>
> - *there no finite spacial extend*
> - *it can not be broken apart.*

Therefore, the category of elementary particles is *not* constant in time

- a little more than a century ago (before Rutherford), *atoms* where considered elementary
- between 1910 and 1950, electrons, photons, protons and neutrons were considered elementary
- in the early 1950s, Robert Hofstadter discovered by elastic electron scattering of protons (hydrogen) that protons have a size of roughly $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$
- as of today, electrons and photons still qualify as elementary

Still, the term *elementary particle physics* also refers to particles that we now know to be unstable or composite, such as protons, neutrons, other baryons and mesons.

2.1.1 Quantum Numbers

According to our observations, elementary particles are completely *indistinguishable*: if the have the same quantum numbers, their states must be either symmetric (bosons) or antisymmetric (fermions) under permutations.

Therefore an elementary particle is completely characterized by its

- representation of the Poincaré group, i. e. mass, spin and parities under space, time and charge inversion
- electric charge
- other more exotic charges: isospin, color, etc.

Of these, the mass, spin and the parities correspond to *spacetime* symmetries, while the rest are called *internal* symmetries.

In the absence of gravity, i. e. in a flat space time, elementary particles can *only* have the spins 0 (scalar), 1/2 (spinor) and 1 (vector), while composite particles can have any spin (cf. Clebsh–Gordan decomposition). If we include gravity, particles with spin 3/2 and 2 (gravitons) become possible.

2.2 Fundamental Interactions

Lecture 02: Wed, 15. 04. 2015

As of today all interactions among elementary particles are described by four fundamental interactions:

2.2.1 Gravity

This is the *only* interactions felt by *all* elementary particles, since it affects space-time itself. Unfortunately, we don't have a good quantum mechanical description yet. Fortunately, its effects on elementary particles are so weak at accessible energy scales that it can savely be ignored.

2.2.2 Electromagnetism

This is described by Quantum Electro Dynamics (QED) to an incredible precision and matches to electromagnetism in the classical limit.

2.2.3 The Strong Force

This has *no* classical analog and affects only baryons and mesons.

2.2.4 Weak Interactions

This also has *no* classical analog.

—3— Symmetries and Groups

3.1 Symmetries

If a charge Q commutes with the Hamiltonian H

$$[H, Q] = 0\,, \tag{3.1}$$

it is conserved

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = \mathrm{i}[H, Q] = 0\,. \tag{3.2}$$

In addition, if such a charge relates two eigenstates of the Hamiltonian H

$$Q\,|1\rangle = |2\rangle \tag{3.3}$$

with

$$H\,|n\rangle = E_n\,|n\rangle\ , \tag{3.4}$$

then

$$E_2\,|2\rangle = H\,|2\rangle = HQ\,|1\rangle = QH\,|1\rangle = QE_1\,|1\rangle = E_1 Q\,|1\rangle = E_1\,|2\rangle \tag{3.5}$$

i. e.

$$E_1 = E_2 \tag{3.6}$$

and the states $|1\rangle$ and $|2\rangle$ are degenerate.

Therefore we will have *multiplets* of degenerate states $\{|i\rangle\}_{i\in I\subset\mathbf{Z}}$, whenever these states form a *representation* (section 3.4) of a *Lie algebra* (section 3.3) of conserved charges $\{Q_j\}_{j\in J\subset\mathbf{Z}}$

$$[H, Q_i] = 0 \tag{3.7a}$$

$$[Q_i, Q_j] = \mathrm{i}\sum_{k\in J} f_{ijk} Q_k\,, \tag{3.7b}$$

i. e.

$$Q_i |j\rangle = \sum_{k \in J} [r(Q_i)]_{jk} |k\rangle \ . \tag{3.8}$$

Since

$$H^2 = M^2 + \vec{P}^2 \,, \tag{3.9}$$

the above reasoning translates from energy levels to masses.

3.2 Lie Groups

In physics[1], symmetries are described as Groups $(G, \circ)$ with G a set and $\circ$ an inner operation

$$\begin{aligned} \circ : G \times G &\to G \\ (x, y) &\mapsto x \circ y \end{aligned} \tag{3.10}$$

with

1. closure: $\forall x, y \in G : x \circ y \in G$,
2. associativity: $x \circ (y \circ z) = (x \circ y) \circ z$,
3. identity element: $\exists e \in G : \forall x \in G : e \circ x = x \circ e = x$,
4. inverse elements: $\forall x \in G : \exists x^{-1} \in G : x \circ x^{-1} = x^{-1} \circ x = e$.

Many examples in physics

- permutations
- reflections
- parity
- translations
- rotations
- Lorentz boosts
- Runge–Lenz vector
- isospin
- ...

[1] And mathematics!

Particularly interesting are *Lie Groups*, i. e. groups, where the set is a *differentiable Manifold* and the composition is differentiable w. r. t. both operands.

Note that the choice of coordinates is not relevant:

$$\begin{aligned} B &= \left\{ b_1(\eta) = \exp\begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \middle| \, \eta \in \mathbf{R} \right\} \\ &= \left\{ b_2(\beta) = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \middle| \beta \in \,]-1,1[\right\} \end{aligned} \tag{3.11}$$

Both times we have the set of all real symmetric 2×2 matrices with unit determinant. The composition laws are given by matrix multiplication[2]:

$$b_1(\eta) \circ b_1(\eta') = b_1(\eta) b_1(\eta') = b_1(\eta + \eta') \tag{3.12a}$$

$$b_2(\beta) \circ b_2(\beta') = b_2(\beta) b_2(\beta') = b_2\left(\frac{\beta + \beta'}{1 + \beta\beta'}\right) . \tag{3.12b}$$

3.3 Lie Algebras

A Lie algebra $(A, [\cdot,\cdot])$ is a K-vector space[3] with a non-associative antisymmetric bilinear inner operation $[\cdot,\cdot]$:

$$\begin{aligned} [\cdot,\cdot] : A \times A &\to A \\ (a,b) &\mapsto [a,b] \end{aligned} \tag{3.13}$$

with

1. closure: $\forall a, b \in A : [a,b] \in A$,
2. antisymmetry: $[a,b] = -[b,a]$
3. bilinearity: $\forall \alpha, \beta \in \mathrm{K} : [\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c]$
4. Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$

---

[2] NB:

$$|\beta| < 1 \wedge |\beta'| < 1 \Rightarrow \left| \frac{\beta + \beta'}{1 + \beta\beta'} \right| < 1$$

[3] $\mathrm{K} = \mathrm{R}$ or C

Since A is a vector space, we can choose a basis and write

$$[a_i, a_j] = \sum_k C_{ijk} a_k \,. \tag{3.14}$$

A Lie algebra is called *simple*, if it has no ideals besides itself and $\{0\}$. Remarkably, *all* simple Lie algebras are known:

$$\mathrm{so}(N), \mathrm{su}(N), \mathrm{sp}(2N), g_2, f_4, e_6, e_7, e_8 \tag{3.15}$$

with $N \in \mathbf{N}$.

The infinitesimal generators of a Lie group form a Lie algebra. Vice versa, the elements of a Lie algebra can be exponentiated to obtain a Lie group (not necessarily the same, but a cover of the original group).

3.4 Representations

A *group homomorphism* f is a map

$$\begin{aligned} f : G &\to G' \\ x &\mapsto f(x) \end{aligned} \tag{3.16}$$

between two groups $(G, \circ)$ and $(G', \circ')$ that is compatible with the group structure

$$f(x) \circ' f(y) = f(x \circ y) \tag{3.17}$$

and therefore

$$f(e) = e' \tag{3.18a}$$

$$f(x^{-1}) = (f(x))^{-1} \,. \tag{3.18b}$$

A *Lie algebra homomorphism* ϕ is a map

$$\begin{aligned} \phi : A &\to A' \\ a &\mapsto \phi(a) \end{aligned} \tag{3.19}$$

between two Lie algebras $(A, [\cdot,\cdot])$ and $(A', [\cdot,\cdot]')$ that is compatible with the Lie algebra structure

$$[\phi(a), \phi(b)]' = \phi([a, b]) \,. \tag{3.20}$$

NB: these need *not* be isomorphisms: $f(x) = e', \forall x$ is a trivial, but well defined group homomorphism and $\phi(a) = 0, \forall a$ is a similarly trivial but also well defined Lie algebra homomorphism.

Lie groups and algebras are abstract objects, which can be made concrete by representations.

A *group representation*

$$R : G \to L \tag{3.21}$$

is a homomorphism from the group $(G, \circ)$ to a group of linear operators $(L, \cdot)$ with $(O_1 \cdot O_2)(v) = O_1(O_2(v))$. The representation is called *unitary* if the operators are unitary. The representation is called *faithful* if $\forall x \neq y : R(x) \neq R(y)$.

A *Lie algebra representation*

$$r : A \to L \tag{3.22}$$

is a homomorphism from the Lie algebra $(A, [\cdot,\cdot])$ to an associative algebra of linear operators $(L, [\cdot,\cdot]')$ with $[O_1, O_2]' = O_1 \cdot O_2 - O_2 \cdot O_1$ or $[O_1, O_2]'(v) = O_1(O_2(v)) - O_2(O_1(v))$, i. e. commutators for Lie brackets.

The Matrix groups $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(2N)$ and their Lie algebras have obvious defining representations.

Every Lie algebra has a *adjoint representation*, using the itself as the linear representation space $a \Leftrightarrow |a\rangle$:

$$r_{\text{adj.}}(a)\,|b\rangle = |[a,b]\rangle \tag{3.23}$$

using the Jacobi identity

$$\begin{aligned}(r_{\text{adj.}}(a)r_{\text{adj.}}(b) - r_{\text{adj.}}(b)r_{\text{adj.}}(a))\,|c\rangle &= |[a,[b,c]] - [b,[a,c]]\rangle \\ &= |[[a,b],c]\rangle = r_{\text{adj.}}([a,b])\,|c\rangle \end{aligned} \tag{3.24}$$

or, using a basis

$$r_{\text{adj.}}(a_i)\,|a_j\rangle = |[a_i,a_j]\rangle = |C_{ijk}a_k\rangle = C_{ijk}\,|a_k\rangle \tag{3.25}$$

we find the matrix elements

$$[r_{\text{adj.}}(a_i)]_{jk} = C_{ijk}\,. \tag{3.26}$$

Using Hausdorff's formula

$$\begin{aligned}\mathrm{e}^a b\,(\mathrm{e}^a)^{-1} &= \mathrm{e}^a b \mathrm{e}^{-a} = \mathrm{e}^{\mathrm{ad}_a} b = \mathrm{e}^{[a,\cdot]} b \\ &= b + [a,b] + \frac{1}{2!}[a,[a,b]] + \frac{1}{3!}[a,[a,[a,b]]] + \ldots\end{aligned} \tag{3.27}$$

we see that the map

$$\begin{aligned} f(x) : A &\to A \\ b &\mapsto x b x^{-1} \end{aligned} \tag{3.28}$$

is well defined and remains *inside* the Lie algebra. It's obviously linear and since

$$f(x)(f(y)(a)) = f(x)\left(yay^{-1}\right) = xyay^{-1}x^{-1} = (xy)a(xy)^{-1} = f(xy)(a) \tag{3.29}$$

it is also a representation, called the *adjoint representation of the group*.

3.4.1 Irreducible Representations

Lecture 03: Tue, 21. 04. 2015

In general, representations can be decomposed: a group representation R is called *reducible*, iff there is a non-trivial invariant subspace $W \subset V$

$$\forall g \in G : R(g)W \subseteq W \tag{3.30}$$

analogously for a Lie algebra representation r

$$\forall a \in A : r(a)W \subseteq W\,. \tag{3.31}$$

If a representation is not reducible, it is called *irreducible*.

Lemma 3.1 (Schur's lemma)**.** *A matrix that commutes with all representatives in an irreducible representation is proportional to the unit matrix.*

3.4.2 Direct Sums

The *direct sum* of representations

$$R(g) = R_1(g) \oplus R_2(g) \oplus \ldots \oplus R_n(g) \tag{3.32}$$

or

$$r(a) = r_1(a) \oplus l_2(a) \oplus \ldots \oplus l_n(a)\,, \tag{3.33}$$

where the action of a direct sum of matrices is defined as

$$(M \oplus N)(v \oplus w) = Mv \oplus Nw \tag{3.34}$$

and the dimension of the direct sum of representation spaces is the sum of the dimensions, is again a representation. Indeed

$$\begin{aligned} R(g)R(g') &= (R_1(g) \oplus R_2(g))(R_1(g') \oplus R_2(g')) \\ &= R_1(g)R_1(g') \oplus R_2(g)R_2(g') = R_1(gg') \oplus R_2(gg') = R(gg') \end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
[r(a), r(a')] &= [r_1(a) \oplus r_2(a), r_1(a') \oplus r_2(a')] \\
&= r_1(a)r_1(a') \oplus r_2(a)r_2(a') - r_1(a')r_1(a) \oplus r_2(a')r_2(a) \\
&= [r_1(a), r_1(a')] \oplus [r_2(a), r_2(a')] = r_1([a, a']) \oplus r_2([a, a']) = r([a, a']) \quad (3.36)
\end{aligned}$$

All representations can be decomposed as a sum of irreducible representations. Note that each irreducible representation can appear more than once.

3.4.3 Tensor Products

The *direct product*, also known as (a.k.a.) *tensor product*, of group representations

$$R(g) = R_1(g) \otimes R_2(g) \otimes \ldots \otimes R_n(g) \quad (3.37)$$

where the action of a direct product of matrices on a product of vectors is defined by

$$(M \otimes N)(v \otimes w) = Mv \otimes Nw \quad (3.38)$$

and extended by linearity to the vectors that can not be written as a product of vectors, is again a representation. The dimension of the direct product of representation spaces is the sum of the dimensions. Indeed

$$\begin{aligned}
R(g)R(g') &= (R_1(g) \otimes R_2(g))(R_1(g') \otimes R_2(g')) \\
&= R_1(g)R_1(g') \otimes R_2(g)R_2(g') = R_1(gg') \otimes R_2(gg') = R(gg') \,. \quad (3.39)
\end{aligned}$$

However the direct product of Lie algebra representations must be defined differently

$$\begin{aligned}
r(a) = r_1(a) \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} + \mathbf{1} \otimes r_2(a) \otimes \ldots \otimes \mathbf{1} + \ldots \\
+ \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes r_n(a) \quad (3.40)
\end{aligned}$$

because for $r = r_1 \otimes \mathbf{1} + \mathbf{1} \otimes r_2$ we have

$$\begin{aligned}
[r(a), r(a')] &= [r_1(a) \otimes \mathbf{1} + \mathbf{1} \otimes r_2(a), r_1(a') \otimes \mathbf{1} + \mathbf{1} \otimes r_2(a')] \\
&= [r_1(a) \otimes \mathbf{1}, r_1(a') \otimes \mathbf{1}] + [\mathbf{1} \otimes r_2(a), \mathbf{1} \otimes r_2(a')] \\
&\quad + \underbrace{[r_1(a) \otimes \mathbf{1}, \mathbf{1} \otimes r_2(a')]}_{=0} + \underbrace{[\mathbf{1} \otimes r_2(a), r_1(a') \otimes \mathbf{1}]}_{=0} \\
&= r_1(a)r_2(a') \otimes \mathbf{1} - r_1(a')r_2(a) \otimes \mathbf{1} + \mathbf{1} \otimes r_2(a)r_2(a') - \mathbf{1} \otimes r_2(a')r_2(a) \\
&= [r_1(a), r_1(a')] \otimes \mathbf{1} + \mathbf{1} \otimes [r_2(a), r_2(a')] \\
&= r_1([a, a'] \otimes \mathbf{1} + \mathbf{1} \otimes [r_2([a, a']) = r([a, a']) \,. \quad (3.41)
\end{aligned}$$

Useful heuristics for (3.40) are

$$e^a \cdot e^b = (1 + a + \ldots) \cdot (1 + b + \ldots) = 1 + a \cdot 1 + 1 \cdot b + \ldots = e^{a \cdot 1 + 1 \cdot b} \tag{3.42}$$

or the product rule

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(f(x) \otimes g(x)) &= \frac{\mathrm{d}f}{\mathrm{d}x}(x) \otimes g(x) + f(x) \otimes \frac{\mathrm{d}g}{\mathrm{d}x}(x) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x} \otimes 1 + 1 \otimes \frac{\mathrm{d}}{\mathrm{d}x}\right)(f(x) \otimes g(x)) \,. \end{aligned} \tag{3.43}$$

Tensor products of (irreducible) representations are in general *not* irreducible, but can be decomposed.

However, there are simple ways to find invariant subspaces. Consider symmetric or antisymmetric tensors in the tensor product of an irreducible representation with itself

$$t^{(S)} = v \otimes v' + v' \otimes v \tag{3.44a}$$

$$t^{(A)} = v \otimes v' - v' \otimes v \tag{3.44b}$$

then it is easy to see that the application of $R(g) \otimes R(g)$ or $r(g) \otimes \mathbf{1} + \mathbf{1} \otimes r(g)$ result in a symmetric or antisymmetric tensor respectively. This observation can be generalized to symmetries under arbitrary permutations of indices with definite signs.

3.4.4 Complex Conjugation

Since

$$\forall M_1, M_2 \in \mathrm{GL}(N, \mathbf{C}) : \overline{M_1 M_2} = \overline{M_1}\,\overline{M_2} \tag{3.45}$$

complex conjugation is a homomorphism from $\mathrm{GL}(N, \mathbf{C})$ to itself. Therefore, if

$$\begin{aligned} R : G &\to \mathrm{GL}(N, \mathbf{C}) \\ g &\mapsto R(g) \end{aligned} \tag{3.46}$$

is a representation, then

$$\begin{aligned} \overline{R} : G &\to \mathrm{GL}(N, \mathbf{C}) \\ g &\mapsto \overline{R(g)} \end{aligned} \tag{3.47}$$

is one as well.

Note that this is in general not true for hermitian conjugation, because

$$\forall M_1, M_2 \in \mathrm{GL}(N, \mathbf{C}) : (M_1 M_2)^\dagger = (M_2)^\dagger (M_1)^\dagger \neq (M_1)^\dagger (M_2)^\dagger \tag{3.48}$$

unless the matrices commute.

Note also, that there are special cases, where $\overline{R} \cong R$ and complex conjugation does *not* give rise to a new representation. This is actually an important ingredient of the minimal SM discussed below.

3.5 Lorentz and Poincaré Group

Lorentz transformations are the linear transformations of the Minkowski space $\mathbf{M}$

$$\begin{aligned} \phi_\Lambda : \mathbf{M} &\to \mathbf{M} \\ x^\mu \mapsto x'^\mu &= \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \end{aligned} \tag{3.49a}$$

that leave the metric invariant

$$\forall x, y \in \mathbf{M} : \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x^\mu y^\nu = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x'^\mu y'^\nu \,. \tag{3.49b}$$

with

$$g^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu} \tag{3.50a}$$

$$g_\mu{}^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g^\mu{}_\nu = \delta_\mu{}^\nu = \delta^\mu{}_\nu \,. \tag{3.50b}$$

This condition can also be written

$$\sum_{\kappa,\lambda=0}^{3} g_{\kappa\lambda} x^\kappa y^\lambda = \sum_{\mu,\nu=0} g_{\mu\nu} x'^\mu y'^\nu = \sum_{\mu,\nu,\kappa,\lambda=0}^{3} g_{\mu\nu} \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda x^\kappa y^\lambda \tag{3.51}$$

and we obtain

$$g_{\kappa\lambda} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \tag{3.52}$$

from comparing coefficients. As a matrix equation this reads

$$g = \Lambda^T g \Lambda \,. \tag{3.53}$$

The 00-component of (3.52) entails with

$$1 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} = \left(\Lambda^{0}{}_{0}\right)^2 - \sum_{i=1}^{3} \left(\Lambda^{i}{}_{0}\right)^2 \leq \left(\Lambda^{0}{}_{0}\right)^2 \tag{3.54}$$

the condition

$$|\Lambda^{0}{}_{0}| \geq 1 \tag{3.55a}$$

and from (3.53) we find

$$\det \Lambda = \pm 1 \,. \tag{3.55b}$$

3.5.1 Lorentz Group

The composition of two Lorentz Transformations (LTs) is obviously again an LT

$$\phi_{\Lambda_2} \circ \phi_{\Lambda_1} = \phi_{\Lambda_3} \,. \tag{3.56}$$

The corresponding matrix Λ is from (3.55a) always invertible and so the inverse transformation

$$\phi_{\Lambda}^{-1} = \phi_{\Lambda^{-1}} \tag{3.57}$$

exists. Thus LTs ϕ_Λ form a group, the *Lorentz Group* $\mathcal{L}$.

From (3.55), we see that $\mathcal{L}$ consists of four *disconnected* components

$$\mathcal{L}_{+}^{\uparrow} = \left\{\Lambda \in \mathcal{L} : \det \Lambda = +1 \wedge \Lambda^{0}{}_{0} \geq 1\right\} \tag{3.58a}$$
$$\mathcal{L}_{-}^{\uparrow} = \left\{\Lambda \in \mathcal{L} : \det \Lambda = -1 \wedge \Lambda^{0}{}_{0} \geq 1\right\} \tag{3.58b}$$
$$\mathcal{L}_{+}^{\downarrow} = \left\{\Lambda \in \mathcal{L} : \det \Lambda = +1 \wedge \Lambda^{0}{}_{0} \leq -1\right\} \tag{3.58c}$$
$$\mathcal{L}_{-}^{\downarrow} = \left\{\Lambda \in \mathcal{L} : \det \Lambda = -1 \wedge \Lambda^{0}{}_{0} \leq -1\right\} \,. \tag{3.58d}$$

Among these, only the proper and orthochronous LTs $\mathcal{L}_{+}^{\uparrow}$ form a subgroup. All other LTs can be written as a product of an element of $\mathcal{L}_{+}^{\uparrow}$ with space and time inversions

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathcal{L}_{-}^{\uparrow} \tag{3.59a}$$

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{L}_{-}^{\downarrow} \tag{3.59b}$$

$$P \circ T = T \circ P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathcal{L}_+^\downarrow \,. \tag{3.59c}$$

Rotations

The rotations

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R^1{}_1 & R^1{}_2 & R^1{}_3 \\ 0 & R^2{}_1 & R^2{}_2 & R^2{}_3 \\ 0 & R^3{}_1 & R^3{}_2 & R^3{}_3 \end{pmatrix} \tag{3.60}$$

with $R \in \mathrm{SO}(3)$, i. e.

$$RR^T = R^T R = \mathbf{1} \tag{3.61a}$$

$$\det R = 1 \tag{3.61b}$$

are obviously proper and orthochronous LTs, since they preserve the length $\vec{x}\vec{y}$ as well as x_0 and y_0 and therefore $x_\mu y^\mu = x_0 y_0 - \vec{x}\vec{y}$.

Lecture 04: Wed, 22. 04. 2015

The SO(3) matrices can be parametrized by Euler angles

$$\begin{aligned} R : [0, 2\pi[\times[0, \pi[\times[0, 2\pi[&\to \mathrm{SO}(3) \\ (\phi, \theta, \psi) &\mapsto R(\phi, \theta, \psi) = R_3(\phi) R_1(\theta) R_3(\psi) \end{aligned} \tag{3.62}$$

with R_i a rotation around the ith axis. Alternatively by three real numbers $\vec{\alpha}$ via

$$\begin{aligned} O : D \subset \mathbf{R}^3 &\to \mathrm{SO}(3) \\ \vec{\alpha} &\mapsto O(\vec{\alpha}) = \mathrm{e}^{\mathrm{i}\vec{\alpha}\vec{T}} \end{aligned} \tag{3.63}$$

with the traceless, hermitian and antisymmetric generators

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix} \tag{3.64a}$$

$$T_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \tag{3.64b}$$

$$T_3 = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.64c}$$

of the so(3) Lie algebra

$$[T_i, T_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} T_k \,. \tag{3.65}$$

Algebraically, (3.63) is the simpler formula, but it is not obvious what $D \subset \mathbf{R}^3$ is without double counting.

However, we observe that multiples

$$\vec{t} = \frac{1}{2}\vec{\sigma} \tag{3.66}$$

of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.67}$$

satisfy the same Lie algebra as (3.65) and generate the SU(2)-matrices via

$$U(\vec{\alpha}) = \mathrm{e}^{\mathrm{i}\vec{\alpha}\vec{\sigma}} \,. \tag{3.68}$$

The group manifold of SU(2) turns out to be geometrically simple. First parametrize *all* 2×2-matrices by four complex numbers $(\chi_0, \vec{\chi})$

$$U = \chi_0 \mathbf{1} + \vec{\chi}\vec{\sigma} \,. \tag{3.69}$$

Then from the hermiticity of the $\vec{\sigma}$

$$U^\dagger = \bar{\chi}_0 \mathbf{1} + \vec{\bar{\chi}}\vec{\sigma} \tag{3.70}$$

and the unitarity condition can be written

$$\mathbf{1} \stackrel{!}{=} UU^\dagger = \left(|\chi_0|^2 + \vec{\chi}\vec{\bar{\chi}}\right) \mathbf{1} + \left(\chi_0 \vec{\bar{\chi}} + \bar{\chi}_0 \vec{\chi} + \mathrm{i}\vec{\chi} \times \vec{\bar{\chi}}\right) \vec{\sigma} \,. \tag{3.71}$$

The condition on the determinant yields

$$1 \stackrel{!}{=} \det U = \chi_0^2 - \vec{\chi}\vec{\chi} \tag{3.72}$$

and we obtain

$$\chi_0^2 - \vec{\chi}\vec{\chi} = 1 \tag{3.73a}$$

$$|\chi_0|^2 + \vec{\chi}\vec{\bar{\chi}} = 1 \tag{3.73b}$$

$$\chi_0\vec{\bar{\chi}} + \bar{\chi}_0\vec{\chi} + \mathrm{i}\vec{\chi} \times \vec{\bar{\chi}} = 0\,. \tag{3.73c}$$

These conditions are solved by

$$\left\{(\chi_0, \vec{\chi}) = (\beta_0, \mathrm{i}\vec{\beta}) : (\beta_0, \vec{\beta}) \in \mathbf{R}^4 \wedge \beta_0^2 + \vec{\beta}\vec{\beta} = 1\right\} \tag{3.74a}$$

and we see that the group manifold of SU(2) homeomorphic to S^3, the three-dimensional hypersphere in $\mathbf{R}^4$.

On the other hand, we observe that

$$\forall \vec{\alpha} \in \mathbf{R}^3, \sqrt{\vec{\alpha}\vec{\alpha}} = 2\pi : \mathrm{e}^{\mathrm{i}\vec{t}\vec{\alpha}} = \mathrm{e}^{\mathrm{i}2\vec{t}\vec{\alpha}/2} = \mathrm{e}^{\mathrm{i}\vec{\sigma}\vec{\alpha}/2} = U(\vec{\alpha}/2) = -\mathbf{1}\,, \tag{3.75}$$

i. e. rotations of angle 2π about any axis are *not* the unit element, but only rotations of angle 4π. Thus the group manifold of the rotation group SO(3) is $\mathrm{SU}(2)/\mathbf{Z}_2 = S^3/\mathbf{Z}_2$, i. e. the three-dimensional hypersphere with opposing points identified. Therefore SU(2) is called a *cover* of SO(3). There is an obvious surjective homomorphism

$$\phi : \mathrm{SU}(2) \to \mathrm{SO}(3) \tag{3.76}$$

and any representation R of SO(3) induces a representation $R \circ \phi$ of SU(2). Note that it is *not* possible to define a global Lie group homomorphism ϕ^{-1} : $\mathrm{SO}(3) \to \mathrm{SU}(2)$ by choosing a member of $\phi^{-1}(g)$ at each point, because such a map will not match smoothly at the boundaries of $\phi^{-1}(\mathrm{SO}(3))$.

The representations of SU(2) that are not representations of SO(3) are physically important because they correspond to the fermionic representations with half integer spin.

Special Lorentz Transformations

As an example for LTs that mix space and time, we can leave x_2 and x_3 invariant:

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & 0 & 0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.77}$$

Then it is easy to see that thet can be parametrized by a single real number β

$$\Lambda(\beta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.78}$$

with

$$|\beta| < 1 \tag{3.79a}$$

und

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \ge 1\,. \tag{3.79b}$$

The composition of two such LTs can be written

$$\Lambda(\beta_1)\Lambda(\beta_2) = \Lambda\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)\,. \tag{3.80}$$

A subtle, but important difference of rotations and special LTs is, that the set SO(3) of rotations is *compact*, since they can be parametrized by the *periodic* Euler angles

$$(\phi,\theta,\psi) \in [0,2\pi[\,\times[0,\pi[\,\times[0,2\pi[\,. \tag{3.81}$$

In contrast, the special LTs form an *open* set

$$\beta \in]-1,+1[\tag{3.82}$$

with the endpoints *excluded*.

By rotating, we find the other special LTs

$$\Lambda(\vec\beta) = \begin{pmatrix} \gamma & -\gamma\beta^1 & -\gamma\beta^2 & -\gamma\beta^2 \\ -\gamma\beta^1 & 1+\Gamma\beta^1\beta^1 & \Gamma\beta^1\beta^2 & \Gamma\beta^3\beta^1 \\ -\gamma\beta^2 & \Gamma\beta^2\beta^1 & 1+\Gamma\beta^2\beta^2 & \Gamma\beta^3\beta^2 \\ -\gamma\beta^3 & \Gamma\beta^3\beta^1 & \Gamma\beta^3\beta^2 & 1+\Gamma\beta^3\beta^3 \end{pmatrix} \tag{3.83a}$$

with

$$\gamma = \frac{1}{\sqrt{1-\vec\beta^2}} \ge 1 \tag{3.83b}$$

$$\Gamma = \frac{\gamma^2}{1+\gamma}\,. \tag{3.83c}$$

Note that the product of two special LTs can in general *not* be written as a *special* LT. Instead

$$\Lambda(\vec\beta_1)\Lambda(\vec\beta_2) = \Lambda\,, \tag{3.84}$$

with Λ a product of a special LT and a rotation.

SL(2, **C**)

There is a geometrically simpler description of the elements of the Lorentz group close to the unit element.

Combining the Pauli matrices $\vec{\sigma}$ with the 2×2 unit matrix $\sigma_0 = \mathbf{1}_{2\times 2}$ in a four vector of 2×2 matrices

$$(\sigma_0, \vec{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \tag{3.85}$$

we can compute

$$\sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} . \tag{3.86}$$

Conversely, given a hermitian 2×2 matrix X, we can use

$$\mathrm{tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu} \tag{3.87}$$

to construct a four vector

$$x^\mu = \frac{1}{2}\,\mathrm{tr}(X\sigma_\mu) \tag{3.88}$$

(note the location of the indices in either side).

We observe

$$\det(\sigma_\mu x^\mu) = g_{\mu\nu} x^\mu x^\nu . \tag{3.89}$$

i. e. that transformations of x keeping the Minkowski length of x invariant will correspond to transformations of X maintaining hermiticity and keeping the determinant invariant. In fact

$$\phi_\alpha : X \mapsto X' = \alpha X \alpha^\dagger \tag{3.90}$$

maintains hermiticity

$$(X')^\dagger = (\alpha X \alpha^\dagger)^\dagger = (\alpha^\dagger)^\dagger X^\dagger \alpha^\dagger = \alpha X \alpha^\dagger = X' \tag{3.91}$$

and if $|\det \alpha| = 1$ also the determinant

$$\det X' = \det(\alpha X \alpha^\dagger) = \det\alpha \det X \det\alpha^* = (\det\alpha)^2 \det X . \tag{3.92}$$

Thus we can identify a neighborhood of the unit element of the Lorentz group with a neighborhood of the unit element of SL(2, **C**), the group of complex 2×2 matrices with unit determinant.

The number of independent real generators agrees. For the Lorentz group we have 6, i. e.

- 3 rotations
- 3 Lorentz boosts

and also in the case of $\mathrm{SL}(2,\mathbf{C})$, because we can write

$$\alpha = \mathrm{e}^{\mathrm{i}\vec\sigma(\vec\theta+\mathrm{i}\vec\eta)} \tag{3.93}$$

with real $\vec\theta$ and $\vec\eta$. To be precise, one can show that we have just defined a homomorphism from $\mathrm{SL}(2,\mathbf{C})$ *onto* $\mathcal{L}_+^\uparrow$.

Representations

As in the case of the rotation group, the representations of $\mathrm{SL}(2,\mathbf{C})$ turn out to be interesting in their own right, because they include the fermionic representations with half integer spin.

The irreducible $R^{(j/2,k/2)}$ representations of $\mathrm{SL}(2,\mathbf{C})$ can be labelled by two non-negative integers j and k. The representation spaces are the tensors in $\mathbf{C}^j\otimes\mathbf{C}^k$ that are totally symmetric in the j undotted and the k dotted indices separately. The j undotted indices transform according to the $\mathrm{SL}(2,\mathbf{C})$ matrix itself, the k dotted indices according to its complex conjugate:

$$\begin{aligned}\psi_{\alpha_1\ldots\alpha_j,\dot\alpha_1\ldots\dot\alpha_k} &\to \left(R^{(j/2,k/2)}\psi\right)_{\alpha_1\ldots\alpha_j,\dot\alpha_1\ldots\dot\alpha_k} \\ &= \sum_{\beta_1\ldots\beta_j,\dot\beta_1\ldots\dot\beta_k=1}^{2} R^{(j/2,k/2)}{}_{\alpha_1\ldots\alpha_j,\dot\alpha_1\ldots\dot\alpha_k}{}^{\beta_1\ldots\beta_j,\dot\beta_1\ldots\dot\beta_k}(A)\psi_{\beta_1\ldots\beta_j,\dot\beta_1\ldots\dot\beta_k} \\ &= \sum_{\beta_1\ldots\beta_j,\dot\beta_1\ldots\dot\beta_k=1}^{2} A_{\alpha_1}{}^{\beta_1}\cdots A_{\alpha_j}{}^{\beta_j}\overline{A_{\dot\alpha_1}{}^{\dot\beta_1}}\cdots\overline{A_{\dot\alpha_k}{}^{\dot\beta_k}}\psi_{\beta_1\ldots\beta_j,\dot\beta_1\ldots\dot\beta_k}\,. \end{aligned} \tag{3.94}$$

3.5.2 Poincaré Group

Lecture 05: Tue, 28. 04. 2015

Adding the space and time translations

$$\begin{aligned}\phi_{\vec a} : \mathbf{M} &\to \mathbf{M} \\ x^\mu &\mapsto x'^\mu = x^\mu + a^\mu\end{aligned} \tag{3.95}$$

to the Lorentz group, we obtain the *Poincaré group* with a semidirect multiplication law

$$\mathcal{P}(\Lambda_1,a_1)\mathcal{P}(\Lambda_2,a_2) = \mathcal{P}(\Lambda_1\Lambda_2, a_1+\Lambda_1 a_2)\,, \tag{3.96}$$

reflecting the noncommutativity of translations and Lorentz transformations. The proper and orthochronous component can be parametrized by 10 parameters:

- 4 space time translations
- 3 rotations
- 3 Lorentz boosts.

Poincaré Algebra

If we represent an element $\mathcal{P}$ of the Poincaré group by exponentiated generators $J^{\mu\nu} = -J^{\nu\mu}$ and P^μ

$$\mathcal{P}(\omega, a) = \mathrm{e}^{\mathrm{i}a_\mu P^\mu} \mathrm{e}^{\mathrm{i}\omega_{\mu\nu} J^{\mu\nu}} \tag{3.97}$$

we obtain the commutation relations of the *Poincaré Algebra*

$$[P_\mu, P_\nu] = 0 \tag{3.98a}$$
$$[J_{\mu\nu}, P_\rho] = \mathrm{i}\,(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \tag{3.98b}$$
$$[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}\,(g_{\mu\rho}J_{\nu\sigma} - g_{\mu\sigma}J_{\nu\rho} - g_{\nu\rho}J_{\mu\sigma} + g_{\nu\sigma}J_{\mu\rho}) \,. \tag{3.98c}$$

Note that the structure of these commutation relations is completely fixed by the fact that the Lorentz generators $J^{\mu\nu}$ are antisymmetric and contain the angular momentum and thus have the dimension of $\hbar$.

Introducing the *Pauli–Lubanski vector*

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \,, \tag{3.99}$$

it is a straightforward calculcation to show that

$$[P_\mu, P_\rho P^\rho] = [J_{\mu\nu}, P_\rho P^\rho] = 0 \tag{3.100a}$$
$$[P_\mu, W_\rho W^\rho] = [J_{\mu\nu}, W_\rho W^\rho] = 0 \,, \tag{3.100b}$$

i. e. that both $P_\rho P^\rho$ and $W_\rho W^\rho$ must be constant *Casimir Operator*s proportional to the unit matrix in each irreducible representation of the Poincaré algebra. This means that the irreducible representations of the Poincaré algebra are labelled by the eigenvalues

$$P_\mu P^\mu = m^2 \tag{3.101a}$$
$$W_\mu W^\mu = -m^2 s(s+1) \tag{3.101b}$$

of these operators. Obviously, m denotes the mass associated to the representation and s the spin.

Representations with $m^2 < 0$ exist, but are pathological and discarded for physical reasons. Representations with $m^2 > 0$ for $2s + 1$ angular momentum multiplets and representations with $m = 0$ are one-dimensional, corresponding to a fixed helicity and two-dimensional if parity is included.

3.5.3 Extensions of the Poincaré Group

Coleman–Mandula Theorem

Coleman and Mandula showed in 1967 that, if

1. the S-matrix is based on a local relativistic QFT in 4 space-time dimensions,
2. there are only a finite number of different particles of a given mass and
3. there is an energy gap between the vacuum and the one particle states,

then the symmetry group is the *direct product* of the Poincaré group and a compact Lie group. I. e., all additional symmetries must commute with the Poincaré group.

In practical terms, this means that all members of a Poincaré group multiplet must have the same "internal" quantum numbers.

NB: there is no anologuous theorem for the Galileo group and non-relativistic multiplets can mix spin and internal symmetries.

Haag–Łopuszański–Sohnius Theorem

There is only one way to avoid the consequences of the Coleman–Mandula theorem, as has been proved by Haag, Łopuszański and Sohnius in 1975: allow *supersymmetries*, i. e. symmetries generated by *fermionic charges*. Furthermore, the number $\mathcal{N}$ of allowed charges is limited:

- $0 \leq \mathcal{N} \leq 2$ if no gravity is involved
- $0 \leq \mathcal{N} \leq 4$ if gravity is involved .

—4— Quark Model

4.1 The Particle Zoo

The advent of particle accelerators in the late 1940 led to a proliferation of observed resonance with different

- masses
- lifetimes, i. e. decay widths,
- decay channels

that called for a systematic classification.

It became obvious that there is a hierarchy of interactions that respect different symmetries:

- strong interactions that respect the most symmetries
- weak and electromagnetic interactions that are, in first approximation, only observable in processes that are *verboten* in strong interactions

4.2 Isospin

4.2.1 Strong Interactions vs. Electromagnetism

Nucleons have spin 1/2 and we observe

- protons, $m_p = 938.3\,\text{MeV}$
- neutrons, $m_n = 939.6\,\text{MeV}$

with

$$\frac{m_n - m_p}{(m_p + m_n)/2} \approx \frac{\alpha}{2\pi} \tag{4.1}$$

suggesting that the mass difference can be attributed to electromagnetic interactions. Similarly pseudoscalar mesons

- charged pions, $m_{\pi^\pm} = 139.57\,\text{MeV}$
- neutral pions, $m_{\pi^0} = 134.98\,\text{MeV}$

with

$$\frac{m_{\pi^\pm} - m_{\pi^0}}{m_\pi} \approx \frac{15\alpha}{\pi}\,. \tag{4.2}$$

Thus it makes sense to try to group mesons and baryons in multiplets of the symmetry group of the strong interactions and this symmetry group must have dublets and triplets in among their representations.

4.2.2 Dublets and Triplets

An obvious candidate is SU(2), a.k.a. *isospin*:

$$\begin{pmatrix} p \\ n \end{pmatrix} \mapsto \begin{pmatrix} p' \\ n' \end{pmatrix} = R_2(g) \begin{pmatrix} p \\ n \end{pmatrix} \tag{4.3a}$$

$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \mapsto \begin{pmatrix} \pi^{+\prime} \\ \pi^{0\prime} \\ \pi^{-\prime} \end{pmatrix} = R_3(g) \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \tag{4.3b}$$

We know from elastic e^-p scattering that protons have a finite size and it make sense to assume that they are composite.

Since

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \tag{4.4a}$$

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4} \tag{4.4b}$$

and $m_p \gg m_\pi$, we can start from the assumption that protons and neutrons are composed of 3 constituent with spin and isospin 1/2 and pions of 2 of them.

These constituents are called *quark*s with the colloquial names

$$\begin{pmatrix} u \\ d \end{pmatrix} \tag{4.5}$$

for up-quarks and down-quarks as in spins.

It is easy see that

$$\forall i = 1, 2, 3 : \sigma_i = \sigma_2(-\overline{\sigma_i})\sigma_2 \tag{4.6}$$

where $\bar{\cdot}$ denotes elementwise complex conjugation and *not* hermitian conjugation. Thus the complex conjugate representation $\bar{\mathbf{2}}$ of SU(2) is equivalent:

$$\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} \in \bar{\mathbf{2}} \cong \mathbf{2} \ni \begin{pmatrix} u \\ d \end{pmatrix} . \tag{4.7}$$

This allows us to construct the triplet from quarks and anti-quarks using familiar techniques.

Raising and Lowering Operators

Lecture 06: Wed, 29. 04. 2015

Using

$$\vec{T} = \frac{\vec{\sigma}}{2} \tag{4.8}$$

we find

$$[T_i, T_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} T_k , \tag{4.9}$$

independently of the representation. For notational simplicity, we will write T_i for $r(T_i)$ hencforth and assume that the representation is specified by the vectors the operators act on.

We observe from Schur's lemma that

$$T^2 = T_1T_1 + T_2T_2 + T_3T_3 \tag{4.10}$$

must be constant in each irreducible representation

$$[T^2, \vec{T}] = 0 . \tag{4.11}$$

Furthermore, there are no two independent linear combinations of the T_i that commute with each other:

$$0 = [\vec{\alpha}\vec{T}, \vec{\beta}\vec{T}] = \mathrm{i} \left(\vec{\alpha} \times \vec{\beta}\right) \vec{T} \Longleftrightarrow \vec{\alpha} \parallel \vec{\beta} \tag{4.12}$$

This means that we can label our irreducible representations by the eigenvalues of the Casimir operator T^2 and the states in each irreducible representation by the eigenvalue of one linear combination, conventionally chosen to be T_3.

Introducing

$$T_\pm = \frac{1}{\sqrt{2}}(T_1 \pm \mathrm{i}T_2) \tag{4.13}$$

we observe

$$[T_3, T_\pm] = \pm T_\pm \tag{4.14a}$$

$$[T_+, T_-] = T_3\,. \tag{4.14b}$$

For eigenvectors $|m\rangle$ of T_3

$$T_3\,|m\rangle = m\,|m\rangle \tag{4.15}$$

we observe

$$T_3T_\pm\,|m\rangle = (m \pm 1)T_\pm\,|m\rangle\ , \tag{4.16}$$

i. e.

$$T_\pm\,|m\rangle \propto |m \pm 1\rangle\ . \tag{4.17}$$

Obviously $T_\pm$ act as raising and lowering operators for the eigenvalue of T_3, respectively. We can use them repeatedly to find *all* states in an irreducible representation, since the representation would be reducible otherwise.

Furthermore, the representation is uniquely characterized by the *highest weight* l, i. e. the largest eigenvalue m of T_3. We know from elementary quantum mechanical that this representation is $2l+1$-dimensional and

$$T^2 = l(l+1)\mathbf{1}_{(2l+1)\times(2l+1)}\,. \tag{4.18}$$

Example

If we write

$$|u\rangle = \begin{pmatrix}1\\0\end{pmatrix} \tag{4.19a}$$

$$|d\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{4.19b}$$

$$|\bar d\rangle = \begin{pmatrix}1\\0\end{pmatrix} \tag{4.19c}$$

$$|\bar u\rangle = \begin{pmatrix}0\\-1\end{pmatrix}\ , \tag{4.19d}$$

we have

$$T_3\,|u\rangle = \frac{1}{2}\,|u\rangle \tag{4.20a}$$

$$T_3 |d\rangle = -\frac{1}{2} |d\rangle \tag{4.20b}$$

$$T_3 |\bar u\rangle = \frac{1}{2} |\bar u\rangle \tag{4.20c}$$

$$T_3 |\bar d\rangle = -\frac{1}{2} |\bar d\rangle \tag{4.20d}$$

and

$$T_- |u\rangle = \frac{1}{\sqrt{2}} |d\rangle \tag{4.21a}$$

$$T_+ |d\rangle = \frac{1}{\sqrt{2}} |u\rangle \tag{4.21b}$$

$$T_- |\bar u\rangle = -\frac{1}{\sqrt{2}} |\bar d\rangle \tag{4.21c}$$

$$T_+ |\bar d\rangle = -\frac{1}{\sqrt{2}} |\bar u\rangle\ . \tag{4.21d}$$

Starting from

$$|\pi^+\rangle = |u\rangle \otimes |\bar d\rangle \tag{4.22}$$

then

$$\begin{aligned} |\pi^0\rangle &= T_- |\pi^+\rangle = (T_- \otimes \mathbf{1} + \mathbf{1} \otimes T_-) |u\rangle \otimes |\bar d\rangle \\ &= T_- |u\rangle \otimes |\bar d\rangle + |u\rangle \otimes T_- |\bar d\rangle = \frac{1}{\sqrt{2}} |d\rangle \otimes |\bar d\rangle - \frac{1}{\sqrt{2}} |u\rangle \otimes |\bar u\rangle \end{aligned} \tag{4.23}$$

and

$$|\pi^-\rangle = T_0 |\pi^0\rangle = |d\rangle \otimes |\bar u\rangle \tag{4.24}$$

Similarly, the spin 3/2, isospin 3/2 quadruplet, i. e. the **4** in

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4} \tag{4.25}$$

is

$$|\Delta^{++}\rangle = |u\rangle \otimes |u\rangle \otimes |u\rangle \tag{4.26a}$$

$$|\Delta^{+}\rangle = \mathcal{S}\left(|u\rangle \otimes |u\rangle \otimes |d\rangle\right) \tag{4.26b}$$

$$|\Delta^{0}\rangle = \mathcal{S}\left(|u\rangle \otimes |d\rangle \otimes |d\rangle\right) \tag{4.26c}$$

$$|\Delta^{-}\rangle = |d\rangle \otimes |d\rangle \otimes |d\rangle\ , \tag{4.26d}$$

where $\mathcal{S}$ denotes symmetrization, e. g.

$$\mathcal{S}\left(|u\rangle \otimes |u\rangle \otimes |d\rangle\right)$$

$$= \frac{1}{\sqrt{3}} \left(|u\rangle \otimes |u\rangle \otimes |d\rangle + |u\rangle \otimes |d\rangle \otimes |u\rangle + |d\rangle \otimes |u\rangle \otimes |u\rangle \right) . \quad (4.27)$$

We have left two choices for the isospin 1/2 dublets

$$|p\rangle = \frac{1}{\sqrt{2}} \left(|u\rangle \otimes |u\rangle \otimes |d\rangle - |u\rangle \otimes |d\rangle \otimes |u\rangle \right) \quad (4.28a)$$

$$|n\rangle = \frac{1}{\sqrt{2}} \left(|d\rangle \otimes |u\rangle \otimes |d\rangle - |d\rangle \otimes |d\rangle \otimes |u\rangle \right) \quad (4.28b)$$

and

$$|p'\rangle = \frac{1}{\sqrt{6}} \left(|u\rangle \otimes |u\rangle \otimes |d\rangle + |u\rangle \otimes |d\rangle \otimes |u\rangle - 2\, |d\rangle \otimes |u\rangle \otimes |u\rangle \right) \quad (4.29a)$$

$$|n'\rangle = \frac{1}{\sqrt{6}} \left(2\, |u\rangle \otimes |d\rangle \otimes |d\rangle - |d\rangle \otimes |d\rangle \otimes |u\rangle - |d\rangle \otimes |u\rangle \otimes |d\rangle \right) \quad (4.29b)$$

or any linear combination of these.

This works by assigning the (additive) electric charges

$$Q\, |u\rangle = \frac{2}{3}\, |u\rangle \quad (4.30a)$$

$$Q\, |d\rangle = -\frac{1}{3}\, |d\rangle \quad (4.30b)$$

$$Q\, |\bar{u}\rangle = -\frac{2}{3}\, |\bar{u}\rangle \quad (4.30c)$$

$$Q\, |\bar{u}\rangle = \frac{1}{3}\, |\bar{u}\rangle . \quad (4.30d)$$

Indeed

$$Q\, |p\rangle = \left(\frac{2}{3} + \frac{2}{3} - \frac{1}{3} \right) |p\rangle = |p\rangle \quad (4.31a)$$

$$Q\, |n\rangle = \left(\frac{1}{3} + \frac{1}{3} - \frac{2}{3} \right) = 0 \quad (4.31b)$$

$$Q\, |\pi^+\rangle = \left(\frac{2}{3} + \frac{1}{3} \right) |\pi^+\rangle = |\pi^+\rangle \quad (4.31c)$$

$$Q\, |\pi^0\rangle = 0 \quad (4.31d)$$

$$Q\, |\pi^-\rangle = \left(-\frac{2}{3} - \frac{1}{3} \right) |\pi^+\rangle = -\, |\pi^-\rangle . \quad (4.31e)$$

In addition, it appears that *baryon number* B

$$B\, |u\rangle = \frac{1}{3}\, |u\rangle \quad (4.32a)$$

$$B\,|d\rangle = \frac{1}{3}\,|d\rangle \tag{4.32b}$$

$$B\,|\bar{u}\rangle = -\frac{1}{3}\,|\bar{u}\rangle \tag{4.32c}$$

$$B\,|\bar{u}\rangle = -\frac{1}{3}\,|\bar{u}\rangle \tag{4.32d}$$

is conserved, leading to the stability of the proton, which is the lightest observed state with $B = 1$.

Statistics

However, there is a problem. If we take this *constituent quark model* seriously, the Hilbert space is a direct product of three factors

$$\mathcal{H} = \underbrace{\mathcal{H}_P}_{=\text{position}} \otimes \underbrace{\mathcal{H}_S}_{=\text{spin}} \otimes \underbrace{\mathcal{H}_F}_{=\text{flavor}}\,, \tag{4.33}$$

where *flavor* denotes the internal quantum numbers, such as isospin.

In the case of the Δ-quadruplet of spin 3/2, isospin 3/2 baryons (which all have been observed in pion-nucleon scattering), the wave functions are *totally symmetric* both in $\mathcal{H}_F$ and in $\mathcal{H}_S$, as can be seen from (4.26) and its analog for spin. Since spin 1/2 quarks must be fermions, this means that the wavefunction in position space $\mathcal{H}_P$

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) \tag{4.34}$$

must be totally antisymmetric for the overall wavefunction to be antisymmetric as well, as required by Fermi statistics.

Thus the wavefunction must have a zero when the relative distances vanish and one would not expect this to be the lowest energy state:

- typically the energy increases with the number of nodes
- an attractive interaction should favor states with the largest overlap.

In fact, the most likely ground state in position space is a product of three S-waves. There are two ways out of this dilemma

- quarks don't satisfy Fermi statistics
- there's another factor in the Hilbert space, where the wave function is totally antisymmetric.

The latter solution turns out to be more efficient, because the new quantum number, *color*, will also provide a description of the strong interaction. The Hilbert space for quarks turns out to be

$$\mathcal{H} = \underbrace{\mathcal{H}_P}_{=\text{position}} \otimes \underbrace{\mathcal{H}_S}_{=\text{spin}} \otimes \underbrace{\mathcal{H}_F}_{=\text{flavor}} \otimes \underbrace{\mathcal{H}_C}_{=\text{color}} , \tag{4.35}$$

where the wavefunctions in $\mathcal{H}_C$ are arranged such that

- the states carry *no* overall color charge
- Fermi statics is satisfied.

We will later see that SU(3) has just the right properties for this.

Constituent Quark Model

By treating the masses of quarks a fit parameters, it is possible to compute static properties of hadrons, like masses and magnetic momenta, in the constituent quark model, when the detailed form of the position space wave function(s) cancel out. The corresponding predictions work surprisingly well.

It is much harder to compute dynamical properties, like decay widths, because they depend on wave function overlaps

$$\langle \text{out}|H_I|\text{in}\rangle , \tag{4.36}$$

which can not be computed from first principle:

- the relevant interaction is strong, precluding the use of perturbation theory
- the form of the interaction potential at long distances is not well known, precluding the use of Schrödinger- or Dirac-equation methods.

Flavor Symmetries

A very powerful approach to strong interactions avoids the use of position space wavefunctions altogether and relies on flavor symmetries alone.

Assuming that the Hamiltonian commutes with flavor symmetries, such as isospin

$$\left[\vec{T}, H\right] = 0 , \tag{4.37}$$

or is a member of a multiplet of tensor operators O^l

$$\left[T_a, O^l_m\right] = \sum_{m'=-l}^{l} [r^l(T_a)]_{mm'} O^l_{m'} , \tag{4.38}$$

it is possible to use (extensions of) the *Wigner–Eckhart theorem* to express all matrix elements among a pair of multiplets by a single number, the reduced matrix element

$$\langle l'||O^l||l''\rangle \tag{4.39}$$

multiplied by appropriate *Clebsh–Gordan coefficients*

$$\langle l',m'|O^l_m|l'',m''\rangle = \delta_{m',m+m''}\,\langle l',m'|l,l'',m,m''\rangle\,\langle l'||O^l||l''\rangle\ . \tag{4.40}$$

For example all scattering amplitudes for pion-nucleon scattering can be expressed by two reduced matrix elements, one for the isospin 1/2 and one for the isospin 3/2 channel.

Since the proof of the Wigner–Eckhart theorem depends only on the commutation relations, it can be copied verbatim from spin to isopin.

4.3 Eight-Fold Way

Lecture 07: Tue, 05. 05. 2015

Soon, it was discovered that the three pions were not the only pseudoskalar mesons: there are also the 4 charged and neutral kaons

$$K^+, K^0, \bar K^0, K^- \tag{4.41}$$

and the neutral η. Here the η and the π^0 are distinguished from the K^0 by the masses

$$m_\eta = 547.9\,\mathrm{MeV} \neq m_{K^0} = m_{\bar K^0} = 497.6\,\mathrm{MeV} \tag{4.42}$$

but the $\bar K^0$ differs from the K^0 by its decays (see below). The masses of the charged kaons are

$$m_{K^\pm} = 493.7\,\mathrm{MeV}\,. \tag{4.43}$$

The kaons where called *strange particles* because they live much longer than other particles that can decay into hadrons. Therefore, it was suggested that there is a new quantum number *strangeness* S that is conserved by the strong interactions and only violated by weak interactions. The difference in the strength will then explain the difference in lifetime.

It turns out that the more useful quantum number is the *hypercharge* Y

$$Y = B + S\,, \tag{4.44}$$

which is conserved by the strong, but violated by the weak interactions. This hypercharge satisfies the Gell-Mann–Nishijima formula

$$Q = T_3 + \frac{Y}{2}\,, \tag{4.45}$$

if we assign

$$S\,|K^+\rangle = |K^+\rangle \tag{4.46a}$$
$$S\,|K^0\rangle = |K^0\rangle \tag{4.46b}$$
$$S\,|\bar{K}^0\rangle = -\,|\bar{K}^0\rangle \tag{4.46c}$$
$$S\,|K^-\rangle = -\,|K^-\rangle \tag{4.46d}$$

and group the kaons in the corresponding two isopin dublets

$$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \qquad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}\,. \tag{4.47}$$

4.3.1 SU(3)

Gell-Mann's stroke of genius was to realize that

$$\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SU}(3) \tag{4.48}$$

as Lie groups, where the U(1) is generated by (a multiple of) the hypercharge Y and that just as the non-strange baryons and mesons fit into almost degenerate iso-spin multiplets, all (at the time) observed baryons and mesons fit into slightly less degenerate SU(3) multiplets.

Gell-Mann Matrices

The group of unitary 3×3-matrices U with unit determinant is generated

$$U = \mathrm{e}^{\mathrm{i}T} \tag{4.49}$$

by traceless, hermitian 3×3-matrices T. A convenient basis for the latter are the *Gell-Mann matrices* $\{\lambda_a\}_{a=1,\ldots,8}$

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}\,, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}\,, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{4.50}$$

with the conventional normalization

$$T_a = \frac{\lambda_a}{2} \tag{4.51}$$

such that

$$\operatorname{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}\,. \tag{4.52}$$

4.3.2 Structure Constants

Not only for SU(3), but for any Lie algebra, we can use (4.52) to define a metric in the adjoint representation

$$\langle T_a | T_b \rangle = 2\operatorname{tr}(T_a^\dagger T_b) = \delta_{ab} \tag{4.53}$$

and to compute the *structure constants* f_{abc} in a given basis

$$[T_a, T_b] = \mathrm{i} \sum_c f_{abc} T_c \tag{4.54}$$

as

$$f_{abc} = -2\mathrm{i}\operatorname{tr}\left([T_a, T_b]\, T_c\right)\,. \tag{4.55}$$

Since (4.55) depends only on the Lie algebra structure, which is independent of the representation, and the normalization (4.52), which can always be arranged. we may compute the structure constants in any representation. If the group has a unitary representation, the generators are hermitian

$$(r(T_a))^\dagger = r(T_a) \tag{4.56}$$

in this representation. We may use it to compute the structure constants and find

$$\begin{aligned} \mathrm{i}\overline{f_{abc}} r(T_c) &= \mathrm{i}\overline{f_{abc}} r(T_c)^\dagger = -(\mathrm{i} f_{abc} r(T_c))^\dagger = -[r(T_a), r(T_b)]^\dagger \\ &= -\left[(r(T_b))^\dagger, (r(T_a))^\dagger\right] = -\left[r(T_b), r(T_a)\right] = [r(T_a), r(T_b)] = \mathrm{i} f_{abc} r(T_c)\,. \end{aligned} \tag{4.57}$$

Thus the structure constants of a Lie algebra are real $f_{abc} = \overline{f_{abc}}$, whenever the Lie group has at least one faithful unitary representation.

4.3.3 Representations

Given a representation $r : L \to \mathrm{GL}(N,\mathbf{C})$, it turns out that there is also a complex conjugate representation

$$\begin{aligned} \tilde{r} : L &\to \mathrm{GL}(N,\mathbf{C}) \\ a &\mapsto -\overline{r(a)} \end{aligned} \tag{4.58}$$

if there is any unitary representation. Indeed, choosing a basis, we can write

$$\tilde{T}_a = -\overline{T_a} \tag{4.59}$$

and find

$$\left[\tilde{T}_a, \tilde{T}_b\right] = \left[\overline{T_a}, \overline{T_b}\right] = \overline{[T_a, T_b]} = \overline{\mathrm{i} f_{abc} T_c} = -\mathrm{i} \overline{f_{abc} T_c} = \mathrm{i} f_{abc} \tilde{T}_c \tag{4.60}$$

since $f_{abc} = \overline{f_{abc}}$.

Given a representation $r : L \to \mathrm{GL}(V)$ and an isomorphism $\phi : V \to W$, we can define a similar representation $r^\phi : L \to \mathrm{GL}(W)$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ r(a)\downarrow & & \downarrow r^\phi(a) = \phi \circ r(a) \circ \phi^{-1} \\ V & \xrightarrow{\phi} & W \end{array} \tag{4.61}$$

commute. Indeed

$$\begin{aligned} \left[r^\phi(a), r^\phi(b)\right] &= r^\phi(a) r^\phi(b) - r^\phi(b) r^\phi(a) \\ &= \phi \circ r(a) \circ \underbrace{\phi^{-1} \circ \phi}_{=\mathrm{id}} \circ r(b) \circ \phi - \phi \circ r(b) \circ \underbrace{\phi^{-1} \circ \phi}_{=\mathrm{id}} \circ r(a) \circ \phi \\ &= \phi \circ [r(a), r(b)] \circ \phi^{-1} = [r(a), r(b)]^\phi \; . \end{aligned} \tag{4.62}$$

4.3.4 Cartan Subalgebra and Rank

A maximal subalgebra of *commuting* elements $[a, b] = 0$ of a Lie algebra is called a *Cartan subalgebra*. It's dimension is called the *rank* of the Lie algebra.

Above, we have shown that the rank of SU(2) is 1. It is very easy to compute the rank for $\mathrm{SU}(N)$: since the members of the Cartan subalgebra commute and are hermitian, they can be diagonalized simultaneously. Thus there is a basis in which they are all diagonal and there can be only N

independent diagonal hermitian matrices. Since the generators of SU(N) are traceless, we must remove the unit matrix from this set and obtain $N-1$ independent diagonal traceless hermitian matrices. This shows

$$\operatorname{rank}(\mathrm{SU}(N)) = N - 1\,, \tag{4.63}$$

which includes the special case rank(SU(2)) = 1.

Below, we will denote the representatives of the generators of the Cartan subalgebra of g by

$$\{H_a\}_{a=1,\ldots,\operatorname{rank}(g)}\,. \tag{4.64}$$

4.3.5 Roots and Weights

The eigenvalues of the generators of the Cartan subalgebra are called *weights*

$$H_a\,|m;r,\eta\rangle = m_a\,|m;r,\eta\rangle = \tag{4.65}$$

and can be combined into a *weight vector*

$$m = (m_1, m_2, \ldots, m_{\operatorname{rank}(g)})\,. \tag{4.66}$$

As eigenvalues of real matrices, the weights are real. They depend on the representation r and possible degenerate eigenvectors might need to be distinguished by additional labels η. In a given basis, we may order the weight vectors lexicographically and define positivity by comparing to $(0,\ldots,0)$ with respect to (wrt) this order.

The weight vectors of the adjoint representation are called *roots* and it can be shown that they are *not* degenerate. The roots of the states corresponding to the Cartan generators vanish

$$H_a\,|H_b\rangle = |[H_a, H_b]\rangle = 0\,. \tag{4.67}$$

The roots of the non-Cartan generators E must not vanish

$$H_a\,|E_m\rangle = m_a\,|E_m\rangle \tag{4.68}$$

which can also be written

$$[H_a, E_m] = m_a E_m\,. \tag{4.69}$$

Note that the E_m are not hermitian

$$m_a E_m^\dagger = [H_a, E_m]^\dagger = [E_m^\dagger, H_a^\dagger] = [E_m^\dagger, H_a] = -[H_a, E_m^\dagger]\,, \tag{4.70}$$

i. e.

$$E_m^\dagger = E_{-m} \,. \tag{4.71}$$

Now observe that the $E_{\pm m}$ act as raising and lowering operators for weights in any representation r

$$\begin{aligned} H_a E_{\pm m} \left| m'; r, \eta \right\rangle &= [H_a, E_{\pm m}] \left| m'; r, \eta \right\rangle + E_{\pm m} H_a \left| m'; r, \eta \right\rangle \\ &= \pm m_a E_{\pm m} \left| m'; r, \eta \right\rangle + E_{\pm m} m'_a \left| m'; r, \eta \right\rangle = (m'_a \pm m_a) E_{\pm m} \left| m'; r, \eta \right\rangle \,. \end{aligned} \tag{4.72}$$

Therefore, acting with E_m on a state with weight $-m$, we get a state with weight $m - m = 0$. In the adjoint representation this means that the result is a combination of Cartan generators

$$E_m \left| E_{-m} \right\rangle = \sum_{a=1}^{\mathrm{rank}(g)} \beta_a \left| H_a \right\rangle \,. \tag{4.73}$$

Even better

$$\begin{aligned} \beta_a = \langle H_a | E_m | E_{-m} \rangle &= \langle H_a | [E_m, E_{-m}] \rangle = 2 \,\mathrm{tr}(H_a^\dagger [E_m, E_{-m}]) \\ &= 2 \,\mathrm{tr}(H_a [E_m, E_{-m}]) = 2 \,\mathrm{tr}(E_{-m} [H_a, E_m]) = 2 m_a \,\mathrm{tr}(E_{-m} E_m) \\ &= 2 m_a \,\mathrm{tr}(E_m^\dagger E_m) = m_a \langle E_m | E_m \rangle = m_a \end{aligned} \tag{4.74}$$

and thus

$$[E_m, E_{-m}] = \sum_{a=1}^{\mathrm{rank}(g)} m_a H_a \tag{4.75}$$

as a generalization of $[L_+, L_-] = L_3$.

In a finite dimensional representation, there must be a state that is annihilated by all raising operators, i. e. all E_m with postive root $m > 0$. This is called the highest weight state.

Unless the raising or lowering operator $E_{\pm m}$ annihilates the state corresponding to the root m', there is another root $m' \pm m$. Therefore it makes sense to define a *simple root* as a root that can not be written as the sum of other simple roots. It then suffices to check that a highest weight state is annihilated by all raising operators corresponding to simple roots.

Lecture 08: Wed, 06. 05. 2015

From (4.69) and (4.75) we can define for each root vector $m > 0$ the operators

$$E_\pm^{(m)} = \frac{1}{|m|} E_{\pm m} \tag{4.76a}$$

$$E_3^{(m)} = \frac{1}{|m|} \sum_a m_a H_a \tag{4.76b}$$

with the length of the root vector

$$|m| = \sqrt{\sum_a m_a^2} \,. \tag{4.77}$$

These operators form a SU(2) Lie algebra written expressed by raising and lowering operators

$$[E_+^{(m)}, E_-^{(m)}] = E_3^{(m)} \tag{4.78a}$$

$$[E_3^{(m)}, E_\pm^{(m)}] = \pm E_\pm^{(m)} \tag{4.78b}$$

and we know all finite dimensional irreducible representations from elementary Quantum Mechanics (QM). This means that the states in each representation must form lines of equidistant points along the direction of each root vector.

4.3.6 *Back to* SU(3)

In our choice of Gell-Mann matrices, a natural choice for a basis of the Cartan subalgebra is

$$H_1 = T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.79a}$$

$$H_2 = T_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{4.79b}$$

and we can compute the weights by reading off the eigenvalues of these matrices from the diagonal elements:

$$(m_1, m_2)_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \tag{4.80a}$$

$$(m_1, m_2)_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \tag{4.80b}$$

$$(m_1, m_2)_3 = \left(0, -\frac{1}{\sqrt{3}}\right) , \tag{4.80c}$$

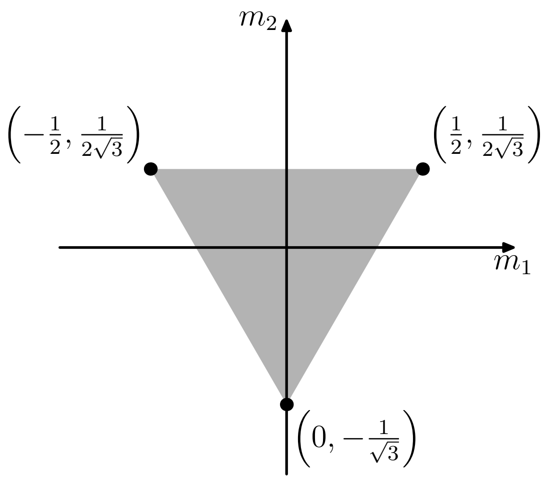


Figure 4.1: **3** *or fundamental representation of* SU(3)*.*

as depicted in figure 4.1. In the complex conjugate representation, all weights have their signs reversed since they are eigenvalues of hermitian operators, as shown in figure 4.2.

Since the roots correspond to raising and lowering operators, they must be differences of weight vectors

$$E_{\pm 1,0} = \pm(m_1, m_2)_1 \mp (m_1, m_2)_2 = (\pm 1, 0) \tag{4.81a}$$

$$E_{\pm 1/2, \pm\sqrt{3}/2} = \mp(m_1, m_2)_2 \pm (m_1, m_2)_3 = \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) \tag{4.81b}$$

$$E_{\pm 1/2, \mp\sqrt{3}/2} = \pm(m_1, m_2)_1 \mp (m_1, m_2)_3 = \left(\pm\frac{1}{2}, \mp\frac{\sqrt{3}}{2}\right) \tag{4.81c}$$

and are depicted in figure 4.3. Together with the two Cartan generators at the origin, they fill out the 8-dimensional adjoint representation of SU(3) in figure 4.4.

The general form of irreducible representations of SU(3) can deduced from the relative orientaten of the roots generating the SU(2) subgroups. They form a triangular grid with hexagonal or triangular boundaries and invariance under the group generated by rotations by $2\pi/3$ and reflections about the m_2-axis. The hexagons can be viewed as triangles with the corners chopped off. Vice versa, the triangles are hexagons with three sides shrunk to a single point. In general two or more of these hexagons or triangles of different size will be combined, like H_1 and H_2 in the **8**, as depicted in figure 4.4.

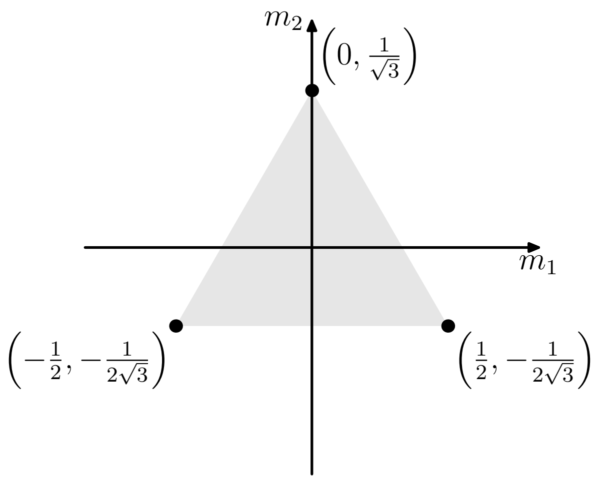


Figure 4.2: $\bar{\mathbf{3}}$ *or anti-fundamental representation of* SU(3).

Since the representations are symmetrical about the m_2 axis and complex conjugation corresponds to an inversion $(m_1, m_2) \mapsto (-m_1, -m_2)$, a representations is equivalent to its complex conjugate, iff it is symmetrical about the m_1 axis.

4.3.7 Quarks, Octets and Decuplets

We can construct all representations of SU(3) out of building blocks from the $\mathbf{3}$ and $\bar{\mathbf{3}}$. As shown in figure 4.5, we call these states *quarks* and antiquarks, respectively. As before, the up and down quarks u and d form an isospin dublet, while the *strange quark* is an isospin singlet.

In oder to avoid irrational coefficients in the Gell-Mann–Nishijima relation (4.45), we rescale the Cartan generators and introduce the usual quantum numbers isospin and hypercharge

$$T_3 = H_1 \tag{4.82a}$$

$$Y = \frac{2}{\sqrt{3}} H_2 \,. \tag{4.82b}$$

Note that this rescaling breaks the triangular symmetries of the representations. We find the quantum numbers depicted in table 4.1.

Starting from the quarks we can now construct quark-antiquark states, graphically, as shown on the Left Hand Side (LHS) of figure 4.6 and identify them with the octet of pseudoscalar mesons as shown on the Right Hand

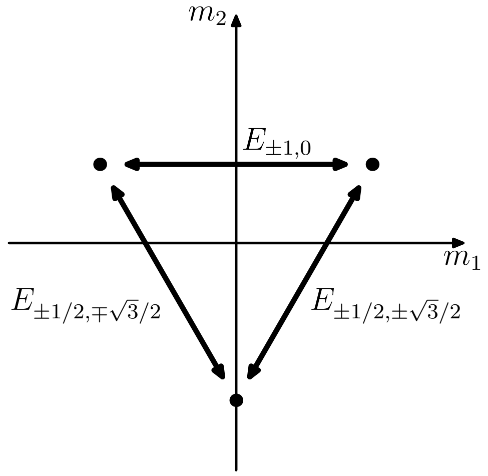


Figure 4.3: *Roots of* SU(3) *derived from* **3** *or fundamental representation. Note that the* $\bar{\mathbf{3}}$ *will yield the same roots since* $E_m^\dagger = E_{-m}$ *has already been included.*

Side (RHS) of figure 4.6. Note that of the three combinations $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ only the two traceless superpositions enter the octet. The orthogonal ninth state forms an SU(3) singlet and is called η'.

This graphical construction reproduces the Clebsh–Gordan decomposition

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} \,. \tag{4.83}$$

Algebrically, we start from the heighest weight state

$$|K^+\rangle = |u\rangle \otimes |\bar{s}\rangle \tag{4.84}$$

and successively apply the lowering operators E_{-m} until we have exhausted all possibilities. A similar approach to the di-quark states will produce

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}} \tag{4.85}$$

but the corresponding states do not appear in nature, because they can not be coupled to color singlets.

On the LHS of figure 4.7 we have depicted all possible three quark states, which have indeed been found as 3/2-baryons, as shown on the RHS of figure 4.7. An early triumph of the quark model, which earned Gell-Mann a trip to Stokholm was the discovery of the Ω^{--} at the predicted mass. The algebraic approach starts again from the highest weight state

$$|\Delta^{++}\rangle = |u\rangle \otimes |u\rangle \otimes |u\rangle \tag{4.86}$$

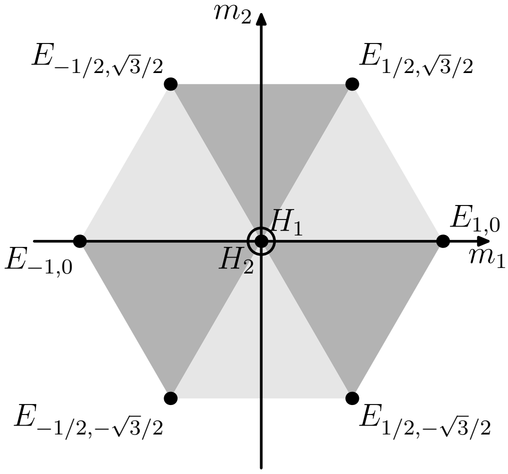


Figure 4.4: *Roots,* $\mathbf{8} \cong \bar{\mathbf{8}}$ *or adjoint representation of* SU(3)*.*

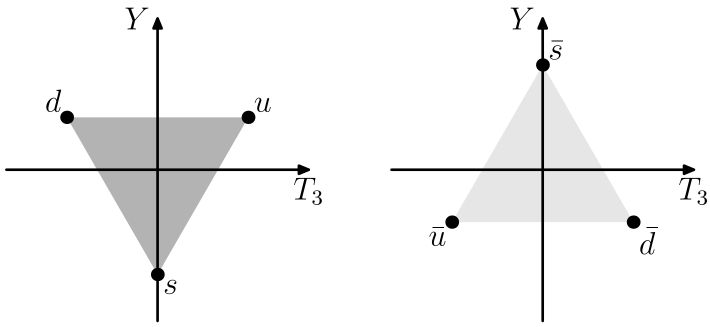


Figure 4.5: *The quarks and antiquarks form a* $\mathbf{3}$ *and a* $\bar{\mathbf{3}}$ *of* SU(3)*.*

and successively applies the lowering operators E_{-m} until we have exhausted all possibilities. This is the first stage of the Clebsh–Gordan decomposition

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \,. \tag{4.87}$$

From this construction, it is obvious that the flavor wave functions of the decuplet states are all totally symmetric under permutations, e. g.

$$|\Xi^{*,0}\rangle = \frac{1}{\sqrt{3}} \left(|u\rangle \otimes |s\rangle \otimes |s\rangle + |s\rangle \otimes |u\rangle \otimes |s\rangle + |s\rangle \otimes |s\rangle \otimes |u\rangle \right) \,. \tag{4.88}$$

As before this causes a problem with Fermi statistics, because the spin wave function for spin 3/2 is also totally symmetric and the position space wave functions are expected to be symmetric in the ground state.

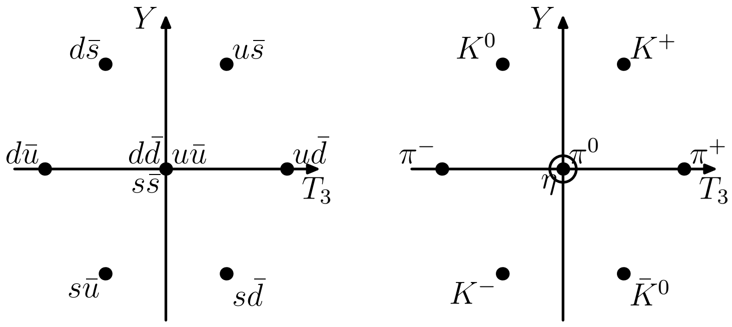


Figure 4.6: SU(3) *octets of quark-antiquark pairs and pseudoscalar mesons.*

This we are let to introduce a new quantum number *color*, corresponding to another, *completely unrelated*, $SU(3)_C$. If there is a chance for confusion, we will denote the flavor SU(3) by $SU(3)_F$. The quarks and antiquarks are assumed to transform as a $\mathbf{3}$ and $\bar{\mathbf{3}}$ under $SU(3)_C$ respectively. It is easy to see that the totally antisymmetric color space wave function

$$|\mathbf{1}\rangle = \frac{1}{\sqrt{6}} \sum_{i,j,k=1}^{3} \epsilon_{ijk} |i\rangle \otimes |j\rangle \otimes |k\rangle \tag{4.89}$$

is indeed a singlet under $SU(3)_C$

$$T|\mathbf{1}\rangle =$$

	T_3	Y	$Q = T_3 + Y/2$
u	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{2}{3}$
d	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{3}$
s	0	$-\frac{2}{3}$	$-\frac{1}{3}$
$\bar{u}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$
$\bar{d}$	$+\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{1}{3}$
$\bar{s}$	0	$+\frac{2}{3}$	$+\frac{1}{3}$

Table 4.1: *Quarks and antiquarks.*

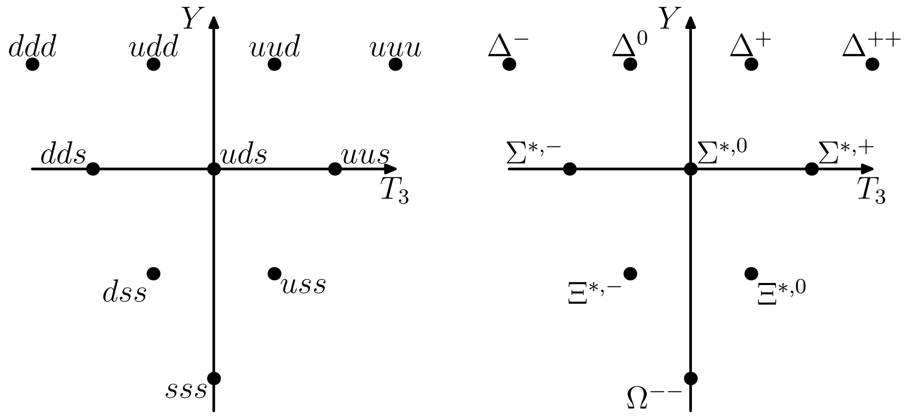


Figure 4.7: SU(3) *decuplets of three quark states and of excited spin*-3/2 *baryons.*

$$\frac{1}{\sqrt{6}}\sum_{i,j,k,l=1}^{3}\epsilon_{ijk}\left(T^i_l\left|l\right\rangle\otimes\left|j\right\rangle\otimes\left|k\right\rangle+T^j_l\left|i\right\rangle\otimes\left|l\right\rangle\otimes\left|k\right\rangle+T^k_l\left|i\right\rangle\otimes\left|j\right\rangle\otimes\left|l\right\rangle\right)$$
$$=\frac{1}{\sqrt{6}}\sum_{i,j,k}^{3}\underbrace{\sum_{l=1}^{3}\left(\epsilon_{ljk}T^l_i+\epsilon_{ilk}T^l_j+\epsilon_{ijl}T^l_k\right)}_{=0}\left|i\right\rangle\otimes\left|j\right\rangle\otimes\left|k\right\rangle=0\,, \tag{4.90}$$

because

$$(T\epsilon)_{ijk}=\sum_{l=1}^{3}\left(\epsilon_{ljk}T^l_i+\epsilon_{ilk}T^l_j+\epsilon_{ijl}T^l_k\right) \tag{4.91}$$

is totally antisymmetric and it suffices to compute

$$(T\epsilon)_{123}=\epsilon_{123}T^1_1+\epsilon_{123}T^2_2+\epsilon_{123}T^3_3=\epsilon_{123}\,\mathrm{tr}(T)=0 \tag{4.92}$$

by the tracelessness of the generators. Note that this is just a "complexified" version of the argument why

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) \tag{4.93}$$

is invariant unter rotations or more generally the determinant theorem. In the case of the pseudoscalar octet, we use the fact that

$$\left|\mathbf{1}\right\rangle=\frac{1}{\sqrt{3}}\sum_{i,j=1}^{3}\delta_{ij}\left|i\right\rangle\otimes\left|\bar{j}\right\rangle \tag{4.94}$$

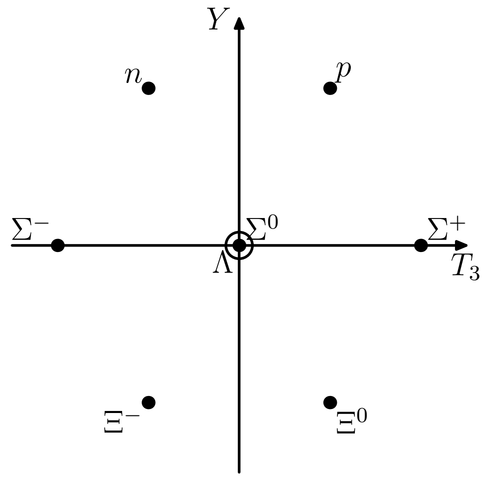


Figure 4.8: SU(3) *octet of baryons.*

is also a $SU(3)_C$ singlet.

We have learned that in the Clebsh–Gordan decomposition

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_M \oplus \mathbf{8}_{M'} \oplus \mathbf{1}_A \,. \tag{4.95}$$

the decuplet is totally symmetric and the singlet is totally antisymmetric. Since there is no totally antisymmetric combination of three spin-1/2 states, the flavor singlet can not be realized with a low lying symmetric position space wavefunction. It turns out that among the flavor octets with the mixed symmetry, only one is realized in combination with total spin 1/2, leading to the SU(3) octet of spin 1/2 baryons depicted in figure 4.8.

Lecture 09: Tue, 12. 05. 2015

The naive approach of assigning masses to quarks and explaining the mass differences by

$$m_s > m_d \approx m_u \tag{4.96}$$

works very well for the **10** of spin 3/2 baryons. However, it does not offer an explanation for the mass splitting of neutral pseudoscalars

$$m_\eta > m_{\pi^0} \tag{4.97}$$

and spin 1/2 baryons

$$m_\Lambda < m_{\Sigma^0} \tag{4.98}$$

that all sit at $T_3 = Y = 0$. The observed mass of the η is hard to explain, because it is a superposition of $u\bar{u}$, $d\bar{u}$ and $s\bar{s}$ contributions. Analogously, one would naively expect $m_\Lambda < m_{\Sigma^0}$, since they share the same quark content uds.

Therefore we need a more systematic approach to describe the breaking of $\mathrm{SU}(3)_F$ together with the conservation of isospin $\mathrm{SU}(2)_F$.

4.3.8 Tensor Methods

We can write a arbitrary quark-antiquark state

$$|\Psi\rangle = \sum_{\substack{f\in\{u,d,s\}\\ \bar{f}'\in\{\bar{u},\bar{d},\bar{s}\}}} \Psi_{\bar{f}'f}\,|f\rangle \otimes |\bar{f}'\rangle\ . \tag{4.99}$$

In the following, we will abbreviate $|f\rangle \otimes |\bar{f}'\rangle = |f\bar{f}'\rangle$. The states from the **8** satisfy

$$\sum_{f\in\{u,d,s\}} \Psi_{\bar{f}f} = 0 \tag{4.100}$$

i. e. they correspond to traceless matrices Ψ. If we parametrize them as

$$\Psi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{4.101}$$

we find

$$\begin{aligned} |\Psi\rangle &= \pi^0\frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right) + \eta\frac{1}{\sqrt{6}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle - 2\,|s\bar{s}\rangle\right) \\ &+ \pi^+\,|u\bar{d}\rangle + K^+\,|u\bar{s}\rangle + \pi^-\,|d\bar{u}\rangle + K^0\,|d\bar{s}\rangle + K^-\,|s\bar{u}\rangle + \bar{K}^0\,|s\bar{d}\rangle\ . \end{aligned} \tag{4.102}$$

Under $\mathrm{SU}(3)_F$, the states transform as

$$|\Psi\rangle \mapsto |\Psi'\rangle = \left(U\otimes\overline{U}\right)|\Psi\rangle = \sum_{\substack{f\in\{u,d,s\}\\ \bar{f}'\in\{\bar{u},\bar{d},\bar{s}\}}} \Psi_{\bar{f}'f} U\,|f\rangle \otimes \overline{U}\,|\bar{f}'\rangle\ , \tag{4.103}$$

where $\overline{U}$ corresponds to U in the complex conjugate representation

$$\overline{U} = \overline{\mathrm{e}^{\mathrm{i}T}} = \mathrm{e}^{-\mathrm{i}\overline{T}} = \mathrm{e}^{\mathrm{i}(-\overline{T})}\,. \tag{4.104}$$

In components

$$U\,|f\rangle = \sum_{f'\in\{u,d,s\}} U_{ff'}\,|f'\rangle \tag{4.105a}$$

$$\overline{U}\,|\bar f\rangle = \sum_{\bar f'\in\{\bar u,\bar d,\bar s\}} \overline{U_{\bar f\bar f'}}\,|\bar f'\rangle \tag{4.105b}$$

and thus

$$|\Psi'\rangle = \sum_{\substack{f_i\in\{u,d,s\}\\ \bar f'_i\in\{\bar u,\bar d,\bar s\}}} \Psi_{\bar f'_1 f_1} U_{f_1 f_2} \overline{U_{\bar f'_1\bar f'_2}}\,|f_2\rangle\otimes|\bar f'_2\rangle = \sum_{\substack{f\in\{u,d,s\}\\ \bar f'\in\{\bar u,\bar d,\bar s\}}} \Psi'_{\bar f' f}\,|f\rangle\otimes|\bar f'\rangle \tag{4.106}$$

with

$$\Psi'_{\bar f' f} = \sum_{\substack{f_1\in\{u,d,s\}\\ \bar f'_1\in\{\bar u,\bar d,\bar s\}}} \overline{U_{\bar f'_1\bar f'}}\Psi_{\bar f'_1 f_1} U_{f_1 f} = \left(U^\dagger\Psi U\right)_{f\bar f'}\,. \tag{4.107}$$

This means that the matrix Ψ transforms bi-unitarily

$$\Psi\mapsto\Psi' = U^\dagger\Psi U\,. \tag{4.108}$$

Hausdorff's formula

$$\Psi\mapsto\Psi' = \mathrm{e}^{-iT}\Psi\mathrm{e}^{\mathrm{i}T} = \mathrm{e}^{-\mathrm{i}\,\mathrm{ad}_T}\Psi = \Psi - \mathrm{i}[T,\Psi]+\ldots \tag{4.109}$$

shows that the matrix Ψ transforms according to the adjoint representation of $\mathrm{SU}(3)_F$. In particular $\mathrm{tr}\,\Psi'=0$ since

$$\mathrm{tr}[A,B] = \mathrm{tr}(AB)-\mathrm{tr}(BA) = 0 \tag{4.110}$$

by cyclic invariance. In general, we can write

$$\mathbf{N}\otimes\bar{\mathbf{N}}\supset\text{adjoint}\,. \tag{4.111}$$

This way we can represent tensors of rank two with one index in the fundamental representation and one index in the complex conjugate representation as square matrices. Then we can use our intuition about the rules of matrix calculus to find quantities invariant under group transformations as traces and determinants. E. g.

$$\mathrm{tr}(AB)\mapsto\mathrm{tr}(U^\dagger AUU^\dagger BU) = \mathrm{tr}(UU^\dagger AUU^\dagger B) = \mathrm{tr}(AB) \tag{4.112}$$

or

$$\det A\mapsto\det(U^\dagger AU) = \det U^\dagger\cdot\det A\cdot\det U = \det A\,. \tag{4.113}$$

In fact, this trick can be generalized to the case

$$\mathbf{N}\otimes\bar{\mathbf{M}} \tag{4.114}$$

with

$$\Psi \mapsto U_M^\dagger \Psi U_N \,, \tag{4.115}$$

where the two factors can be differenct representations of the same or even different groups. In this case the matrices are no longer square and can be combined only if the dimensions match. Unfortunately, there is no useful generalization to tensor of rank three and higher.

Not that without going into the details of the quark compostion, we can write the analogous matrix for the spin-1/2 baryons, since we assume that they also transform according to the **8** of $\mathrm{SU}(3)_F$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} . \tag{4.116}$$

4.3.9 Gell-Mann–Okubo Formula

Similar to the transformation properties of states as tensors in a representation r

$$T\,|r,i\rangle = \sum_j |r,i\rangle\,[r(T)]_{ji} \,, \tag{4.117}$$

we can also classify the transformation properties of operators and introduce *tensor operators*

$$[T, O_i^r] = \sum_j [r(T)]_{ji} O_j^r \tag{4.118}$$

in an irreducible representation r. Obviously, just as all states can be expressed as a superposition of states from irreducible representations of a symmetry group, all operators can be expressed as a sum of tensor operators.

In particular any Hamiltonian can be decomposed according to $\mathrm{SU}(3)_F$ transformation properties

$$H = H^{\mathbf{1}} + H^{\mathbf{3}} + H^{\bar{\mathbf{3}}} + H^{\mathbf{6}} + H^{\bar{\mathbf{6}}} + H^{\mathbf{8}} + H^{\mathbf{10}} + H^{\overline{\mathbf{10}}} + \ldots \,. \tag{4.119}$$

In the case of a matrix element between two states in the **8**-representation, the lowest orders in the expansion are

$$M = \langle\Psi|H|\Psi\rangle = \langle\Psi|H^{\mathbf{1}}|\Psi\rangle + \langle\Psi|H^{\mathbf{8}}|\Psi\rangle + \ldots \,, \tag{4.120}$$

since these are the lowest *real* representations of $\mathrm{SU}(3)$ and expectation values of the hamiltonian should be real. Indeed, there are singlets in

$$\mathbf{8}\otimes\mathbf{8}\otimes\mathbf{1} \qquad \mathbf{8}\otimes\mathbf{8}\otimes\mathbf{8} \,,$$

but there are no singlets in

$$\begin{array}{rr} \mathbf{8}\otimes\mathbf{8}\otimes\mathbf{3} & \mathbf{8}\otimes\mathbf{8}\otimes\bar{\mathbf{3}} \\ \mathbf{8}\otimes\mathbf{8}\otimes\mathbf{6} & \mathbf{8}\otimes\mathbf{8}\otimes\bar{\mathbf{6}} \\ \mathbf{8}\otimes\mathbf{8}\otimes\mathbf{10} & \mathbf{8}\otimes\mathbf{8}\otimes\overline{\mathbf{10}}\,. \end{array}$$

In the case of $H^{\mathbf{1}} \propto \mathbf{1}$, we need to form an invariant that is bilinear in Ψ and $\Psi^\dagger$. Due to the cyclic invariance of the trace, there is only a single such combination

$$\langle\Psi|H^{\mathbf{1}}|\Psi\rangle = 2\alpha\,\mathrm{tr}\left(\Psi^\dagger\Psi\right)\,, \tag{4.122}$$

since $\mathrm{tr}\left(\Psi^\dagger\right)\mathrm{tr}\left(\Psi\right) = 0\,.$ The normalization condition

$$\mathrm{tr}(T_aT_b) = \frac{1}{2}\delta_{ab} \tag{4.123}$$

means that this term contributes the *same* mass α to all particles in the octet.

Again by cyclic invariance of the trace, we can form only two independent $\mathrm{SU}(3)_F$-invariant combinations that are linear in Ψ, $\Psi^\dagger$ and $H_{\mathbf{8}}$ simultaneously:

$$\langle\Psi|H^{\mathbf{8}}|\Psi\rangle = \beta\,\mathrm{tr}\left(\Psi^\dagger T_8\Psi\right) + \gamma\,\mathrm{tr}\left(\Psi^\dagger\Psi T_8\right)\,, \tag{4.124}$$

where we have already used

$$H^{\mathbf{8}} \propto T_8\,, \tag{4.125}$$

because $H^{\mathbf{8}}$ should commute with all isospin generators. Note that

$$\mathrm{tr}\left(\Psi^\dagger\Psi\right)\mathrm{tr}\left(H^{\mathbf{8}}\right) = 0 = \mathrm{tr}\left(\Psi^\dagger H^{\mathbf{8}}\right)\mathrm{tr}\left(\Psi\right)\,, \tag{4.126}$$

since all matrices are traceless. Equation (4.124) is a generalization of the *Wigner–Eckhart Theorem.*

Therefore we can express all masses in the octet by just three real parameters: α, β, γ. On The Other Hand (OTOH), there are four degenerate isospin multiplets in the spin 1/2 octet: (p,n), $(\Sigma^+,\Sigma^0,\Sigma^-)$, (Ξ^0,Ξ^-), (Λ) and we should expect one non-trivial relation.

Indeed

$$\mathrm{tr}\left(\Psi^\dagger\Psi T_8\right) = \frac{1}{2\sqrt{3}}\left([\Psi^\dagger\Psi]_{11} + [\Psi^\dagger\Psi]_{22} - 2[\Psi^\dagger\Psi]_{33}\right) \tag{4.127a}$$

$$\mathrm{tr}\left(\Psi^\dagger T_8\Psi\right) = \frac{1}{2\sqrt{3}}\left([\Psi\Psi^\dagger]_{11} + [\Psi\Psi^\dagger]_{22} - 2[\Psi\Psi^\dagger]_{33}\right)\,, \tag{4.127b}$$

where (4.127a) contains sums of squares of columns of (4.116)

$$[\Psi^\dagger\Psi]_{11} = \frac{1}{2}\left|\Sigma^0 + \frac{1}{\sqrt{3}}\Lambda\right|^2 + \left|\Sigma^-\right|^2 + \left|\Xi^-\right|^2 \tag{4.128a}$$

$$[\Psi^\dagger\Psi]_{22} = \left|\Sigma^+\right|^2 + \frac{1}{2}\left|\Sigma^0 - \frac{1}{\sqrt{3}}\Lambda\right|^2 + \left|\Xi^0\right|^2 \tag{4.128b}$$

$$[\Psi^\dagger\Psi]_{33} = |p|^2 + |n|^2 + \frac{2}{3}|\Lambda|^2 \tag{4.128c}$$

and (4.127b) contains sums of squares of rows of (4.116)

$$[\Psi\Psi^\dagger]_{11} = \frac{1}{2}\left|\Sigma^0 + \frac{1}{\sqrt{3}}\Lambda\right|^2 + \left|\Sigma^+\right|^2 + |p|^2 \tag{4.129a}$$

$$[\Psi\Psi^\dagger]_{22} = \left|\Sigma^-\right|^2 + \frac{1}{2}\left|\Sigma^0 - \frac{1}{\sqrt{3}}\Lambda\right|^2 + |n|^2 \tag{4.129b}$$

$$[\Psi\Psi^\dagger]_{33} = \left|\Xi^-\right|^2 + \left|\Xi^0\right|^2 + \frac{2}{3}|\Lambda|^2 \,. \tag{4.129c}$$

Summing up, we find

$$\begin{aligned}
&2\sqrt{3}\,\langle\Psi|H^{\mathbf{8}}|\Psi\rangle \\
&= \beta\left(\left|\Sigma^0\right|^2 + \frac{1}{3}|\Lambda|^2 + \left|\Sigma^+\right|^2 + |p|^2 + \left|\Sigma^-\right|^2 + |n|^2 - 2\left|\Xi^-\right|^2 - 2\left|\Xi^0\right|^2 - \frac{4}{3}|\Lambda|^2\right) \\
&+\gamma\left(\left|\Sigma^0\right|^2 + \frac{1}{3}|\Lambda|^2 + \left|\Sigma^-\right|^2 + \left|\Xi^-\right|^2 + \left|\Sigma^+\right|^2 + \left|\Xi^0\right|^2 - 2|p|^2 - 2|n|^2 - \frac{4}{3}|\Lambda|^2\right) \\
&= \beta\left(|\Sigma|^2 + |N|^2 - 2|\Xi|^2 - |\Lambda|^2\right) + \gamma\left(|\Sigma|^2 + |\Xi|^2 - 2|N|^2 - |\Lambda|^2\right)
\end{aligned} \tag{4.130}$$

with the isospin multiplets

$$|N|^2 = |p|^2 + |n|^2 \tag{4.131a}$$

$$|\Xi|^2 = \left|\Xi^0\right|^2 + \left|\Xi^-\right|^2 \tag{4.131b}$$

$$|\Sigma|^2 = \left|\Sigma^+\right|^2 + \left|\Sigma^0\right|^2 + \left|\Sigma^-\right|^2 \,. \tag{4.131c}$$

We can now read off the masses

$$m_N = \alpha + \frac{\beta - 2\gamma}{2\sqrt{3}} \tag{4.132a}$$

$$m_\Xi = \alpha + \frac{\gamma - 2\beta}{2\sqrt{3}} \tag{4.132b}$$

$$m_\Sigma = \alpha + \frac{\beta + \gamma}{2\sqrt{3}} \tag{4.132c}$$

$$m_\Lambda = \alpha - \frac{\beta + \gamma}{2\sqrt{3}} \,. \tag{4.132d}$$

From this, we can read off the *Gell-Mann–Okubo relation*

$$2m_N + 2m_\Xi = 3m_\Lambda + m_\Sigma\,, \tag{4.133}$$

since both sides are equal to $-(\beta+\gamma)/\sqrt{3}$.

Solving (4.133) for m_Λ and inserting the current (2014) values from table 4.2, we find

$$m_\Lambda = \frac{1}{3}\left(2m_N + 2m_\Xi - m_\Sigma\right) = 1107\,\text{GeV}\,, \tag{4.134}$$

which is within 1% of the experimental result. This is unreasonably good, since we have normalized with the large common contribution from the $\text{SU}(3)_F$ singlet $H^\mathbf{1}$ of about 1 GeV. Nevertheless, even if we look only at the mass splittung

$$m_{\Sigma^0} - m_\Lambda = \begin{cases} 77\,\text{MeV} & \text{experiment} \\ 86\,\text{MeV} & \text{Gell-Mann–Okubo (4.134)} \end{cases}, \tag{4.135}$$

our prediction is only 11% off, which is *very* good for a symmetry argument without *any* dynamical assumptions.

In chapter 8 below, we will find a dynamical explanation for the success of the symmetry argument in Quantum Chromo Dynamics (QCD), the theory of strong interactions. In QCD, there is a $\text{SU}(N_f)_F$ flavor symmetry for N_f flavors of quarks. This symmetry is only broken by (small) quark masses

$$m_s > m_d \approx m_u \tag{4.136}$$

and the bulk of the hadron masses is generated dynamically.

Isospin multiplet	$m_{\text{avg.}}/\text{MeV}$	m/MeV
$N = (p, n)$	939	$(938.272, 939.565)$
$(\Sigma^+, \Sigma^0, \Sigma^-)$	1193	$(1189.4, 1192.6, 1197.4)$
(Ξ^0, Ξ^-)	1318	$(1315, 1321.7)$
(Λ)	1116	(1115.68)

Table 4.2: *Masses of the* SU(3) *octet of baryons from* `http://pdg.lbl.gov`.

4.4 Heavy Quarks

There are three more quarks: *charm* (c), *bottom* (b) and *top* (t), that are (much) heavier than the three discussed so far. The top quark decays so fast, that it can not form bound states.

The bound states of *charm* and *bottom* bottom quarks can be organized in $\mathrm{SU}(4)_F$ and $\mathrm{SU}(5)_F$ multiplets, but these symmetries are so badly broken, that little quantitative predictions can be obtained.

4.4.1 Quarkonia

It turns out that QCD (see chapter 8) becomes weakly interacting at high energies and the binding energies are much smaller than the masses of the heavy quarks. This means that we can compute the spectra of $c\bar{c}$ and $b\bar{b}$ bound states, the so called *charmonium* and *bottomonium* states with *non-relativistic* Schroedinger wave functions from a potential

$$V_{q\bar{q}}(r) = \frac{\alpha_S}{|\vec{r}_q - \vec{r}_{\bar{q}}|} + \ldots\,, \tag{4.137}$$

similar to positronium. The results of these calculations have been confirmed with spectactular success in the spectroscopy of such states.

—5—
Gauge Theories

Lecture 10: Wed, 13. 05. 2015

5.1 Basics of Quantum Field Theory

Quantum Field Theory (QFT) plays a dual role:

- "quantum mechanics" of classical field theory, e. g. quantized radiation field in quantum electrodynamics
- quantum mechanics for (infinitely) many particles with creation and annihilation

are described by the *same* formalism.

5.1.1 Classical Field Theory

Configuration space: linear space of all functions ϕ

$$\begin{aligned} \phi : \mathbf{M} = \mathbf{R}^4 &\to \mathbf{C} \\ x &\mapsto \phi(x) \end{aligned} \tag{5.1}$$

or rather of all distributions, since we often encounter singularities, e. g. in the Coulomb potential of point charges.

The dynamics of the fields ϕ is governed by second order Partial Differential Equations (PDEs), e. g. the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0 \tag{5.2}$$

with appropriate Cauchy data for $\phi(x)$ and $\partial_0\phi(x)$ on a spacelike hypersurface, e. g. $x_0 = 0$.

Action Principle, Euler-Lagrange-Equations

Since the study of coupled nonlinear PDEs is complicated and in particular symmetries are not manifest for multi-component fields, it helps to derive the equation of motion from an action principle:

$$\delta S(\phi_1,\ldots,\phi_n) = \sum_{i=1}^{n} \int \mathrm{d}^4x \, \frac{\delta S}{\delta\phi_i}(\phi_1,\ldots\phi_n,x)\delta\phi_i(x) = 0 \tag{5.3}$$

for all variations $\{\delta\phi_i\}_{i=1,\ldots,n}$ and therefore

$$\frac{\delta S}{\delta\phi_i}(\phi_1,\ldots\phi_n,x) = 0\,. \tag{5.4}$$

For example the local action for a real field ϕ

$$S(\phi) = \int \mathrm{d}^4x \, \mathcal{L}\left(\phi(x),\partial_\mu\phi(x)\right) \tag{5.5}$$

with *Lagrangian*

$$\mathcal{L}\left(\phi(x),\partial_\mu\phi(x)\right) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)\phi(x) - V(\phi(x)) \tag{5.6}$$

leads to

$$\begin{aligned}
0 \stackrel{!}{=} \delta S(\phi) &= \int \mathrm{d}^4x \, \frac{\delta S}{\delta\phi}(\phi,x)\delta\phi(x) \\
&= \int \mathrm{d}^4x \, \left[\frac{\partial\mathcal{L}}{\partial\phi}(\phi(x),\partial_\mu\phi(x))\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(\phi(x),\partial_\mu\phi(x))\delta\partial_\mu\phi(x)\right] \\
&= \int \mathrm{d}^4x \, \left[\frac{\partial\mathcal{L}}{\partial\phi}(\phi(x),\partial_\mu\phi(x))\delta\phi(x) - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(\phi(x),\partial_\mu\phi(x))\delta\phi(x)\right] \\
&= \int \mathrm{d}^4x \, \left[\frac{\partial\mathcal{L}}{\partial\phi}(\phi(x),\partial_\mu\phi(x)) - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}(\phi(x),\partial_\mu\phi(x))\right]\delta\phi(x) \\
&= \int \mathrm{d}^4x \, \left[-\Box\phi(x) - m^2\phi(x) - V'(\phi(x))\right]\delta\phi(x)\,,
\end{aligned} \tag{5.7}$$

i. e.

$$\left(\Box + m^2\right)\phi = -V'(\phi) \tag{5.8}$$

or for a general Lagrangian

$$\frac{\partial\mathcal{L}}{\partial\phi_i}(\phi_1(x),\ldots,\phi_n(x),\partial_\mu\phi_1(x),\ldots,\partial\phi_n(x))$$

$$- \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}(\phi_1(x), \ldots, \phi_n(x), \partial_\mu \phi_1(x), \ldots, \partial \phi_n(x)) = 0 \,. \quad (5.9)$$

All interesting field equations are second order in time and space, since higher orders lead to problems with causality. The second order field equations have to be combined with Cauchy data for the fields $\{\phi_i(x)\}_{i=1\ldots,n}$ and their first time derivatives $\{\partial_0 \phi_i(x)\}_{i=1\ldots,n}$ on a space-like hypersurface ("Cauchy surface").

5.1.2 *Quantum Field Theory*

In QFT, the classical fields $\{\phi_i(x)\}_{i=1\ldots,n} : \mathbf{M} \to \mathbf{R}$ are promoted to field operators and it can be shown that *all* observables, in particular cross sections an decay rates, can be recovered from vacuum expectation values (vevs) of time-ordered products of field operators, a.k.a. *greensfunctions*

$$\langle 0| \mathrm{T}\left(\phi_{i_1}(x_1)\phi_{i_2}(x_2)\cdots\phi_{i_n}(x_n)\right)|0\rangle \,, \quad (5.10)$$

where

$$\mathrm{T}\phi_i(x)\phi_j(y) = \Theta(x^0 - y^0)\phi_i(x)\phi_j(y) \pm \Theta(y^0 - x^0)\phi_j(y)\phi_i(x) \,, \quad (5.11)$$

i. e. fields are ordered from right to left according to time. The negative sign appears iff both fields are fermions. This time ordering is familiar from time-dependent perturbation theory in non-relativistic QM:

$$U(t, t_0) = \mathrm{T}\left(\mathrm{e}^{-\mathrm{i}\int_{t_0}^t \mathrm{d}t' H_I(t')}\right) \,. \quad (5.12)$$

Generating Functionals

Compact expression containing *all* greensfunctions of interacting (Heisenberg) fields of a theory

$$\begin{aligned} Z : C^\infty(\mathbf{R}^4) &\to \mathbf{C} \\ j &\mapsto Z(j) = \langle 0| \mathrm{T}[\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \phi(x)j(x)}]|0\rangle \end{aligned} \quad (5.13)$$

such that

$$\langle 0| \mathrm{T}[\phi(x_1)\ldots\phi(x_n)]|0\rangle = \lim_{j\to 0} \frac{\delta}{\mathrm{i}\delta j(x_1)} \cdots \frac{\delta}{\mathrm{i}\delta j(x_n)} Z(j) \quad (5.14)$$

with obvious generalization for more than one field:

$$\begin{aligned} Z : \left(C^\infty(\mathbf{R}^4)\right)^{\otimes n} &\to \mathbf{C} \\ (j_1, \ldots, j_n) &\mapsto Z(j_1, \ldots, j_n) = \langle 0| \mathrm{T}[\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \sum_{i=1}^n \phi_i(x)j_i(x)}]|0\rangle \,. \end{aligned} \quad (5.15)$$

For a free scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{5.16}$$

we can compute the 2-point greensfunction exactly

$$\langle 0|\, \mathrm{T}[\phi(x)\phi(y)]|0\rangle = -\mathrm{i}G_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-y)}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon} \tag{5.17}$$

and we find a *closed* expression for the generating functional:

$$Z^0(j) = \mathrm{e}^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)G_F(x-y)j(y)}\,. \tag{5.18}$$

E. g.

$$\begin{aligned}
\langle 0|\, \mathrm{T}[\phi(x_1)]|0\rangle &= \lim_{j\to 0}\frac{\delta}{\mathrm{i}\delta j(x_1)}Z(j) && (5.19\mathrm{a})\\
&= \lim_{j\to 0}\int \mathrm{d}^4x_2\, G_F(x_1-x_2)j(x_2) = 0 &&\\
\langle 0|\, \mathrm{T}[\phi(x_1)\phi(x_2)]|0\rangle &= -\mathrm{i}G_F(x_1-x_2) && (5.19\mathrm{b})\\
\langle 0|\, \mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)]|0\rangle &= 0 && (5.19\mathrm{c})\\
\langle 0|\, \mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle &= -G_F(x_1-x_2)G_F(x_3-x_4) &&\\
&\quad - G_F(x_1-x_3)G_F(x_2-x_4) &&\\
&\quad - G_F(x_1-x_4)G_F(x_2-x_3) && (5.19\mathrm{d})\\
&\dots && (5.19\mathrm{e})\\
\langle 0|\, \mathrm{T}[\phi(x_1)\dots\phi(x_{2n+1})]|0\rangle &= 0 && (5.19\mathrm{f})\\
&\dots && (5.19\mathrm{g})
\end{aligned}$$

5.1.3 Pathintegral and Feynman Rules

The generating functional of all greensfunctions can be expressed as an integral over all field configurations[1] that are compatible with the boundary conditions in the past and in the future:

$$Z(j) = \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S(\varphi)+\mathrm{i}\int \mathrm{d}^4x\,\varphi(x)j(x)}\,. \tag{5.20}$$

Separating the action into a free and interaction part

$$S = S_0 + S_I\,, \tag{5.21}$$

[1]A mathematically rigorous definition of the integration measure $\mathcal{D}\varphi$ in (5.20) is *not* trivial and has so far only been achieved in $2+1$ space-time dimensions.

we can write

$$Z(j) = \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_I(\varphi)} \mathrm{e}^{\mathrm{i}S_0(\varphi)+\mathrm{i}\int \mathrm{d}^4x\, \varphi(x)j(x)} \tag{5.22}$$

and using

$$\mathrm{T}\left(\phi(x)\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x'\, \phi(x')j(x')}\right) = \frac{\delta}{\mathrm{i}\delta j(x)} \mathrm{T}\left(\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x'\, \phi(x')j(x')}\right), \tag{5.23}$$

we can pull out the interaction

$$\begin{aligned} Z(j) &= \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} \mathrm{e}^{\mathrm{i}S_0(\varphi)+\mathrm{i}\int \mathrm{d}^4x\, \varphi(x)j(x)} \\ &= \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_0(\varphi)+\mathrm{i}\int \mathrm{d}^4x\, \varphi(x)j(x)} = \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} Z^0(j), \end{aligned} \tag{5.24}$$

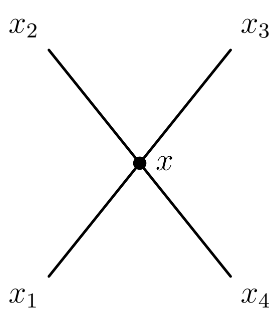
with Z^0 the generating functional of free or non-interacting greensfunctions.

With (5.24), we have given a *formal*, highly non-rigorous, derivation of the Feynman rules for the perturbative computation of greensfunctions.

For $V(\phi) = g\phi^4/4!$ with $\lim_{j\to 0}$ implied and "disc." refering to disconnected diagrams we find

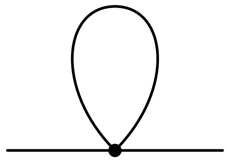
1.

$$\begin{aligned} \langle 0|\, \mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle &= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_4)} Z(j) \\ &= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_4)} \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} Z^0(j) \\ = \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_4)} &\mathrm{i} \int \mathrm{d}^4x\, \frac{g}{4!} \left(\frac{\delta}{\mathrm{i}\delta j(x)}\right)^4 Z^0(j) + \mathcal{O}(g^2) + \text{disc.} \\ = \mathrm{i} \int \mathrm{d}^4x\, g G_F(x_1-x) G_F(x_2-x) &G_F(x_3-x) G_F(x_4-x) + \mathcal{O}(g^2) + \text{disc.} \end{aligned} \tag{5.25}$$



2.

$$\langle 0|\,\mathrm{T}[\phi(x_1)\phi(x_2)]|0\rangle = \frac{\delta}{\mathrm{i}\delta j(x_1)}\frac{\delta}{\mathrm{i}\delta j(x_2)}Z(j) = \frac{\delta}{\mathrm{i}\delta j(x_1)}\frac{\delta}{\mathrm{i}\delta j(x_2)}\mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})}Z^0(j)$$
$$= \frac{\delta}{\mathrm{i}\delta j(x_1)}\frac{\delta}{\mathrm{i}\delta j(x_2)}\mathrm{i}\int \mathrm{d}^4x\,\frac{g}{4!}\left(\frac{\delta}{\mathrm{i}\delta j(x)}\right)^4 Z^0(j) + \mathcal{O}(g^2)$$
$$= \mathrm{i}\int \mathrm{d}^4x\,\frac{g}{2}G_F(x_1-x)G_F(x_2-x)G_F(x-x) + \mathcal{O}(g^2) \quad (5.26)$$



NB: $G_F(x-x) = G_F(0)$ is not well defined and leads to divergencies in perturbation theory, whose proper treatment is the subject of renormalization theory.

5.2 Gauge Invariant Actions

5.2.1 Global Transformations

Given a symmetry group G and a finite dimensional representation R, we can easily construct invariant actions for multiplets of fields transforming under this representation

$$U(\alpha)\in G: \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\\ \dots\\ \phi_n(x)\end{pmatrix} \mapsto \begin{pmatrix}\phi'_1(x)\\ \phi'_2(x)\\ \dots\\ \phi'_n(x)\end{pmatrix} = R(U(\alpha))\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\\ \dots\\ \phi_n(x)\end{pmatrix} \quad (5.27)$$

or, in components,

$$\phi_i(x)\mapsto\phi'_i(x) = \sum_{j=1}^{n}\left[R(U(\alpha))\right]_{ij}\phi_j(x) \quad (5.28)$$

or, combining the components to vectors,

$$\phi(x)\mapsto\phi'(x) = R(U(\alpha))\phi(x) \quad (5.29)$$

or, if there's no danger of mistaking the group for the representation,

$$\phi(x)\mapsto\phi'(x) = U(\alpha)\phi(x)\,. \quad (5.30)$$

However, while R is in many cases the defining representation, there are important examples for other representations in particle physics.

Parametrizing the group elements

$$U(\alpha) = \mathrm{e}^{\mathrm{i}T_a\alpha_a} = \mathrm{e}^{\mathrm{i}\alpha} \tag{5.31}$$

with $\{T_a\}$ a basis of the corresponding Lie algebra, we can often concentrate on infinitesimal transformations:

$$\phi(x) \mapsto \phi'(x) = \phi(x) + \delta\phi(x) \tag{5.32}$$

with

$$\delta\phi_i(x) = \mathrm{i}\sum_a \sum_{j=1}^n \alpha_a \left[r(T_a)\right]_{ij} \phi_j(x) = \mathrm{i}\sum_{j=1}^n \left[r(\alpha)\right]_{ij} \phi_j(x) \tag{5.33}$$

or

$$\delta\phi(x) = \mathrm{i}\sum_a \alpha_a r(T_a)\phi(x) = \mathrm{i}r(\alpha)\phi(x) \tag{5.34}$$

or

$$\delta\phi(x) = \mathrm{i}\sum_a \alpha_a T_a\phi(x) = \mathrm{i}\alpha\phi(x)\,. \tag{5.35}$$

Mass terms in a complex unitary representation,

$$\phi^\dagger(x)\phi(x) = \sum_{i=1}^n \phi_i^*(x)\phi_i(x) \tag{5.36}$$

and in a real orthogonal representation

$$\phi^T(x)\phi(x) = \sum_{i=1}^n \phi_i(x)\phi_i(x)\,, \tag{5.37}$$

are obviously invariant:

$$\phi^\dagger(x)\phi(x) \mapsto (\phi')^\dagger(x)\phi'(x) = \phi^\dagger(x)\phi(x) \tag{5.38}$$

and

$$\phi^T(x)\phi(x) \mapsto (\phi')^T(x)\phi'(x) = \phi^T(x)\phi(x)\,. \tag{5.39}$$

Since

$$\partial_\mu\phi'(x) = \partial_\mu\left(R(U(\alpha))\phi(x)\right) = R(U(\alpha))\partial_\mu\phi(x) \tag{5.40}$$

derivatives transform just like the fields and kinetic terms are invariant as well. Using this we can easily write invariant Lagrangians

$$\mathcal{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\phi^\dagger\phi - P(\phi^\dagger\phi) \tag{5.41}$$

and

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^T(\partial^\mu \phi) - \frac{m^2}{2}\phi^T\phi - P\left(\phi^T\phi\right) . \tag{5.42}$$

There are of course many more interaction terms, e. g.

$$\mathcal{L}_{333} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\phi_i\phi'_j\phi''_k \tag{5.43a}$$

$$\mathcal{L}_{\bar{2}23} = \sum_{i,j=1}^{2}\sum_{k=1}^{3} \sigma^k_{ij}\psi^*_i\psi'_j\phi_k \tag{5.43b}$$

$$\mathcal{L}_{333} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\chi_i\chi'_j\chi'_k \tag{5.43c}$$

$$\ldots \tag{5.43d}$$

for the ϕ in SU(2) triplets, the ψ in SU(2) dublets and the χ in SU(3) triplets.

5.2.2 Noether's Theorem

Lecture 11: Tue, 19. 05. 2015

Assume that an action S is invariant under a field dependent transformation with infinitesimal generators

$$\delta\phi(x) = \Delta(\phi)(x) \tag{5.44a}$$

$$\delta S(\phi) = 0\,, \tag{5.44b}$$

where ϕ in general denotes a multiplet of fields. For internal symmetries we will always have

$$\Delta(\phi) = \Delta' \circ \phi\,, \tag{5.45}$$

but for space-time symmetries, $\Delta(\phi)$ must be allowed to depend on values of the field ϕ at other space time points

$$\Delta(\phi) = \Delta' \circ \phi \circ f^{-1}\,, \tag{5.46}$$

with $f : \mathbf{M} \to \mathbf{M}$. Note, that an active transformation of coordinates

$$\begin{aligned} f : \mathbf{M} &\to \mathbf{M} \\ x &\mapsto x' = f(x) \end{aligned} \tag{5.47}$$

corresponds to a transformation of fields

$$\begin{aligned} f^* : C^\infty(\mathbf{M}) &\to C^\infty(\mathbf{M}) \\ \phi &\mapsto f^*(\phi) = \phi \circ f^{-1}\,, \end{aligned} \tag{5.48}$$

in order to keep the observable values of the field invariant

$$\phi(x) \mapsto (\phi \circ f^{-1})(f(x)) = \phi(f^{-1}(f(x)) = \phi(x) \,. \tag{5.49}$$

In any case, in this section we will *exclude* space-time dependence of the form

$$\Delta(\phi, x) \tag{5.50}$$

that can not be factored like (5.46).

Note that the invariance (5.44b) of the action S means that the Lagrangian $\mathcal{L}$ transforms into a total derivative

$$\delta\mathcal{L}(\phi) = \partial_\mu \sigma^\mu_\delta(\phi) \,. \tag{5.44b'}$$

First consider the variation of $\mathcal{L}$ under an *arbitrary* transformation $\delta\phi$

$$\begin{aligned}\delta\mathcal{L} &= \sum_i \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\partial_\mu\phi_i \\ &= \sum_i \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right)}_{=0 \text{ by equations of motion}} \delta\phi_i + \partial_\mu \left(\sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) \,, \end{aligned} \tag{5.51}$$

i. e. along the solutions of the equations of motion we have

$$\delta\mathcal{L}(\phi) = \partial_\mu \left(\sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) \,. \tag{5.52}$$

Since OTOH for a symmetry transformation $\delta\mathcal{L} = \partial_\mu\sigma^\mu_\delta$ (5.44b'), we must have

$$\partial_\mu \left(\sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\right) = \partial_\mu\sigma^\mu_\delta \,, \tag{5.53}$$

or *current conservation*

$$\partial_\mu j^\mu_\delta = 0 \tag{5.54}$$

for the *conserved current*

$$j^\mu_\delta = \sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i - \sigma^\mu_\delta \,. \tag{5.55}$$

Space-Time Symmetries

Translations $x^\mu \mapsto x'^\mu_\epsilon = X^\mu - \epsilon^\mu$, i. e.

$$\delta_\epsilon x^\mu = -\epsilon^\mu \tag{5.56}$$

or

$$\delta_\epsilon \phi(x) = \epsilon^\mu \partial_\mu \phi(x) \tag{5.57}$$

and, of course

$$\delta_\epsilon \mathcal{L} = \epsilon^\mu \partial_\mu \mathcal{L} \,. \tag{5.58}$$

Therefore

$$\sigma^\mu_\epsilon = \epsilon^\mu \mathcal{L} \tag{5.59}$$

and we find conserved currents

$$j^\mu_\epsilon = \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \epsilon^\nu \partial_\nu \phi_i - \epsilon^\nu \mathcal{L} \,. \tag{5.60}$$

Introduzing unit vectors

$$\nu = 0, 1, 2, 3 : \epsilon^\mu_{(\nu)} = \delta^\mu_\nu \,, \tag{5.61}$$

we find the conserved *energy momentum tensor*

$$T^{\mu\nu} := j^\mu_{\epsilon_{(\nu)}} = \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \partial^\nu \phi_i - g^{\mu\nu} \mathcal{L} \tag{5.62}$$

with

$$\partial_\mu T^{\mu\nu} = 0 \,. \tag{5.63}$$

The name is justified by noticing that energy and momentum are the integrals over space

$$E = \int \mathrm{d}^3 x \, T^{00} = H \tag{5.64a}$$

$$P^i = \int \mathrm{d}^3 x \, T^{0i} \,. \tag{5.64b}$$

Analogously, one can obtains the conserved charges generating the Lorentz group by studying infinitesimal Lorentz transformations

$$\delta_\omega \phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi \tag{5.65}$$

as discussed in the exercises.

Internal Symmetries

If the Lagrangian in invariant under an internal symmetry

$$\delta\phi(x) = \Delta(\phi(x)) \tag{5.66}$$

that commutes with translations and Lorentz translations, the term σ^μ_δ in absent from (5.44b') and (5.55)

$$j^\mu_\delta = \sum_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i\,. \tag{5.67}$$

In particular, if we have a Lie algebra representation r

$$\delta_a\phi(x) = \mathrm{i}r(T_a)\phi(x) \tag{5.68}$$

or in components

$$\delta_a\phi_i(x) = \mathrm{i}\sum_j [r(T_a)]_{ij}\phi_j(x)\,, \tag{5.69}$$

we find

$$j^\mu_a = \mathrm{i}\sum_{ij} \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}[r(T_a)]_{ij}\phi_j\,. \tag{5.70}$$

U(1) *Examples*

- The Lagrangian of a free complex scalar

  $$\mathcal{L} = \overline{\partial_\mu\phi}\partial^\mu\phi - m^2\overline{\phi}\phi \tag{5.71}$$

  is invariant under U(1) transformations

  $$\phi \mapsto \mathrm{e}^{\mathrm{i}\alpha}\phi \tag{5.72a}$$

  $$\overline{\phi} \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\overline{\phi} \tag{5.72b}$$

  with

  $$\delta_\alpha\phi = \mathrm{i}\alpha\phi \tag{5.73a}$$

  $$\delta_\alpha\overline{\phi} = -\mathrm{i}\alpha\overline{\phi}\,. \tag{5.73b}$$

  Therefore

  $$-\alpha j^\mu_\alpha = \overline{\partial^\mu\phi}\mathrm{i}\alpha\phi - \mathrm{i}\alpha\overline{\phi}\partial^\mu\phi =: -\mathrm{i}\alpha\left(\overline{\phi}\overleftrightarrow{\partial^\mu}\phi\right) \tag{5.74}$$

  is the corresponding conserved Noether current.

- The Lagrangian of a free spin-1/2 field

  $$\mathcal{L} = \bar\psi\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi \tag{5.75}$$

  is also invariant under U(1) transformations with Noether current

  $$\alpha j^\mu_\alpha = \bar\psi\mathrm{i}\alpha\gamma^\mu\psi = \mathrm{i}\alpha\bar\psi\gamma^\mu\psi\,. \tag{5.76}$$

5.2.3 Local Transformations

Considering local, x-dependent, transformations with

$$U(x) = \mathrm{e}^{\mathrm{i}T_a\alpha_a(x)} = \mathrm{e}^{\mathrm{i}\alpha(x)} \tag{5.77}$$

we find that derivatives no longer transform covariantly

$$\begin{aligned}\partial_\mu\phi'(x) = \partial_\mu\left(U(x)\phi(x)\right) &= U(x)\partial_\mu\phi(x) + \partial_\mu U(x)\phi(x) \\ &= U(x)\left[\partial_\mu + U^{-1}(x)\left(\partial_\mu U(x)\right)\right]\phi(x)\,. \end{aligned}\tag{5.78}$$

According to

$$\begin{aligned}U^{-1}(x)\partial_\mu U(x) &= \mathrm{e}^{-\mathrm{i}\alpha(x)}\partial_\mu\mathrm{e}^{\mathrm{i}\alpha(x)} = \mathrm{e}^{-\mathrm{i}[\alpha(x),\cdot]}\partial_\mu = \mathrm{e}^{-\mathrm{i}\mathrm{ad}_{\alpha(x)}}\partial_\mu \\ &= \partial_\mu - \mathrm{i}[\alpha(x),\partial_\mu] - \frac{1}{2!}[\alpha(x),[\alpha(x),\partial_\mu]] + \ldots \\ &= \partial_\mu + \mathrm{i}\partial_\mu\alpha(x) + \frac{1}{2!}[\alpha(x),\partial_\mu\alpha(x)] - \frac{\mathrm{i}}{3!}[\alpha(x),[\alpha(x),\partial_\mu\alpha(x)]] + \ldots \\ &= \partial_\mu + U^{-1}(x)\left(\partial_\mu U(x)\right)\,, \end{aligned}\tag{5.79}$$

the additional term is composed of multiple commutators of generators and their derivatives. Therefore it is defined *in* the *Lie algebra* representation and can be cancelled by a field in the same Lie algebra representation!

5.2.4 Covariant Derivative

Define a *covariant derivate*

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu(x) \tag{5.80}$$

such that

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu(x) \to D'_\mu = U(x)D_\mu U^{-1}(x) = \partial_\mu - \mathrm{i}A'_\mu(x) \tag{5.81}$$

and demand the transformation property of the Lie algebra valued *connection*

$$A_\mu(x) = T_a A^a_\mu(x) \tag{5.82}$$

accordingly

$$\begin{aligned}\partial_\mu - \mathrm{i}A'_\mu(x) = U(x)\left(\partial_\mu - \mathrm{i}A_\mu(x)\right)U^{-1}(x) &= U(x)\partial_\mu U(x) - \mathrm{i}U(x)A_\mu(x)U^{-1}(x) \\ &= \partial_\mu + U(x)\left(\partial_\mu U^{-1}(x)\right) - \mathrm{i}U(x)A_\mu(x)U^{-1}(x) \end{aligned}\tag{5.83}$$

i. e.

$$\begin{aligned} A_\mu(x) \to A'_\mu(x) &= U(x)A_\mu(x)U^{-1}(x) + \mathrm{i}U(x)\left(\partial_\mu U^{-1}(x)\right) \\ &= A_\mu(x) + \mathrm{i}[\alpha(x), A_\mu(x)] - \frac{1}{2!}[\alpha(x), [\alpha(x), A_\mu(x)]] + \ldots \\ &+ \partial_\mu\alpha(x) + \frac{\mathrm{i}}{2!}[\alpha(x), \partial_\mu\alpha(x)] - \frac{1}{3!}[\alpha(x), [\alpha(x), \partial_\mu\alpha(x)]] + \ldots \end{aligned} \tag{5.84}$$

NB: more precisely, D_μ depends on the representation

$$D^r_\mu = \partial_\mu - \mathrm{i}r(A_\mu(x)) \tag{5.85}$$

e. g.

$$D^{\text{adj.}}_\mu = \partial_\mu - \mathrm{i}[A_\mu(x), \cdot] = \partial_\mu - \mathrm{i}A^a_\mu(x)[T_a, \cdot] \tag{5.86}$$

and in

$$D^r_\mu = \partial_\mu - \mathrm{i}r(A_\mu(x)) \to D^{r\prime}_\mu = R(U(x))D^r_\mu R(U^{-1}(x)) = \partial_\mu - \mathrm{i}r(A'_\mu(x)) \tag{5.87}$$

the representations r and R must match. However, by Hausdorff's formula,

$$A_\mu(x) \to A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + \mathrm{i}U(x)\left(\partial_\mu U^{-1}(x)\right) \tag{5.88}$$

is representation independent and we can use the *same* gauge connection for *all* representations.

NB: for the special case of abelian transformations

$$[\alpha(x), \alpha'(x)] = [\alpha(x), \partial_\mu\alpha'(x)] = 0 \tag{5.89}$$

we find

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x) \tag{5.90}$$

to *all orders*, i. e. the gauge transformations of electrodynamics.

D_μ is called a *covariant derivative*, because it transforms as an adjoint

$$r(D_\mu) \to r(D'_\mu) = R(U(x))r(D_\mu)R(U^{-1}(x)) \tag{5.91}$$

and we find

$$\begin{aligned} &r(D_\mu)\phi_R(x) \to r(D'_\mu)\phi'_R(x) \\ &= R(U(x))r(D_\mu)R(U^{-1}(x))R(U(x))\phi_R(x) = R(U(x))r(D_\mu)\phi_R(x) \end{aligned} \tag{5.92}$$

iff the representations r and R match.

If we introduce the convention that *the appropriate representation is implied, depending on which field D_μ is acting*, we can drop r and R consistently in

$$D_\mu \to D'_\mu = U(x) D_\mu U^{-1}(x) \tag{5.93}$$

and

$$D_\mu \phi(x) \to D'_\mu \phi'(x) = U(x) D_\mu U^{-1}(x) U(x) \phi(x) = U(x) D_\mu \phi(x) \,. \tag{5.94}$$

We will adapt this convention from now on!

This way we can easily write invariant Lagrangians for matter fields

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - P(\phi^\dagger \phi) \,, \tag{5.95}$$

but the connection $A_\mu(x)$ is still an external field. We need dynamics for it.

5.2.5 Field Strength

The Ricci identity

$$F_{\mu\nu} = \mathrm{i} \left[D_\mu, D_\nu \right] = F^a_{\mu\nu} T_a \tag{5.96}$$

can be used to *define* a new object $F_{\mu\nu}$, *en detail*

$$\begin{aligned} F_{\mu\nu} = \mathrm{i} \left[\partial_\mu - \mathrm{i} A_\mu, \partial_\nu - \mathrm{i} A_\nu \right] &= \mathrm{i} \left[\partial_\mu, \partial_\nu \right] + \left[\partial_\mu, A_\nu \right] + \left[A_\mu, \partial_\nu \right] - \mathrm{i} \left[A_\mu, A_\nu \right] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} \left[A_\mu, A_\nu \right] \,, \end{aligned} \tag{5.97}$$

that transforms like an adjoint

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \tag{5.98}$$

because

$$\left[D_\mu, D_\nu \right] \to \left[D'_\mu, D'_\nu \right] = \left[U D_\mu U^{-1}, U D_\nu U^{-1} \right] = U \left[D_\mu, D_\nu \right] U^{-1} \tag{5.99}$$

Finally

$$F_{\mu\nu} F^{\mu\nu} \to U F_{\mu\nu} F^{\mu\nu} U^{-1} \tag{5.100}$$

and by the cyclic invariance of the trace

$$\mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu} \right) \to \mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu} \right) \tag{5.101}$$

we find a viable candidate for a Lagrangian for A_μ

$$\mathcal{L} = -\frac{1}{2} \mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu} \right) = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \tag{5.102}$$

independent of the representation with normalization fixed by

$$\mathrm{tr} \left(T_a T_b \right) = \frac{1}{2} \delta_{ab} \,. \tag{5.103}$$

5.2.6 Building Blocks

Lecture 12: Wed, 20. 05. 2015

This way, gauge theory lagrangians are like Lego bricks: just plug matching blocks together so that pairs of $U^{-1}(x)$ and $U(x)$ cancel:

$$\phi, D_\mu, \psi, \not{D}, F_{\mu\nu}\,, \tag{5.104}$$

where the covariant derivative for fermions acts in the tensor product of Dirac spinors and gauge group representation

$$\not{D} = \mathbf{1}_R \otimes \gamma^\mu \partial_\mu - \mathrm{i} r(A_\mu(x)) \otimes \gamma^\mu = r(D_\mu) \otimes \gamma^\mu \tag{5.105}$$

Typical terms are for bosons

$$\phi^\dagger \cdots D_\mu \cdots F_{\rho\sigma} \cdots \phi\,, \tag{5.106a}$$

fermions

$$\bar\psi \cdots D_\mu \cdots F_{\rho\sigma} \cdots \gamma_\nu \cdots \psi \tag{5.106b}$$

and gauge bosons

$$\mathrm{tr}(F_{\mu\nu} \cdots D_\lambda \cdots F_{\rho\sigma}) \tag{5.106c}$$

but more complicated structures like

$$\sum_{abc} C^{abc} \left(\phi^T T_a D_\mu \phi\right) \left(\phi^T T_b D_\nu \phi\right) \left(\bar\psi T_c F^{\mu\nu} \not{D} \psi\right) \tag{5.107}$$

are also possible.

Note that due to (5.96), of the three combinations

$$F_{\mu\nu},\ D_\mu D_\nu,\ D_\nu D_\mu \tag{5.108}$$

only two are independent!

5.3 Quantization

5.3.1 Perturbative Expansion

So far, we have no small parameter in our action, that would allow a perturbative expansion. Therefore, we perform a *simultaneous* rescaling of our gauge potential A_μ and gauge lagrangian

$$D_\mu = \partial_\mu - \mathrm{i} A_\mu \to \partial_\mu - \mathrm{i} g A_\mu \tag{5.109a}$$

$$F_{\mu\nu} \to gF_{\mu\nu} = g\left(\partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu, A_\nu\right]\right) \tag{5.109b}$$

$$\mathrm{tr}\left(\cdots\right) \to \frac{1}{g^2}\,\mathrm{tr}\left(\cdots\right)\,. \tag{5.109c}$$

Then

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} \tag{5.110}$$

with summation over the adjoint representation index a implied and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\,. \tag{5.111}$$

Therefore, we can collect terms according to the powers of g

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \tag{5.112}$$

with

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a,\nu} - \partial^\nu A^{a,\mu}\right)\\ &= -\frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\partial^\mu A^{a,\nu} \qquad (5.113a)\\ \mathcal{L}_1 &= -\frac{g}{2}f^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)A^{b,\mu}A^{c,\nu}\\ &= -gf^{abc}\partial_\mu A^a_\nu A^{b,\mu}A^{c,\nu} \qquad (5.113b)\\ \mathcal{L}_2 &= \frac{g^2}{4}f^{abc}f^{ab'c'}A^b_\mu A^c_\nu A^{b',\mu}A^{c',\nu} \qquad (5.113c)\end{aligned}$$

5.3.2 Propagator

We obtain the greensfunctions by "inverting" the free equations of motion

$$\frac{\delta}{\delta A^a_\nu}\mathcal{L}_0 = \partial^\mu\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) = \left(\Box g_{\nu\mu} - \partial_\nu\partial_\mu\right)A^{a,\mu} \stackrel{!}{=} 0\,. \tag{5.114}$$

Making the *ansatz*

$$D_{\mu\nu}(x) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\tilde D_{\mu\nu}(k)\mathrm{e}^{-\mathrm{i}kx} \tag{5.115}$$

we find

$$\left(\Box g_{\rho\mu} - \partial_\rho\partial_\mu\right)D^\mu{}_\nu(x) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\left(-k^2g_{\rho\mu} + k_\rho k_\mu\right)\tilde D^\mu{}_\nu(k)\mathrm{e}^{-\mathrm{i}kx} \stackrel{!}{=} \mathrm{i}g_{\rho\nu}\delta^4(x) \tag{5.116}$$

i. e.

$$\left(-k^2 g_{\rho\mu} + k_\rho k_\mu\right) \tilde D^\mu{}_\nu(k) \stackrel{!}{=} \mathrm{i} g_{\rho\nu} \,. \tag{5.117}$$

This is a simple algebraic equation, that can, in principle, be solved by another ansatz

$$\tilde D^{\mu\nu}(k) = g^{\mu\nu} \tilde D_1(k^2) + k^\mu k^\nu \tilde D_2(k^2) \tag{5.118}$$

because k is the only vector in the game. However

$$\begin{aligned}\mathrm{i} g_{\rho\nu} &\stackrel{!}{=} \left(-k^2 g_{\rho\mu} + k_\rho k_\mu\right) \left(g^\mu{}_\nu \tilde D_1(k^2) + k^\mu k_\nu \tilde D_2(k^2)\right) \\ &= -k^2 g_{\rho\nu} \tilde D_1(k^2) + k_\rho k_\nu \tilde D_1(k^2) - k^2 k_\rho k_\nu \tilde D_2(k^2) + k_\rho k_\nu k^2 \tilde D_2(k^2) \\ &= \left(-k^2 g_{\rho\nu} + k_\rho k_\nu\right) \tilde D_1(k^2) \end{aligned} \tag{5.119}$$

has no solution. The reason is, of course, that

$$-k^2 \mathbf{1} + k \otimes k \tag{5.120}$$

has an eigenvector k with eigenvalue 0 and is therefore not invertible.

The only way out is to add a term quadratic in A^a_μ to the free Lagrangian $\mathcal{L}_0$ that makes the matrix appearing in the equations of motion invertible. There are three choices

$$\mathcal{L}_m = \frac{m^2}{2} A^a_\mu A^{a,\mu} \tag{5.121a}$$

$$\mathcal{L}_{\text{gf}} = -\frac{\alpha}{2} \partial^\mu A^a_\mu \partial^\nu A^a_\nu \tag{5.121b}$$

$$\mathcal{L}_Z = -\frac{Z}{4} \partial_\mu A^a_\nu \partial^\mu A^{a,\nu} \,. \tag{5.121c}$$

Of these, $\mathcal{L}_Z$ can be shown to be redundant, because adding it is equivalent to adding $\mathcal{L}_{\text{gf}}$ and rescaling A^a_μ. Such a rescaling cancels in all observables.

All three terms are *not* gauge invariant

$$\delta\mathcal{L}_m \neq 0 \tag{5.122a}$$

$$\delta\mathcal{L}_{\text{gf}} \neq 0 \tag{5.122b}$$

$$\delta\mathcal{L}_Z \neq 0 \,, \tag{5.122c}$$

showing that the existence of a propagator and gauge invariance are mutually incompatible. While there is, a priori, nothing wrong with breaking gauge invariance this way, it is not desirable, because:

1. we loose a predictive principle for constructing interactions
2. interacting vector particles without gauge invariance lead to unphysical predictions (see my lecture on advanced QFT).

The first issue is mostly aesthetics, but the second is serious, of course.

Lecture 13: Wed, 27. 05. 2015

One of the problems of the explicit mass term is that the resulting propagator

$$\tilde{D}_{\mu\nu}(k) = \mathrm{i}\frac{-g_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 - m^2 + \mathrm{i}\epsilon} \tag{5.123}$$

does *not* fall off for $k_\mu \to \infty$. This makes it impossible to compute loop diagrams consistently and causes a violation of unitarity in the scattering of vector bosons with longitudinal polarization at high energies.

5.3.3 Faddeev-Popov Procedure

There's nothing wrong with the path integral

$$Z(j) = \int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A) - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \tag{5.124}$$

with the gauge invariant Yang-Mills action

$$S_{\mathrm{YM}}(A) = -\frac{1}{4}\int \mathrm{d}^4x\, F_{a,\mu\nu}F_a^{\mu\nu} \tag{5.125}$$

and it is used with great success in nonperturbative calculations on the lattice (to be precise an equivalent form that reduces to S_{YM} in the continuum limit).

However, we can not evaluate it in perturbation theory, because it has no propagator, unless we fix the gauge. We could obtain a propagator by fixing the gauge by brute force

$$Z_{\mathbf{BF}}(j,\chi) = \int \mathcal{D}A\, \delta\left(\mathcal{G}(A) - \chi\right) \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A) - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \tag{5.126}$$

but that would not guarantee that the physics remains unchanged. Instead, we should properly separate the gauge degrees of freedom in the functional integral and integrate once over each orbit, i. e. equivalence classes under

$$A_\mu \leftrightarrow UA_\mu U^{-1} + \mathrm{i}U\partial_\mu U^{-1}\,, \tag{5.127}$$

with the *same weight*. Just using the δ-distribution does *not* guarantee this:

$$\int \mathrm{d}x\, f(x)\delta(g(x)) = \sum_{x:g(x)=0} \frac{f(x)}{|\det g'(x)|}\,. \tag{5.128}$$

However

$$\int \mathrm{d}x\, f(x)\delta(g(x))|\det g'(x)| = \sum_{x:g(x)=0} f(x) \tag{5.129}$$

depends *only* on the zeros of g, *not* on any other property of g.

Thus we obtain a better gauge fixed path integral

$$Z_{\mathbf{FP}}(j,\chi) = \int \mathcal{D}A\,\delta\left(\mathcal{G}(A)-\chi\right)\det\left(\frac{\delta\mathcal{G}(A)}{\delta g}\right)\mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A)-\mathrm{i}\int\mathrm{d}^4x\,j_{a,\mu}A^\mu_a} \tag{5.130}$$

where $\delta\mathcal{G}(A)/\delta g$ is the functional derivative of the gauge fixing functional w.r.t. gauge transformations.

Since the generating functional does not depend on χ, we can get rid of the δ-distribution by integrating over χ with a suitable weight, e.g.

$$\begin{aligned} Z_{\mathbf{FP}}(j) &= \int \mathcal{D}\chi \mathrm{e}^{-\mathrm{i}\int\mathrm{d}^4x\,\frac{1}{2\alpha}\chi^2} Z_{\mathbf{FP}}(j,\chi) \\ &= \int \mathcal{D}A\ \det\left(\frac{\delta\mathcal{G}(A)}{\delta g}\right)\mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A)-\mathrm{i}\int\mathrm{d}^4x\,\frac{1}{2\alpha}(\mathcal{G}(A))^2-\mathrm{i}\int\mathrm{d}^4x\,j_{a,\mu}A^\mu_a}\,. \end{aligned} \tag{5.131}$$

The functional determinant $\det\delta\mathcal{G}(A)/\delta g$ can be written as a *fermionic* gaussian path integral

$$\det\frac{\delta\mathcal{G}(A)}{\delta g} = \int \mathcal{D}\bar\eta\mathcal{D}\eta\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}^4x\,\bar\eta\frac{\delta\mathcal{G}(A)}{\delta g}\eta}\,. \tag{5.132}$$

If we choose

$$\mathcal{G}(A) = \partial_\mu A^\mu\,, \tag{5.133}$$

this path integral turns out to be a generating functional for Faddeev-Popov ghosts. In fact, since

$$\bar\eta\frac{\delta\mathcal{G}(A)}{\delta g}\eta = \bar\eta\partial_\mu D^\mu\eta = \bar\eta\partial\delta_B A_\mu\,. \tag{5.134}$$

we find

$$Z_{\mathbf{FP}}(j) = \int \mathcal{D}A\mathcal{D}\bar\eta\mathcal{D}\eta\,\mathrm{e}^{\mathrm{i}S_{\mathrm{BRST}}(A)-\mathrm{i}\int\mathrm{d}^4x\,j_{a,\mu}A^\mu_a}\,. \tag{5.135}$$

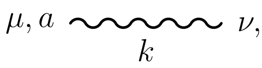
with

$$\mathcal{L}_{\mathrm{BRST}} = -\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - 2\operatorname{tr}\left(\frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 + \partial_\mu\bar\eta D^\mu\eta\right)\,. \tag{5.136}$$

Many other choices for the gauge fixing functional (5.133) are possible, but most will lead to generating functionals that involve additional interactions or will not have a simple propagator and are therefore less useful for perturbation theory.

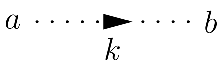
5.3.4 Feynman Rules

If $0 < |\alpha| < \infty$, we can construct a gauge propagator from (5.136):



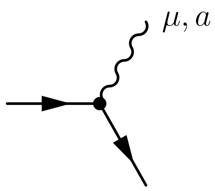
$$= \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\epsilon}\left(-g_{\mu\nu} + (1-\alpha)\frac{k_\mu k_\nu}{k^2 + \mathrm{i}\epsilon}\right) \tag{5.137a}$$

while the ghost propagator is simply

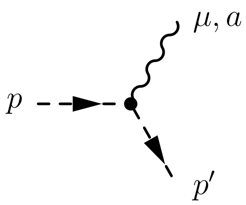


$$= -\frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\epsilon} \tag{5.137b}$$

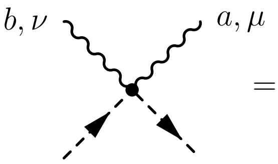
And vertices



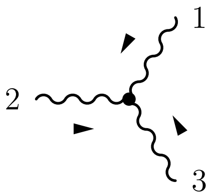
$$= \mathrm{i}g\gamma_\mu T_a \tag{5.137c}$$



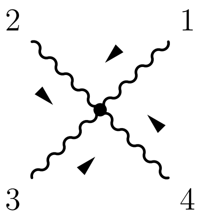
$$= \mathrm{i}g(p + p')_\mu T_a \tag{5.137d}$$



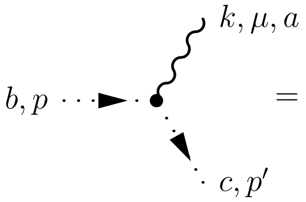
$$= \mathrm{i}g^2 g_{\mu\nu}\left(T_a T_b + T_b T_a\right) \tag{5.137e}$$



$$= \begin{array}{l} g f_{a_1 a_2 a_3} g_{\mu_1\mu_2}(k^1_{\mu_3} - k^2_{\mu_3}) \\ + g f_{a_1 a_2 a_3} g_{\mu_2\mu_3}(k^2_{\mu_1} - k^3_{\mu_1}) \\ + g f_{a_1 a_2 a_3} g_{\mu_3\mu_1}(k^3_{\mu_2} - k^1_{\mu_2}) \end{array} \tag{5.137f}$$



$$= \begin{array}{l} -\mathrm{i}g^2 f_{a_1 a_2 b} f_{a_3 a_4 b}(g_{\mu_1\mu_3} g_{\mu_4\mu_2} - g_{\mu_1\mu_4} g_{\mu_2\mu_3}) \\ -\mathrm{i}g^2 f_{a_1 a_3 b} f_{a_4 a_2 b}(g_{\mu_1\mu_4} g_{\mu_2\mu_3} - g_{\mu_1\mu_2} g_{\mu_3\mu_4}) \\ -\mathrm{i}g^2 f_{a_1 a_4 b} f_{a_2 a_3 b}(g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_4\mu_2}) \end{array} \tag{5.137g}$$



$$= \quad g p'_\mu f_{abc} \,, \tag{5.137h}$$

where the ghost-gauge vertex is indeed *not* symmetric in the momenta.

5.4 Massive Gauge Bosons

From the propagator (5.137a), we see that the poles are *only* at $p^2 = 0$, i. e. there are only massless particles with

$$E = |\vec{p}| \tag{5.138}$$

in the spectrum, independent of the choice of α. Since massless force carriers correspond to an infinite range of the interaction

$$V(r) \propto \frac{\mathrm{e}^{-mr}}{r} \,, \tag{5.139}$$

it appears that gauge theories are mathematically interesting, but only useful for describing electromagnetism. Because they are not observed at macroscopic scales, both the strong and weak interactions can have only a finite range.

In the case of the strong interactions, we will see in chapter 8 below that the interaction is so strong at long distances, that perturbation theory can not be used and that the spectrum can consequently not be inferred from the propagator. It will turn out that the massless states are absent from the physical spectrum.

In the case of weak interactions, perturbation theory is expected to be reliable and we have no reason for excluding the massless states. Fortunately, there is a way out, since we can introduce a term that *looks like* a mass term, but is generated dynamically.

Consider a theory of gauge bosons coupled to a scalar field ϕ (ignoring gauge fixing and ghost terms)

$$\mathcal{L} = -\frac{1}{2} \operatorname{tr} \left(F_{\mu\nu} F^{\mu\nu} \right) + \left(D_\mu \phi \right)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi \,. \tag{5.140}$$

Here, we have chosen Without Loss Of Generality (WLOG) a complex representation for ϕ, the same idea will work with real representations as well.

Let us parametrize

$$\phi(x) = \phi_0 + \phi_{\text{lin.}}(x) \tag{5.141a}$$

or

$$\phi(x) = \mathrm{e}^{\mathrm{i}\phi_{\text{exp.}}(x)/v}\phi_0 = \phi_0 + \frac{\mathrm{i}}{v}\phi_{\text{exp.}}(x)\phi_0 + \ldots\,, \tag{5.141b}$$

where ϕ_0 is a *constant vector* and $\phi_{\text{lin.}}(x)$ is a dynamical *vector* in the same representation as $\phi(x)$, while $\phi_{\text{exp.}}(x)$ is a dynamical *matrix* in this representation. Then

$$\begin{aligned} D_\mu\phi &= -\mathrm{i}gA_\mu\phi_0 + \partial_\mu\phi_{\text{lin.}} - \mathrm{i}gA_\mu\phi_{\text{lin.}} \\ &= -\mathrm{i}gA_\mu\phi_0 + \partial_\mu\phi_{\text{lin.}} + \text{terms with 2 or more fields} \end{aligned} \tag{5.142a}$$

or

$$\begin{aligned} D_\mu\phi &= -\mathrm{i}gA_\mu\mathrm{e}^{\mathrm{i}\phi_{\text{exp.}}/v}\phi_0 + \left(\partial_\mu\mathrm{e}^{\mathrm{i}\phi_{\text{exp.}}/v}\right)\phi_0 \\ &= -\mathrm{i}gA_\mu\phi_0 + \frac{\mathrm{i}}{v}\left(\partial_\mu\phi_{\text{exp.}}\right)\phi_0 + \text{terms with 2 or more fields}\,. \end{aligned} \tag{5.142b}$$

Therefore, the *gauge invariant* kinetic term of ϕ becomes

$$\begin{aligned} \left(D_\mu\phi\right)^\dagger D^\mu\phi &= \left(-\mathrm{i}gA_\mu\phi_0 + \partial_\mu\phi_{\text{lin.}}\right)^\dagger\left(-\mathrm{i}gA^\mu\phi_0 + \partial^\mu\phi_{\text{lin.}}\right) + \ldots \\ &= \left(\partial_\mu\phi_{\text{lin.}}\right)^\dagger\partial^\mu\phi_{\text{lin.}} + g^2\phi_0^\dagger A_\mu A^\mu\phi_0 \\ &\quad - \mathrm{i}g\left(\partial_\mu\phi_{\text{lin.}}\right)^\dagger A^\mu\phi_0 + \mathrm{i}g\left(A^\mu\phi_0\right)^\dagger\partial_\mu\phi_{\text{lin.}} \\ &\quad + \text{terms with 3 or more fields} \end{aligned} \tag{5.143a}$$

or

$$\begin{aligned} &\left(D_\mu\phi\right)^\dagger D^\mu\phi = \\ &\quad\left(-\mathrm{i}gA_\mu\phi_0 + \frac{\mathrm{i}}{v}\left(\partial_\mu\phi_{\text{exp.}}\right)\phi_0\right)^\dagger\left(-\mathrm{i}gA^\mu\phi_0 + \frac{\mathrm{i}}{v}\left(\partial^\mu\phi_{\text{exp.}}\right)\phi_0\right) + \ldots \\ &\quad= \frac{1}{v^2}\phi_0^\dagger\left(\partial_\mu\phi_{\text{exp.}}\right)^\dagger\partial^\mu\phi_{\text{exp.}}\phi_0 + g^2\phi_0^\dagger A_\mu A^\mu\phi_0 \\ &\quad\quad - \frac{g}{v}\phi_0^\dagger\left(\left(\partial_\mu\phi_{\text{exp.}}\right)^\dagger A^\mu + A^{\mu\,\dagger}\partial_\mu\phi_{\text{exp.}}\right)\phi_0 \\ &\quad\quad + \text{terms with 3 or more fields}\,. \end{aligned} \tag{5.143b}$$

In both approaches, we find a term

$$\begin{aligned} \mathcal{L}_M &= g^2\phi_0^\dagger A_\mu A^\mu\phi_0 = g^2\phi_0^\dagger T_aT_b\phi_0A_\mu^aA^{b\,\mu} \\ &= \frac{g^2}{2}\phi_0^\dagger\left(T_aT_b + T_bT_a\right)\phi_0A_\mu^aA^{b\,\mu} = \frac{1}{2}M_{ab}^2A_\mu^aA^{b\,\mu} \end{aligned} \tag{5.144}$$

with a symmetric *mass matrix*

$$M_{ab}^2 = \frac{1}{2}\phi_0^\dagger \left(T_a T_b + T_b T_a\right)\phi_0 \tag{5.145}$$

for the gauge bosons. Since M^2 is symmetric, it can be diagonalized by an orthogonal transformation, with the square roots of the eigenvalues corresponding to the masses of gauge bosons. We must make sure that the eigenvalues are non-negative, of course.

The terms

$$\mathcal{L}_{\text{kin.}} = \left(\partial_\mu \phi_{\text{lin.}}\right)^\dagger \partial^\mu \phi_{\text{lin.}} \tag{5.146a}$$

and

$$\mathcal{L}_{\text{kin.}} = \frac{1}{v^2}\phi_0^\dagger \left(\partial_\mu \phi_{\text{exp.}}\right)^\dagger \partial^\mu \phi_{\text{exp.}}\phi_0 \tag{5.146b}$$

are kinetic terms for $\phi_{\text{lin.}}$ or $\phi_{\text{exp.}}$, respectively. In the case of (5.146b), we must check if all degrees of freedom in $\phi_{\text{exp.}}$ contribute.

Finally, there are the terms mixing the $\partial_\mu A^\mu$ part of the gauge bosons with $\phi_{\text{lin.}}$

$$\begin{aligned}\mathcal{L}_{\text{mix.}} &= -\mathrm{i}g\left(\partial_\mu \phi_{\text{lin.}}\right)^\dagger A^\mu \phi_0 + \mathrm{i}g\left(A^\mu \phi_0\right)^\dagger \partial_\mu \phi_{\text{lin.}} \\ &= \mathrm{i}g\phi_{\text{lin.}}^\dagger \partial_\mu A^\mu \phi_0 - \mathrm{i}g\phi_0^\dagger \partial_\mu A^\mu \phi_{\text{lin.}}\end{aligned} \tag{5.147a}$$

or $\phi_{\text{exp.}}$

$$\mathcal{L}_{\text{mix.}} = \frac{g}{v}\phi_0^\dagger \left(\phi_{\text{exp.}}^\dagger \partial_\mu A^\mu + \partial_\mu A^\mu \phi_{\text{exp.}}\right)\phi_0 \tag{5.147b}$$

respectively. In section 6.2.2 we will show that these terms can be cancelled by a clever choice of gauge fixing functional

$$\mathcal{G}(A) = \partial_\mu A^\mu - g\alpha\phi_{\text{lin.}}^\dagger \phi_0 \,.$$

Of course, we haven't changed the physics by just renaming the degrees of freedom from ϕ to $\phi_{\text{lin.}}$ or $\phi_{\text{exp.}}$ respectively. Therefore, the complete lagrangian is still gauge invariant. Nevertheless, $\mathcal{L}_M$ looks like a mass term for the gauge bosons, which is *verboten* by gauge invariance. Thus the remaining interaction pieces, containing 3 and more fields are organized exactly in such a way that they cancel the gauge non-invariance of $\mathcal{L}_M$. If we can show that these terms don't contribute to physical observables, we have found a way to give masses to some gauge bosons. Note that it is irrelevant, whether this an exact statement or a approximation that is better than the experimental accuracy.

An obvious realization would be

$$\phi(x) = \phi_0 \tag{5.148}$$

exactly, but this breaks gauge invariance explicitely and is not different from adding a mass term by hand. To avoid the negative consequences at high energies, we *need* the dynamical fields $\phi_{\text{lin.}}(x)$ or $\phi_{\text{exp.}}(x)$. A more modest approach is therefore to require

$$|\partial_\mu \phi_{\text{lin.}}(x)| \ll |\phi_0|^2 \tag{5.149a}$$

or

$$|\partial_\mu \phi_{\text{exp.}}(x)| \ll v|\phi_0|\,, \tag{5.149b}$$

with a suitable norm $|\cdot|$. An example familiar from condensed matter physics is a *Heisenberg magnet*, where Hamiltonian is invariant under rotations, but the ground state below the Courie temperature has a magnetization pointing in one direction, breaking the rotation invariance.

—6— Spontaneous Symmetry Breaking

Lecture 14: Tue, 02. 06. 2015

6.1 Goldstone's Theorem

Consider a multiplet ϕ of scalar fields, transforming under a real orthogonal representation R of a global, i. e. not gauged, symmetry group G. Assume that the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^T\partial^\mu\phi - \frac{\lambda}{4}\left(\phi^T\phi - v^2\right)^2 \\ = \frac{1}{2}\partial_\mu\phi^T\partial^\mu\phi + \frac{\lambda v^2}{2}\phi^T\phi - \frac{\lambda}{4}\left(\phi^T\phi\right)^2 - \frac{\lambda v^4}{4} \tag{6.1}$$

with $\lambda > 0$. In the unexpanded form, the potential

$$V(\phi) = \frac{\lambda}{4}\left(\phi^T\phi - v^2\right)^2 \tag{6.2}$$

is seen to be positive, with no flat directions, as can also be seen in figure 6.1. In the expanded form of the Lagrangian, the constant term $\lambda v^4/4$ can be ignored, because it doesn't add to the equations of motion or the Feynman rules. However, in the equations of motion

$$\underbrace{\left(\Box - \lambda v^2\right)}_{=\Box + m^2}\phi + \lambda\left(\phi^T\phi\right)\phi = 0\,, \tag{6.3}$$

we observe that the "mass" term $m^2 = -\lambda v^2 \le 0$ is *negative*. Therefore, perturbation theory around the value $\phi = 0$ is not well defined, since it is a local maximum of the potential (cf. figure 6.1).

We are therefore led to expand around a minimum ϕ_0 of the potential with

$$\phi_0^T\phi_0 = v^2\,, \tag{6.4}$$

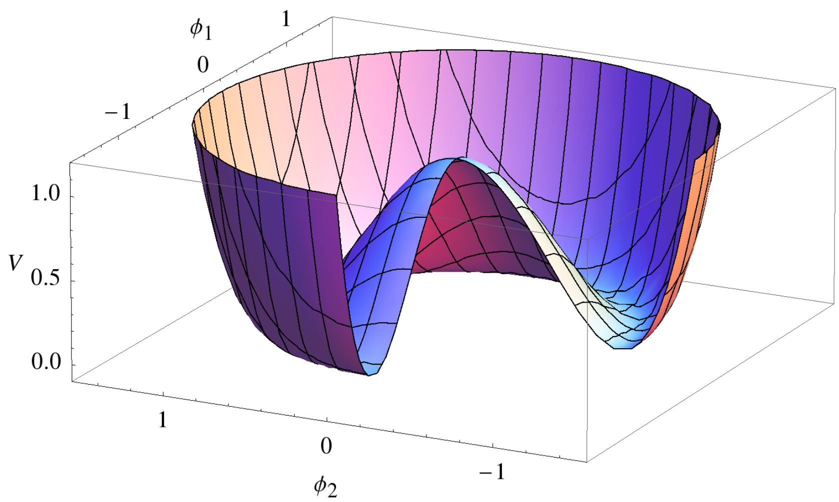


Figure 6.1: *Potential* $V(\phi_1,\phi_2) = (\phi_1^2+\phi_2^2-v^2)^2/4$.

i. e.

$$\phi(x) = \phi_0 + \chi(x) \tag{6.5a}$$

$$\partial_\mu\phi(x) = \partial_\mu\chi(x)\,, \tag{6.5b}$$

and to treat χ as the new dynamical variable. Then

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\partial_\mu\chi^T\partial^\mu\chi - \frac{\lambda}{4}\left((\phi_0+\chi)^T(\phi_0+\chi)-v^2\right)^2 \\ = \frac{1}{2}\partial_\mu\chi^T\partial^\mu\chi - \frac{\lambda}{4}\left(2\phi_0^T\chi+\chi^T\chi\right)^2 \\ = \frac{1}{2}\partial_\mu\chi^T\partial^\mu\chi - \chi^T\underbrace{\left(\lambda\phi_0\otimes\phi_0^T\right)}_{=\frac{M}{2}}\chi - \lambda\phi_0^T\chi\chi^T\chi - \frac{\lambda}{4}\left(\chi^T\chi\right)^2\end{aligned} \tag{6.6}$$

using

$$\phi_0^T\chi = \chi^T\phi_0 \tag{6.7}$$

and introducing the mass matrix

$$M = 2\lambda\phi_0\otimes\phi_0^T\,. \tag{6.8}$$

By construction, this matrix has one eigenvector with positive eigenvalue

$$M\phi_0 = 2\lambda\phi_0\otimes\phi_0^T\phi_0 = \left(2\lambda\phi_0^T\phi_0\right)\phi_0 = 2\lambda v^2\phi_0 \tag{6.9}$$

and all other eigenvalues zero. Therefore, we find one mode with mass $m = 2\lambda v^2$ and all other modes *massless*.

This scenario turns out the be the general case. In fact one can prove *without* arguments from perturbation theory

Theorem 6.1 (Goldstone)**.** *Whenever a continuous global symmetry is broken by the ground state, i. e. when ground state expectation values are not invariant under this symmetry, there are massless excitations in the spectrum. These excitations have the quantum numbers of the generators of the broken symmetry.*

In our example, the symmetry group G is broken to the subgroup $H \subset G$ of transformations that leave

$$\phi_0 = \langle 0|\phi(x)|0\rangle \tag{6.10}$$

invariant

$$\forall h \in H : h\phi_0 = \phi_0\,. \tag{6.11}$$

As a Lie group, H is generated by generators $\{T_a\}_{a=1,\ldots,\dim(H)}$

$$[T_a, T_b] = \mathrm{i}f_{abc}T_c \tag{6.12a}$$

$$T_a\phi_0 = 0\,. \tag{6.12b}$$

The remaining generators $\{X_i\}_{i=1,\ldots,\dim(G)-\dim(H)}$ generate the coset G/H. The correspond to broken symmetries and give rise to the *Goldstone bosons*. They do *not* close as a Lie algebra, because their commutators can contain contributions from $H \subset G$

$$[X_i, X_j] = \mathrm{i}f_{ijk}X_k + \mathrm{i}f_{ija}T_a\,. \tag{6.13}$$

In many cases of interest G/H turns out to be a *symmetric space*, in which case (6.13) simplifies to

$$[X_i, X_j] = \mathrm{i}f_{ija}T_a\,. \tag{6.13'}$$

In any case, it can be shown that mixed commutators are such that the generators of H act linearly on the generators of G/H

$$[T_a, X_i] = \mathrm{i}f_{aij}X_j\,. \tag{6.14}$$

We can decomposean arbitrary group element g as

$$g = \mathrm{e}^{\mathrm{i}\xi_i X_i}\mathrm{e}^{\mathrm{i}\eta_i T_i} \tag{6.15}$$

and find

$$g\phi_0 = \mathrm{e}^{\mathrm{i}\xi_i X_i}\phi_0 \tag{6.16}$$

This way, we seem to have moved into the wrong direction. While attempting to give masses to gauge bosons, we have introduced new massless particles. Fortunately, Goldstone's theorem applies to *global* symmetries only – not to gauge symmetries.

6.2 Higgs Mechanism

Repeating the above example with a gauged symmetry

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2}\left(D_\mu\phi\right)^\dagger D^\mu\phi - \frac{\lambda}{4}\left(\phi^T\phi - v^2\right)^2 , \tag{6.17}$$

where the shape of the potential for ϕ could be more complicated of course, we are again led to expand around a minimum ϕ_0 of the potential with

$$\phi_0^T\phi_0 = v^2 . \tag{6.18}$$

6.2.1 Unitarity Gauge

If we can reach all possible values for ϕ by a transformation from G applied to ϕ_0, we may use an element of G/H and employ an exponential representation

$$\phi(x) = \mathrm{e}^{\mathrm{i}\chi(x)/v}\phi_0 \tag{6.19}$$

with

$$\chi(x) = \sum_{i=1}^{\dim(G)-\dim(H)} \chi_i(x)X_i . \tag{6.20}$$

Then we can use the properties of the covariant derivative to write

$$D_\mu\phi(x) = \mathrm{e}^{\mathrm{i}\chi(x)/v}D_\mu\phi_0 = \mathrm{e}^{\mathrm{i}\chi(x)/v}\left(-\mathrm{i}gA_\mu(x)\phi_0\right) = -\mathrm{i}g\mathrm{e}^{\mathrm{i}\chi(x)/v}A_\mu(x)\phi_0 \tag{6.21}$$

and we can absorb $\chi(x)$ by a gauge transformation

$$\begin{pmatrix} \phi(x) \\ D_\mu\phi(x) \\ A_\mu(x) \\ F_{\mu\nu}(x) \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\chi(x)/v}\phi(x) \\ \mathrm{e}^{-\mathrm{i}\chi(x)/v}D_\mu\phi(x) \\ A'_\mu(x) \\ F'_{\mu\nu}(x) \end{pmatrix} = \begin{pmatrix} \phi_0 \\ -\mathrm{i}gA_\mu\phi_0 \\ \mathrm{e}^{-\mathrm{i}\chi(x)/v}A_\mu(x)\mathrm{e}^{\mathrm{i}\chi(x)/v} + \frac{\mathrm{i}}{g}\mathrm{e}^{-\mathrm{i}\chi(x)/v}\left(\partial_\mu\mathrm{e}^{\mathrm{i}\chi(x)/v}\right) \\ \mathrm{e}^{-\mathrm{i}\chi(x)/v}F_{\mu\nu}(x)\mathrm{e}^{\mathrm{i}\chi(x)/v} \end{pmatrix} \tag{6.22}$$

to end up with

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{g^2}{2}\left(A_\mu\phi_0\right)^\dagger\left(A^\mu\phi_0\right) . \tag{6.23}$$

In this Lagrangian, χ does not appear anymore, because we have used the gauge freedom to remove it. In fact, we have used up all our gauge freedom corresponding to the generators X_i of G/H and choosen the so-called *unitarity gauge*, in which there are massive gauge bosons with mass matrix

$$M_{ij}^2 = \frac{g^2}{2}\phi_0^\dagger \left(X_i X_j + X_j X_i\right) \phi_0 \tag{6.24}$$

since $X_i\phi_0 \neq 0$. Only the gauge bosons corresponding to the generators T_a of H will remain massless, due to $T_a\phi_0 \neq 0$.

Therefore, it is possible to construct a theory of gauge bosons coupled to Goldstone bosons that looks, in a particular gauge, just like a theory of massive gauge bosons, without Goldstone bosons. The counting of degrees of freedom is consistent, because the massive gauge bosons need an addional, longitudinal polarization state, which is provided by the Goldstone bosons.

The choice of of unitarity gauge is useful for determining the physical degrees of freedom, because we have identified the Goldstone bosons as redundant, unphysical, degrees of freedom. For actual calculations, it is not always useful, because the propagator

$$\mathrm{i}\frac{-g_{\mu\nu} + k_\mu k_\nu/M^2}{k^2 - M^2 + \mathrm{i}\epsilon} \tag{6.25}$$

does not fall off at high energies. Also the polarization vector for longitudinal gauge bosons

$$\epsilon_{(L)}^\mu(k) \approx \frac{k^\mu}{m} \tag{6.26}$$

causes the scattering amplitudes for the scattering of longitudinal gauge bosons to rise too fast at high energies. However, we are free to choose other gauges, where the Goldstone bosons are still separate from the gauge bosons, which don't have longitudinal polarization states. These will be discussed in section 6.2.2.

The problem in model building for a given gauge group G is now to find a real or complex representation of G and a G-invariant potential so that a scalar field ϕ can be expanded about a stable point ϕ_0 with a subgroup $H \in G$ with $H\phi_0 = \phi_0$ such that the generators X_i of G/H provide exactly the needed Goldstone bosons. In addition eigenvalues of the mass matrix

$$M_{ij}^2 = \frac{g^2}{2}\phi_0^\dagger \left(X_i X_j + X_j X_i\right) \phi_0 \tag{6.27}$$

must match. Given a gauge group G, this is a straightforward problem, because all possible subgroups H are known.

In principle, one can sidestep the construction of a potential and write a *non-linear* Lagrangian for fields with values in G/H directly

$$\mathcal{L}_{\mathrm{NL}} = -\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2}\left(D_\mu(\mathrm{e}^{\mathrm{i}\chi(x)/v}\phi_0)\right)^\dagger D^\mu(\mathrm{e}^{\mathrm{i}\chi(x)/v}\phi_0)\,, \tag{6.28}$$

but the price to pay is that the expansion of $\mathcal{L}_{\mathrm{NL}}$ in terms of χ contains arbitrarily high powers of ϕ.

6.2.2 R_ξ-Gauge

Lecture 15: Wed, 03. 06. 2015

For definiteness, let us consider the abelian *Higgs-Kibble Model*

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{D_\mu\phi}D^\mu\phi - \frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2\,, \tag{6.29a}$$

with $\phi : \mathbf{M} \to \mathbf{C}$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.29b}$$

$$D_\mu\phi = \partial_\mu\phi - \mathrm{i}gA_\mu\phi\,, \tag{6.29c}$$

which is invariant under U(1) gauge transformations

$$\begin{pmatrix}\phi\\ A_\mu\end{pmatrix} \mapsto \begin{pmatrix}\mathrm{e}^{\mathrm{i}\omega}\phi\\ A_\mu + \frac{1}{g}\partial_\mu\omega\end{pmatrix}\,. \tag{6.29d}$$

WLOG, expand around $\phi = \phi_0 = v$

$$\phi(x) = v + \frac{\phi_1(x) + \mathrm{i}\phi_2(x)}{\sqrt{2}}\,, \tag{6.30}$$

with $\phi_i : \mathbf{M} \to \mathbf{R}$. Then

$$\begin{aligned}\overline{D_\mu\phi}D^\mu\phi &= \left|-\mathrm{i}gvA_\mu + \partial_\mu\frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}} - \mathrm{i}A_\mu\frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}}\right|^2\\ &= g^2v^2A_\mu A^\mu + \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) - \sqrt{2}\mathrm{i}gvA_\mu\partial^\mu\phi_2 + \ldots\,,\end{aligned} \tag{6.31}$$

where we have not spelled out the terms with more than two fields, and

$$-\frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2 = -\frac{\lambda}{2}\left(\frac{1}{2}\left(\phi_1^2 + \phi_2^2\right) + \sqrt{2}v\phi_1\right)^2$$

$$= -\lambda v^2\phi_1^2 - \frac{\lambda v}{\sqrt{2}}\left(\phi_1^2+\phi_2^2\right)\phi_1 - \frac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2 . \tag{6.32}$$

Thus the bilinear part of the Lagrangian is

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{2\lambda v^2}{2}\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{2g^2v^2}{2}A_\mu A^\mu + \sqrt{2}\mathrm{i}gv\phi_2\partial^\mu A_\mu . \tag{6.33}$$

Therefore $\mathcal{L}$ appearently describes a

- a massive scalar field ϕ_1 with mass $m_1 = \sqrt{2\lambda}v$,
- a massless scalar field ϕ_2, and
- a massive gauge field A_μ with mass $M = \sqrt{2}gv$,

as long as we ignore the mixing of $\partial_\mu A^\mu$ with ϕ_2. Diagonalizing this part is inconvenient, because we would have to separate polarization states of A_μ. It's easier to choose a clever gauge fixing

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\alpha}\left(\mathcal{G}(A,\phi)\right)^2 \tag{6.34}$$

with

$$\mathcal{G}(A,\phi) = \partial_\mu A^\mu + \sqrt{2}\mathrm{i}\alpha gv\phi_2 . \tag{6.35}$$

Then, if we suppress the Lagrangian for the Faddeev–Popov ghosts, because they can be shown to decouple in an abelian gauge theory,

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 - \sqrt{2}\mathrm{i}gv\phi_2\partial_\mu A^\mu - \alpha g^2v^2\phi_2^2 \tag{6.36}$$

and

$$\mathcal{L}_2 + \mathcal{L}_{\mathrm{gf}} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{2\lambda v^2}{2}\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{2\alpha g^2v^2}{2}\phi_2^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 + \frac{2g^2v^2}{2}A_\mu A^\mu . \tag{6.37}$$

Now there is no more mixing term left and we see that $\mathcal{L} + \mathcal{L}_{\mathrm{gf}}$ describes a

- a massive scalar field ϕ_1 with mass $m_1 = \sqrt{2\lambda}v$,
- a massive scalar field ϕ_2 with mass $m_2 = \sqrt{2\alpha}gv$,

- a massive gauge field A_μ with mass $M = \sqrt{2}gv$.

Indeed, in the exercises, we have seen that the *Stueckelberg Lagrangian*

$$\mathcal{L}_\text{S} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\partial^\mu A_\mu\right)^2 + \frac{M^2}{2}A_\mu A^\mu \tag{6.38}$$

gives rise to the propagator

$$\tilde D_{\mu\nu}(k) = \mathrm{i}\frac{-g_{\mu\nu} + k_\mu k_\nu/M^2}{k^2 - M^2 + \mathrm{i}\epsilon} - \mathrm{i}\frac{k_\mu k_\nu/M^2}{k^2 - \alpha M^2 + \mathrm{i}\epsilon} \tag{6.39a}$$

$$= \mathrm{i}\frac{-g_{\mu\nu} + (1-\alpha)k_\mu k_\nu/(k^2 - \alpha M^2 + \mathrm{i}\epsilon)}{k^2 - M^2 + \mathrm{i}\epsilon}\,. \tag{6.39b}$$

The two forms of the *Stueckelberg propagator* can be used to highlight different aspects

- (6.39a) shows that the physical, transversal degrees of freedom satisfying $\partial_\mu A^\mu = 0$ are propagated with a mass M, while the unphysical degrees of freedom are propagated with the mass $\sqrt{\alpha}M$,

- (6.39b) shows that
$$\lim_{k_\mu\to\infty} \tilde D_{\mu\nu}(k) = 0\,. \tag{6.40}$$

Of course, the Goldstone boson ϕ_2 also has the mass $\sqrt{\alpha}M$, allowing the cancellation of its contributions with the unphysical part of A_μ.

There are two illuminating special choices for α

- $|\alpha| \to \infty$: in this limit, the unphysical parts of A_μ and the Goldstone boson ϕ_2 become infinitely heavy and therefore decouple from the other states. This limit corresponds to the *unitarity gauge* discussed in section 6.2.1 and

- $\alpha \to 1$: everything propagates with the same mass M.

—7— (Elektroweak) Standard Model

7.1 Observations

Weak and elektromagnetic interactions appear to be very different:

- electromagnetic interactions
  - infinite range, i. e. massless particle exhange
  - doesn't change the flavor
  - coupling of photons A_μ to *vector* currents

    $$j_\mu(x) = \bar\psi(x)\gamma_\mu\psi(x) \tag{7.1}$$

    from the angular dependence of scattering cross sections
  - coupling to left-handed and right-handed fermions

    $$\psi_L = \frac{1}{2}(1-\gamma_5)\psi \tag{7.2a}$$

    $$\psi_R = \frac{1}{2}(1+\gamma_5)\psi \tag{7.2b}$$

    with the same strength
- *charged currents*: e. g.

  $$\mu^- \to \nu_\mu e^- \bar\nu_e \tag{7.3a}$$

  $$n \to p e^- \bar\nu_e \tag{7.3b}$$

  - parity violation from the angular dependence of scattering cross sections: coupling *only* to the left-handed part of fermions

    $$j_\mu(x) = 2\bar\psi'_L(x)\gamma_\mu\psi_L(x) = \bar\psi'(x)\gamma_\mu(1-\gamma_5)\psi(x) \tag{7.4}$$

also called $V - A$, vector–minus–axialvector

$$
\begin{aligned}
j_\mu(x) &= \bar\psi'(x)\gamma_\mu\psi(x) - \bar\psi'(x)\gamma_\mu\gamma_5\psi(x) \\
&= \bar\psi'(x)\gamma_\mu\psi(x) + \bar\psi'(x)\gamma_5\gamma_\mu\psi(x) \qquad (7.5)
\end{aligned}
$$

– compatible with Fermi's phenomenological *pointlike* current–current interaction

$$
\begin{aligned}
\mathcal{L}_I &= \frac{G_F}{\sqrt{2}} j_L^\mu j_{L,\mu} \\
&= \frac{G_F}{\sqrt{2}} \bar\psi'(x)\gamma_\mu(1-\gamma_5)\psi(x)\bar\psi'''(x)\gamma^\mu(1-\gamma_5)\psi'(x) \qquad (7.6)
\end{aligned}
$$

with

$$G_F = 1.166 \cdot 10^{-5}\,\mathrm{GeV}^{-2} \approx \left(\frac{1}{300\,\mathrm{GeV}}\right)^2 \qquad (7.7)$$

7.2 Problem

Computing the cross section for neutrino scattering reveals a problem with the Fermi interaction. Since neutrinos are massless, the *only* dimensionfull parameters entering the expression for the total cross section are the Fermi constant G_F and the center of mass energy E. Furthermore, in lowest order perturbation theory, we must have

$$\sigma \propto G_F^2 \qquad (7.8)$$

since the scattering amplitude is proportional to G_F. Since the cross section has the dimensions of an area, the only way to get the dimensions right is then.

$$\sigma = \text{const.} \cdot G_F^2 E^2\,. \qquad (7.9)$$

Therefore the cross section grows without bound and must become unphysically large, violating unitarity, eventually. A detailed calculation shows that this will happen at $E \approx \mathcal{O}(100\,\mathrm{GeV})$.

7.2.1 Solution

Dampen the interaction at high energies

$$G_F \to G_F f(E) \propto \frac{G_F}{E^2}\,. \qquad (7.10)$$

The most sensible approach to $f(E)$ is to replace the pointlike interaction of currents by the exchange of a heavy particle W between currents

$$\frac{G_F}{\sqrt{2}} j_\mu j^\mu = \frac{g^2}{8M_W^2} j_\mu j^\mu = \left(\frac{g}{2\sqrt{2}} j_\mu\right) \frac{g^{\mu\nu}}{M_W^2} \left(\frac{g}{2\sqrt{2}} j_\nu\right) \to - \left(\frac{g}{\sqrt{2}} j_\mu\right) \frac{g^{\mu\nu}}{p^2 - M_W^2} \left(\frac{g}{\sqrt{2}} j_\nu\right) . \tag{7.11}$$

Thus we would like to interpret W as a massive gauge boson with

$$M_W = \sqrt{\frac{\sqrt{2} g^2}{8G_F}} = \frac{g}{2} \cdot 246\,\text{GeV} . \tag{7.12}$$

The interactions are Charged Current (CC) interactions changing neutrinos into electrons, down-quarks into up-quarks and vice versa. Therefore, we should expect that the gauge symmetry is non-abelian, because the couplings act like shifting operators. The simplest possibility is then a SU(2) gauge group for the weak interactions.

Since the SU(2) has three gauge bosons W_1^μ, W_2^μ and W_3^μ and we can form charged combinations

$$W_\pm^\mu = \frac{W_1^\mu \mp \mathrm{i} W_2^\mu}{\sqrt{2}} \tag{7.13}$$

it is very tempting to identify W_3 with the photon and to give mass only to W_1 and W_2 by breaking SU(2)/U(1). However, this can not work, because the photon couples to right-handed currents, whereas the $W_\pm^\mu$ don't and the gauge symmetries must commute with the space-time symmetries, including parity.

7.3 $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y / \mathrm{U}(1)_Q$

Lecture 16: Tue, 09. 06. 2015

Thus we are led to propose a gauge group and breaking pattern

$$G/H = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Q / \mathrm{U}(1)_Q$$

for the electromagnetic and weak interactions. The third gauge boson W_3^μ should also be heavy and correspond to a new Neutral Current (NC) interaction. The electromagnetic $\mathrm{U}(1)_Q$ remains unbroken, because photons are massless. It turns out that this doesn't quite work for explaining the experimental results and the better approach is

$$G/H = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y/\mathrm{U}(1)_Q\,, \tag{7.14}$$

where $\mathrm{SU}(2)_L$ is the gauge group of the *weak isospin* T, $\mathrm{U}(1)_Y$ is the gauge group of the hypercharge Y and the unbroken electromagnetic $\mathrm{U}(1)_Q$ appears as a mixture of T_3 and Y after symmetry breaking

$$Q = T_3 + \frac{Y}{2} \tag{7.15}$$

as in the Gell-Mann–Nishijima formula.

The gauge lagrangian before symmetry breaking is then

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{7.16}$$

where the three $\mathrm{SU}(2)_L$ gauge bosons form a triplet

$$\vec{W}_\mu = (W^1_\mu, W^2_\mu, W^3_\mu) \tag{7.17}$$

with

$$\vec{W}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - g\vec{W}_\mu \times \vec{W}_\nu \tag{7.18}$$

and the $\mathrm{U}(1)_Y$ gauge boson is a singlet

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{7.19}$$

The corresponding convention for the signs and magnitude of the coupling constants in the covariant derivative is

$$D_\mu = \partial_\mu + \mathrm{i}g\vec{W}_\mu\vec{T} + \mathrm{i}g' B_\mu \frac{Y}{2}\,. \tag{7.20}$$

7.3.1 Matter Fields

Having identified the gauge structure[1], we have to place the fermionic matter fields in the corresponding multiplets, as in table 7.1. Note that we have already added to $\mathrm{SU}(3)_C$ quantum numbers for the strong interactions, to be discussed in chapter 8.

By placing the left-handed part of the leptons and quarks in $\mathrm{SU}(2)_L$ dublets and the right handed part into singlets, we make shure that only the former couple to the $\mathrm{SU}(2)_L$ gauge bosons. Note that the righthanded

[1] Or rather a candidate for it, since we have to still work out the details

neutrinos ν_R^e, ν_R^μ, and ν_r^τ are singlets under *all* gauge quantum numbers and decouble therefore completely. We will need them only if you want to give masses to the neutrinos.

Note that since

$$\bar\psi(\mathrm{i}\partial\!\!\!/ - m)\psi = \bar\psi_L \mathrm{i}\partial\!\!\!/\psi_L + \bar\psi_R \mathrm{i}\partial\!\!\!/\psi_R - m\bar\psi_R\psi_L - m\bar\psi_L\psi_R\,, \tag{7.21}$$

a non-vanishing Dirac mass term $\bar\psi_R\psi_L + \bar\psi_L\psi_R$ requires the left- and right-handed fermions to have the same charges. Therefore all matter fields in table 7.1 must be strictly massless and we must still find a way to give them masses.

7.3.2 Higgs Fields

Finally, we need a Higgs field and corresponding interaction to break the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ to $\mathrm{U}(1)_Q$. The minimal approach adds just a

			$(C,T)_Y$	$Q = T_3 + Y/2$
$\begin{pmatrix}\nu_L^e \\ e_L^-\end{pmatrix}$	$\begin{pmatrix}\nu_L^\mu \\ \mu_L^-\end{pmatrix}$	$\begin{pmatrix}\nu_L^\tau \\ \tau_L^-\end{pmatrix}$	$(\mathbf{1},\mathbf{2})_{-1}$	$\begin{pmatrix}0 \\ -1\end{pmatrix}$
ν_R^e	ν_R^μ	ν_r^τ	$(\mathbf{1},\mathbf{1})_0$	0
e_R^-	μ_R^-	τ_r^-	$(\mathbf{1},\mathbf{1})_{-2}$	-1
$\begin{pmatrix}u_L \\ d_L\end{pmatrix}$	$\begin{pmatrix}c_L \\ s_L\end{pmatrix}$	$\begin{pmatrix}t_L \\ b_L\end{pmatrix}$	$(\mathbf{3},\mathbf{2})_{1/3}$	$\begin{pmatrix}+2/3 \\ -1/3\end{pmatrix}$
u_R	c_R	t_r	$(\mathbf{3},\mathbf{1})_{4/3}$	$+2/3$
d_R	s_R	b_r	$(\mathbf{3},\mathbf{1})_{-2/3}$	$-1/3$

Table 7.1: *Matter fields in the* $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y/\mathrm{SU}(3)_C \times \mathrm{U}(1)_Q$ *Standard Model.*

doublet from the $(\mathbf{1},\mathbf{2})_1$ representation

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_3 + \mathrm{i}\phi_4 \\ \phi_1 + \mathrm{i}\phi_2 \end{pmatrix}, \tag{7.22}$$

where both ϕ^+ and ϕ^0 are *complex* fields, while the $\{\phi_i\}_{i=1,2,3,4}$ are real. Using this parameterization, we see that

$$\phi^\dagger\phi = |\phi^+|^2 + |\phi^0|^2 = \frac{1}{2}\sum_{i=1}^{4}\phi_i^2 = \frac{1}{2}\Phi^T\Phi\,, \tag{7.23}$$

with

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \tag{7.24}$$

is not only invariant under $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ transformations, but under the larger group SO(4) of fourdimensional orthogonal transformations in 4 dimensions.

As is shown in the exercises, the Lie algebras of SO(4) and $\mathrm{SU}(2)\times\mathrm{SU}(2)$ are isomorphic. Therefore, as long as the potential depends only on $\phi^\dagger\phi$, the global symmetry of the Higgs sector is $\mathrm{SU}(2)_L \times \mathrm{SU}(2) \supset \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, though only the smaller group is gauged. This will have important consequences, because the symmetry breaking

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)/\mathrm{SU}(2)_{\text{custodial}} \tag{7.25}$$

leaves an SU(2) *custodial symmetry* that relates the masses of the charged and neutral massive gauge bosons and protects theses relations from radiative corrections. We will see below that the electromagnetic couplings and the fermionic matter sector violate this custodial symmetry, but in a controlled way.

We postulate a Higgs potential

$$V(\phi) = \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2 \tag{7.26}$$

with degenerate minima $\phi_0^\dagger\phi_0 = v^2/2$ or

$$\Phi_0^T\Phi_0 = v^2\,. \tag{7.27}$$

We choose

$$\Phi_0 = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.28a}$$

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{7.28b}$$

as the value to expand the Higgs field about. Note that

$$T_3 \phi_0 = -\frac{1}{2} \phi_0 \tag{7.29a}$$

$$Y \phi_0 = \phi_0 \tag{7.29b}$$

$$Q \phi_0 = \left(T_3 + \frac{Y}{2} \right) \phi_0 = 0 \tag{7.29c}$$

and the electric charge Q turns out *not* to be broken, as desired. Furthermore, all $\mathrm{SO}(3) \cong \mathrm{SU}(2)_{\text{custodial}}$ transformations of the lower three components of Φ leave Φ_0 invariant.

Expanding around ϕ_0

$$\phi = \phi_0 + \chi \tag{7.30}$$

with

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_3 + \mathrm{i}\chi_4 \\ \chi_1 + \mathrm{i}\chi_2 \end{pmatrix} , \tag{7.31}$$

we can use the fact that a *arbitrary* ϕ can be parameterized

$$\phi = \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\vec\alpha\vec T} \begin{pmatrix} 0 \\ v + h \end{pmatrix} \tag{7.32}$$

with real $\vec\alpha$ and h and employ unitarity gauge

$$\phi \to \mathrm{e}^{-\mathrm{i}\vec\alpha\vec T} \phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix} . \tag{7.33}$$

In the $(\mathbf{1}, \mathbf{2})_1$ representation, the covariant derivative reads

$$\begin{aligned} D_\mu = \partial_\mu + \mathrm{i}g\vec W_\mu \vec T + \mathrm{i}g' B_\mu \frac{Y}{2} &= \mathbf{1}\partial_\mu + \frac{\mathrm{i}}{2} \begin{pmatrix} gW^3_\mu + g'B_\mu & g(W^1_\mu - \mathrm{i}W^2_\mu) \\ g(W^1_\mu + \mathrm{i}W^2_\mu) & -gW^3_\mu + g'B_\mu \end{pmatrix} \\ &= \mathbf{1}\partial_\mu + \frac{\mathrm{i}}{2} \begin{pmatrix} gW^3_\mu + g'B_\mu & \sqrt{2}gW^+_\mu \\ \sqrt{2}gW^-_\mu & -gW^3_\mu + g'B_\mu \end{pmatrix} \end{aligned} \tag{7.34}$$

and we find

$$D_\mu\phi = \frac{1}{\sqrt{2}} D_\mu \begin{pmatrix} 0 \\ v+h \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\mathrm{i}g}{\sqrt{2}} W^+_\mu (v+h) \\ \partial_\mu h + \frac{\mathrm{i}}{2}(-gW^3_\mu + g'B_\mu)(v+h) \end{pmatrix} \tag{7.35}$$

The kinetic term of the Higgs field becomes then

$$\begin{aligned} |D_\mu\phi|^2 &= \frac{g^2}{4} W^-_\mu W^{+,\mu} \\ &\quad + \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{8}(-gW^3_\mu + g'B_\mu)(-gW^{3,\mu} + g'B^\mu)(v+h)^2 \,, \end{aligned} \tag{7.36}$$

from which we read off masses

$$M^2_{W^\pm} = \frac{g^2v^2}{4} \tag{7.37a}$$

$$M^2_{W^3,B} = \frac{v^2}{4} \begin{pmatrix} g^2 & -gg' \\ -gg' & (g')^2 \end{pmatrix} \,. \tag{7.37b}$$

The (squared) mass matrix $M^2_{W^3,B}$ has one vanishing eigenvalue with corresponding eigenvector

$$\frac{1}{\sqrt{g^2+(g')^2}} \begin{pmatrix} g' \\ g \end{pmatrix} = \begin{pmatrix} \sin\theta_w \\ \cos\theta_w \end{pmatrix} \tag{7.38a}$$

and the orthogonal eigenvector

$$\frac{1}{\sqrt{g^2+(g')^2}} \begin{pmatrix} g \\ -g' \end{pmatrix} = \begin{pmatrix} \cos\theta_w \\ -\sin\theta_w \end{pmatrix} \tag{7.38b}$$

belongs to the other eigenvalue

$$M^2_Z = \frac{v^2}{4}(g^2+(g')^2) = \frac{M^2_W}{\cos^2\theta_w} \ge M^2_W \,. \tag{7.39}$$

In (7.38) we have introduced the so-called *weak mixing angle* or *Weinberg angle* θ_w. Therefore, the mass eigenstates are

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{7.40}$$

and W^3_μ and B_μ are expressed as

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \,. \tag{7.41}$$

Lecture 17: Wed, 10. 06. 2015

Inserting this into the corresponding part of the covariant derivative, we find

$$\begin{aligned} gW_\mu^3 T_3 + g' B_\mu \frac{Y}{2} &= \sqrt{g^2+(g')^2}\cos\theta_w(\cos\theta_w Z_\mu + \sin\theta_w A_\mu)T_3 \\ &\quad + \sqrt{g^2+(g')^2}\sin\theta_w(-\sin\theta_w Z_\mu + \cos\theta_w A_\mu)\frac{Y}{2} \\ &= A_\mu \sqrt{g^2+(g')^2}\sin\theta_w\cos\theta_w\left(T_3+\frac{Y}{2}\right) \\ &\quad + Z_\mu\sqrt{g^2+(g')^2}\left(\cos^2\theta_w T_3 - \sin^2\theta_w\frac{Y}{2}\right) \\ &\stackrel{!}{=} A_\mu eQ + Z_\mu\sqrt{g^2+(g')^2}\underbrace{\left(\cos^2\theta_w T_3 - \sin^2\theta_w\frac{Y}{2}\right)}_{=T_3-\sin^2\theta_w Q}, \end{aligned} \tag{7.42}$$

i. e.

$$e = \sqrt{g^2+(g')^2}\sin\theta_w\cos\theta_w = g\sin\theta_w = g'\cos\theta_w \tag{7.43}$$

since the photon must couple with strength e to the electric charge Q. Finally

$$gW_\mu^3 T_3 + g' B_\mu \frac{Y}{2} = A_\mu eQ + Z_\mu \frac{e}{\sin\theta_w\cos\theta_w}\left(T_3 - \sin^2\theta_w Q\right) \tag{7.44}$$

and we see that the Z_μ couples to the combination $T_3 - \sin^2\theta_w Q$ which only vanishes for right-handed neutrinos, but gives a non-zero NC for left-handed neutrinos.

In particular, we obtain the non-trivial prediction

$$1 - \frac{M_W^2}{M_Z^2} = 1 - \cos^2\theta_w = \sin^2\theta_w = \frac{e^2}{g^2} \tag{7.45}$$

relating the ratio of the masses of the weak gauge bosons to the ratio of the coupling of weak and electromagnetic interactions.

Finally, from expanding the Higgs potential

$$V(\phi) = \frac{\lambda}{2}\left(\frac{1}{2}(v+h)^2 - v^2\right)^2 = \frac{\lambda v^2}{2}h^2 + \ldots, \tag{7.46}$$

we find the mass of the scalar particle remaining after symmetry breaking, the Higgs boson

$$m_H = \sqrt{\lambda}v\,. \tag{7.47}$$

while it is proportial to v, the parameter λ only appears in the Higgs self-couplings and is therefore essentially a free parameter that is only required to be positive to ensure vacuum stability and not too large to allow the application of perturbation theory.

We can summarize a set of input parameters of the gauge-Higgs sector

$$\alpha_{\text{QED}} = \frac{e^2}{4\pi} = \frac{1}{137.036} \tag{7.48a}$$
$$M_Z = 91.1876(21)\,\text{GeV} \tag{7.48b}$$
$$M_W = 80.385(15)\,\text{GeV} \tag{7.48c}$$
$$m_H = 125.4\,\text{GeV}\,. \tag{7.48d}$$

The other parameters are not independent and may not be changed without breaking symmetries:

$$\sin^2\theta_w = 1 - \frac{M_W^2}{M_Z^2} \approx 0.23 \tag{7.49a}$$
$$g = \frac{e}{\sin\theta} \tag{7.49b}$$
$$g' = \frac{e}{\cos\theta} \tag{7.49c}$$
$$v = \frac{2M_W}{g} = \frac{2\sin\theta_w M_W}{e} = \frac{\sin\theta_w M_W}{\sqrt{\alpha\pi}} = 255.3\,\text{GeV} \tag{7.49d}$$
$$\lambda = \frac{m_H^2}{v^2} \tag{7.49e}$$

Not that relations in (7.49) are subject to renormalization and receive calculable corrections in higher orders.

Of course one may use other subsets of the parameters as input parameters, e. g. the parameters in the lagrangian before symmetry breaking

$$\{g, g', \lambda, v\}\,, \tag{7.50}$$

and derive the rest from them, but some sets are better than others, because the radiative corrections are better under control and/or the input parameters can be measured more precisely.

7.3.3 Yukawa Couplings

If we want to produce mass terms for fermions

$$m\left(\bar\psi_R\psi_L + \bar\psi_L\psi_R\right)\,, \tag{7.51}$$

we need to couple

$$\bar\psi_R \psi_L \in (\mathbf{1},\mathbf{2})_{Y_L-Y_R} \tag{7.52}$$

to a scalar field that can be expanded about an appropriate non-vanishing value. In particular, the scalar field(s) and their conjugate(s) must transform according to $(\mathbf{1},\bar{\mathbf{2}})_{Y_R-Y_L}$ and their conjugates. Introducing the doublets

$$\Psi_L = \begin{pmatrix} \nu^\ell_L \\ \ell^-_L \end{pmatrix} \tag{7.53a}$$

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}\,, \tag{7.53b}$$

we find the representations for the four different combinations in each family

$$\overline{\nu^\ell_R}\Psi_L \in (\mathbf{1},\mathbf{2})_{-1} \tag{7.54a}$$

$$\overline{\ell^-_R}\Psi_L \in (\mathbf{1},\mathbf{2})_{+1} \tag{7.54b}$$

$$\overline{u_R} q_L \in (\mathbf{1},\mathbf{2})_{-1} \tag{7.54c}$$

$$\overline{d_R} q_L \in (\mathbf{1},\mathbf{2})_{+1}\,. \tag{7.54d}$$

Thus it appears that we need *two different* Higgs doublets

$$\phi \in (\mathbf{1},\mathbf{2})_{+1} \tag{7.55a}$$

$$\tilde\phi \in (\mathbf{1},\mathbf{2})_{-1}\,. \tag{7.55b}$$

In the case of the leptons, one could avoid $\overline{\phi}$ by not giving Dirac masses to neutrinos and attibute the observed neutrino masses to Majorana masses. This solution is not available for quarks, because they carry charges and can therefore not be their own antiparticle.

Indeed, in the case of the popular supersymmetric extensions of the SM, one is forced to introduce the second Higgs doublet $\tilde\phi$ in order not to break supersymmetry explicitely. However in the non-supersymmetric case, one can make use of the fact that for SU(2)

$$\mathbf{2} \cong \bar{\mathbf{2}} \tag{7.56}$$

and construct the $(\mathbf{1},\mathbf{2})_{-1}$ as the complex conjugate of the $(\mathbf{1},\mathbf{2})_{+1}$

$$\tilde\phi = \mathrm{i}\sigma^2\overline{\phi} = \begin{pmatrix} \overline{\phi^0} \\ -\overline{\phi^+} \end{pmatrix} = \begin{pmatrix} \overline{\phi^0} \\ -\phi^- \end{pmatrix}\,. \tag{7.57}$$

For now, we will concentrate on the minimal one Higgs doublet model, but the many versions of the two Higgs doublet model have not been excluded so far.

Since there are three families of fermions with identical quantum numbers

$$\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \nu^e \\ \nu^\mu \\ \nu^\tau \end{pmatrix} \tag{7.58a}$$

$$\ell = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \tag{7.58b}$$

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} u \\ c \\ t \end{pmatrix} \tag{7.58c}$$

$$D = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \tag{7.58d}$$

we must allow off-diagonal Yukawa couplings

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} = &- \sum_{i,j=1}^{3} Y^\ell_{ij} \overline{\Psi_{i,L}} \ell^-_{j,R} \phi - \sum_{i,j=1}^{3} Y^\nu_{ij} \overline{\Psi_{i,L}} \nu_{j,R} \tilde\phi \\ &- \sum_{i,j=1}^{3} Y^D_{ij} \overline{Q_{i,L}} D_{j,R} \phi - \sum_{i,j=1}^{3} Y^U_{ij} \overline{Q_{i,L}} U_{j,R} \tilde\phi + \text{h. c.} \\ = &- \left(\overline{\Psi_L} Y^\ell \ell^-_R\right) \phi - \left(\overline{\Psi_L} Y^\nu \nu_R\right) \tilde\phi - \left(\overline{Q_L} Y^D D_R\right) \phi - \left(\overline{Q_L} Y^U U_R\right) \tilde\phi + \text{h. c.}\,,\end{aligned} \tag{7.59}$$

where we have introduced an obvious matrix notation.

After symmetry breaking we expand and choose unitarity gauge again

$$\phi = \frac{1}{\sqrt2} \begin{pmatrix} 0 \\ v+h \end{pmatrix} \tag{7.60a}$$

$$\tilde\phi = \frac{1}{\sqrt2} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \tag{7.60b}$$

and find the mass terms

$$\begin{aligned}\mathcal{L}_{\text{mass}} = &-\frac{v}{\sqrt2} \left(\overline{\ell^-_L} Y^\ell \ell^-_R\right) - \frac{v}{\sqrt2} \left(\overline{\nu_L} Y^\nu \nu_R\right) \\ &- \frac{v}{\sqrt2} \left(\overline{D_L} Y^D D_R\right) - \frac{v}{\sqrt2} \left(\overline{U_L} Y^U U_R\right) + \text{h. c.}\,,\end{aligned} \tag{7.61}$$

that are non-diagonal, in general. The matrices Y^U, Y^D, Y^ν and Y^ℓ are not guaranteed to be normal, i. e. diagonalizable. Fortunately, we have

Theorem 7.1. *An arbitrary quadratric matrix M can be factorized in the form*

$$M = L\Delta R^\dagger \,, \tag{7.62}$$

where L and R are unitary and Δ is diagonal with real, non-negative entries.

Proof. $MM^\dagger$ is a positive self-adjoint matrix and can be diagonalized by a unitary transformation

$$L^\dagger MM^\dagger L = \Delta^2 \,, \tag{7.63}$$

where the diagonal entries d_n^2 of Δ^2 are non-negative. One may therefore choose the positive square rootd_n of the diagonal entries to define a square root Δ of Δ^2. Finally, the matrix

$$C = M^\dagger L \tag{7.64}$$

satisfies

$$C^\dagger C = \Delta^2 \tag{7.65}$$

and consists therefore of pairwise orthogonal column vectors v_n with

$$v_n^\dagger v_m = d_n^2 \delta_{nm} \,. \tag{7.66}$$

This means that there is a unitary matrix R with

$$C = R\Delta \tag{7.67}$$

and we find

$$M^\dagger = R\Delta L^\dagger \,. \tag{7.68}$$

□

Lecture 18: Tue, 16. 06. 2015

We may now use theorem 7.1 to diagonalize the mass matrices

$$\forall\lambda \in \{\nu, \ell, U, D\} : \frac{v}{\sqrt{2}} Y^\lambda = L^\lambda M^\lambda \left(R^\lambda\right)^\dagger \,. \tag{7.69}$$

Since we observe the particles in the broken symmetry phase, we should identify the observed fermions with the states in a basis where the mass matrices are diagonal. Therefore, we should absorb the matrices L^λ and R^λ in the definition of the fermion states

$$\forall\lambda \in \{\nu, \ell, U, D\} : (R^\lambda)^\dagger \lambda_R \to \lambda_R \tag{7.70a}$$

$$\overline{\lambda_L} L^\lambda \to \overline{\lambda_L} \,. \tag{7.70b}$$

Since the kinetic terms and the NC gauge couplings are propotional to the unit matrix in the family indices and never connect up-type quarks with down-type quarks or neutrinos with charged leptons, such transformations cancel out

$$\bar\psi_L i\partial\!\!\!/ \psi_L = \bar\psi'_L L i\partial\!\!\!/ L^\dagger \psi'_L = \bar\psi'_L i\partial\!\!\!/ \psi'_L \,, \text{ etc.} \tag{7.71}$$

Thus we can perform these transformations for everything but the CC interactions without leaving a trace. Indeed is the absence of Flavor Changing Neutral Currents (FCNC) in the lagrangian one of the most stringent requirement in modelbuilding, because the observed FCNC are *very* small and can be explained by suppressed loop corrections.

If we assume for simplicity that $Y^\nu = 0$ because the neutrinos are almost massless, we can choose L^ν and R^ν arbitrarily, in particular $L^\nu = L^\ell$. Thus $(L^\nu)^\dagger L^\ell = \mathbf{1}$ and the ℓ-ν CC interactions

$$\mathcal{L}_{\text{CC},\ell\nu} = -\frac{g}{\sqrt{2}} \overline{\nu}_L W\!\!\!\!/^+ \ell_L - \frac{g}{\sqrt{2}} \bar\ell_L W\!\!\!\!/^- \nu_L \tag{7.72}$$

are not affected by the change of basis. In the CC interactions of quarks

$$\mathcal{L}_{\text{CC},UD} = -\frac{g}{\sqrt{2}} \overline{U}_L W\!\!\!\!/^+ D_L - \frac{g}{\sqrt{2}} \overline{D}_L W\!\!\!\!/^- U_L \tag{7.73}$$

however, both L^D and L^U are fixed by Y^D and Y^U, respectively. Thus we will in general have $L^D \neq L^U$ and in particular a non-trivial *Cabibbo–Kobayashi–Maskawa matrix*

$$V_{\text{CKM}} = (L^U)^\dagger L^D \neq \mathbf{1} \,. \tag{7.74}$$

Therefore, after the change to a basis in which the masses are diagonal, we find that the CC interaction will connect different generations

$$\begin{aligned} \mathcal{L}_{\text{CC},UD} &= -\frac{g}{\sqrt{2}} \overline{U}_L (L^U)^\dagger W\!\!\!\!/^+ L^D D_L - \frac{g}{\sqrt{2}} \overline{D}_L (L^D)^\dagger W\!\!\!\!/^- L^U U_L \\ &= -\frac{g}{\sqrt{2}} \overline{U}_L W\!\!\!\!/^+ V_{\text{CKM}} D_L - \frac{g}{\sqrt{2}} \overline{D}_L W\!\!\!\!/^- V_{\text{CKM}}^\dagger U_L \,. \end{aligned} \tag{7.75}$$

As the product of two unitary matrices, the CKM-matrix

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{7.76}$$

is unitary itself. Not all of the N^2 complex or $2N^2$ real parameters of such a $N \times N$ mixing matrix are independent. There are N^2 real constraints from unitarity

$$V^\dagger V = \mathbf{1} = \left(V^\dagger V\right)^\dagger . \tag{7.77}$$

Of the remaining N^2 real parameters, $N(N-1)/2$ are rotation parameters and the remaining $N(N+1)/2$ are phases. Of these phases, $2N-1$ can be absorbed in the choice of relative phases for $2N$ quark fields

$$\begin{pmatrix} U_i \\ D_j \\ V_{ij} \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{i\alpha_i^U} U_i \\ \mathrm{e}^{i\alpha_j^D} D_j \\ \mathrm{e}^{i\alpha_i^U - \alpha_j^D} V_{ij} \end{pmatrix} . \tag{7.78}$$

Thus we are left with

- $N(N-1)/2 = 3$ angles and
- $(N-1)(N-2)/2 = 1$ phase(s) .

Currently, there is no theory explaining the values for the entries of the CKM matrix and the four physical parameters have to be measured experimentally. One finds a hierarchical structure, as expressed in the approximate *Wolfenstein parametrization*

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - \mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{7.79}$$

with

$$V_{\mathrm{CKM}}^\dagger V_{\mathrm{CKM}} = \mathbf{1} + \mathcal{O}(\lambda^4) , \tag{7.80}$$

to better than 0.25%. The measured values are

$$\lambda = 0.2254 \tag{7.81a}$$
$$A = 0.8 \tag{7.81b}$$
$$\rho = 0.12 \tag{7.81c}$$
$$\eta = 0.53 \tag{7.81d}$$

. The flavor sector contributes most of the free parameters in the SM that have to be measured. If the neutrinos were massless, there would be 13 parameters

- 6 quark masses

- 3 charged lepton masses
- 3 angles and 1 phase in the CKM matrix,

but the neutrino sector adds

- 3 neutrino masses
- 3 angles and 1 phase in the PMNS (Pontecorvo–Maki–Nakagawa–Sakata) matrix,

bringing the total up to 20.

A lot of experimental effort has gone into measuring as many elements of the CKM matrix as possible, in order to test its unitarity. No evidence for a violation of unitarity has been found. The complex phase in the CKM matrix is significant, because it is the only source for CP violation in the standard model.

7.3.4 Interactions

Lecture 19: Wed, 17. 06. 2015

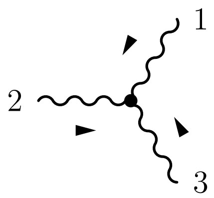
After fixing the free parameters of the SM from the quadratic mass terms in the lagrangian and from the CC and electromagnetic gauge interactions, there are no more ambiguities and the can express all other interactions through these parameters.

Gauge Boson Selfinteractions

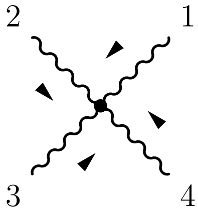
The gauge boson selfinteractions are determined completely by the gauge lagrangian

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \,. \tag{7.82}$$

We know Lorentz structure of the self interactions of the $\vec{W}_\mu$ gauge boson already from (5.137f)



$$= \begin{array}{l} -g\epsilon_{a_1a_2a_3}g_{\mu_1\mu_2}(k^1_{\mu_3} - k^2_{\mu_3}) \\ -g\epsilon_{a_1a_2a_3}g_{\mu_2\mu_3}(k^2_{\mu_1} - k^3_{\mu_1}) \\ -g\epsilon_{a_1a_2a_3}g_{\mu_3\mu_1}(k^3_{\mu_2} - k^1_{\mu_2}) \end{array} \tag{7.83a}$$

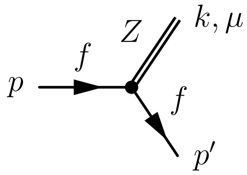


$$= \begin{aligned} &-\mathrm{i}g^2\epsilon_{a_1a_2b}\epsilon_{a_3a_4b}(g_{\mu_1\mu_3}g_{\mu_4\mu_2} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}) \\ &-\mathrm{i}g^2\epsilon_{a_1a_3b}\epsilon_{a_4a_2b}(g_{\mu_1\mu_4}g_{\mu_2\mu_3} - g_{\mu_1\mu_2}g_{\mu_3\mu_4}) \,. \\ &-\mathrm{i}g^2\epsilon_{a_1a_4b}\epsilon_{a_2a_3b}(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_3}g_{\mu_4\mu_2}) \end{aligned} \tag{7.83b}$$

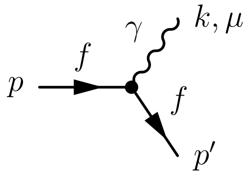
Since the mass eigenstates Z_μ and A_μ are linear superpositions of W_μ^3 and B_μ, it is straightforward to work out the coupling constants entering the vertices coupling $W_\mu^\pm$, Z_μ and A_μ, see the exercises.

Gauge Bosons with Matter: Fermions and Scalar Bosons

The interactions of fermions with gauge bosons derive from the NC and CC parts of the lagrangian. The NC Feynman rules are



$$= -\mathrm{i}\frac{g}{2\cos\theta_w}\left(g_V^f\gamma_\mu - g_A^f\gamma_\mu\gamma_5\right) \tag{7.84a}$$



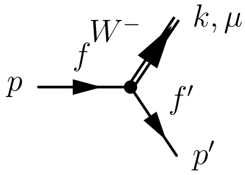
$$= -\mathrm{i}eQ_f\gamma_\mu \,, \tag{7.84b}$$

where the vector and axial vector couplings of the Z are are determined by the isospin and electric charge quantum numbers of the left and right handed parts

$$g_V = T_3 - 2Q\sin^2\theta_w \tag{7.85a}$$

$$g_A = T_3 \,. \tag{7.85b}$$

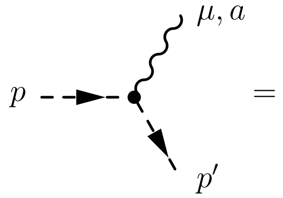
Note that hypercharge and isospin have been arranged so that the photon couples to left and right handed parts with the same strengh and there remains a pure vector coupling. Note again the absence of FCNC. In contrast, the CC Feynman rules



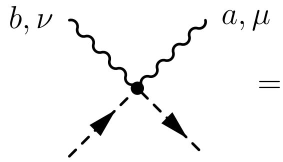
$$= -\mathrm{i}\frac{g}{\sqrt{2}}V_{ff'}\tau^+\gamma_\mu\frac{1-\gamma_5}{2} \,. \tag{7.86}$$

couple only left handed fermions, but are mixing the generations

Even though there are no scalar matter particles in the SM, they appear in extensions of the SM:



$$= \quad -\mathrm{i}g(p+p')_\mu T_a \tag{7.87a}$$



$$= \quad \mathrm{i}g^2 g_{\mu\nu} \left(T_a T_b + T_b T_a\right). \tag{7.87b}$$

Gauge Bosons with Higgs Bosons

The couplings of gauge bosons with higgs bosons are derived by evaluating the kinetic term

$$|D_\mu \phi|^2 , \tag{7.88}$$

as in the exercises.

In unitarity gauge, one finds that there are no couplings of the kind hhA, hhZ or $hhW^\pm$. This is obvious for hhA and $hhW^\pm$ from charge conservation and the vanishing electric charge of the physical Higgs boson. Furthermore, couplings of the form

$$hh\partial^\mu Z_\mu \tag{7.89}$$

can not contribute because they are “pure gauge” and

$$\left(h \overleftrightarrow{\partial_\mu} h\right) Z^\mu \tag{7.90}$$

vanish for neutral scalars h.

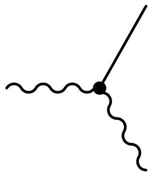
The structure of the other couplings can be understood, by observing that, in unitarity gauge, h alway appears in the combination $v+h$. Therefore

$$M_W = \frac{gv}{2} \to \frac{g}{2}(v+h) = M_W\left(1+\frac{h}{v}\right) \tag{7.91}$$

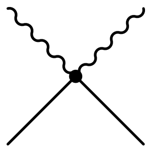
and

$$M_W^2 W_\mu^+ W^{-,\mu} \to M_W^2 W_\mu^+ W^{-,\mu} + \frac{2M_W^2}{v} h W_\mu^+ W^{-,\mu} + \frac{M_W^2}{v^2} h^2 W_\mu^+ W^{-,\mu} \tag{7.92}$$

and analogously for hZZ and $hhZZ$. From this argument, the absence of hAA and $hhAA$ should be obvious.



$$\propto \qquad \frac{M}{v} \tag{7.93a}$$



$$\propto \qquad \frac{M^2}{v^2}\,. \tag{7.93b}$$

Matter Fermions with Higgs Bosons

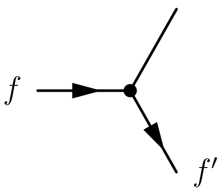
For the fermion couplings we can again argue

$$m_f = Yv \to Y(v+h) = m_f\left(1+\frac{h}{v}\right) \tag{7.94}$$

and find that their couplings to Higgs bosons are fixed by the ratio

$$Y = \frac{m}{v}\,. \tag{7.95}$$

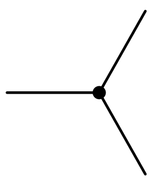
Therefore a testable prediction arises that the couplings of gauge bosons and fermions are both propotional to there masses



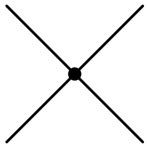
$$\propto \qquad \frac{m_f}{v}\,. \tag{7.96}$$

Higgs Boson Selfinteractions

Expanding the quartic potential, we will find hhh, $hhhh$, as in the exercises:



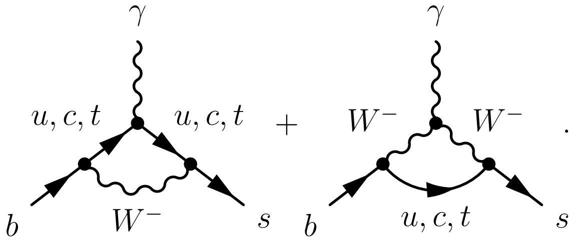
$$\propto \qquad m_h \tag{7.97a}$$



. (7.97b)

7.4 GIM-Mechanism

The absence of FCNC becomes rather subtle in higher orders of perturbation theory, because two consecutive CC interactions can result in a NC interaction. Consider the *penguin diagram* giving a contribution to $b \to s\gamma$



. (7.98)

There are three sets of diagrams with different up-type quarks in the loop. They differ only by the V_{CKM} factors and by the mass of the quark

$$\mathcal{M}(b \to s\gamma) = V_{ts}^* V_{tb} \mathcal{M}(m_t) + V_{cs}^* V_{cb} \mathcal{M}(m_s) + V_{us}^* V_{ub} \mathcal{M}(m_u) . \tag{7.99}$$

Taking a closer look at the integral in the expression for the left loop diagram

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma_\rho (1-\gamma_5) \frac{1}{\not{k} + \not{p}_s - m_q + \mathrm{i}\epsilon} \gamma_\mu \frac{1}{\not{k} + \not{p}_b - m_q + \mathrm{i}\epsilon} \gamma_\sigma (1-\gamma_5) \times \frac{g^{\rho\sigma}}{k^2 - M_W^2 + \mathrm{i}\epsilon} , \tag{7.100}$$

it appears to be logarithmically divergent, because there are only four powers of k in the denominator. In this case everything would break down, because we would need a counterterm corresponding to a FCNC, losing the prediction of small FCNC [2].

However, if the masses were degenerate,

$$\mathcal{M}(b \to s\gamma)|_{m_t = m_s = m_u = m} = \underbrace{(V_{ts}^* V_{tb} + V_{cs}^* V_{cb} + V_{us}^* V_{ub})}_{=(V^\dagger V)_{sb} = 0} \mathcal{M}(m) \tag{7.101}$$

[2] Even if the divergencies would cancel between the left and right diagram, there would remain the same problem for FCNC involving gluons, since the gluon does not couple to the W and only the left diagram contributes to $b \to sg$.

by the unitarity of V_{CKM}. From this, one can argue that

$$\mathcal{M}(b \to s\gamma) \propto \Delta m \tag{7.102}$$

and by dimensional analysis, there must be one more power of k in the denominator for $k \gg m_q, M_W$, making the diagram finite. Actually, one can do even better. Expanding quark the propagator

$$\frac{\mathrm{i}}{\not p - m + \mathrm{i}\epsilon} = \frac{\mathrm{i}}{\not p + \mathrm{i}\epsilon} + \frac{\mathrm{i}}{\not p + \mathrm{i}\epsilon}(-\mathrm{i}m)\frac{\mathrm{i}}{\not p + \mathrm{i}\epsilon} + \mathcal{O}(m^2)\,, \tag{7.103}$$

we observe that the each power of m can be treated as the insertion of a vertex $-\mathrm{i}m$, *without* a gamma matrix, i.e. flipping the chirality. Therefore, since only left handed quarks couple to the $W^\pm$, we need an *even* number of such insertion and the terms linear in Δm must vanish as well. Thus

$$\mathcal{M}(b \to s\gamma) \propto \Delta m^2\,, \tag{7.104}$$

which is often a much stronger suppression. Such cancellations due to the unitarity of the CKM matrix are known as the *GIM mechanism*, named after Glashow, Illiopoulos and Maiani.

Lecture 20: Tue, 23. 06. 2015

These processes are observed in nature in decays like

$$B \to K^*\gamma \tag{7.105a}$$
$$B \to K\ell^+\ell^- \tag{7.105b}$$
$$B \to K\nu\bar\nu \tag{7.105c}$$

which have branching ratios

$$\mathrm{Br}(B \to X) = \frac{\Gamma(B \to X)}{\Gamma_{\text{tot.}}(B)} \tag{7.106}$$

of the order $\mathcal{O}(10^{-7})$–$\mathcal{O}(10^{-6})$. In the quark picture these correspond to decays of mesons or baryons with other quarks acting as spectators

$$|\overline{B^0}\rangle = |b\rangle \otimes |\bar d\rangle \to |s\rangle \otimes |\bar d\rangle \otimes |\gamma\rangle = |\overline{K^{0,(*)}}\rangle \otimes |\gamma\rangle \tag{7.107a}$$
$$|B^-\rangle = |b\rangle \otimes |\bar u\rangle \to |s\rangle \otimes |\bar d\rangle \otimes |\gamma\rangle = |K^{-,(*)}\rangle \otimes |\gamma\rangle \tag{7.107b}$$
$$|\overline{B_s}\rangle = |b\rangle \otimes |\bar s\rangle \to |s\rangle \otimes |\bar s\rangle \otimes |\gamma\rangle = |\phi\rangle \otimes |\gamma\rangle \tag{7.107c}$$
$$\ldots \to \ldots\,.$$

Due to the strong GIM suppression of the SM contributions, many deviations from the SM will have a strong impact on the branching ratios of such rare decays. Therefore they provide a fertile testing ground.

Note that the computation of the quark decay $b \to s\gamma$ can be performed reliably in perturbation theory. However, the computation of the effected of the non-perturbative hadronic binding forces is still impossible. Nevertheless, one can use the wealth of existing data for the allowed CC decay $b \to cW^-$ appearing in $B \to D\ell^+\nu$ to extract the hadronic matrix elements, which are independent of the flavor c or s in the final state, except for the mass dependence.

7.5 K^0-$\overline{K^0}$ Mixing, etc.

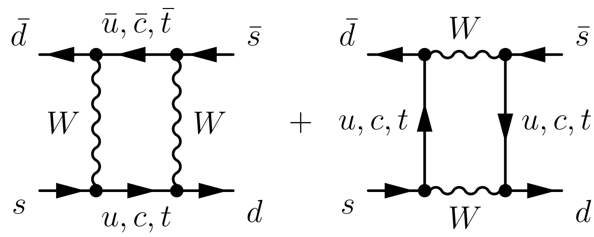
7.5.1 Box diagrams

Even more dramatic implications of the GIM mechanism, can be found in processes like

$$|\overline{K^0}\rangle = |s\rangle \otimes |\bar{d}\rangle \to |d\rangle \otimes |\bar{s}\rangle = |K^0\rangle \tag{7.108a}$$

$$|\overline{B^0}\rangle = |b\rangle \otimes |\bar{d}\rangle \to |d\rangle \otimes |\bar{b}\rangle = |B^0\rangle\ , \tag{7.108b}$$

that are again prohibited at tree level. Nevertheless, we can draw one-loop diagrams



(7.109)

that contribute to these processes. These diagrams are already finite by power counting and the GIM mechanism is not required for renormalizability. Nevertheless, we observe that

$$\mathcal{M}(s\bar{d} \to d\bar{s}) = \sum_{q,q' \in \{u,c,t\}} V^*_{qd} V_{qs} \mathcal{M}(m_q, m_{q'}) V^*_{q's} V_{q'd} \tag{7.110}$$

vanishes by unitarity if all masses of up-type quarks are equal. Again, since all couplings are left handed, the suppression factor is given by the difference of the square of the masses. But this time there are two such factors, so we can extract four powers of the masses

Before the charm quark was discovered, it's existence with roughly the correct mass, was predicted since it was required in a two generation version of the GIM mechanism to keep the size of the amplitude for $K^0 \leftrightarrow \overline{K^0}$ transitions close to the observed values. This was possible, because on one hand, even if the box is finite without the GIM mechanism, it's numerical size would be enhanced by $(m_W/m_s)^4 = \mathcal{O}(10^{10})$. On the other hand, the $K^0 \leftrightarrow \overline{K^0}$ transition matrix element can be measured rather precisely, due to some fortunate numerical accidents, to be described in the following.

7.5.2 Mixing

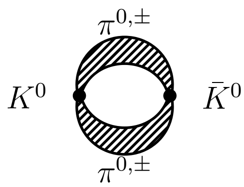
Even before computing the box diagrams (7.109), we observe that K^0 and $\overline{K^0}$ can decay into the *same* final states

$$\left.\begin{matrix} K^0 \\ \overline{K^0} \end{matrix}\right\} \to \begin{cases} \pi^0\pi^0 \\ \pi^+\pi^- \\ \pi^0\pi^0\pi^0 \\ \pi^+\pi^-\pi^0 \\ \vdots \end{cases} \tag{7.111}$$

and we must expect them to couple, e. g. in nonrelativistic perturbation theory

$$\sum_{\pi\pi} \langle K^0|H_w|\pi\pi\rangle \frac{1}{E_{K^0} - E_{\pi\pi}} \langle \pi\pi|H_w|\bar{K}^0\rangle \tag{7.112}$$

which can be visualized by pseudo Feynman diagrams



. (7.113)

In addition to the short distance box (7.109) there are additional penguin diagrams and a correct combination of all contributions without double counting is complicated.

7.5.3 Experimental Observations

Recall that there are two different neutral Kaons $K^0 = d\bar{s}$ and $\overline{K^0} = s\bar{d}$ that can be distinguished by their semileptonic decays

$$K^0 \to \pi^- e^+ \nu_e \qquad (d\bar{s} \to d\bar{u}W^+) \qquad K^0 \not\to \pi^+ e^- \bar{\nu}_e \tag{7.114a}$$

$$\overline{K^0} \to \pi^+ e^- \bar{\nu}_e \qquad (s\bar{d} \to u\bar{d}W^-) \qquad \overline{K^0} \not\to \pi^- e^+ \nu_e \tag{7.114b}$$

However, the dominant hadronic decay modes are flavor-neutral

$$\left.\begin{matrix} K^0 \\ \overline{K^0} \end{matrix}\right\} \to \begin{cases} \pi^0\pi^0 \\ \pi^+\pi^- \end{cases} . \tag{7.115}$$

There is a helpful numerical accident

$$m_{K^0} = 498\,\mathrm{MeV} \tag{7.116a}$$
$$m_{\pi^\pm} = 140\,\mathrm{MeV} \tag{7.116b}$$
$$m_{\pi^0} = 135\,\mathrm{MeV} \tag{7.116c}$$

that leads to

$$m_{K^0} \approx 3m_\pi + 3\cdot 29\,\mathrm{MeV} \tag{7.117a}$$
$$m_{K^0} \approx 2m_\pi + 2\cdot 111\,\mathrm{MeV}\,. \tag{7.117b}$$

Therefore, $\pi\pi$ and $\pi\pi\pi$ are the *only* open hadronic decay channels and $\pi\pi$ dominates because of the much larger available phase space

$$\Gamma(K^0 \to \pi\pi) \approx 575 \cdot \Gamma(K^0 \to \pi\pi\pi)\,. \tag{7.118}$$

7.5.4 CP Conservation

The pions and kaons are pseudoscalar

$$P\,|\pi\rangle = -\,|\pi\rangle \tag{7.119a}$$
$$P\,|K\rangle = -\,|K\rangle \tag{7.119b}$$

and we choose the phases for charge conjugation as

$$C\,|\pi^\pm\rangle = |\pi^\mp\rangle \tag{7.120a}$$
$$C\,|\pi^0\rangle = |\pi^0\rangle \tag{7.120b}$$
$$C\,|K^0\rangle = |\overline{K^0}\rangle \tag{7.120c}$$
$$C\,|\overline{K^0}\rangle = |K^0\rangle\ . \tag{7.120d}$$

Thus we can form find the CP eigenstates

$$CP\,|K^0_{CP=\pm1}\rangle = \pm\,|K^0_{CP=\pm1}\rangle \tag{7.121}$$

as

$$|K^0_{CP=\pm1}\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle \mp |\overline{K^0}\rangle\right)\ . \tag{7.122}$$

The lightest decay channel is CP *even*

$$CP\,|\pi\pi; l\rangle = +(-1)^2(-1)^l\,|\pi\pi; l\rangle\ , \tag{7.123}$$

i. e.

$$CP\,|\pi\pi; l = 0\rangle = +\,|\pi\pi; l = 0\rangle\ . \tag{7.124}$$

Therefore, if CP is conserved, the long lived state K_L^0 is CP odd and the shor lived state K_S^0 is CP even

$$|K_S^0\rangle = |K_{CP=+1}^0\rangle \tag{7.125}$$

$$|K_L^0\rangle = |K_{CP=-1}^0\rangle\ . \tag{7.126}$$

However, one observes some $K^0 \to \pi\pi$ decays even after all K_S^0 have decayed (recall that $\Gamma_S \approx 575 \cdot \Gamma_L$). Therefore there are $K_L^0 \to \pi\pi$ decays and CP must be violated.

There are two ways CP can be violated in $K_L^0 \to \pi\pi$

1. *direct CP violation* in the $\Delta S = 1$ decay Hamiltonian

$$\langle\pi\pi|H_w|K_L^0\rangle \neq 0 \tag{7.127}$$

   or

2. *indirect CP violation* from the *mixing* of CP eigenstates

$$|K_L^0\rangle \neq |K_{CP=-1}^0\rangle\ , \tag{7.128}$$

   i. e. a $\Delta S = 2$ interaction

$$\langle K_{CP=+1}^0|H_w|K_{CP=-1}^0\rangle \neq 0\,. \tag{7.129}$$

After the discovery of CP violation, a possible, if unsatisfacotory, explanation was a new "superweak" $\Delta S = 2$ interaction, that violates CP, while the "normal" weak interactions conserve CP. It took around 35 years to prove the existence of direct CP violation experimentally.

7.5.5 $K^0 - \overline{K^0}$ Oscillations

Lecture 21: Wed, 24. 06. 2015

For a given momentum, the Hilbert space is two dimensional and we can introduce a matrix notation

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = a(t) \left| K^0 \right\rangle + b(t) \left| \overline{K^0} \right\rangle . \tag{7.130}$$

The kaons decay and the time evolution is *not* unitary in this subspace of the full Hilbert space. Therefore, we expect a non-hermitean Hamiltonian

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} M - \mathrm{i}\frac{\Gamma}{2} & M_{12} - \mathrm{i}\frac{\Gamma_{12}}{2} \\ \overline{M_{12} - \mathrm{i}\frac{\Gamma_{12}}{2}} & M - \mathrm{i}\frac{\Gamma}{2} \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \tag{7.131}$$

with complex M and Γ. The two-level system (7.131) will be solved exactly in the exercises. We can compute the eigenstates of this Hamiltonian as a superposition of CP eigenstates

$$\left| K_L \right\rangle = \frac{1}{\sqrt{1 + |\bar\epsilon|^2}} \left(\left| K^0_{CP=-1} \right\rangle + \bar\epsilon \left| K^0_{CP=+1} \right\rangle \right) \tag{7.132a}$$

$$\left| K_S \right\rangle = \frac{1}{\sqrt{1 + |\bar\epsilon|^2}} \left(\left| K^0_{CP=+1} \right\rangle + \bar\epsilon \left| K^0_{CP=-1} \right\rangle \right) \tag{7.132b}$$

with

$$\bar\epsilon \approx \frac{\mathrm{i}}{2} \frac{\mathrm{Im}\, M_{12} - \mathrm{i}\, \mathrm{Im}\, \Gamma_{12}/2}{\mathrm{Re}\, M_{12} - \mathrm{i}\, \mathrm{Re}\, \Gamma_{12}/2} \tag{7.133a}$$

$$M_{12} - \mathrm{i}\frac{\Gamma_{12}}{2} = \left\langle K^0 \right| H_w \left| \overline{K^0} \right\rangle . \tag{7.133b}$$

In order to obtain CP violation from mixing, we must have

$$\mathrm{Im}\, M_{12} \neq 0 \tag{7.134}$$

or

$$\mathrm{Im}\, \Gamma_{12} \neq 0 \,. \tag{7.135}$$

Note that these imaginary parts *must not* be confused with the non-hermition part of the Hamiltonian describing the decays.

There is another helpful numerical accident: the difference of the eigenvalues

$$2\, \mathrm{Re}\, M_{12} \approx \Delta m = m_L - m_S \approx 3.5 \cdot 10^{-6}\, \mathrm{eV}\, (\text{sic!}) \tag{7.136}$$

is tiny and of the same order as the K_S^0 decay width

$$\Gamma_S = \frac{1}{\tau_S} \approx 7.35 \cdot 10^{-6}\,\text{eV} \approx 2\Delta m\,. \tag{7.137}$$

Therefore we will see spectacular oscillation and interference phenomena on *macroscopic* scales. In addition, the tiny CP violating contributions are enhanced by a small denominator in perturbation theory.

Irrespective of a small CP-violating contribution, we expect the following behaviour of a K^0 beam, which can be produced, e. g., by scattering K^+s on protons

- The K^0 beam will oscillate into a $\overline{K^0}$ beam and back. This can be observed by detecting the different semileptonic decays along the beam line.
- Subsequently, the K_S^0 component will die out. This can be observed by detecting the decline of observed $K^0 \to \pi\pi$ decays along the beam line.
- If the beam later passes through matter, the K_L^0 component will loose phase coherence, since the K^0 and $\overline{K^0}$ interact differently. Therefore, the K_S^0 part is regenerated. This can be observed by detecting $K^0 \to \pi\pi$ decays again after the passage through matter.

7.5.6 CP Violation

In order to distinguish indirect from direct CP violation, we need to separate the effects of strong and weak interactions. Since the strong interactions conserve isospin, we should decompose the amplitudes according to isospin (see [2] for more details and further discussion).

The final state of the pions must be symmetric, because they are bosons. The s-wave can therefore have only isospin 0 or 2

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1}_S \oplus \mathbf{3}_A \oplus \mathbf{5}_S\,. \tag{7.138}$$

In terms of the corresponding amplitudes, we can write

$$A_{K^0\to\pi^+\pi^-} = A_0 \mathrm{e}^{\mathrm{i}\xi_0} \mathrm{e}^{\mathrm{i}\delta_0} + \frac{A_2}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\xi_2} \mathrm{e}^{\mathrm{i}\delta_2} \tag{7.139a}$$

$$A_{K^0\to\pi^0\pi^0} = A_0 \mathrm{e}^{\mathrm{i}\xi_0} \mathrm{e}^{\mathrm{i}\delta_0} - \sqrt{2} A_2 \mathrm{e}^{\mathrm{i}\xi_2} \mathrm{e}^{\mathrm{i}\delta_2} \tag{7.139b}$$

$$A_{\overline{K^0}\to\pi^+\pi^-} = -A_0 \mathrm{e}^{-\mathrm{i}\xi_0} \mathrm{e}^{\mathrm{i}\delta_0} - \frac{A_2}{\sqrt{2}} \mathrm{e}^{-\mathrm{i}\xi_2} \mathrm{e}^{\mathrm{i}\delta_2} \tag{7.139c}$$

$$A_{\overline{K^0}\to\pi^0\pi^0} = -A_0 \mathrm{e}^{-\mathrm{i}\xi_0}\mathrm{e}^{\mathrm{i}\delta_0} + \sqrt{2}A_2\mathrm{e}^{-\mathrm{i}\xi_2}\mathrm{e}^{\mathrm{i}\delta_2}\,, \tag{7.139d}$$

where

- A_I: modulus of the *weak* decay amplitude into the state with isospin I,
- ξ_I: phase of the *weak* decay amplitude into the state with isospin I,
- δ_I: *elastic* scattering phase of the strong interactions in the final state with isospin I.

Note that the signs in (7.139c) and (7.139d) are such that $\xi_I = 0$ if CP is conserved

$$\langle\pi\pi|H_w|K^0\rangle = \underbrace{\langle\pi\pi|\,(CP)^{-1}}_{=\langle\pi\pi|}\underbrace{CPH_w(CP)^{-1}}_{=H_w}\underbrace{CPketK^0}_{=-|\overline{K^0}\rangle} = -\langle\pi\pi|H_w|\overline{K^0}\rangle\ . \tag{7.140}$$

Also note that ξ_I changes the sign for the charge conjugate state, whereas δ_I remains the same, because the final state is the same and the strong interactions are invariant under CP.

We can now parametrize the amount of CP violation by two numbers

$$\eta_{+-} = \frac{\langle\pi^+\pi^-|H_w|K_L^0\rangle}{\langle\pi^+\pi^-|H_w|K_S^0\rangle} = \epsilon + \epsilon' \tag{7.141a}$$

$$\eta_{00} = \frac{\langle\pi^0\pi^0|H_w|K_L^0\rangle}{\langle\pi^0\pi^0|H_w|K_S^0\rangle} = \epsilon - 2\epsilon'\,. \tag{7.141b}$$

Using (7.139), one finds

$$\epsilon = \bar\epsilon + \mathrm{i}\xi_0 \tag{7.142a}$$

$$\epsilon' = \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(\delta_2-\delta_0)}}{\sqrt{2}}\frac{A_2}{A_0}(\xi_2-\xi_0)\,. \tag{7.142b}$$

This means that a nonvanishing ϵ can be attributed to the mixing alone, while a nonvanishing ϵ' needs an interference of the isoscalar and isotensor channels and is independent of the mixing.

If $\epsilon' \ll \epsilon$, which turns out to be correct,

$$\frac{|\eta_{+-}|^2}{|\eta_{00}|^2} = 1 + 6\cdot\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) + O\left(\left(\frac{\epsilon'}{\epsilon}\right)^2\right) \tag{7.143}$$

and we can observe ϵ'/ϵ in a double ratio

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) \approx \frac{1}{6}\left(\frac{|\eta_{+-}|^2}{|\eta_{00}|^2} - 1\right) = \frac{1}{6}\left(\frac{\left(\frac{\Gamma(K_L^0\to\pi^+\pi^-)}{\Gamma(K_S^0\to\pi^+\pi^-)}\right)}{\left(\frac{\Gamma(K_L^0\to\pi^0\pi^0)}{\Gamma(K_S^0\to\pi^0\pi^0)}\right)} - 1\right) \tag{7.144}$$

where systematic errors can cancel. Indeed in 1999 finally a consistent measurement was made

$$\frac{\epsilon'}{\epsilon} \approx 1.6 \cdot 10^{-3}\,, \tag{7.145}$$

showing that the weak $\Delta S = 1$ interaction violates CP and a "superweak" $\Delta S = 2$ is not necessary.

7.5.7 B^0-$\overline{B^0}$ and D^0-$\overline{D^0}$

Similar phenomena have been observed in the mixing of neutral B-, B_s- and D-mesons. The formalism is identical and the results confirm the CKM picture. However, the oscillation frequencies, decay rates and their ratios are very different, leading to qualitatively different phenomenologies.

The observed strength of the B^0-$\overline{B^0}$ oscillations, which are driven mostly by the heavy top quark gave the first indication that the top quark could be heavier than the W boson.

The study of D^0-$\overline{D^0}$ oscillations is complicated by two facts of life:

- the masses of the down-type quarks are must closer together and the contribution of the box diagrams is therefore smaller
- the long distance contributions from $D^0 \to K\pi \to \overline{D^0}$ are harder to compute or estimate.

7.6 Higgs Production and Decay

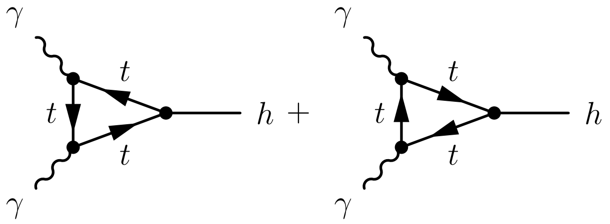
Lecture 22: Tue, 30. 06. 2015

We have learned that in the SM the coupling of Higgs bosons to other particles is proportial to their mass, either via Yukawa couplings or the gauge couplings. This has two consequences

- the Higgs boson will decay predominantly into the heaviest particles available. For $m_h = 126\,\mathrm{GeV}$, these are the ZZ and W^+W^- final states.

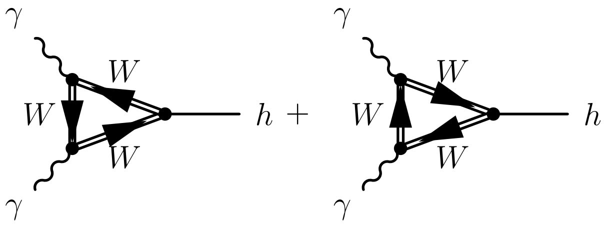
- it is not obvious how to produce Higgs bosons from "ordinary" matter, because the production cross section will be suppressed by a factor of $(m/v)^2 = (m/(4m_W))^2$ relative to other processes.

However, one must not forget loop diagrams for $\gamma\gamma \to h$ involving top quarks



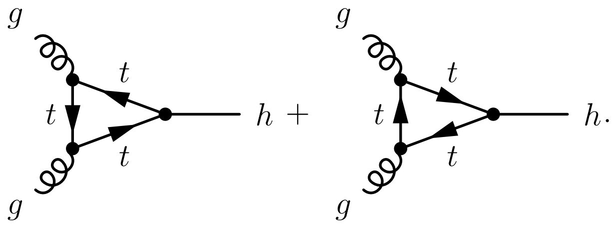
(7.146a)

and $W^\pm$



(7.146b)

as well as top quarks for $gg \to h$



(7.147)

Here the $t\bar{t}h$-coupling is large and can overcome the loop suppression. From naive power counting, the top-quark loops are linearly divergent

$$\int \mathrm{d}^4k \left(\frac{1}{\not{k}}\right)^3 \tag{7.148}$$

and the W loops are logarithmically divergent

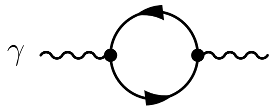
$$\int \mathrm{d}^4k \, k_\mu k_\nu \left(\frac{1}{k^2}\right)^3 . \tag{7.149}$$

If this was indeed the case, the standard model would be in serious trouble, because direct couplings of the Higgs to massless particles of the form

$$\mathcal{L} = c_{h\gamma\gamma} h A_\mu A^\mu + c_{hgg} h G_\mu G^\mu \tag{7.150}$$

are forbidden by the unbroken gauge invariances, but would be required as counter terms.

Fortunately, the naive power counting is not correct. In QED, the diagram



$$= \Pi_{\mu\nu}(p) \tag{7.151}$$

appears to be quadratically divergent, but a more careful analysis reveals that the Ward identity

$$p^\mu \Pi_{\mu\nu}(p) = 0 \tag{7.152}$$

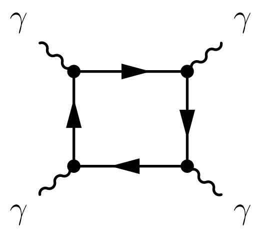
forces $\Pi_{\mu\nu}(p)$ to have the form

$$\Pi_{\mu\nu}(p) = \left(p_\mu p_\nu - p^2 g_{\mu\nu}\right) \Pi(p)\,. \tag{7.153}$$

Since $\Pi(p)$ must, by dimensional analysis, have two additional powers of the loop momentum in the denominator, the diagram is only logarithmically divergent. In addition the counterterm has the form

$$\mathcal{L} = c_{\gamma\gamma} F_{\mu\nu} F^{\mu\nu}\,, \tag{7.154}$$

which is compatible with gauge invariance. Analogously, the diagrams for light-by-light scattering



$$+ \text{ permutations} \tag{7.155}$$

appear to be logarithmically divergent, requiring a A^4-counterterm that is not part of QED. Here it can be shown that the Ward identities allow to extract four powers of momentum, making the diagram convergent. It corresponds to an effective interaction of the form

$$\mathcal{L}_{\text{Euler-Heisenberg}} = \frac{c_{\gamma\gamma\gamma\gamma}}{m_e^4} \left(\frac{\alpha}{\pi}\right)^2 F_{\mu\nu} F^{\nu\kappa} F_{\kappa\lambda} F^{\lambda\mu} + \text{ permutations}\,. \tag{7.156}$$

The same arguments can be used to argue that the effective $h\gamma\gamma$- and hgg- vertices must carry two additional powers of the momenta and are thus finite. Indeed the lowest order effective interactions that we can write down are

$$\mathcal{L}_{h\gamma\gamma} = c_{h\gamma\gamma} \frac{\alpha}{4\pi} \frac{h}{v} F_{\mu\nu} F^{\mu\nu} \tag{7.157a}$$

$$\mathcal{L}_{hgg} = c_{hgg} \frac{\alpha_s}{4\pi} \frac{h}{v} G_{\mu\nu} G^{\mu\nu} , \tag{7.157b}$$

where the $\mathcal{O}(1)$ coefficients $c_{h\gamma\gamma}$ and $c_{h\gamma\gamma}$ can be obtained from performing the loop integrals. Here $F_{\mu\nu}F^{\mu\nu}$ and $G_{\mu\nu}G^{\mu\nu}$ are the only bilinear terms allowed by QED and QCD gauge invariance, the Higgs boson is know to couple in the combination h/v and $\alpha/(4\pi) = e^2/(16\pi^2)$ is the combination of two coupling constants and a loop factor. The mass of the top quark drops out for $m_t \to \infty$, because after extracting the two powers of momentum

$$\int^{m_t} \mathrm{d}^4 k \, \frac{1}{k^5} \propto \frac{1}{m_t} , \tag{7.158}$$

which is cancelled by the m_t/v in the Yukawa coupling.

The existence of these couplings allows the production of Higgs bososn in the fusion of two gluons and its decay into to photons

$$gg \to h \to \gamma\gamma . \tag{7.159}$$

While the corresponding cross section is smaller than

$$gg \to h \to ZZ, W^+W^- , \tag{7.160}$$

it is competetive after accounting for experimental uncertainties.

7.7 Anomaly Cancellation

- classical and quantum symmetries, renormalization
- triangle diagrams
- axial current conservation
- global $\mathrm{SU}(3)_A$: $\pi \to \gamma\gamma$
- gauged $\mathrm{SU}(2)_L$: SM
- $\sum Q \stackrel{!}{=} 0$

—8— Quantum Chromo Dynamics

8.1 Lowest Order Perturbation Theory

- Lagrangian
- spectrum (massless particles)

8.2 Lattice Gauge Theory

- Wilson action

$$S_{\text{Wilson}} = \sum_{\square} \operatorname{tr}\left(P \exp\left(\int_{\square} \mathrm{d}x_\mu A^\mu(x) \right) \right) \tag{8.1}$$

- Monte Carlo
- two point functions and mass gap
- static $\propto |\vec{r}|$ potential
- confinement
- quark gluon plasma

Lecture 23: Wed, 01. 07. 2015

- chiral symmetry breaking
- color strings

8.3 Asymptotic Freedom

- running coupling
- dimensional transmutation, Λ_{QCD}

8.3.1 Jets

- $e^+e^- \to 3j$: qqg-vertex
- clustering, N-jet rate vs. distance measures
- infrared and collinear singularities, KLN-theorem
- final state parton showers
- infrared safe observables
- matrix element and shower matching

8.3.2 Parton Model

- impulse approximation
- justification by asymptotic freedom for large momentum transfer, operator product expansion
- parton distribution functions
- DGLAP evolution
- initial state parton showers

8.4 Hadronization

- independent fragmentation ruled out
- Lund model: string effect, antennae, etc.
- Herwig: clusters

—A— Acronyms

a.k.a. also known as

d.o.f. degrees of freedom

CC Charged Current

FAPP For All Practical Purposes

FCNC Flavor Changing Neutral Currents

iff if and only if

LT Lorentz Transformation

LHS Left Hand Side

NC Neutral Current

OTOH On The Other Hand

PDE Partial Differential Equation

QED Quantum Electro Dynamics

QCD Quantum Chromo Dynamics

QFT Quantum Field Theory

QM Quantum Mechanics

RHS Right Hand Side

SM Standard Model

SSB Spontaneous Symmetry Breaking

vev vacuum expectation value

WLOG Without Loss Of Generality

wrt with respect to

—B— Bibliography

[1] S. Chatrchyan et al., JINST **7**, P10002 (2012).

[2] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press, 1992.

Index