

7. Problemset “Theoretical Particle Physics”

June 5, 2015

SO(4) and $SU(2) \times SU(2)$

7.1 $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$

1. Find a basis for the Lie algebra $\mathfrak{so}(4)$ generating the group $SO(4)$ of orthogonal 4×4 -matrices with unit determinant.
2. Show that generators of $\mathfrak{so}(4)$ can be expressed by linear combinations of generators of two copies of $\mathfrak{su}(2)$. In other words, show that $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$.

7.2 $SO(4)$ vs. $SU(2) \times SU(2)$

Just as $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ does not imply that the groups are isomorphic (just $SO(3) \cong SU(2)/\mathbf{Z}_2$), we can *not* expect $SO(4) \cong SU(2) \times SU(2)$.

1. Find the exact relationship of $SO(4)$ and $SU(2) \times SU(2)$.

7.3 $SO(4)$ vs. $SU(2) \times SU(2)$ redux

Given a set X and a group G , we say that G acts on X from the left, if there is a map

$$\begin{aligned} \triangleright : G \times X &\rightarrow X \\ (g, x) &\mapsto g \triangleright x \end{aligned} \tag{1}$$

and from the right

$$\begin{aligned} \triangleleft : G \times X &\rightarrow X \\ (g, x) &\mapsto x \triangleleft g, \end{aligned} \tag{2}$$

iff these actions are compatible with the group structure

$$(gg') \triangleright x = g \triangleright (g' \triangleright x) \tag{3}$$

and

$$(gg') \triangleleft x = (x \triangleleft g') \triangleleft g \tag{4}$$

etc.

1. Find canonical left and right actions for the special case $X = G$.

2. Show that

$$\begin{aligned} \blacktriangleright: (G \times H) \times X &\rightarrow X \\ ((g, h), x) &\mapsto (g \otimes h) \blacktriangleright x = g \triangleright x \triangleleft h \end{aligned} \tag{5}$$

defines a left action of $G \times G$ on G . Does this construction also work for an action of $G \times H$ on X ?

3. Find a parameterization of the matrices in the set $\text{SU}(2)$ by $\alpha \in \mathbf{R}^4$ with $\alpha^2 = 1$. Use it to show that the action

$$(\text{SU}(2) \otimes \text{SU}(2)) \blacktriangleright \text{SU}(2). \tag{6}$$

can also be understood as an action

$$\text{SO}(4) \blacktriangleright \text{SU}(2). \tag{7}$$

What does this mean for the relationship of $\text{SO}(4)$ and $\text{SU}(2) \times \text{SU}(2)$?