Quantum Field Theory 2.0

Thorsten Ohl
Institut für Theoretische Physik und Astrophysik
Universität Würzburg
Hubland Campus Nord
Emil-Hilb-Weg 22
97074 Würzburg
Germany

Personal Manuscript!
Use at your own peril!

July 5, 2016

Abstract

A set of lectures on semi-advanced quantum field theory.

CONTENTS

1 QUANTUM FIELD THEORY

—1—

QUANTUM FIELD THEORY

Lecture 01: Tue, 11. 04. 2016

[1] [2]

Quantum Field Theory (QFT) plays a dual role:

- "quantum mechanics" of classical field theory, e. g. quantized radiation field in quantum electrodynamics
- quantum mechanics for (infinitely) many particles with creation and annihilation

are described by the *same* formalism[1].

1.1 Classical Field Theory

Configuration space: linear space of all functions ϕ

$$\begin{aligned} \phi : M &\to \mathbf{C} \\ x &\mapsto \phi(x) \end{aligned} \tag{1.1}$$

or rather of all distributions, since we often encounter singularities, e. g. in the Coulomb potential of point charges. Mathematically, the space of all (tempered) distributions is the dual of the space of smooth testfunctions, that (fall off faster than any power for $|x| \to \infty$) have compact support:

$$\begin{aligned} \phi : C^\infty(M) &\to \mathbf{C} \\ f &\mapsto \phi(f) = \int_M \mathrm{d}\mu(x) f(x)\phi(x)\,. \end{aligned} \tag{1.2}$$

---

[1] The second interpretation requires the notion of particle, however, which is not available in general curved background geometries.

In this lecture: $M = \mathbf{R}^4$ with Lorentzian inner product

$$xy = x_\mu x^\mu = x^\mu x_\mu = x^0 y^0 - \vec{x}\vec{y} \tag{1.3}$$

for general curved M *much* more complicated, see Niemeyer/Ohl lecture.

The dynamics of the fields ϕ is governed by second order Partial Differential Equations (PDE), e. g. the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0 \tag{1.4}$$

with appropriate Cauchy data for $\phi(x)$ and $\partial_0\phi(x)$ on a spacelike hypersurface, e. g. $x_0 = 0$.

1.1.1 Action Principle, Euler-Lagrange-Equations

Since the study of coupled nonlinear PDEs is complicated and in particular symmetries are not manifest for multi-component fields, it helps to derive the equation of motion from an action principle:

$$\delta S(\phi_1, \ldots, \phi_n) = \sum_{i=1}^{n} \int \mathrm{d}^4x \, \frac{\delta S}{\delta \phi_i}(\phi_1, \ldots \phi_n, x) \delta\phi_i(x) = 0 \tag{1.5}$$

for all variations $\{\delta\phi_i\}_{i=1,\ldots,n}$ and therefore

$$\frac{\delta S}{\delta \phi_i}(\phi_1, \ldots \phi_n, x) = 0 \,. \tag{1.6}$$

For example the local action for a real field ϕ

$$S(\phi) = \int \mathrm{d}^4x \, \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)\phi(x) - V(\phi(x)) \right) \tag{1.7}$$

leads to

$$0 = \frac{\delta S}{\delta \phi}(\phi, x) = -\Box\phi(x) - m^2\phi(x) - V'(\phi(x)) \,. \tag{1.8}$$

All interesting field equations are second order in time and space, since higher orders lead to problems with causality. The second order field equations have to be combined with Cauchy data for the fields $\{\phi_i(x)\}_{i=1\ldots,n}$ and their first time derivatives $\{\partial_0\phi_i(x)\}_{i=1\ldots,n}$ on a space-like hypersurface ("Cauchy surface").

1.1.2 Canonical Formalism

Second order PDEs can always be reformulated as a larger system of first order PDEs in time.

Classical canonical dynamics for real Klein-Gordon field

$$\begin{aligned} S &= \int \mathrm{d}t \, L(t) && (1.9a) \\ L(t) &= \int_{x^0=t} \mathrm{d}^3\vec{x} \, \mathcal{L}(x) && (1.9b) \\ \mathcal{L}(x) &= \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) - V(\phi(x)) && (1.9c) \end{aligned}$$

canonically conjugate momentum

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}(x) = \partial^0\phi(x) \tag{1.10}$$

Hamiltonian from Legendre transformation

$$\begin{aligned} H(t) &= \int_{x_0=t} \mathrm{d}^3\vec{x}\Big(\pi(x)\partial_0\phi(x) - \mathcal{L}(x)\Big) \\ &= \int_{x_0=t} \mathrm{d}^3\vec{x}\,\frac{1}{2}\Big(\pi^2(x) + \vec{\nabla}\phi(x)\vec{\nabla}\phi(x) + m^2\phi^2(x) + V(\phi(x))\Big) \end{aligned} \tag{1.11}$$

Equations of motion

$$\begin{aligned} \dot{\phi}(t,\vec{x}) &= \{\phi(x), H(t)\} && (1.12a) \\ \dot{\pi}(t,\vec{x}) &= \{\pi(x), H(t)\} && (1.12b) \end{aligned}$$

with Poisson bracket

$$\{f,g\} = \int \mathrm{d}^3x \left(\frac{\delta f}{\delta\phi}(t,\vec{x})\frac{\delta g}{\delta\pi}(t,\vec{x}) - \frac{\delta f}{\delta\pi}(t,\vec{x})\frac{\delta g}{\delta\phi}(t,\vec{x})\right) . \tag{1.13}$$

Equivalent definition: denote the space of all (nonlinear) functionals of ϕ and π with

$$\mathcal{C} = C^\infty(\mathbf{R}^3) \times C^\infty(\mathbf{R}^3) \to \mathbf{C} \,. \tag{1.14}$$

Then the binary operation

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C} \times \mathcal{C} &\to \mathcal{C} \\ (f,g) &\mapsto \{f,g\} \end{aligned} \tag{1.15}$$

is an antisymmetric *derivation*, i. e.

$$\{f,g\} = -\{g,f\} \tag{1.16a}$$

$$\{f,gh\} = g\{f,h\} + \{f,g\}h \tag{1.16b}$$

$$\{f,\alpha g+\beta h\} = \alpha\{f,g\} + \beta\{f,h\} \tag{1.16c}$$

for $\alpha,\beta\in\mathbf{C}$, and we define

$$\{\phi(t,\vec x),\pi(t,\vec y)\} = \delta^3(\vec x-\vec y) \tag{1.17a}$$

$$\{\phi(t,\vec x),\phi(t,\vec y)\} = \{\pi(t,\vec x),\pi(t,\vec y)\} = 0\,. \tag{1.17b}$$

The Poisson braket also satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \tag{1.18}$$

and consequently forms a Lie algebra.

The first order in time canonical equations of motion (1.12) have a unique solution, if initial conitions for the field ϕ and the momentum π are given on a space-like Cauchy surface.

1.2 Quantization

1.2.1 Canonical Quantization

Promote fields to operators in a suitable Hilbert space (more precisely: operator valued distributions) and replace Poisson brackets by commutators

$$[\phi_i(t,\vec x),\pi_j(t,\vec y)] = \mathrm{i}\delta_{ij}\delta^3(\vec x-\vec y) \tag{1.19a}$$

$$[\phi_i(t,\vec x),\phi_j(t,\vec y)] = [\pi_i(t,\vec x),\pi_j(t,\vec y)] = 0 \tag{1.19b}$$

Perturbation theory: split Hamiltonian

$$H = H_0 + V \tag{1.20a}$$

$$H_0 = \int_{x_0=t} \mathrm{d}^3\vec x\,\frac{1}{2}\Big(\pi^2(x) + \vec\nabla\phi(x)\vec\nabla\phi(x) + m^2\phi^2(x)\Big) \tag{1.20b}$$

linear equations of motion resulting from H_0 (“free wave equation”) can be solved by Fourier transform

$$\phi_i(x) = \int\widetilde{\mathrm{d}k}\,\Big(a_i(k)\mathrm{e}^{-\mathrm{i}kx} + a_i^\dagger(k)\mathrm{e}^{\mathrm{i}kx}\Big) \tag{1.21a}$$

$$\pi_i(x) = -\mathrm{i} \int \widetilde{\mathrm{d}k}\, k_0 \left(a_i(k)\mathrm{e}^{-\mathrm{i}kx} - a_i^\dagger(k)\mathrm{e}^{\mathrm{i}kx}\right) \tag{1.21b}$$

with

$$\widetilde{\mathrm{d}k} = \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2k_0}\Big|_{k_0=+\sqrt{\vec{k}^2+m^2}} = \frac{\mathrm{d}^4 k}{(2\pi)^4} 2\pi\Theta(k_0)\delta(k^2-m^2) \tag{1.22}$$

The commutation relations are realized by

$$[a_i(k), a_j^\dagger(k')] = (2\pi)^3 2k_0 \delta_{ij}\delta^3(\vec{k}-\vec{k}') \tag{1.23a}$$

$$[a_i(k), a_j(k')] = [a_i^\dagger(k), a_j^\dagger(k')] = 0 \tag{1.23b}$$

acting on a unique normalized vacuum state $|0\rangle$:

$$\forall i,k : a_i(k)\,|0\rangle = 0, \qquad \langle 0|0\rangle = 1 \tag{1.24}$$

one-particle states

$$|i,k\rangle = a_i^\dagger(k)\,|0\rangle \tag{1.25}$$

normalization

$$\begin{aligned}\langle i,k|j,k'\rangle &= \langle 0|a_i(k)a_j^\dagger(k')|0\rangle \\ &= \langle 0|a_i^\dagger(k')a_j(k)|0\rangle + (2\pi)^3 2k_0\delta_{ij}\delta^3(\vec{k}-\vec{k}')\,\langle 0|0\rangle = (2\pi)^3 2k_0\delta_{ij}\delta^3(\vec{k}-\vec{k}')\end{aligned} \tag{1.26}$$

The n-particle states (only one field, for simplicity)

$$|k_1,k_2,\ldots,k_n\rangle = a^\dagger(k_1)a^\dagger(k_2)\cdots a^\dagger(k_n)\,|0\rangle \tag{1.27}$$

span the n-particle Hilbert space $\mathcal{H}_n = \mathcal{H}^{\otimes_S n}$ and the commutation relations guarantee that only symmetrical states appear for bosons.

With

$$\mathcal{H}_0 = \mathcal{H}_0 = \{c\,|0\rangle : c \in \mathbf{C}\} \tag{1.28}$$

we recover (in the sense of distributions)

$$a^\dagger(k) : \mathcal{H}_n \to \mathcal{H}_{n+1} \tag{1.29a}$$

$$a(k) : \mathcal{H}_n \to \mathcal{H}_{n-1} \tag{1.29b}$$

and the operators act in the Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n = \mathcal{H}_0 \oplus \mathcal{H} \otimes_S \mathcal{H} \oplus \mathcal{H} \otimes_S \mathcal{H} \otimes_S \mathcal{H} \oplus \ldots \tag{1.30}$$

$$a^\dagger(k) : \mathcal{F} \to \mathcal{F} \tag{1.31a}$$

$$a(k) : \mathcal{F} \to \mathcal{F} \tag{1.31b}$$

1.2.2 Perturbation Theory, Interaction Picture and the Gell-Man Low Formula

Split the Hamiltonian in two parts

$$H = H_0 + V \tag{1.32}$$

and define "free fields"

$$\phi^0(x) = \int \widetilde{\mathrm{d}k}\, \left(a(k)\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(k)\mathrm{e}^{\mathrm{i}kx}\right) \tag{1.33}$$

that are defined in a Fock space which is generated by $|0^0\rangle$ with $a(k)\,|0^0\rangle = 0$. Their time evolution is such that

$$\frac{\partial}{\partial x_0}\phi^0(x) = \mathrm{i}[H_0, \phi^0(x)] \tag{1.34}$$

and consequently

$$\phi^0(y) = \mathrm{e}^{\mathrm{i}P^0_\mu(y-x)^\mu}\phi^0(x)\mathrm{e}^{-\mathrm{i}P^0_\mu(y-x)^\mu} \tag{1.35}$$

with $P^0_0 = H_0$. Compare this with the Heisenberg fields $\phi(x)$ with

$$\frac{\partial}{\partial x_0}\phi(x) = \mathrm{i}[H, \phi(x)] \tag{1.36}$$

and

$$\phi(y) = \mathrm{e}^{\mathrm{i}P_\mu(y-x)^\mu}\phi(x)\mathrm{e}^{-\mathrm{i}P_\mu(y-x)^\mu} \tag{1.37}$$

with $P_0 = H$. If H is not quadratic, i. e. the equations of motion not linear, there is no simple splitting in creation and annihilation parts.

Lecture 02: Thu, 14. 04. 2016

Compatibility of matrix elements of Heisenberg and Schrödinger picture field operators

$$\begin{aligned}\langle A|\phi(\vec{x},t)|B\rangle &= \langle A|\mathrm{e}^{\mathrm{i}Ht}\phi(\vec{x},0)\mathrm{e}^{-\mathrm{i}Ht}|B\rangle \\ &= \langle A^S(0)|\mathrm{e}^{\mathrm{i}Ht}\phi^S(\vec{x})\mathrm{e}^{-\mathrm{i}Ht}|B^S(0)\rangle = \langle A^S(t)|\phi^S(\vec{x})|B^S(t)\rangle\end{aligned} \tag{1.38}$$

can be extended to the interaction picture

$$\langle A|\phi(\vec{x},t)|B\rangle = \langle A^0(0)|\phi(\vec{x},t)|B^0(0)\rangle$$

$$= \langle A^0(0)|\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t}\phi^0(x)\mathrm{e}^{\mathrm{i}H_0t}\mathrm{e}^{-\mathrm{i}Ht}|B^0(0)\rangle = \langle A^0(0)|U(t,0)^\dagger\phi^0(x)U(t,0)|B^0(0)\rangle \tag{1.39}$$

where the (formally) unitary operator

$$U(t,t_0) = \mathrm{e}^{\mathrm{i}H_0(t-t_0)}\mathrm{e}^{-\mathrm{i}H(t-t_0)} = ((U(t,t_0))^\dagger)^{-1} \tag{1.40}$$

satisfies

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0) = V(t)U(t,t_0) \tag{1.41a}$$

$$U(t_0,t_0) = \mathbf{1} \tag{1.41b}$$

with a *time dependent* interaction

$$V(t) = \mathrm{e}^{\mathrm{i}H_0t}V(0)\mathrm{e}^{-\mathrm{i}H_0t}\,. \tag{1.42}$$

Note that the time dependence can be described by the time dependence of the interaction picture “free fields”

$$V(t) = V\left(\phi^0\big|_{x_0=t}\right) \tag{1.43}$$

All interaction picture matrix elements can be evaluated using Dyson’s formula

$$U(t,t_0) = \mathrm{T}\,\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t}\mathrm{d}x_0\,\int\mathrm{d}^3\vec x\,V(\phi^0(x))} \tag{1.44}$$

as solution of the Schrödinger equation for the time evolution operator in the interaction picture.

Therefore, if we assume that $|0^0\rangle$ and $|0\rangle$ agree for $t\to-\infty$ we can write

$$\begin{aligned}
&\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \\
&\quad\langle 0^0|\,U^\dagger(t_1,-\infty)\phi^0(x_1)U(t_1,-\infty)U^\dagger(t_2,-\infty)\phi^0(x_2)U(t_2,-\infty) \\
&\qquad\cdots U^\dagger(t_n,-\infty)\phi^0(x_n)U(t_n,-\infty)\,|0^0\rangle \\
&\quad= \langle 0^0|\,U^\dagger(\infty,-\infty)U(\infty,t_1)\phi^0(x_1)U(t_1,t_2)\phi^0(x_2)U(t_2,t_3) \\
&\qquad\cdots U(t_{n-1},t_n)\phi^0(x_n)U(t_n,-\infty)\,|0^0\rangle \\
&\quad= \langle 0^0|U^\dagger(\infty,-\infty)|0^0\rangle\,\langle 0^0|\,U(\infty,t_1)\phi^0(x_1)U(t_1,t_2)\phi^0(x_2)U(t_2,t_3) \\
&\qquad\cdots U(t_{n-1},t_n)\phi^0(x_n)U(t_n,-\infty)\,|0^0\rangle = \\
&\frac{\langle 0^0|U(\infty,t_1)\phi^0(x_1)U(t_1,t_2)\phi^0(x_2)U(t_2,t_3)\cdots U(t_{n-1},t_n)\phi^0(x_n)U(t_n,-\infty)|0^0\rangle}{\langle 0^0|U(\infty,-\infty)|0^0\rangle}\,.
\end{aligned} \tag{1.45}$$

Here we have used that due to energy conservation and unitarity

$$U(\infty,-\infty)\,|0^0\rangle = \mathrm{e}^{-\mathrm{i}\varphi}\,|0^0\rangle \tag{1.46}$$

and therefore

$$\begin{aligned}\langle 0^0|\,U^\dagger(\infty,-\infty) &= \langle 0^0|U^\dagger(\infty,-\infty)|0^0\rangle\,\langle 0^0| = \mathrm{e}^{\mathrm{i}\varphi}\,\langle 0^0| \\ &= \frac{1}{\mathrm{e}^{-\mathrm{i}\varphi}}\,\langle 0^0| = \frac{1}{\langle 0^0|U(\infty,-\infty)|0^0\rangle}\,\langle 0^0|\ .\end{aligned} \tag{1.47}$$

This simplifies in the time ordered case

$$\begin{aligned}&\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = \\ &= \frac{\langle 0^0|\mathrm{T}U(\infty,t_1)\phi^0(x_1)U(t_1,t_2)\phi^0(x_2)U(t_2,t_3)\ldots U(t_{n-1},t_n)\phi^0(x_n)U(t_n,-\infty)|0^0\rangle}{\langle 0^0|U(\infty,-\infty)|0^0\rangle} = \\ &\qquad \frac{\langle 0^0|\mathrm{T}U(\infty,-\infty)\phi^0(x_1)\phi^0(x_2)\cdots\phi^0(x_n)|0^0\rangle}{\langle 0^0|U(\infty,-\infty)|0^0\rangle}\,,\end{aligned} \tag{1.48}$$

which is just the Gell-Man Low formula for Green's functions

$$\langle 0|\,\mathrm{T}\,\phi(x_1)\ldots\phi(x_n)|0\rangle = \frac{\langle 0^0|\,\mathrm{T}\,\mathrm{e}^{-\mathrm{i}\int\mathrm{d}^4x\,V(\phi^0(x))}\phi^0(x_1)\ldots\phi^0(x_n)|0^0\rangle}{\langle 0^0|\,\mathrm{T}\,\mathrm{e}^{-\mathrm{i}\int\mathrm{d}^4x\,V(\phi^0(x))}|0^0\rangle} \tag{1.49}$$

and leads with Wick's theorem to the Feynman rules.

1.2.3 Generating Functionals

Compact expression containing *all* Green's functions of interacting (Heisenberg) fields of a theory

$$\begin{aligned} Z : C^\infty(\mathbf{R}^4) &\to \mathbf{C} \\ j &\mapsto Z(j) = \langle 0|\,\mathrm{T}\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}^4x\,\phi(x)j(x)}|0\rangle\end{aligned} \tag{1.50}$$

such that

$$\langle 0|\,\mathrm{T}\,\phi(x_1)\ldots\phi(x_n)|0\rangle = \lim_{j\to 0}\frac{\delta}{\mathrm{i}\delta j(x_1)}\cdots\frac{\delta}{\mathrm{i}\delta j(x_n)}Z(j) \tag{1.51}$$

with obvious generalization for more than one field:

$$\begin{aligned} Z : \left(C^\infty(\mathbf{R}^4)\right)^{\otimes n} &\to \mathbf{C} \\ (j_1,\ldots,j_n) &\mapsto Z(j_1,\ldots,j_n) = \langle 0|\,\mathrm{T}\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}^4x\,\sum_{i=1}^n\phi_i(x)j_i(x)}|0\rangle\ .\end{aligned} \tag{1.52}$$

Free Fields

For a free scalar field ϕ with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{1.53}$$

we can compute the 2-point Green's function exactly

$$\langle 0|\,\mathrm{T}\,\phi(x)\phi(y)|0\rangle = -\mathrm{i}G_F(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-y)}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon} \tag{1.54}$$

and we find a *closed* expression for the generating functional:

$$Z^0(j) = \mathrm{e}^{\frac{\mathrm{i}}{2}\int\mathrm{d}^4x\mathrm{d}^4y\,j(x)G_F(x-y)j(y)}\,. \tag{1.55}$$

E. g.

$$\begin{aligned}
\langle 0|0\rangle &= \lim_{j\to0} Z(j) = 1 && (1.56a)\\
\langle 0|\,\mathrm{T}\,\phi(x_1)|0\rangle &= \lim_{j\to0}\frac{\delta}{\mathrm{i}\delta j(x_1)}Z(j) && (1.56b)\\
&= \lim_{j\to0}\int\mathrm{d}^4x_2\,G_F(x_1-x_2)j(x_2)Z(j) = 0 &&\\
\langle 0|\,\mathrm{T}\,\phi(x_1)\phi(x_2)|0\rangle &= -\mathrm{i}G_F(x_1-x_2) && (1.56c)\\
\langle 0|\,\mathrm{T}\,\phi(x_1)\phi(x_2)\phi(x_3)|0\rangle &= 0 && (1.56d)\\
\langle 0|\,\mathrm{T}\,\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle &= -G_F(x_1-x_2)G_F(x_3-x_4) &&\\
&\quad - G_F(x_1-x_3)G_F(x_2-x_4) &&\\
&\quad - G_F(x_1-x_4)G_F(x_2-x_3) && (1.56e)\\
&\ldots && (1.56f)\\
\langle 0|\,\mathrm{T}\,\phi(x_1)\ldots\phi(x_{2n+1})|0\rangle &= 0 && (1.56g)\\
&\ldots && (1.56h)
\end{aligned}$$

Interacting Fields and Feynman Rules

Ignoring the vacuum-to-vacuum diagrams in the denominator of the Gell-Man Low formula (1.49), we can use

$$\mathrm{T}\,\phi^0(x)\mathrm{e}^{\mathrm{i}\int\mathrm{d}^4x'\,\phi^0(x')j(x')} = \frac{\delta}{\mathrm{i}\delta j(x)}\,\mathrm{T}\,\mathrm{e}^{\mathrm{i}\int\mathrm{d}^4x'\,\phi^0(x')j(x')} \tag{1.57}$$

to formally write the generating functional for interacting fields

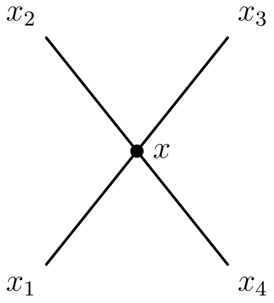
$$
\begin{aligned}
Z(j) &= \langle 0^0 |\, \mathrm{T}\, \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, V(\phi^0(x))} \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \phi^0(x) j(x)} | 0^0 \rangle \\
&= \langle 0^0 |\, \mathrm{T}\, \mathrm{e}^{\mathrm{i}S_I(\phi^0)} \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \phi^0(x) j(x)} | 0^0 \rangle = \langle 0^0 |\, \mathrm{T}\, \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \phi^0(x) j(x)} | 0^0 \rangle \\
&= \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} \langle 0^0 |\, \mathrm{T}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \phi^0(x) j(x)} | 0^0 \rangle = \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} Z^0(j) \qquad (1.58)
\end{aligned}
$$

which also results in the Feynman rules.

Examples: for $V(\phi) = g\phi^4/4!$ with $\lim_{j\to 0}$ implied

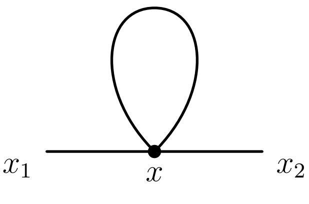
1.

$$
\begin{aligned}
\langle 0 |\, \mathrm{T}\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle &= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_4)} Z(j) \\
&= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_4)} \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} Z^0(j) \\
&= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_4)} \mathrm{i} \int \mathrm{d}^4x\, \frac{g}{4!} \left(\frac{\delta}{\mathrm{i}\delta j(x)} \right)^4 Z^0(j) + \mathcal{O}(g^2) + \text{disc.} \\
&= \mathrm{i} \int \mathrm{d}^4x\, g G_F(x_1 - x) G_F(x_2 - x) G_F(x_3 - x) G_F(x_4 - x) + \mathcal{O}(g^2) + \text{disc.} \qquad (1.59)
\end{aligned}
$$



2.

$$
\begin{aligned}
\langle 0 |\, \mathrm{T}\, \phi(x_1)\phi(x_2) | 0 \rangle &= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} Z(j) = \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \mathrm{e}^{\mathrm{i}S_I(\frac{\delta}{\mathrm{i}\delta j})} Z^0(j) \\
&= \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \mathrm{i} \int \mathrm{d}^4x\, \frac{g}{4!} \left(\frac{\delta}{\mathrm{i}\delta j(x)} \right)^4 Z^0(j) + \mathcal{O}(g^2) \\
&= \mathrm{i} \int \mathrm{d}^4x\, \frac{g}{2} G_F(x_1 - x) G_F(x_2 - x) G_F(x - x) + \mathcal{O}(g^2) \qquad (1.60)
\end{aligned}
$$



NB: $G_F(x-x) = G_F(0)$ is not well defined and leads to divergencies in perturbation theory, which will be the subject of chapter 2.

1.3 Pathintegral

The generating functional of all Green's functions can be expressed as an integral over all field configurations that are compatible with the boundary conditions in the past and in the future:

$$Z(j) = \int \mathcal{D}\varphi \, \mathrm{e}^{\mathrm{i}S(\varphi)+\mathrm{i}\int \mathrm{d}^4x\, \varphi(x)j(x)} \,. \tag{1.61}$$

A mathematically rigorous definition of the integration measure $\mathcal{D}\varphi$ in (1.61) is *not* trivial and has so far only been achieved in $2+1$ space-time dimensions.

1.3.1 Gaussian Integrals

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-x^2} = \sqrt{\pi} \tag{1.62}$$

(proof: compute I^2 and use polar coordinates). By translation invariance and rescaling, we find

$$I(a,b) = \int \mathrm{d}x \, \mathrm{e}^{-a(x-b)^2} = \sqrt{\frac{\pi}{a}} \tag{1.63}$$

and in higher dimensions

$$I(A) = \int \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}(x,Ax)} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \tag{1.64}$$

with $(x,Ay) = \sum_{i=1}^n x_i A_{ij} y_j$ and A real symmetric and positive (proof: A can be diagonalized with an orthogonal transformation that leaves the measure $\mathrm{d}^n x$ invariant). Finally,

$$I(A,j) = \int \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}(x,Ax)+(j,x)} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \, \mathrm{e}^{\frac{1}{2}(j,A^{-1}j)} \tag{1.65}$$

Proof: complete the square: $x \to y = x - A^{-1}j$

$$I(A,j) = \int \mathrm{d}^n y \, \mathrm{e}^{-\frac{1}{2}(y,Ay)+\frac{1}{2}(j,A^{-1}j)} = \mathrm{e}^{\frac{1}{2}(j,A^{-1}j)} \int \mathrm{d}^n y \, \mathrm{e}^{-\frac{1}{2}(y,Ay)} \tag{1.66}$$

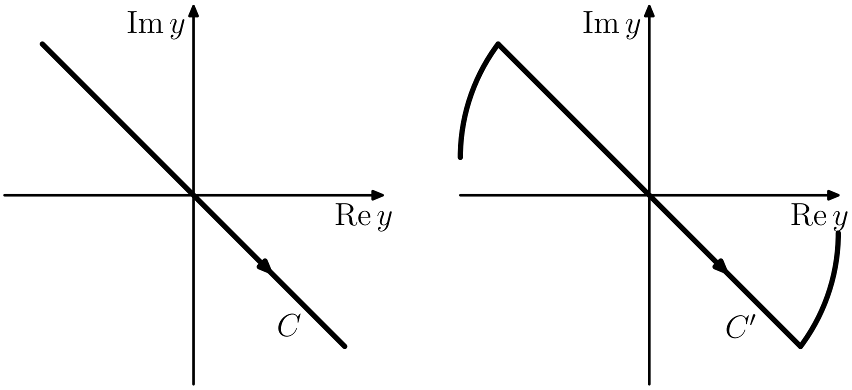


Figure 1.1: *Integration paths in (1.67)*

.

and use (1.64).

Also for imaginary exponents (see figure 1.1)

$$\begin{aligned}I'(a) = I(-\mathrm{i}a,0) &= \int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}ax^2} = \lim_{\epsilon\to0+}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}(a+\mathrm{i}\epsilon)x^2} \\ &\overset{y=\mathrm{e}^{-\mathrm{i}\pi/4}x}{=} \lim_{\epsilon\to0+}\mathrm{e}^{\mathrm{i}\pi/4}\int_C \mathrm{d}y\,\mathrm{e}^{-(a+\mathrm{i}\epsilon)y^2} = \lim_{\epsilon\to0+}\mathrm{e}^{\mathrm{i}\pi/4}\int_{C'}\mathrm{d}y\,\mathrm{e}^{-(a+\mathrm{i}\epsilon)y^2} \\ &= \lim_{\epsilon\to0+}\mathrm{e}^{\mathrm{i}\pi/4}\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-(a+\mathrm{i}\epsilon)y^2} = \lim_{\epsilon\to0+}\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\frac{\pi}{(a+\mathrm{i}\epsilon)}}\,. \end{aligned} \tag{1.67}$$

1.3.2 Functional Integrals

Lecture 03: Tue, 19. 04. 2016

In particular

$$\frac{I(A,j)}{I(A_0,0)} = \sqrt{\frac{\det A_0}{\det A}}\,\mathrm{e}^{\frac{1}{2}(j,A^{-1}j)} \tag{1.68}$$

which can (formally) be extended to the infinite dimensional case

$$Z_E(j) = \frac{\int\mathrm{d}\mu(x)\,\mathrm{e}^{-\frac{1}{2}(x,Ax)+(j,x)}}{\int\mathrm{d}\mu(x)\,\mathrm{e}^{-\frac{1}{2}(x,A_0x)}} = \sqrt{\frac{\det A_0}{\det A}}\,\mathrm{e}^{\frac{1}{2}(j,A^{-1}j)} \tag{1.69}$$

with $\mathrm{d}\mu(x)$ a suitable measure and $(\cdot,\cdot)$ a suitable inner product and A a linear operator. Initially, A is selfadoint and positive, but the formulae can again be extended by analytic continuation:

$$\begin{aligned}I'(A,j) = I(-\mathrm{i}A,\mathrm{i}j) = \int \mathrm{d}^n x\, \mathrm{e}^{\frac{\mathrm{i}}{2}(x,Ax)+\mathrm{i}(j,x)} = \int \mathrm{d}^n x\, \mathrm{e}^{\frac{\mathrm{i}}{2}(x,(A+\mathrm{i}\epsilon)x)+\mathrm{i}(j,x)} \\ = \int \mathrm{d}^n y\, \mathrm{e}^{\frac{\mathrm{i}}{2}(y,(A+\mathrm{i}\epsilon)y)+\frac{\mathrm{i}}{2}(j,(A+\mathrm{i}\epsilon)^{-1}j)-\mathrm{i}(j,(A+\mathrm{i}\epsilon)^{-1}j)} = \frac{(2\pi)^{n/2}\mathrm{e}^{n\mathrm{i}\pi/4}}{\sqrt{\det(A+\mathrm{i}\epsilon)}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(j,(A+\mathrm{i}\epsilon)^{-1}j)}\end{aligned} \tag{1.70}$$

i. e.

$$Z(j) = \frac{\int \mathrm{d}\mu(x)\, \mathrm{e}^{\frac{\mathrm{i}}{2}(x,Ax)+\mathrm{i}(j,x)}}{\int \mathrm{d}\mu(x)\, \mathrm{e}^{\frac{\mathrm{i}}{2}(x,A_0x)}} = \sqrt{\frac{\det(A_0+\mathrm{i}\epsilon)}{\det(A+\mathrm{i}\epsilon)}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(j,(A+\mathrm{i}\epsilon)^{-1}j)} \tag{1.71}$$

Free Fields

Example:

$$A = -\Box - m^2 \tag{1.72a}$$

$$(f,g) = \int \mathrm{d}^4x\, f^*(x)g(x) \tag{1.72b}$$

$$((A+\mathrm{i}\epsilon)^{-1}f)(x) = -\int \mathrm{d}^4y\, G_F(x-y)f(y),\ (\Box+m^2)G_F(x) = \delta^4(x) \tag{1.72c}$$

Then

$$Z(j) = \frac{\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S(\varphi)+\mathrm{i}\int \mathrm{d}^4x\,\varphi(x)j(x)}}{\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S(\varphi)}} = \mathrm{e}^{\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)G_F(x-y)j(y)} \tag{1.73}$$

1.3.3 Formal Derivation in the Schrödinger Picture

Using quantum mechanics as an example with obvious generalization to QFT.

Hamiltonian Path Integral

$$\begin{aligned}\langle q+\delta q|\mathrm{e}^{-\mathrm{i}H\delta t}|q\rangle &= \int \mathrm{d}p\ \langle q+\delta q|p\rangle\,\langle p|\mathrm{e}^{-\mathrm{i}H\delta t}|q\rangle \\ &= \int \mathrm{d}p\ \langle q+\delta q|p\rangle\,\langle p|q\rangle\,\mathrm{e}^{-\mathrm{i}H(q,p)\delta t} = \int \frac{\mathrm{d}p}{2\pi}\,\mathrm{e}^{\mathrm{i}\delta qp}\mathrm{e}^{-\mathrm{i}H(q,p)\delta t}\end{aligned} \tag{1.74}$$

where

$$\langle p|H|q\rangle = H(q,p)\,\langle p|q\rangle \tag{1.75}$$

for *normal ordered* H, i. e. all p to the left of all q. Therefore

$$\langle q + \dot{q}\delta t | \mathrm{e}^{-\mathrm{i}H\delta t} | q \rangle = \int \frac{\mathrm{d}p}{2\pi} \, \mathrm{e}^{\mathrm{i}(p\dot{q} - H(q,p))\delta t} \,. \tag{1.76}$$

Finite intervals

$$\begin{aligned} &\langle q_N | \mathrm{T}\mathrm{e}^{-\mathrm{i}\int \mathrm{d}t\, H} | q_0 \rangle = \\ &\int \mathrm{d}q_{N-1} \ldots \mathrm{d}q_1 \; \langle q_N | \mathrm{T}\mathrm{e}^{-\mathrm{i}\int_{t_{N-1}}^{t_N} \mathrm{d}t\, H} | q_{N-1} \rangle \, \langle q_{N-1} | \cdots | q_1 \rangle \, \langle q_1 | \mathrm{T}\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t_1} \mathrm{d}t\, H} | q_0 \rangle \\ &\qquad = \int \frac{\mathrm{d}q_{N-1}\mathrm{d}p_{N-1}}{2\pi} \ldots \frac{\mathrm{d}q_1 \mathrm{d}p_1}{2\pi} \, \mathrm{e}^{\mathrm{i}\int_{t_{N-1}}^{t_N} \mathrm{d}t\, (p\dot{q} - H)} \ldots \mathrm{e}^{\mathrm{i}\int_{t_0}^{t_1} \mathrm{d}t\, (p\dot{q} - H)} \end{aligned} \tag{1.77}$$

$$\langle q_2 | \mathrm{T}\mathrm{e}^{-\mathrm{i}\int \mathrm{d}t\, H} | q_1 \rangle = \mathcal{N} \int \mathcal{D}q \mathcal{D}p \, \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\, (p\dot{q} - H(q,p))} \tag{1.78}$$

Lagrangian Path Integral

Quadratic Hamiltonians (non-trivial in other cases)

$$\begin{aligned} \langle q + \dot{q}\delta t | \mathrm{e}^{-\mathrm{i}\left(\frac{1}{2a}p^2 + V(q)\right)\delta t} | q \rangle &= \int \frac{\mathrm{d}p}{2\pi} \, \mathrm{e}^{\mathrm{i}\left(p\dot{q} - \frac{1}{2a}p^2 - V(q)\right))\delta t} \\ &= \mathrm{e}^{\mathrm{i}\pi/4} \sqrt{\frac{-a}{2\pi\delta t}} \, \mathrm{e}^{\mathrm{i}\left(\frac{a}{2}\dot{q}^2 - V(q)\right)\delta t} \end{aligned} \tag{1.79}$$

i. e.

$$\langle q + \dot{q}\delta t | \mathrm{e}^{-\mathrm{i}H\delta t} | q \rangle = \mathcal{N}' \mathrm{e}^{\mathrm{i}S(q,\dot{q})} \tag{1.80}$$

and

$$\langle q_2 | \mathrm{T}\mathrm{e}^{-\mathrm{i}\int \mathrm{d}t\, H} | q_1 \rangle = \mathcal{N}'' \int \mathcal{D}q \, \mathrm{e}^{\mathrm{i}S(q,\dot{q})} \tag{1.81}$$

1.3.4 Applications

Simple Lagrangian:

$$\mathcal{L}(\phi, \chi) = \mathcal{L}_\phi(\phi) + \mathcal{L}_\chi(\chi) + \mathcal{L}_I(\phi, \chi) \tag{1.82}$$

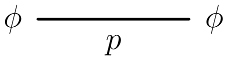
with

$$\mathcal{L}_\phi(\phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m_\phi^2}{2}\phi^2 \tag{1.83a}$$

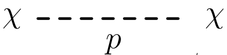
$$\mathcal{L}_\chi(\chi) = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{m_\chi^2}{2}\chi^2 \tag{1.83b}$$

$$\mathcal{L}_I(\phi,\chi) = -\frac{g}{2}\phi^2\chi \tag{1.83c}$$

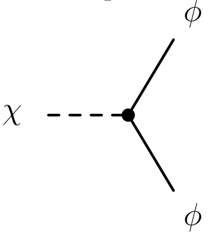
Feynman rules:



$$= \frac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon} \tag{1.84a}$$



$$= \frac{\mathrm{i}}{p^2 - m_\chi^2 + \mathrm{i}\epsilon} \tag{1.84b}$$



$$= -\mathrm{i}g \tag{1.84c}$$

Lagrangian path integral for the generating functional:

$$Z(j_\phi, j_\chi) = \frac{\int\mathcal{D}\varphi\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S(\varphi,\chi)+\mathrm{i}\int\mathrm{d}^4x\,(\varphi(x)j_\phi(x)+\chi(x)j_\chi(x))}}{\int\mathcal{D}\varphi\mathcal{D}\chi\,\mathrm{e}^{\mathrm{i}S(\varphi,\chi)}} \tag{1.85}$$

Hubbard-Stratonovich Transformation

If no insertions of χ are required (i. e. no external χs), we can set $j_\chi = 0$:

$$\begin{aligned} Z(j_\phi, 0) &= \frac{\int\mathcal{D}\varphi\,\mathrm{e}^{\mathrm{i}S_\phi(\varphi)+\mathrm{i}\int\mathrm{d}^4x\,\varphi(x)j_\phi(x)}\;\int\mathcal{D}\chi\,\mathrm{e}^{\mathrm{i}S_\chi(\chi)+\mathrm{i}S_I(\varphi,\chi)}}{\int\mathcal{D}\varphi\,\mathrm{e}^{\mathrm{i}S_\phi(\varphi)}\;\int\mathcal{D}\chi\,\mathrm{e}^{\mathrm{i}S_\chi(\chi)+\mathrm{i}S_I(\varphi,\chi)}} \\ &= \frac{\int\mathcal{D}\varphi\,\mathrm{e}^{\mathrm{i}S_\phi(\varphi)+\mathrm{i}\Gamma_1(\varphi)+\mathrm{i}\int\mathrm{d}^4x\,\varphi(x)j_\phi(x)}}{\int\mathcal{D}\varphi\,\mathrm{e}^{\mathrm{i}S_\phi(\varphi)+\mathrm{i}\Gamma_1(\varphi)}} \end{aligned} \tag{1.86}$$

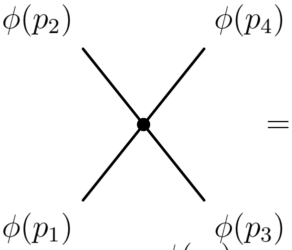
with

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\Gamma_1(\varphi)} &= \frac{\int\mathcal{D}\chi\,\mathrm{e}^{\mathrm{i}S_\chi(\chi)+\mathrm{i}S_I(\varphi,\chi)}}{\int\mathcal{D}\chi\,\mathrm{e}^{\mathrm{i}S_\chi(\chi)}} \\ &= Z\left(j_\chi = -\frac{g}{2}\varphi^2\right) = \exp\left(\mathrm{i}\frac{g^2}{8}\int\mathrm{d}^4x\mathrm{d}^4y\,\varphi^2(x)G_F(x-y)\varphi^2(y)\right) \end{aligned} \tag{1.87}$$

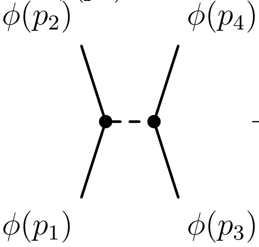
i. e.

$$S_\phi(\varphi)+\Gamma_1(\varphi) = \int\mathrm{d}^4x\,\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m_\phi^2}{2}\varphi^2\right) + \frac{g^2}{8}\int\mathrm{d}^4x\mathrm{d}^4y\,\varphi^2(x)G_F(x-y)\varphi^2(y) \tag{1.88}$$

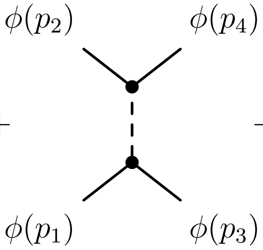
Nonlocal, but perfectly well defined effective interaction:



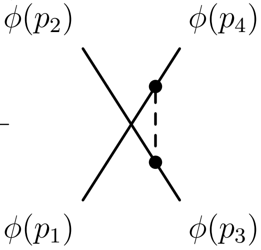
$$= -\mathrm{i}g^2 \left(\begin{matrix} \dfrac{1}{(p_1+p_2)^2 - m_\chi^2 + \mathrm{i}\epsilon} \\ + \dfrac{1}{(p_1+p_3)^2 - m_\chi^2 + \mathrm{i}\epsilon} \\ + \dfrac{1}{(p_1+p_4)^2 - m_\chi^2 + \mathrm{i}\epsilon} \end{matrix} \right) =$$



$+$



$+$



(1.89)

Coleman-Weinberg Potential

If no insertions of ϕ are required (i. e. no external ϕ), we can set $j_\phi = 0$:

$$Z(0, j_\chi) = \frac{\int\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi) + \mathrm{i}\int \mathrm{d}^4x\, \chi(x) j_\chi(x)} \int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_\phi(\varphi) + \mathrm{i}S_I(\varphi,\chi)}}{\int\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi)} \int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_\phi(\varphi) + \mathrm{i}S_I(\varphi,\chi)}} \\ = \frac{\int\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi) + \mathrm{i}\Gamma_2(\chi) + \mathrm{i}\int \mathrm{d}^4x\, \chi(x) j_\chi(x)}}{\int\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi) + \mathrm{i}\Gamma_2(\chi)}} \tag{1.90}$$

with

$$\mathrm{e}^{\mathrm{i}\Gamma_2(\chi)} = \frac{\int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_\phi(\varphi) + \mathrm{i}S_I(\varphi,\chi)}}{\int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S_\phi(\varphi)}} = \sqrt{\frac{\det\left(\Box + m_\phi^2 - \mathrm{i}\epsilon\right)}{\det\left(\Box + m_\phi^2 + g\chi - \mathrm{i}\epsilon\right)}} \\ = \frac{1}{\sqrt{\det\left(1 + g\frac{\chi}{\Box + m_\phi^2 - \mathrm{i}\epsilon}\right)}} = \exp\left(-\frac{1}{2}\operatorname{tr}\ln\left(1 + g\frac{\chi}{\Box + m_\phi^2 - \mathrm{i}\epsilon}\right)\right) \tag{1.91}$$

using

$$\det \mathrm{e}^A = \mathrm{e}^{\operatorname{tr} A} \tag{1.92}$$

i. e.

$$S_\chi(\chi) + \Gamma_2(\chi) = \int \mathrm{d}^4x\, \left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{m_\chi^2}{2}\chi^2\right) + \frac{\mathrm{i}}{2}\operatorname{tr}\ln\left(1 + g\frac{\chi}{\Box + m_\phi^2 - \mathrm{i}\epsilon}\right) \tag{1.93}$$

Using

$$\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \tag{1.94}$$

we can write

$$\begin{aligned}\Gamma_2(\chi) = \mathrm{i}\frac{g}{2}\operatorname{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right) - \mathrm{i}\frac{g^2}{4}\operatorname{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right)\\ + \mathrm{i}\frac{g^3}{6}\operatorname{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right) + \ldots\end{aligned} \tag{1.95}$$

The trace of integral kernels

$$\left(\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}f\right)(x) = \int \mathrm{d}^4y\, G_F(x-y)f(y) \tag{1.96}$$

or

$$\left\langle x\middle|\left(\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right)\middle|y\right\rangle = G_F(x-y) \tag{1.97}$$

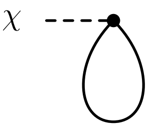
therefore

$$\operatorname{tr}(\chi G_F) = \int \mathrm{d}^4x\ \langle x|\chi G_F|x\rangle = \int \mathrm{d}^4x\,\chi(x)G_F(x-x) \tag{1.98a}$$

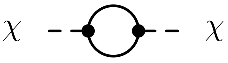
$$\begin{aligned}\operatorname{tr}(\chi G_F\chi G_F)) &= \int \mathrm{d}^4x\mathrm{d}^4y\ \langle x|\chi G_F|y\rangle\,\langle y|\chi G_F|x\rangle\\ &= \int \mathrm{d}^4x\mathrm{d}^4y\,\chi(x)G_F(x-y)\chi(y)G_F(y-x)\end{aligned} \tag{1.98b}$$

etc.

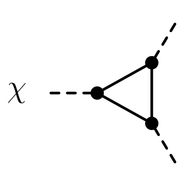
After Fourier transform, we find the following effective vertices:



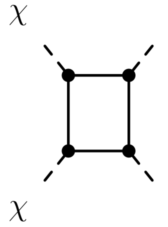
$$= -\frac{g}{2}\operatorname{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right) \tag{1.99a}$$



$$= \frac{g^2}{2}\operatorname{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right) \tag{1.99b}$$



$$= -\frac{g^3}{2}\,\mathrm{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\right) \tag{1.99c}$$



$$= \frac{g^4}{2}\,\mathrm{tr}\left(\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\chi\frac{1}{\Box+m_\phi^2-\mathrm{i}\epsilon}\cdots\right) \tag{1.99d}$$

$$\ldots = \ldots \tag{1.99e}$$

1.3.5 Fermions

Lecture 04: Thu, 21. 04. 2016

So far, we've only dealt with bosons and the Green's functions will always have bosonic symmetry, never fermionic antisymmetry.

Grassmann Numbers

Introduce a set of *anticommuting* numbers

$$\{\theta_i\} : \theta_i\theta_j = -\theta_j\theta_i \tag{1.100a}$$

and in particular

$$\theta_i^2 = 0\,. \tag{1.100b}$$

The polynomials in θ_i form the *Grassmann algebra.*

Grassmann Calculus

Naturally $(a, b \in \mathbf{C})$

$$\frac{\partial}{\partial\theta_i}(a+b\theta_j) = b\delta_{ij} \tag{1.101}$$

and the derivatives must anticommute themselves:

$$\left[\frac{\partial}{\partial\theta_i}, (a+b\theta_j)\right]_+ = b\delta_{ij} \tag{1.102a}$$

$$\left[\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right]_+ = 0 \tag{1.102b}$$

Less familiar

$$\int \mathrm{d}\theta_i \left(a + b\theta_j\right) = b\delta_{ij} \tag{1.103}$$

i. e. *integration and differentiation are the same*! Might be surprising, but its the only linear functional that makes sense and preserves the *Grassmann parity* (even or odd number of θ_i in a product).

Gaussian Grassmann Integration

All power series terminate, e. g.:

$$\int \mathrm{d}\theta \mathrm{d}\bar{\theta}\, \mathrm{e}^{\bar{\theta}M\theta} = \int \mathrm{d}\bar{\theta}\mathrm{d}\theta \, \left(1 + \bar{\theta}M\theta\right) = M \tag{1.104}$$

more generally

$$\begin{aligned}\int \mathrm{d}\theta_1 \mathrm{d}\bar{\theta}_1 \ldots \mathrm{d}\theta_n \mathrm{d}\bar{\theta}_n \, \mathrm{e}^{\sum_{ij} \bar{\theta}_i M_{ij} \theta_j} \\ = \int \mathrm{d}\theta_1 \mathrm{d}\bar{\theta}_1 \ldots \mathrm{d}\theta_n \mathrm{d}\bar{\theta}_n \, \frac{1}{n!} \left(\sum_{ij} \bar{\theta}_i M_{ij} \theta_j \right)^n \\ = \int \mathrm{d}\theta_1 \mathrm{d}\bar{\theta}_1 \ldots \mathrm{d}\theta_n \mathrm{d}\bar{\theta}_n \, \frac{1}{n!} \left(\prod_i \bar{\theta}_i \theta_i M_{ii} + \text{permutations} \right) \\ = \det M\end{aligned} \tag{1.105}$$

e. g. for two pairs

$$\left(\bar{\theta}M\theta\right) = \bar{\theta}_1\theta_1 M_{11} + \bar{\theta}_1\theta_2 M_{12} + \bar{\theta}_2\theta_1 M_{21} + \bar{\theta}_2\theta_2 M_{22} \tag{1.106}$$

and

$$\begin{aligned}\frac{1}{2}\left(\bar{\theta}M\theta\right)^2 &= \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 M_{11}M_{22} + \bar{\theta}_1\theta_2\bar{\theta}_2\theta_1 M_{12}M_{21} \\ &= \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 M_{11}M_{22} - \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 M_{12}M_{21} \\ &= \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 \left(M_{11}M_{22} - M_{12}M_{21}\right) = \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 \, \det M\end{aligned} \tag{1.107}$$

Finally

$$\begin{aligned}\int \mathrm{d}\theta_1 \mathrm{d}\bar{\theta}_1 \ldots \mathrm{d}\theta_n \mathrm{d}\bar{\theta}_n \, \mathrm{e}^{\sum_{ij} \bar{\theta}_i M_{ij} \theta_j + \sum_i \bar{\chi}_i \theta_i + \sum_i \bar{\theta}_i \chi_i} \\ = \det M \cdot \mathrm{e}^{-\sum_{ij} \bar{\chi}_i (M^{-1})_{ij} \chi_j}\end{aligned} \tag{1.108}$$

after completing the square as before.

Path Integral

$$Z(\eta,\bar\eta) = \int \mathcal{D}\psi\mathcal{D}\bar\psi\, \mathrm{e}^{\mathrm{i}S(\psi,\bar\psi)+\mathrm{i}\int \mathrm{d}^4x\,(\bar\psi(x)\eta(x)+\bar\eta(x)\psi(x))} \tag{1.109}$$

Example

Simple Lagrangian:

$$\mathcal{L}(\psi,\chi) = \mathcal{L}_\psi(\psi) + \mathcal{L}_\chi(\chi) + \mathcal{L}_I(\psi,\chi) \tag{1.110}$$

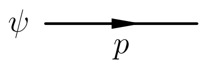
with

$$\mathcal{L}_\psi(\psi) = \bar\psi\,(\mathrm{i}\not\partial - m_\psi)\,\psi \tag{1.111a}$$

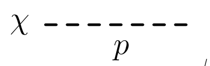
$$\mathcal{L}_\chi(\chi) = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{m_\chi^2}{2}\chi^2 \tag{1.111b}$$

$$\mathcal{L}_I(\psi,\chi) = -g\bar\psi\psi\chi \tag{1.111c}$$

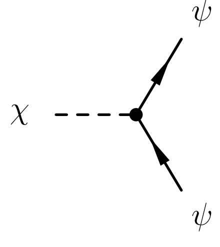
Feynman rules:



$$\psi = \frac{\mathrm{i}}{\not p - m_\psi + \mathrm{i}\epsilon} \tag{1.112a}$$



$$\chi = \frac{\mathrm{i}}{p^2 - m_\chi^2 + \mathrm{i}\epsilon} \tag{1.112b}$$



$$= -\mathrm{i}g \tag{1.112c}$$

Lagrangian path integral for the generating functional:

$$Z(\eta,\bar\eta,j) = \frac{\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S(\psi,\chi)+\mathrm{i}\int \mathrm{d}^4x\,(\bar\psi(x)\eta(x)+\bar\eta(x)\psi(x)+\chi(x)j(x))}}{\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S(\psi,\chi)}} \tag{1.113}$$

If no insertions of ψ are required (i. e. no external ψ), we can set $\eta = \bar\eta = 0$:

$$\begin{aligned} Z(0,0,j) &= \frac{\int \mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi)+\mathrm{i}\int \mathrm{d}^4x\,\chi(x)j(x)}\, \int \mathcal{D}\psi\mathcal{D}\bar\psi\, \mathrm{e}^{\mathrm{i}S_\psi(\psi)+\mathrm{i}S_I(\psi,\chi)}}{\int \mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi)}\, \int \mathcal{D}\psi\mathcal{D}\bar\psi\, \mathrm{e}^{\mathrm{i}S_\psi(\psi)+\mathrm{i}S_I(\psi,\chi)}} \\ &= \frac{\int \mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi)+\mathrm{i}\Gamma_3(\chi)+\mathrm{i}\int \mathrm{d}^4x\,\chi(x)j(x)}}{\int \mathcal{D}\chi\, \mathrm{e}^{\mathrm{i}S_\chi(\chi)+\mathrm{i}\Gamma_3(\chi)}} \end{aligned} \tag{1.114}$$

with

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\Gamma_3(\chi)} &= \frac{\int\mathcal{D}\psi\mathcal{D}\bar\psi\,\mathrm{e}^{\mathrm{i}S_\psi(\psi)+\mathrm{i}S_I(\psi,\chi)}}{\int\mathcal{D}\psi\mathcal{D}\bar\psi\,\mathrm{e}^{\mathrm{i}S_\psi(\psi)}} = \frac{\det\left(\mathrm{i}\partial\!\!\!/ - m_\psi - g\chi + \mathrm{i}\epsilon\right)}{\det\left(\mathrm{i}\partial\!\!\!/ - m_\psi + \mathrm{i}\epsilon\right)} \\ &= \det\left(1 - g\frac{\chi}{\mathrm{i}\partial\!\!\!/ - m_\psi + \mathrm{i}\epsilon}\right) = \exp\left(+\operatorname{tr}\ln\left(1 - g\frac{\chi}{\mathrm{i}\partial\!\!\!/ - m_\psi + \mathrm{i}\epsilon}\right)\right)\end{aligned} \tag{1.115}$$

i. e.

$$S_\chi(\chi) + \Gamma_3(\chi) = \int\mathrm{d}^4x\,\left(\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{m_\chi^2}{2}\chi^2\right) - \mathrm{i}\operatorname{tr}\ln\left(1 - g\frac{\chi}{\mathrm{i}\partial\!\!\!/ - m_\psi + \mathrm{i}\epsilon}\right) \tag{1.116}$$

where the trace that describes the one-loop diagrams has received a factor of (-2): (-1) from Fermi statistics and 2 for the distinction of particles and antiparticles.

1.4 LSZ Reduction Formulae

1.4.1 Källén-Lehmann Representation

The momentum operator P_μ

$$\phi(x) = \mathrm{e}^{\mathrm{i}P_\mu(x-y)^\mu}\phi(y)\mathrm{e}^{-\mathrm{i}P_\mu(x-y)^\mu}\,. \tag{1.117}$$

with a translation invariant ground state

$$P_\mu\left|0\right\rangle = 0 \tag{1.118}$$

possesses a resolution of unity

$$\mathbf{1} = \sum_\alpha\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\left|p,\alpha\right\rangle\left\langle p,\alpha\right| \tag{1.119a}$$

$$P_\mu\left|p,\alpha\right\rangle = p_\mu\left|p,\alpha\right\rangle \tag{1.119b}$$

where we can restrict the integral by causality and the energy condition to

$$p_0 \ge 0 \land p^2 \ge 0\,. \tag{1.120}$$

Then we find

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|\mathrm{e}^{\mathrm{i}P_\mu x^\mu}\phi(0)\mathrm{e}^{-\mathrm{i}P_\mu x^\mu}\mathrm{e}^{\mathrm{i}P_\mu y^\mu}\phi(0)\mathrm{e}^{-\mathrm{i}P_\mu y^\mu}|0\rangle \\ &= \sum_\alpha\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\langle 0|\mathrm{e}^{\mathrm{i}P_\mu x^\mu}\phi(0)\mathrm{e}^{-\mathrm{i}P_\mu x^\mu}|p,\alpha\rangle\,\langle p,\alpha|\mathrm{e}^{\mathrm{i}P_\mu y^\mu}\phi(0)\mathrm{e}^{-\mathrm{i}P_\mu y^\mu}|0\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_\alpha \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left\langle 0 | \phi(0) \mathrm{e}^{-\mathrm{i}p_\mu x^\mu} | p, \alpha \right\rangle \left\langle p, \alpha | \mathrm{e}^{\mathrm{i}p_\mu y^\mu} \phi(0) | 0 \right\rangle \\
&= \sum_\alpha \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p_\mu (x-y)^\mu} \left| \langle 0 | \phi(0) | p, \alpha \rangle \right|^2 \\
&= \int_0^\infty \mathrm{d}m^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} (2\pi) \Theta(p_0) \delta(p^2 - m^2) \mathrm{e}^{-\mathrm{i}p_\mu (x-y)^\mu} \sum_\alpha \frac{1}{2\pi} \left| \langle 0 | \phi(0) | p, \alpha \rangle \right|^2 \\
&= \int_0^\infty \mathrm{d}m^2 \, \rho(m^2) \int \widetilde{\mathrm{d}p}_m \mathrm{e}^{-\mathrm{i}p_\mu (x-y)^\mu} \,, \qquad (1.121)
\end{aligned}
$$

where in the last line we have introduced the spectral density

$$
\rho(p^2) = \sum_\alpha \frac{1}{2\pi} \left| \langle 0 | \phi(0) | p, \alpha \rangle \right|^2 \geq 0 \qquad (1.122)
$$

and used the fact that, by Lorentz covariance, it can *only* depend on the invariant mass p^2 of the states $|p, \alpha\rangle$. Similarly

$$
\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int_0^\infty \mathrm{d}m^2 \, \rho(m^2) \int \widetilde{\mathrm{d}p}_m \left(\mathrm{e}^{-\mathrm{i}p_\mu (x-y)^\mu} - \mathrm{e}^{\mathrm{i}p_\mu (x-y)^\mu} \right) \qquad (1.123)
$$

and we observe that

$$
\langle 0 | [\phi_m^{(0)}(x), \phi_m^{(0)}(y)] | 0 \rangle = \int \widetilde{\mathrm{d}p}_m \left(\mathrm{e}^{-\mathrm{i}p_\mu (x-y)^\mu} - \mathrm{e}^{\mathrm{i}p_\mu (x-y)^\mu} \right) = \mathrm{i}\Delta(x-y; m) \qquad (1.124)
$$

for a *free* field $\phi_m^{(0)}$ of mass m

$$
(\Box + m^2) \phi_m^{(0)}(x) = 0 \,. \qquad (1.125)
$$

Therefore we arrive at the Källén-Lehmann representation of the interacting commutator function

$$
\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int_0^\infty \mathrm{d}m^2 \, \rho(m^2) \mathrm{i}\Delta(x-y; m) \,. \qquad (1.126)
$$

The same reasoning can be repeated for the interacting propagator

$$
\begin{aligned}
\langle 0 | \mathrm{T} \, \phi(x) \phi(y) | 0 \rangle &= \int_0^\infty \mathrm{d}m^2 \, \rho(m^2) \langle 0 | \mathrm{T} \, \phi_m^{(0)}(x) \phi_m^{(0)}(y) | 0 \rangle \\
&= - \int_0^\infty \mathrm{d}m^2 \, \rho(m^2) \mathrm{i}G_F(x-y; m) \,. \qquad (1.127)
\end{aligned}
$$

with the *same* spectral density ρ from (1.122).

The Spectral Density ρ

Requiring canonical commutation relations for the interacting field

$$\left[\phi(x), \frac{\partial}{\partial y_0}\phi(y)\right]_{x_0=y_0} = \mathrm{i}\delta^3(\vec{x}-\vec{y}) \tag{1.128}$$

and from

$$\left.\frac{\partial}{\partial y_0}\Delta(x-y)\right|_{x_0=y_0} = \delta^3(\vec{x}-\vec{y}) \tag{1.129}$$

we derive

$$\int_0^\infty \mathrm{d}m^2\,\rho(m^2) = 1\,. \tag{1.130}$$

Assuming only massive particles, with no interactions, we expect the following spectrum

- isolated multiplets of one-particle states at m_i^2, labeled by the remaining quantum numbers α,
- continua of two-particle states starting at $(m_i + m_j)^2$,
- continua of three-particle states starting at $(m_i + m_j + m_k)^2$, etc.

In the case of interactions, there will be additional bound states and the thresholds will be lowered. In the case of massless particles, there will be no mass gaps.

We can therefore write (for one field of mass $m^2 > 0$, for simplicity)

$$\rho(s) = Z\delta(s-m^2) + \rho_{\text{cont.}}(s) \tag{1.131}$$

with the consequence

$$Z + \int_{4m^2-\delta}^\infty \mathrm{d}s\,\rho_{\text{cont.}}(s) = 1 \tag{1.132}$$

or

$$0 \le Z = 1 - \int_{4m^2-\delta}^\infty \mathrm{d}s\,\rho_{\text{cont.}}(s) \le 1\,. \tag{1.133}$$

Thus

$$\langle 0|\,\mathrm{T}\,\phi(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}p(x-y)}\left[\frac{\mathrm{i}Z}{p^2-m^2+\mathrm{i}\epsilon} + \int_{4m^2-\delta}^\infty \mathrm{d}s\,\frac{\mathrm{i}\rho_{\text{cont.}}(s)}{p^2-s+\mathrm{i}\epsilon}\right]. \tag{1.134}$$

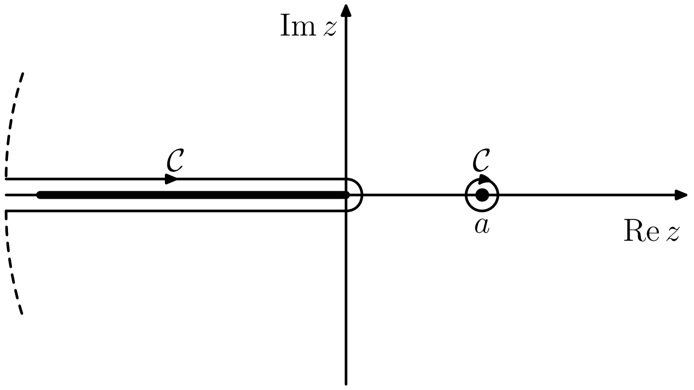


Figure 1.2: *Integration paths in (1.135) for (1.136)*

.

Lecture 05: Tue, 26. 04. 2016

Cauchy's integral formula

$$f(z) = \int_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{f(\zeta)}{\zeta - z} \tag{1.135}$$

for $\mathcal{C}$ from figure 1.2 (or later from figure 1.3). Consider a simplified example

$$f(z) = \frac{\ln z}{z - a}, \qquad \text{(with } a > 0\text{)} \tag{1.136}$$

for the principal branch of the logarithm

$$\ln(z \pm \mathrm{i}\epsilon) = \ln|z| \pm \mathrm{i}\pi\Theta(-z) \tag{1.137}$$

from

$$z = r\mathrm{e}^{\mathrm{i}\phi}\,. \tag{1.138}$$

The function f is meromorphic: it has a pole at a and a branch cut from 0 to $-\infty$, as shown in figure 1.2.

We can directly verify (1.135) for this example

$$\begin{aligned} f(z) &= \int_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{\ln\zeta}{(\zeta-a)(\zeta-z)} = -\frac{\ln a}{a-z} + \int_{-\infty}^{0} \frac{\mathrm{d}x}{2\pi\mathrm{i}} \frac{\ln(x+\mathrm{i}\epsilon) - \ln(x-\mathrm{i}\epsilon)}{(x-a)(x-z)} \\ &= \frac{\ln a}{z-a} + \int_{-\infty}^{0} \frac{\mathrm{d}x}{2\pi\mathrm{i}} \frac{\mathrm{disc}[\ln x]}{(x-a)(x-z)} = \frac{\ln a}{z-a} + \int_{-\infty}^{0} \frac{\mathrm{d}x}{(x-a)(x-z)} \end{aligned}$$

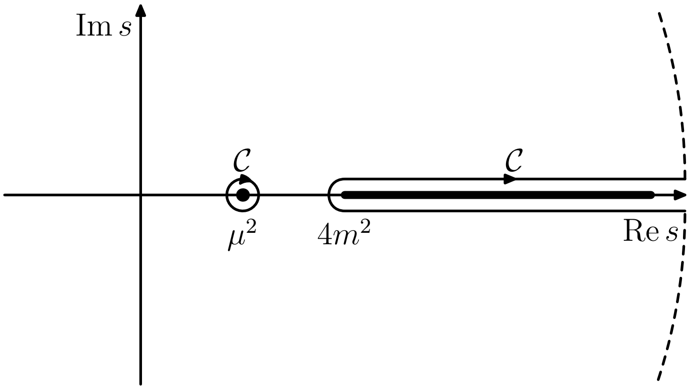


Figure 1.3: *Integration paths in (1.141)*

.

$$
\begin{aligned}
&= \frac{\ln a}{z-a} + \frac{1}{z-a}\int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{x-z} - \frac{1}{x-a}\right) \\
&= \frac{\ln a}{z-a} + \frac{1}{z-a}\left[\ln(x-z) - \ln(x-a)\right]_{-\infty}^{0} \\
&= \frac{\ln a}{z-a} + \frac{1}{z-a}\left[\ln\frac{x-z}{x-a}\right]_{-\infty}^{0} = \frac{\ln a}{z-a} + \frac{1}{z-a}\ln\frac{z}{a} = \frac{\ln z}{z-a}
\end{aligned}
\tag{1.139}
$$

In more realistic examples (e. g. in higher orders of QED perturbation theory) the function

$$
\ln\left(\frac{4m^2}{s\pm\mathrm{i}\epsilon} - 1\right) = \ln\left(\frac{4m^2}{s} - 1 \mp \mathrm{i}\epsilon\right) = \ln\left|1 - \frac{4m^2}{s}\right| \mp \mathrm{i}\pi\Theta(s-4m^2)
\tag{1.140}
$$

appears. This causes a branch cut from $4m^2$ to $+\infty$, as in figure 1.3 and we find

$$
f(s) = f(m^2) + \int_{4m^2}^{\infty} \frac{\mathrm{d}s'}{2\pi\mathrm{i}}\,\frac{\mathrm{disc} f(s')}{s'-s+\mathrm{i}\epsilon}\,.
\tag{1.141}
$$

We wll see below that the discontinuity (i. e. the spectral density) is the imaginary part of a forward scattering amlitude and can be derived from a tocal production cross section using unitarity via the optical theorem.

Asymptotic Limit

Asymptotically, all but the lowest mass states will be damped by oscillations (Riemann-Lebesque-Lemma)

$$
\phi(x) \to \sqrt{Z}\phi_{\text{in}}(x) \qquad \text{for } x_0 \to -\infty
\tag{1.142}
$$

and we know that

$$0 \leq Z < 1 \tag{1.143}$$

unless there are *no* interactions.

1.4.2 LSZ

Asymptotically free fields

$$\phi(x) \to \begin{cases} \sqrt{Z}\phi_{\text{in}}(x) & \text{for } x_0 \to -\infty \\ \sqrt{Z}\phi_{\text{out}}(x) & \text{for } x_0 \to +\infty \end{cases} \tag{1.144}$$

can be expressed in terms of creation and annihilation operators

$$\phi_{\text{in/out}}(x) = \int \widetilde{\mathrm{d}k} \left(a_{\text{in/out}}(k)\mathrm{e}^{-\mathrm{i}kx} + a^\dagger_{\text{in/out}}(k)\mathrm{e}^{\mathrm{i}kx} \right) \tag{1.145}$$

and vice versa[2]

$$a_{\text{in/out}}(k) = \mathrm{i} \int \mathrm{d}^3\vec{x} \left(\mathrm{e}^{\mathrm{i}kx} \overleftrightarrow{\partial_0} \phi_{\text{in/out}}(x) \right) \tag{1.147a}$$

$$a^\dagger_{\text{in/out}}(k) = -\mathrm{i} \int \mathrm{d}^3\vec{x} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \phi_{\text{in/out}}(x) \right) \tag{1.147b}$$

independent of $x_0 = t$! Consider multi particle scattering matrix elements

$$\begin{aligned} &\langle q_1, q_2, \ldots, q_n; \text{out} | k, p_1, p_2, \ldots, p_m; \text{in} \rangle \\ &\qquad = \langle q_1, \ldots, q_n; \text{out} | a^\dagger_{\text{in}}(k) | p_1, \ldots, p_m; \text{in} \rangle \\ &\qquad = -\mathrm{i} \int \mathrm{d}^3\vec{x} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \langle q_1, \ldots, q_n; \text{out} | \phi_{\text{in}}(x) | p_1, \ldots, p_m; \text{in} \rangle \right) \\ &\qquad = \frac{1}{\mathrm{i}\sqrt{Z}} \lim_{x_0 \to -\infty} \int \mathrm{d}^3\vec{x} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \langle q_1, \ldots, q_n; \text{out} | \phi(x) | p_1, \ldots, p_m; \text{in} \rangle \right) . \end{aligned} \tag{1.148}$$

Analogously

$$\begin{aligned} &\langle q_1, \ldots, q_n; \text{out} | a^\dagger_{\text{out}}(k) | p_1, \ldots, p_m; \text{in} \rangle \\ &\qquad = -\mathrm{i} \int \mathrm{d}^3\vec{x} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \langle q_1, \ldots, q_n; \text{out} | \phi_{\text{out}}(x) | p_1, \ldots, p_m; \text{in} \rangle \right) \end{aligned}$$

---

[2] Notation:

$$f(x) \overleftrightarrow{\partial_\mu} g(x) = f(x) \left(\partial_\mu g(x) \right) - \left(\partial_\mu f(x) \right) g(x) \tag{1.146}$$

$$= \frac{1}{\mathrm{i}\sqrt{Z}} \lim_{x_0\to+\infty} \int \mathrm{d}^3\vec{x} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \langle q_1,\ldots,q_n;\mathrm{out}|\phi(x)|p_1,\ldots,p_m;\mathrm{in}\rangle \right) \tag{1.149}$$

and using

$$\lim_{x_0\to+\infty} \int \mathrm{d}^3\vec{x} f(x) - \lim_{x_0\to-\infty} \int \mathrm{d}^3\vec{x} f(x) = \int \mathrm{d}^4x \frac{\partial}{\partial x_0} f(x) \tag{1.150}$$

we can write

$$\begin{aligned} &\langle q_1,q_2,\ldots,q_n;\mathrm{out}|k,p_1,p_2,\ldots,p_m;\mathrm{in}\rangle \\ &\qquad - \langle q_1,\ldots,q_n;\mathrm{out}|a^\dagger_{\mathrm{out}}(k)|p_1,\ldots,p_m;\mathrm{in}\rangle \\ &= \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x \frac{\partial}{\partial x_0} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \langle q_1,\ldots,q_n;\mathrm{out}|\phi(x)|p_1,\ldots,p_m;\mathrm{in}\rangle \;. \right) \end{aligned} \tag{1.151}$$

Using

$$\partial_0^2 \mathrm{e}^{-ikx} = -k_0^2 \mathrm{e}^{-ikx} = -\left(k^2 + \vec{k}^2\right) \mathrm{e}^{-ikx} = \left(\Delta - k^2\right) \mathrm{e}^{-ikx} \tag{1.152}$$

and spatial partial integration for wave packets, we can derive

$$\begin{aligned} &\int \mathrm{d}^4x \frac{\partial}{\partial x_0} \left(\mathrm{e}^{-\mathrm{i}kx} \overleftrightarrow{\partial_0} \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle \right) \\ &\quad = \int \mathrm{d}^4x \left(\mathrm{e}^{-\mathrm{i}kx} \partial_0^2 \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle - \partial_0^2 \mathrm{e}^{-\mathrm{i}kx} \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle \right) \\ &= \int \mathrm{d}^4x \left(\mathrm{e}^{-\mathrm{i}kx} \partial_0^2 \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle - \left(\Delta - m^2\right) \mathrm{e}^{-\mathrm{i}kx} \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle \right) \\ &= \int \mathrm{d}^4x \left(\mathrm{e}^{-\mathrm{i}kx} \partial_0^2 \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle - \mathrm{e}^{-\mathrm{i}kx} \left(\Delta - m^2\right) \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle \right) \\ &\qquad\qquad = \int \mathrm{d}^4x \mathrm{e}^{-\mathrm{i}kx} \left(\Box + m^2\right) \langle A;\mathrm{out}|\phi(x)|B;\mathrm{in}\rangle \;. \end{aligned} \tag{1.153}$$

The term

$$\begin{aligned} &\langle q_1,\ldots,q_n;\mathrm{out}|a^\dagger_{\mathrm{out}}(k)|p_1,\ldots,p_m;\mathrm{in}\rangle \\ &\quad = \sum_{i=1}^n (2\pi)^3 2k_0 \delta^3(\vec{k}-\vec{q_i}) \langle q_1,\ldots,\widehat{q_i},\ldots,q_n;\mathrm{out}|p_1,\ldots,p_m;\mathrm{in}\rangle \end{aligned} \tag{1.154}$$

is a disconnected contribution that vanishes unless one particle doesn't participate in the interactions at all. It can be ignored for 2 → n scattering. Therefore we can extract the connected part of the matrix element from the expectation value:

$$\langle q_1, q_2, \ldots, q_n; \mathrm{out}|k, p_1, p_2, \ldots, p_m; \mathrm{in}\rangle = \text{disconnected} + \\ \frac{\mathrm{i}}{\sqrt{Z}} \int \mathrm{d}^4x \mathrm{e}^{-\mathrm{i}kx} \left(\Box + m^2\right) \langle q_1, \ldots, q_n; \mathrm{out}|\phi(x)|p_1, \ldots, p_m; \mathrm{in}\rangle \ . \quad (1.155)$$

Note that

- $\Box + m^2$ cancels the pole of the external propagator so that we can go on shell with the Green's functions, vice versa
- $\Box + m^2$ will vanish on shell, unless there is a pole that it can cancel.

This way, we project on the asymptotic one-particle states.

Repeating this procedure, we obtain the Lehmann-Symanzik-Zimmermann reduction formula

$$\langle q_1, q_2, \ldots, q_n; \mathrm{out}|p_1, p_2, \ldots, p_m; \mathrm{in}\rangle = \text{disconnected} + \\ \left(\frac{\mathrm{i}}{\sqrt{Z}}\right)^{n+m} \int \prod_{i=1}^{m} \mathrm{d}^4x_i \prod_{j=1}^{n} \mathrm{d}^4y_j \mathrm{e}^{-\mathrm{i}\left(\sum_{i=1}^{m} p_i x_i - \sum_{j=1}^{n} q_j y_j\right)} \prod_{i=1}^{m} \left(\Box_{x_i} + m^2\right) \\ \prod_{j=1}^{n} \left(\Box_{y_j} + m^2\right) \langle 0; \mathrm{out}|\mathrm{T}\phi(x_1)\ldots\phi(x_m)\phi(y_1)\ldots\phi(y_n)|0; \mathrm{in}\rangle \ , \quad (1.156)$$

where the time ordered product guarantees that the asymptotic fields $\phi_{\mathrm{in/out}}(x)$ act on the external states during the derivation.

1.5 Generating Functionals Revisited

1.5.1 Connected Green's Functions

Consider the generating functional of all *connected* Green's functions $Z_c(j)$. Then all Green's functions can be obtained by exponentiation:

$$Z(j) = \mathrm{e}^{Z_c(j)} \quad (1.157)$$

In more detail

$$Z(j) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(Z_c(j)\right)^k \quad (1.158)$$

and

$$Z(j) = \sum_{n=0}^{\infty} G^{(n)}(j) \quad (1.159a)$$

$$Z_c(j) = \sum_{n=1}^{\infty} G_c^{(n)}(j) \tag{1.159b}$$

with

$$G^{(n)}(j) = \frac{\mathrm{i}^n}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, G(x_1,\ldots,x_n) j(x_1) \ldots j(x_n) \tag{1.160a}$$

$$G_c^{(n)}(j) = \frac{\mathrm{i}^n}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, G_c(x_1,\ldots,x_n) j(x_1) \ldots j(x_n) \tag{1.160b}$$

respectively (NB: $G_c^{(0)}(j) = 0$, i. e. there's no connected vacuum bubble). Order by order

$$\begin{aligned} Z(j) &= 1 + G^{(1)} + G^{(2)} + G^{(3)} + \ldots \\ &= 1 + \left(G_c^{(1)} + G_c^{(2)} + G_c^{(3)} + \ldots\right) + \frac{1}{2!}\left(G_c^{(1)} + G_c^{(2)} + G_c^{(3)} + \ldots\right)^2 \\ &\qquad + \frac{1}{3!}\left(G_c^{(1)} + G_c^{(2)} + G_c^{(3)} + \ldots\right)^3 + \ldots \\ &= 1 + G_c^{(1)} + G_c^{(2)} + G_c^{(3)} + \frac{1}{2!}\left(G_c^{(1)}\right)^2 + G_c^{(1)}G_c^{(2)} + \frac{1}{3!}\left(\left(G_c^{(1)}\right)^3\right) + \ldots \end{aligned} \tag{1.161}$$

and therefore

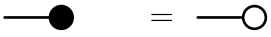
$$G^{(1)} = G_c^{(1)} \tag{1.162a}$$

$$G^{(2)} = G_c^{(2)} + \frac{1}{2!}\left(G_c^{(1)}\right)^2 \tag{1.162b}$$

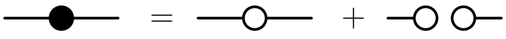
$$G^{(3)} = G_c^{(3)} + G_c^{(1)}G_c^{(2)} + \frac{1}{3!}\left(G_c^{(1)}\right)^3 \tag{1.162c}$$

$$\ldots = \ldots ,$$

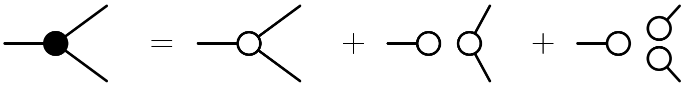
i. e.



(1.163a)



(1.163b)



(1.163c)

$$\ldots = \ldots ,$$

All orders

$$\sum_{n=0}^{\infty} G^{(n)}(j) = Z(j) = \exp\left(\sum_{n=1}^{\infty} G_c^{(n)}(j)\right) = \prod_{n=1}^{\infty} \mathrm{e}^{G_c^{(n)}(j)} \tag{1.164}$$

and we can equate the coefficients of $j(x_1)\ldots j(x_m)$ on both sides

$$G^{(n)}(x_1,\ldots,x_m) = \lim_{j\to 0} \frac{\delta}{\mathrm{i}\delta j(x_1)} \cdots \frac{\delta}{\mathrm{i}\delta j(x_m)} \prod_{n=1}^{\infty} \mathrm{e}^{G_c^{(n)}(j)} \,. \tag{1.165}$$

Using the Leibniz rule and

$$\frac{\delta}{\mathrm{i}\delta j(x)} \mathrm{e}^{G_c^{(n)}(j)} = \frac{\delta G_c^{(n)}(j)}{\mathrm{i}\delta j(x)} \mathrm{e}^{G_c^{(n)}(j)} \,, \tag{1.166}$$

we see that each functional derivative is applied to each factor and we get the sum of all ways to distribute the external legs among the $G_c^{(n)}$.

$$\lim_{j\to 0} \frac{\delta}{\mathrm{i}\delta j(x_1)} \prod_{n=1}^{\infty} \mathrm{e}^{G_c^{(n)}(j)} = \frac{\delta}{\mathrm{i}\delta j(x_1)} G_c^{(1)}(j) \tag{1.167a}$$

$$\begin{aligned}\lim_{j\to 0} \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \prod_{n=1}^{\infty} \mathrm{e}^{G_c^{(n)}(j)} &= \frac{\delta}{\mathrm{i}\delta j(x_1)} G_c^{(1)}(j) \frac{\delta}{\mathrm{i}\delta j(x_2)} G_c^{(1)}(j) \\ &\quad + \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} G_c^{(2)}(j)\end{aligned} \tag{1.167b}$$

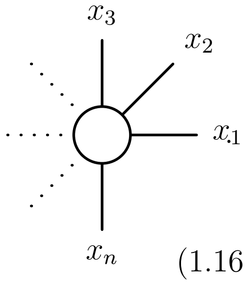
$$\begin{aligned}\lim_{j\to 0} \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \prod_{n=1}^{\infty} \mathrm{e}^{G_c^{(n)}(j)} &= \frac{\delta}{\mathrm{i}\delta j(x_1)} G_c^{(1)}(j) \frac{\delta}{\mathrm{i}\delta j(x_2)} G_c^{(1)}(j) \frac{\delta}{\mathrm{i}\delta j(x_3)} G_c^{(1)}(j) \\ &\quad + \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_2)} G_c^{(2)}(j) \frac{\delta}{\mathrm{i}\delta j(x_3)} G_c^{(1)}(j) \\ &\quad + \frac{\delta}{\mathrm{i}\delta j(x_2)} \frac{\delta}{\mathrm{i}\delta j(x_3)} G_c^{(2)}(j) \frac{\delta}{\mathrm{i}\delta j(x_1)} G_c^{(1)}(j) \\ &\quad + \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_3)} G_c^{(2)}(j) \frac{\delta}{\mathrm{i}\delta j(x_2)} G_c^{(1)}(j) \\ &\quad + \frac{\delta}{\mathrm{i}\delta j(x_1)} \frac{\delta}{\mathrm{i}\delta j(x_3)} \frac{\delta}{\mathrm{i}\delta j(x_2)} G_c^{(3)}(j)\end{aligned} \tag{1.167c}$$

1.5.2 Amputated Green's Functions

Lecture 06: Thu, 28. 04. 2016

Consider a general n-point function $G^{(n)}(x_1, x_2, \ldots, x_n)$

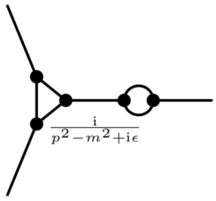
$$G^{(n)}(x_1, x_2, \ldots, x_n) = \langle 0 | \mathrm{T}\, \phi(x_1)\phi(x_2)\ldots\phi(x_n) | 0 \rangle = \tag{1.168}$$



In a diagrammatical expansion, *all* the outer lines are connected to the rest of the diagrams by free propagators, which amount to simple $\mathrm{i}/(p^2 - m^2 + \mathrm{i}\epsilon)$ factors.

These factors are not just trivial, but pose serious problems, when taking the external particles *on-shell*, i. e. in the limit $p^2 \to m^2$. In fact, we have seen, that they must be removed (a. k. a. *truncated*, *amputated*) when physical scattering amplitudes are calculated from the Green's functions.

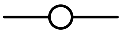
However, in the diagram



$$p^2 = m^2 \tag{1.169}$$

this does not suffice, because there is an internal propagator that blows up for on-shell external lines. Therefore, we should amputate the *full* two-point function

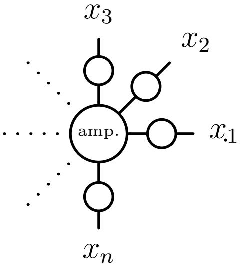
$$G^{(2)}(x - y) = \langle 0 | \mathrm{T}\, \phi(x)\phi(y) | 0 \rangle = \tag{1.170}$$



for *each* external line, in order to define the *amputated* (a. k. a. *truncated*) Green's functions:

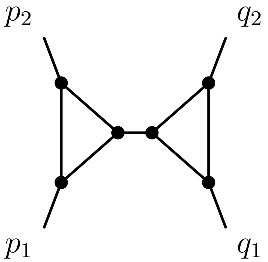
$$G^{(n)}(x_1, x_2, \ldots, x_n) = \int \prod_{i=1}^{n} \mathrm{d}^4 y_i G^{(2)}(x_i, y_i) G^{(n)}_{\text{amp.}}(y_1, y_2, \ldots, y_n)$$

$$= \tag{1.171}$$



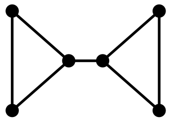
1.5.3 1PI Green's Functions and Effective Action

After amputating



(1.172)

we find a diagram that is a trivial, i. e. singly connected, composite of two *interesting* pieces that are multiply connected



$$\propto \frac{\mathrm{i}}{(p_1+p_2)^2 - m^2 + \mathrm{i}\epsilon}\,. \tag{1.173}$$

Can we derive an expression for the generating functional of these interesting pieces?

Legendre transformation

$$\tilde\Gamma(\varphi) = \mathrm{i}\int \mathrm{d}^4x\,\varphi(x)j(x) - Z_c(j) = \mathrm{i}\int \mathrm{d}^4x\,\varphi(x)j(\varphi)(x) - Z_c(j(\varphi)) \tag{1.174}$$

with $j(\varphi)$ from solving

$$\varphi(x) = \frac{\delta Z_c(j)}{\mathrm{i}\delta j(x)} \tag{1.175}$$

(NB: $j \neq 0$!). The derivative of (1.174) yields

$$\begin{aligned}\frac{\delta\tilde\Gamma(\varphi)}{\mathrm{i}\delta\varphi(x)} &= j(\varphi)(x) + \int \mathrm{d}^4y\,\varphi(y)\frac{\delta j(\varphi)(y)}{\delta\varphi(x)} - \int \mathrm{d}^4y\,\frac{\delta j(\varphi)(y)}{\delta\varphi(x)}\,\frac{\delta Z_c(j)}{\mathrm{i}\delta j(y)}\Big|_{j=j(\varphi)} \\ &= j(\varphi)(x) + \int \mathrm{d}^4y\,\frac{\delta j(\varphi)(y)}{\delta\varphi(x)}\left[\phi(y) - \frac{\delta Z_c(j)}{\mathrm{i}\delta j(y)}\Big|_{j=j(\varphi)}\right] = j(\varphi)(x)\,.\end{aligned} \tag{1.176}$$

And with the shorthand notation

$$f * g = \int \mathrm{d}^4x\,f(x)g(x) \tag{1.177}$$

we can write symmetrically

$$\mathrm{i}j * \varphi = \tilde\Gamma(\varphi) + Z_c(j) \tag{1.178a}$$

$$\varphi(x) = \frac{\delta Z_c(j)}{\mathrm{i}\delta j(x)} \tag{1.178b}$$

$$j(x) = \frac{\delta\tilde\Gamma(\varphi)}{\mathrm{i}\delta\varphi(x)} \tag{1.178c}$$

NB: (1.178a) is to be understood as one of

$$\mathrm{i}j(\varphi) * \varphi = \tilde\Gamma(\varphi) + Z_c(j(\varphi)) \tag{1.179a}$$

$$\mathrm{i}j * \varphi(j) = \tilde\Gamma(\varphi(j)) + Z_c(j) \tag{1.179b}$$

Dependence on an external parameter α: from (1.178a)

$$\begin{aligned}&\mathrm{i}\frac{\mathrm{d}\varphi}{\mathrm{d}\alpha} * j + \mathrm{i}\varphi * \frac{\mathrm{d}j}{\mathrm{d}\alpha}\\ &\quad = \frac{\mathrm{d}\tilde\Gamma}{\mathrm{d}\alpha} + \frac{\mathrm{d}Z_c}{\mathrm{d}\alpha} = \left.\frac{\partial\tilde\Gamma}{\partial\alpha}\right|_{\varphi=\text{const.}} + \frac{\mathrm{d}\varphi}{\mathrm{d}\alpha} * \underbrace{\frac{\delta\tilde\Gamma(\varphi)}{\delta\varphi}}_{\mathrm{i}j} + \left.\frac{\partial Z_c}{\partial\alpha}\right|_{j=\text{const.}} + \frac{\mathrm{d}j}{\mathrm{d}\alpha} * \underbrace{\frac{\delta Z_c(j)}{\delta j}}_{\mathrm{i}\varphi}\end{aligned} \tag{1.180}$$

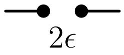
therefore

$$\left.\frac{\partial\tilde\Gamma}{\partial\alpha}\right|_{\varphi=\text{const.}} = -\left.\frac{\partial Z_c}{\partial\alpha}\right|_{j=\text{const.}} \tag{1.181}$$

Adding a *disconnected*[3] insertion $S \to S + S_I$ with

$$S_I(\phi) = \epsilon \int \mathrm{d}^4x\, \phi(x) \int \mathrm{d}^4y\, \phi(y) \tag{1.184}$$

and the corresponding Feynman rule



(1.185)

gives the ϵ-expanded generating functional

$$Z(j,\epsilon) = \exp\left(\mathrm{i}S_I\left(\frac{\delta}{\mathrm{i}\delta j}\right)\right) Z(j,0)$$

---

[3] NB:

$$S_I'(\phi) = \frac{\epsilon}{2}\int \mathrm{d}^4x\, \phi^2(x) \tag{1.182}$$

corresponds to

(1.183)

$$
\begin{aligned}
&= \left(1 - \mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \, \frac{\delta^2}{\delta j(x)\delta j(y)} + \mathcal{O}(\epsilon^2)\right) Z(j,0) \\
&= Z(j,0) \left(1 - \mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \, \frac{1}{Z(j,0)} \frac{\delta^2 Z(j,0)}{\delta j(x)\delta j(y)} + \mathcal{O}(\epsilon^2)\right) \qquad (1.186)
\end{aligned}
$$

Application of the chain rule

$$
\frac{1}{Z(j)} \frac{\delta^2 Z(j)}{\delta j(x)\delta j(y)} = \frac{\delta Z_c(j)}{\delta j(x)} \frac{\delta Z_c(j)}{\delta j(y)} + \frac{\delta^2 Z_c(j)}{\delta j(x)\delta j(y)} \qquad (1.187)
$$

we find with $\ln(a(1+b)) = \ln a + \ln(1+b) = \ln a + b + \mathcal{O}(b^2)$ the *explicit* ϵ-dependence in first order

$$
\begin{aligned}
Z_c(j,\epsilon) &= \ln Z(j,\epsilon) = \ln Z(j,0) - \mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \, \frac{1}{Z(j,0)} \frac{\delta^2 Z(j,0)}{\delta j(x)\delta j(y)} + \mathcal{O}(\epsilon^2) \\
&= Z_c(j,0) - \underbrace{\mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\frac{\delta Z_c(j,0)}{\delta j(x)} \frac{\delta Z_c(j,0)}{\delta j(y)} + \frac{\delta^2 Z_c(j,0)}{\delta j(x)\delta j(y)} \right)}_{-\epsilon \frac{\partial Z_c}{\partial \epsilon}} + \mathcal{O}(\epsilon^2) \,.
\end{aligned}
\qquad (1.188)
$$

Using

$$
\frac{\partial \tilde\Gamma}{\partial \epsilon} = -\frac{\partial Z_c}{\partial \epsilon} \qquad (1.189)
$$

we find

$$
\begin{aligned}
\tilde\Gamma(\varphi,\epsilon) - \tilde\Gamma(\varphi,0) &= \epsilon \frac{\partial \tilde\Gamma}{\partial \epsilon} + \mathcal{O}(\epsilon^2) \\
&= \mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \left(\frac{\delta Z_c(j,0)}{\delta j(x)} \frac{\delta Z_c(j)}{\delta j(y)} + \frac{\delta^2 Z_c(j,0)}{\delta j(x)\delta j(y)} \right) + \mathcal{O}(\epsilon^2) \\
&= -\mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \, \left(\varphi(x)\varphi(y) + G_c(x,y,j)\right) + \mathcal{O}(\epsilon^2) \\
&= -\mathrm{i}S_I(\varphi) - \mathrm{i}\epsilon \int \mathrm{d}^4x \int \mathrm{d}^4y \, G_c(x,y,j) + \mathcal{O}(\epsilon^2) \qquad (1.190)
\end{aligned}
$$

and infer that the $\mathcal{O}(\epsilon)$ contribution, which is obtained from the diagrams in $\tilde\Gamma(\varphi,0)$ by cutting a single line in all possible ways, remains connected, since it is given as a functional derivative of $Z_c(j,0)$. Note that $S_I(\varphi)$ contributes a single disconnected diagram (1.185), which is not obtained from cutting a line in $\tilde\Gamma(\varphi,0)$.

Therefore, $\mathrm{i}\tilde{\Gamma}$ is the generating functional for one particle irreducible (1PI) Green's Functions

$$\Gamma(\varphi) = \mathrm{i}\tilde{\Gamma}(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \, \Gamma(x_1, \ldots, x_n) \varphi(x_1) \cdots \varphi(x_n) \tag{1.191}$$

and $\Gamma = \mathrm{i}\tilde{\Gamma}$ can be interpreted as an *effective action*, because *all* Feynman diagrams are obtained by calculating tree diagrams with the vertices derived from Γ.

1.5.4 Free fields

Indeed, for free fields, we can explicitely perform the Legendre transform and check that all signs and factors of i are correct. Starting from

$$Z_c(j) = \frac{\mathrm{i}}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, j(x) G_F(x-y) j(y) \tag{1.192}$$

we find

$$\varphi(x) = \int \mathrm{d}^4y \, G_F(x-y) j(y) \tag{1.193}$$

i. e.

$$j(x) = \left(\Box + m^2\right) \varphi(x) \tag{1.194}$$

and therefore

$$\begin{aligned} &\tilde{\Gamma}(\varphi) = \\ &\mathrm{i} \int \mathrm{d}^4x \, \varphi(x) \left(\Box + m^2\right) \varphi(x) - \frac{\mathrm{i}}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, G_F(x-y) \left(\Box + m^2\right) \varphi(x) \left(\Box + m^2\right) \varphi(y) \\ &\qquad = \mathrm{i} \int \mathrm{d}^4x \, \varphi(x) \left(\Box + m^2\right) \varphi(x) - \frac{\mathrm{i}}{2} \int \mathrm{d}^4x \, \varphi(x) \left(\Box + m^2\right) \varphi(x) \\ &\qquad = \frac{\mathrm{i}}{2} \int \mathrm{d}^4x \, \varphi(x) \left(\Box + m^2\right) \varphi(x) \end{aligned} \tag{1.195}$$

i. e.

$$\Gamma(\varphi) = -\frac{1}{2} \int \mathrm{d}^4x \, \varphi(x) \left(\Box + m^2\right) \varphi(x) = S_0(\varphi) \tag{1.196}$$

the action of a free scalar field.

1.5.5 Semiclassical Expansion

Method of Stationary Phase

Consider the asymptotic limit $\lambda \to \infty$ of the family of integrals

$$I(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x) \mathrm{e}^{\mathrm{i}\lambda\phi(x)} \tag{1.197}$$

with $f, \phi : \mathbf{R} \to \mathbf{R}$ and f much more slowly varying than $\lambda\phi$ (which will eventually be the case for $\lambda \to \infty$). Assume that the phase function ϕ is stationary only at a single point x_0

$$\phi'(x_0) = 0 \tag{1.198}$$

and that

$$\phi''(x_0) \neq 0\,. \tag{1.199}$$

Now Taylor-expand around this point

$$\phi(x) \approx \phi(x_0) + \frac{1}{2}\phi''(x_0)(x - x_0)^2\,. \tag{1.200}$$

Then, with $f(x) \approx f(x_0)$

$$\begin{aligned} I(\lambda) \approx f(x_0)\mathrm{e}^{\mathrm{i}\lambda\phi(x_0)} &\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\frac{\mathrm{i}\lambda\phi''(x_0)}{2}(x-x_0)^2} \\ &= f(x_0)\mathrm{e}^{\mathrm{i}\lambda\phi(x_0)} \sqrt{\frac{2\pi}{|\lambda\phi''(x_0)|}}\, \mathrm{e}^{\mathrm{sgn}(\lambda\phi''(x_0))\mathrm{i}\pi/4} \end{aligned} \tag{1.201}$$

and one can show by more careful considerations[4] (e. g. arguments leading to formula (IV.4.8.1) in [3, 4]) that the error is $\mathcal{O}(1/\lambda)$.

Expansion in Powers of $\hbar$

So far, we have used our standard units with $\hbar = 1$. However, it is easy to see that the propagators come with one power of $\hbar$

$$\langle 0|\, \mathrm{T}\, \phi(x)\phi(y)|0\rangle = \mathcal{O}(\hbar)\,, \tag{1.202}$$

because of their linear relation to the commutator function

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = \mathrm{i}\hbar D(x - y)\,. \tag{1.203}$$

---

[4] Break the integration region into a piece containing x_0 and the rest. The latter contribution vanishes asymptotically by the Riemann-Lebesgue Lemma and the former contributions can by computed by multiple partial integrations.

In addition, the vertices in the Feynman rules come with a factor $1/\hbar$ to make the exponent

$$-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\, H_I(t) \tag{1.204}$$

in the Gell-Man-Low formula dimensionless. Since the number of loops (i. e. momentum integrations) in a diagram is related to the number of vertices (i. e. momentum constraints) and the number of internal propagators (i. e. momenta) by

$$L = I - (V-1) \tag{1.205}$$

we see that any L-loop Feynman diagram is

$$\mathcal{O}\left(\frac{\hbar^I}{\hbar^V}\right) = \mathcal{O}(\hbar^{L-1})\,. \tag{1.206}$$

The loop expansion is therefore an expansion in $\hbar$.

In the path integral

$$Z(j) = \int \mathcal{D}\varphi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(S(\varphi)+\varphi*j)}\,, \tag{1.207}$$

the limit $\hbar \to 0$ leads to a stationary phase approximation with the dominant contribution coming from the vicinity of solutions $\varphi_0(j)$ of the *classical* equation of motion in the presence of the current j

$$\left.\frac{\delta S}{\delta\varphi}\right|_{\varphi=\varphi_0(j)} + j = 0\,. \tag{1.208}$$

The terms quadratic in the fluctuations around $\varphi_0(j)$ will yield the leading corrections of $\mathcal{O}(\hbar)$ via a Gaussian path integral.

1.5.6 1-Loop Effective Action

We start by rewriting the path integral for the generating functional

$$Z(j) = \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S(\varphi)+\mathrm{i}\varphi*j} \tag{1.209}$$

with a shift of the integration variable

$$\varphi \to \varphi + \varphi_0(j) \tag{1.210a}$$

$$\mathcal{D}\varphi \to \mathcal{D}\varphi \tag{1.210b}$$

by the solution $\varphi_0(j)$ of the classical equation of motion (1.208). Note that such a constant shift leaves the integration measure unchanged. Next we expand the action to second order in the integration variable φ

$$\begin{aligned}S(\varphi+\varphi_0) = S(\varphi_0) &+ \int \mathrm{d}^4x \left.\frac{\delta S}{\delta\varphi(x)}\right|_{\varphi=\varphi_0} \varphi(x) \\ &+ \int \mathrm{d}^4x\mathrm{d}^4y\, \frac{1}{2}\varphi(x) \left.\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}\right|_{\varphi=\varphi_0} \varphi(y) + S_{\varphi_0}(\varphi) \\ &= S(\varphi_0) - j*\varphi + \frac{1}{2}\varphi * K_{\varphi_0}\varphi + S_{\varphi_0}(\varphi) \end{aligned} \quad (1.211)$$

where remainder term $S_{\varphi_0}(\varphi)$ of $\mathcal{O}(\varphi^3)$ is *defined* by (1.211) and the term $j*\phi$ is only correct for $\phi_0 = \phi_0(j)$. Note that the inverse propagator

$$\begin{aligned}K_{\varphi_0}(x)\delta^4(x-y) = \left.\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}\right|_{\varphi=\varphi_0} &= \frac{\delta}{\delta\varphi(x)}\left.\left(\frac{\partial\mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi}(y) + \ldots\right)\right|_{\varphi=\varphi_0} \\ &= \left.\left(\frac{\partial^2\mathcal{L}(\varphi,\partial\varphi)}{\partial\varphi^2}(x) + \ldots\right)\right|_{\varphi=\varphi_0} \delta^4(x-y) \end{aligned} \quad (1.212)$$

is local for

$$S(\varphi) = \int \mathrm{d}^4x\, \mathcal{L}(\varphi(x),\partial\varphi(x)) \quad (1.213)$$

and we write

$$\left(K_{\varphi_0}\phi\right)(x) = K_{\varphi_0}(x)\phi(x)\,. \quad (1.214)$$

(1.211) can also be written as

$$S(\varphi+\varphi_0) + (\varphi+\varphi_0)*j = S(\varphi_0) + \varphi_0 * j + \frac{1}{2}\varphi * K_{\varphi_0}\varphi + S_{\varphi_0}(\varphi)\,. \quad (1.215)$$

Therefore, we can rewrite the generating functional as

$$Z(j) = \int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S(\varphi+\varphi_0(j))+\mathrm{i}\varphi*j+\mathrm{i}\varphi_0(j)*j} = Z_0(j)\int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi*K_{\varphi_0(j)}\varphi+\mathrm{i}S_{\varphi_0(j)}(\varphi)}\,, \quad (1.216)$$

where the pieces independent of φ

$$\ln Z_0(j) = Z_{0,c}(j) = \mathrm{i}S(\varphi_0(j)) + \mathrm{i}\varphi_0(j)*j \quad (1.217)$$

have been pulled out of the path integral.

Lecture 07: Tue, 03. 05. 2016

The terms in the path integral proper can now be rearranged

$$\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi * K_{\varphi_0}\varphi + \mathrm{i}S_{\varphi_0}(\varphi)} = \\ \int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi * K_{\varphi_0}\varphi} \frac{\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi * K_{\varphi_0}\varphi + \mathrm{i}S_{\varphi_0}(\varphi)}}{\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\varphi * K_{\varphi_0}\varphi}} = \mathrm{e}^{-\frac{1}{2}\operatorname{tr}\ln K_{\varphi_0} + Z_{2,c}(j)} \tag{1.218}$$

with the generating functionals

$$Z_2(j) = \mathrm{e}^{Z_{2,c}(j)} = \frac{\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi * K_{\varphi_0(j)}\varphi + \mathrm{i}S_{\varphi_0(j)}(\varphi)}}{\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi * K_{\varphi_0(j)}\varphi}} \tag{1.219}$$

and the Gaussian path integral

$$\int \mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\varphi * K_{\varphi_0}\varphi} = \frac{1}{\sqrt{\det K_{\varphi_0}}} = \mathrm{e}^{-\frac{1}{2}\operatorname{tr}\ln K_{\varphi_0}}\,. \tag{1.220}$$

The generating functional for connected Green's Functions can now be expressed as

$$Z_c(j) = \ln Z(j) = Z_{0,c}(j) + Z_{1,c}(j) + Z_{2,c}(j) \tag{1.221}$$

with

$$Z_{1,c}(j) = -\frac{1}{2}\operatorname{tr}\ln K_{\varphi_0(j)}\,. \tag{1.222}$$

The defining equations (1.222) and (1.219) show that both $Z_{1,c}(j)$ and $Z_{2,c}(j)$ depend on j *only* through their dependence on $\varphi_0(j)$:

$$\forall n \in \{1,2\}:\; Z_{n,c}(j) = \hat{Z}_{n,c}(\varphi_0(j))\,. \tag{1.223}$$

We can therefore write

$$Z_c(j) = \hat{Z}_c(\varphi_0(j)) \tag{1.224}$$

with

$$\hat{Z}_c(\varphi_0) = \mathrm{i}S(\varphi_0) + \mathrm{i}\varphi_0 * j(\varphi_0) + \hat{Z}_{1,c}(\varphi_0) + \hat{Z}_{2,c}(\varphi_0) \tag{1.225}$$

So far and in particular in (1.219) and (1.221), we have used our standard units with $\hbar = 1$. Reintroducing factors of $\hbar$, the exponent in the numerator reads

$$\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^4x\, \frac{1}{2}\varphi K_{\varphi_0}\varphi + \frac{\mathrm{i}}{\hbar}S_{\varphi_0}(\varphi) \tag{1.226}$$

and we are lead to rescale the integration variable (*not* the classical field $\varphi_0(j)$)

$$\varphi \to \sqrt{\hbar}\,\varphi \tag{1.227}$$

in order to absorb all powers of $\hbar$ in the quadratic piece:

$$\mathrm{i}\int \mathrm{d}^4x\,\frac{1}{2}\varphi K_{\varphi_0}\varphi + \frac{\mathrm{i}}{\hbar}S_{\varphi_0}(\sqrt{\hbar}\,\varphi)\,. \tag{1.228}$$

Since I is, by construction, $\mathcal{O}(\varphi^3)$, we find

$$\frac{\mathrm{i}}{\hbar}S_{\varphi_0}(\sqrt{\hbar}\,\varphi) = \mathcal{O}\left(\sqrt{\hbar}\right)\,. \tag{1.229}$$

Because the exponent in the denominator is independent of $\hbar$ and the powers of $\hbar$ in the measures cancel, we conclude

$$Z_{2,c}(j) = \mathcal{O}\left(\sqrt{\hbar}\right) \tag{1.230}$$

and since there are only integer powers in a perturbative loop expansion

$$Z_{2,c}(j) = \mathcal{O}(\hbar)\,. \tag{1.231}$$

This motivates the split in 1.218, because

$$Z_{n,c}(j) = \mathcal{O}(\hbar^{n-1})\,. \tag{1.232}$$

In order to obtain the effective action, we must compute the Legendre transform of Z_c:

$$\Gamma(\bar\varphi) = -\mathrm{i}Z_c(j(\bar\varphi)) - j(\bar\varphi) * \bar\varphi \tag{1.233}$$

with $j(\bar\varphi)$ determined from solving

$$\bar\varphi = \left.\frac{\delta Z_c}{\mathrm{i}\delta j}\right|_{j=j(\bar\varphi)} \tag{1.234}$$

for $j(\bar\varphi)$. For the classical approximation, we observe using (1.208) that indeed

$$\frac{\delta Z_{0,c}}{\mathrm{i}\delta j} = \frac{\delta S}{\delta\varphi_0} * \frac{\delta\varphi_0}{\delta j} + \frac{\delta\varphi_0}{\delta j} * j + \varphi_0(j) = \varphi_0(j) \tag{1.235}$$

and we can expand the argument $\bar\varphi$ of the effective action in the same way we have shifted the integration variable φ:

$$-\varphi_1 = \bar\varphi - \varphi_0(j) = \mathcal{O}(\hbar)\,. \tag{1.236}$$

In the following we have to be careful about the functional dependencies, even if readability forces us not to spell them out explicitely everytime. We can now start to rewrite the effective action in a form that can be expanded in powers of $\hbar$

$$\begin{aligned}\Gamma(\bar\varphi) = -\mathrm{i}Z_c(j) - j * \bar\varphi &= \overbrace{-\mathrm{i}Z_{0,c}(j)}^{= S(\varphi_0) + \varphi_0 * j} -\mathrm{i}Z_{1,c}(j) - \mathrm{i}Z_{2,c}(j) - j * \bar\varphi \\ &= S(\varphi_0) + \varphi_1 * j - \mathrm{i}Z_{1,c}(j) - \mathrm{i}Z_{2,c}(j) \end{aligned} \tag{1.237}$$

where j is to be understood as $j(\bar\varphi)$ from (1.234) and φ_0 as $\varphi_0(j(\bar\varphi))$ from (1.208) and (1.234). The relation between j and ϕ_0 from (1.208) can be assumed to be one-to-one in perturbation theory. Therefore, we can also write

$$j(\bar\varphi) = \hat\jmath(\varphi_0) = \hat\jmath(\varphi_0(j(\bar\varphi))) \tag{1.238}$$

and find

$$\begin{aligned}\Gamma(\bar\varphi) &= S(\varphi_0) + \varphi_1 * \hat\jmath(\varphi_0) - \mathrm{i}\hat Z_{1,c}(\varphi_0) - \mathrm{i}\hat Z_{2,c}(\varphi_0) \\ &= S(\bar\varphi + \varphi_1) + \varphi_1 * \hat\jmath(\bar\varphi + \varphi_1) - \mathrm{i}\hat Z_{1,c}(\bar\varphi + \varphi_1) - \mathrm{i}\hat Z_{2,c}(\bar\varphi + \varphi_1)\,. \end{aligned} \tag{1.239}$$

When expanding in powers of $\hbar$, we must remember that the pieces $Z_{n,c}$ of the generating functional are dimensionless and must be multiplied by $\hbar$ to obtain an action. Therefore, we will use

$$S = \mathcal{O}(\hbar^0) \tag{1.240a}$$
$$\varphi_1 = \mathcal{O}(\hbar) \tag{1.240b}$$
$$\hbar Z_{n,c} = \mathcal{O}(\hbar^n) \tag{1.240c}$$

in the expansion. Expanding S and $\hat\jmath$ to first order in $\hbar$, we find

$$\begin{aligned}\Gamma(\bar\varphi) &= S(\bar\varphi) + \varphi_1 * \left.\frac{\delta S}{\delta\varphi}\right|_{\bar\varphi} + \varphi_1 * \hat\jmath(\bar\varphi) - \mathrm{i}\hat Z_{1,c}(\varphi_0) + \mathcal{O}(\hbar^2) \\ &= S(\bar\varphi) - \mathrm{i}\hat Z_{1,c}(\varphi_0) + \mathcal{O}(\hbar^2) = S(\bar\varphi) + \frac{\mathrm{i}\hbar}{2}\operatorname{tr}\ln K_{\varphi_0} + \mathcal{O}(\hbar^2) \end{aligned} \tag{1.241}$$

where we have used that $\hat\jmath$ is the inverse of $\varphi_0(j)$ from (1.208) and satisfies the functional equation

$$\left.\frac{\delta S}{\delta\varphi}\right|_{\varphi} = \hat\jmath(\varphi)\,. \tag{1.242}$$

Note that

$$\hbar \operatorname{tr}\ln K_{\varphi_0} = \hbar\operatorname{tr}\ln K_{\bar\varphi} + \mathcal{O}(\hbar^2)\,. \tag{1.243}$$

In summary, we have derived the following closed expression for the one-loop approximation to the effective action, i. e. the generating functional for 1PI Green's Functions

$$\Gamma(\bar\varphi) = S(\bar\varphi) + \frac{\mathrm{i}\hbar}{2}\operatorname{tr}\ln\left(\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi=\bar\varphi}\right) + \mathcal{O}(\hbar^2)\,. \tag{1.244}$$

When evaluating the trace, one must take into account the normalization factor in the functional integral, which can be written $1/Z(0)$. Therefore one must subtract a term

$$+\frac{\mathrm{i}\hbar}{2}\operatorname{tr}\ln\left(\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi=0}\right)$$

in the exponent, leading to

$$\Gamma(\bar\varphi) = S(\bar\varphi) + \frac{\mathrm{i}\hbar}{2}\operatorname{tr}\ln\left(\left(\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi=0}\right)^{-1}\left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\varphi=\bar\varphi}\right) + \mathcal{O}(\hbar^2)\,. \tag{1.245}$$

1.5.7 Effective Action at Higher Orders

One can continue with the second order in the expansion

$$\begin{aligned}\Gamma(\bar\varphi) &= S(\bar\varphi) + \varphi_1 * \left.\frac{\delta S}{\delta\varphi}\right|_{\bar\varphi} + \frac{1}{2}\varphi_1 * \left.\frac{\delta^2 S}{\delta\varphi^2}\right|_{\bar\varphi}\varphi_1 + \varphi_1 * \hat\jmath(\bar\varphi) + \varphi_1 * \left.\frac{\delta\hat\jmath}{\delta\varphi}\right|_{\bar\varphi}\varphi_1 \\ &\quad - \mathrm{i}\hat Z_{1,c}(\bar\varphi) - \mathrm{i}\varphi_1 * \left.\frac{\delta\hat Z_{1,c}}{\delta\varphi}\right|_{\bar\varphi} - \mathrm{i}\hat Z_{2,c}(\bar\varphi) + \mathcal{O}(\hbar^3) \\ &= S(\bar\varphi) - \mathrm{i}\hat Z_{1,c}(\bar\varphi) - \mathrm{i}\hat Z_{2,c}(\bar\varphi) + \frac{1}{2}\varphi_1 * K_{\varphi_0}\varphi_1 \\ &\quad + \varphi_1 * \left.\frac{\delta\hat\jmath}{\delta\varphi}\right|_{\bar\varphi}\varphi_1 - \mathrm{i}\varphi_1 * \left.\frac{\delta\hat Z_{1,c}}{\delta\varphi}\right|_{\bar\varphi} + \mathcal{O}(\hbar^3)\,. \end{aligned} \tag{1.246}$$

and show [5] that all terms of order $\hbar^2$ and higher that don't contain derivatives can be computed from a *finite* set of vacuum Feynman diagrams for each order.

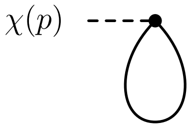
—2—

RENORMALIZATION

2.1 Examples

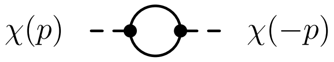
Cf. (1.99)

$$\mathrm{i}\Gamma^{(1)}(p)(2\pi)^4\delta^4(p) = \tag{2.1a}$$



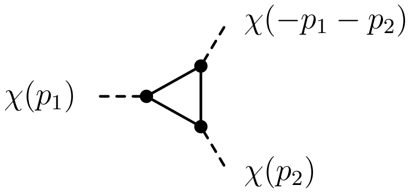
$$= \frac{\mathrm{i}g}{2!}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}$$

$$\mathrm{i}\Gamma^{(2)}(p) = \tag{2.1b}$$



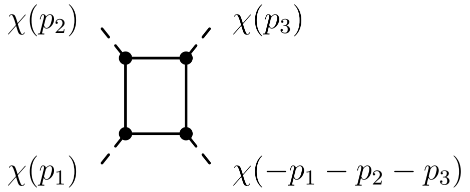
$$= \frac{(\mathrm{i}g)^2}{2!}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\,\frac{\mathrm{i}}{(k+p)^2-m^2+\mathrm{i}\epsilon}$$

$$\mathrm{i}\Gamma^{(3)}(p_1,p_2) = \tag{2.1c}$$



$$= (\mathrm{i}g)^3\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\,\frac{\mathrm{i}}{(k+p_1)^2-m^2+\mathrm{i}\epsilon}\,\frac{\mathrm{i}}{(k+p_1+p_2)^2-m^2+\mathrm{i}\epsilon}$$

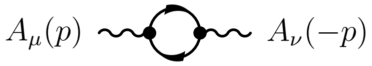
$$\mathrm{i}\Gamma^{(4)}(p_1,p_2,p_3) = \tag{2.1d}$$



$$\begin{aligned} &= (\mathrm{i}g)^4 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon} \frac{\mathrm{i}}{(k+p_1)^2-m^2+\mathrm{i}\epsilon} \times \\ &\qquad \frac{\mathrm{i}}{(k+p_1+p_2)^2-m^2+\mathrm{i}\epsilon} \frac{\mathrm{i}}{(k+p_1+p_2+p_3)^2-m^2+\mathrm{i}\epsilon} \\ \ldots &= \ldots \end{aligned}$$

More general: QED

$$\mathrm{i}\Gamma^{(2)}(p) = \tag{2.2a}$$



$$\begin{aligned} &= e^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma_\mu\left(\not k + m\right)\gamma_\nu\left(\not k + \not p + m\right)\right]}{(k^2-m^2+\mathrm{i}\epsilon)\left((k+p)^2-m^2+\mathrm{i}\epsilon\right)} \\ \ldots &= \ldots \end{aligned}$$

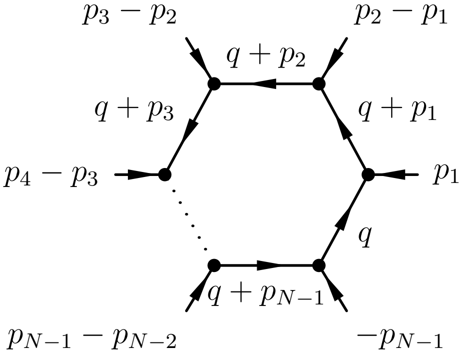
2.2 General Tensor Integrals (1-Loop)

… i. e. we have to allow momenta in the numerator as well:

$$\begin{aligned} &\tilde{T}^{(N)}_{\mu_1\mu_2\ldots\mu_M}(p_1,p_2,\ldots,p_{N-1};m_0,m_1,\ldots,m_{N-1}) = \\ &\int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_M}}{(q^2-m_0^2+\mathrm{i}\epsilon)\left((q+p_1)^2-m_1^2+\mathrm{i}\epsilon\right)\cdots\big((q+p_{N-1})^2-m_{N-1}^2+\mathrm{i}\epsilon\big)} \end{aligned} \tag{2.3}$$

graphically:

$$\tilde{T}^{(N)} = \qquad . \tag{2.4}$$



For later convenience, we generalize from 4 to D dimensions and extract a prefactor

$$T^{(N)}_{\mu_1\mu_2\ldots\mu_M}(p_1, p_2, \ldots, p_{N-1}; m_0, m_1, \ldots, m_{N-1}; D) = \\ \frac{16\pi^2}{\mathrm{i}} \mu^{4-D} \tilde{T}^{(N)}_{\mu_1\mu_2\ldots\mu_M}(p_1, p_2, \ldots, p_{N-1}; m_0, m_1, \ldots, m_{N-1}) \Bigg|_{\text{“}4 \to D\text{”}} = \\ \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_M}}{(q^2 - m_0^2 + \mathrm{i}\epsilon)\left((q+p_1)^2 - m_1^2 + \mathrm{i}\epsilon\right) \cdots \left((q+p_{N-1})^2 - m_{N-1}^2 + \mathrm{i}\epsilon\right)} \,. \tag{2.5}$$

Notational conventions for one, two, three and four point integrals:

- $T^{(1)}_{\mu_1\mu_2\ldots\mu_M} = A_{\mu_1\mu_2\ldots\mu_M}(m_0)$,
- $T^{(2)}_{\mu_1\mu_2\ldots\mu_M} = B_{\mu_1\mu_2\ldots\mu_M}(p_1; m_0, m_1)$,
- $T^{(3)}_{\mu_1\mu_2\ldots\mu_M} = C_{\mu_1\mu_2\ldots\mu_M}(p_1, p_2; m_0, m_1, m_2)$,
- $T^{(4)}_{\mu_1\mu_2\ldots\mu_M} = D_{\mu_1\mu_2\ldots\mu_M}(p_1, p_2, p_3; m_0, m_1, m_2, m_3)$

and for the *scalar integrals* ($M = 0$)

- $A(m_0) = A_0(m_0)$,
- $B(p_1; m_0, m_1) = B_0(p_1; m_0, m_1)$,
- $C(p_1, p_2; m_0, m_1, m_2) = C_0(p_1, p_2; m_0, m_1, m_2)$,
- $D(p_1, p_2, p_3; m_0, m_1, m_2, m_3) = D_0(p_1, p_2, p_3; m_0, m_1, m_2, m_3)\,.$

The only vectors and tensors that can appear are the momenta p_i and the metric g. Since the integrand is totally symmetric, the totally antisymmetric ϵ-tensor can not appear. Therefore we can expand the tensor integrals in covariants

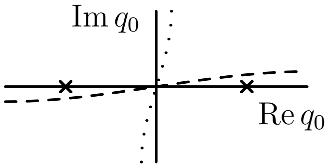
$$
\begin{aligned}
B^\mu(p_1;m_0,m_1) &= p_1^\mu B_1(p_1;m_0,m_1) &\text{(2.6a)}\\
C^\mu(p_1,p_2;m_0,m_1,m_2) &= p_1^\mu C_1(p_1,p_2;m_0,m_1,m_2) + p_2^\mu C_2(p_1,p_2;m_0,m_1,m_2) &\text{(2.6b)}\\
\ldots &= \ldots &\text{(2.6c)}
\end{aligned}
$$

and

$$
\begin{aligned}
B^{\mu\nu}(p_1;m_0,m_1) &= p_1^\mu p_1^\nu B_{11}(p_1;m_0,m_1) + g^{\mu\nu} B_{00}(p_1;m_0,m_1) &\text{(2.7a)}\\
C^{\mu\nu}(p_1,p_2;m_0,m_1,m_2) &= p_1^\mu p_1^\nu C_{11}(p_1,p_2;m_0,m_1,m_2) \\
&\quad + \left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu\right) C_{12}(p_1,p_2;m_0,m_1,m_2) \\
&\quad + p_2^\mu p_2^\nu C_{22}(p_1,p_2;m_0,m_1,m_2) \\
&\quad + g^{\mu\nu} C_{00}(p_1,p_2;m_0,m_1,m_2) &\text{(2.7b)}\\
\ldots &= \ldots &\text{(2.7c)}
\end{aligned}
$$

2.2.1 Wick Rotation

The q_0-integration contour in the loop integrals can be deformed from the dashed curves to the dotted curve



without crossing poles or cuts. With the subsequent substitution

$$(q^0,\vec q) \to (iq_E^0,\vec q_E)\,, \tag{2.8}$$

the Minkowski-"length" becomes a *euclidean* length

$$q^2 = (q^0)^2 - \vec q^{\,2} = -(q_E^0)^2 - \vec q^{\,2} = -q_E^2\,. \tag{2.9}$$

2.2.2 D−Dimensional Integration

In the following, we will assume

$$n > \max\left\{1, \frac{D}{2}\right\} \tag{2.10a}$$

$$a > 0 \tag{2.10b}$$

and continue analytically, if necessary. Using the Wick rotation we can the rewrite the integral

$$\begin{aligned} I_n(a) &= \int \frac{\mathrm{d}^D q}{(q^2 - a + \mathrm{i}\epsilon)^n} = \int_{-\infty}^{\infty} \mathrm{d}q_0 \int \frac{\mathrm{d}^{(D-1)}\vec{q}}{(q_0^2 - \vec{q}^2 - a + \mathrm{i}\epsilon)^n} \\ &= \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}q_0 \int \frac{\mathrm{d}^{(D-1)}\vec{q}}{(q_0^2 - \vec{q}^2 - a + \mathrm{i}\epsilon)^n} = \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}q_{E,0} \int \frac{\mathrm{d}^{(D-1)}\vec{q}_E}{(-q_{E,0}^2 - \vec{q}_E^2 - a + \mathrm{i}\epsilon)^n} \\ &= (-1)^n \mathrm{i} \int \frac{\mathrm{d}^D q_E}{(q_E^2 + a - \mathrm{i}\epsilon)^n} \end{aligned} \tag{2.11}$$

Lecture 08: Tue, 10. 05. 2016

and introducing D-dimensional polar coordinates

$$\int \mathrm{d}^D q_E = \int \mathrm{d}\Omega_D \int_0^\infty |q_E|^{D-1} \mathrm{d}|q_E| = \frac{1}{2} \int \mathrm{d}\Omega_D \int_0^\infty \left(q_E^2\right)^{\frac{D}{2}-1} \mathrm{d}q_E^2 \tag{2.12}$$

with

$$\int \mathrm{d}\Omega_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \tag{2.13}$$

we find[1]

$$\begin{aligned} I_n(a) &= (-1)^n \mathrm{i} \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \mathrm{d}q_E^2 \frac{(q_E^2)^{\frac{D}{2}-1}}{(q_E^2 + a - \mathrm{i}\epsilon)^n} \\ &= (-1)^n \mathrm{i} \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} (a - \mathrm{i}\epsilon)^{\frac{D}{2}-n} \int_0^\infty \mathrm{d}x \frac{x^{\frac{D}{2}-1}}{(x+1)^n} \\ &= (-1)^n \mathrm{i} \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} (a - \mathrm{i}\epsilon)^{\frac{D}{2}-n} B\left(\frac{D}{2}, n - \frac{D}{2}\right) \end{aligned}$$

---

[1] Euler's Beta-function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,.$$

$$= (-1)^n \mathrm{i}\pi^{\frac{D}{2}} \frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)} (a - \mathrm{i}\epsilon)^{\frac{D}{2}-n} \,. \tag{2.14}$$

From the properties of Euler's Γ-function

- $\Gamma(z)$ is analytical everywhere, except for simple poles at $0, -1, -2, \ldots$
- $1/\Gamma(z)$ is analytical everywhere
- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(n+1) = n!$ for $n \in \mathbf{N_0}$
- $\Gamma(1/2) = \sqrt{\pi}$
- Laurent expansion at the origin

  $$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{2.15}$$

  with $\gamma_E = 0.5772\ldots$

we can derive the analytical continuation of $I_n(a)$ in D and a and we find that logarithmic UV divergencies appear as poles in $\epsilon = 2 - \frac{D}{2}$ and quadratic divergencies as poles in $2 - D$.

2.2.3 Scalar Integrals

A_0

Using these formulae, we find

$$\begin{aligned} A_0(m_0) &= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{1}{q^2 - m_0^2 + \mathrm{i}\varepsilon} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} I_1(m_0^2) \\ &= -m_0^2 \left(\frac{m_0^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right) = -m_0^2 \left(\frac{4\pi\mu^2}{m_0^2}\right)^{\epsilon} \Gamma(\epsilon - 1) \end{aligned} \tag{2.16}$$

with the conventional definition

$$D = 4 - 2\epsilon \,. \tag{2.17}$$

For $D \to 4$, i. e. $\epsilon \to 0$, we can expand

$$\left(\frac{4\pi\mu^2}{m_0^2}\right)^{\epsilon} = 1 + \epsilon \ln \frac{4\pi\mu^2}{m_0^2} + \mathcal{O}(\epsilon^2) \tag{2.18a}$$

$$\begin{aligned}\Gamma(\epsilon-1) &= \frac{1}{\epsilon-1}\Gamma(\epsilon) = -\left(1+\epsilon+\mathcal{O}(\epsilon^2)\right)\left(\frac{1}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\\ &= -\left(\frac{1}{\epsilon}-\gamma_E+1\right)+\mathcal{O}(\epsilon)\end{aligned} \tag{2.18b}$$

and therefore

$$\begin{aligned}A_0(m_0;D) &= m_0^2\left(\underbrace{\frac{1}{\epsilon}-\gamma_E+\ln(4\pi)}_{\Delta}+\ln\frac{\mu^2}{m_0^2}+1\right)+\mathcal{O}(\epsilon)\\ &= m_0^2\left(\Delta+\ln\frac{\mu^2}{m_0^2}+1\right)+\mathcal{O}(\epsilon)\end{aligned} \tag{2.19}$$

B_0

Similarly

$$B_0(p_1;m_0,m_1) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int\mathrm{d}^Dq\,\frac{1}{(q^2-m_0^2+\mathrm{i}\varepsilon)\left((q+p_1)^2-m_1^2+\mathrm{i}\varepsilon\right)} \tag{2.20}$$

but before we can use (2.14) we need to combine denominators using *Feynman parameters*

$$\frac{1}{xy} = \int_0^1\frac{\mathrm{d}\xi}{\left((1-\xi)x+\xi y\right)^2}\,. \tag{2.21}$$

Completing the square

$$\begin{aligned}&\frac{1}{(q^2-m_0^2+\mathrm{i}\varepsilon)\left((q+p_1)^2-m_1^2+\mathrm{i}\varepsilon\right)} =\\ &\qquad\int_0^1\frac{\mathrm{d}\xi}{\left((1-\xi)\left(q^2-m_0^2+\mathrm{i}\varepsilon\right)+\xi\left((q+p_1)^2-m_1^2+\mathrm{i}\varepsilon\right)\right)^2}\\ &\qquad= \int_0^1\frac{\mathrm{d}\xi}{\left(q^2+\xi 2qp_1+\xi(p_1^2-m_1^2+m_0^2)-m_0^2+\mathrm{i}\varepsilon\right)^2}\\ &\quad= \int_0^1\frac{\mathrm{d}\xi}{\left(\underbrace{(q+\xi p_1)}_{q'}{}^2-\underbrace{\left(\xi^2p_1^2-\xi(p_1^2-m_1^2+m_0^2)+m_0^2\right)}_{a}+\mathrm{i}\varepsilon\right)^2}\\ &\qquad\qquad= \int_0^1\frac{\mathrm{d}\xi}{\left((q')^2-a+\mathrm{i}\varepsilon\right)^2}\end{aligned} \tag{2.22}$$

we can substitute q' for q with unit Jacobian

$$\begin{aligned} B_0(p_1; m_0, m_1) &= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int_0^1 \mathrm{d}\xi \int \mathrm{d}^D q' \, \frac{1}{\left((q')^2 - a(\xi) + \mathrm{i}\varepsilon\right)^2} \\ &= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int_0^1 \mathrm{d}\xi \, I_2(a(\xi)) \\ &= (4\pi)^\epsilon \Gamma(\epsilon) \int_0^1 \mathrm{d}\xi \left(\frac{\xi^2 p_1^2 - \xi(p_1^2 - m_1^2 + m_0^2) + m_0^2 - \mathrm{i}\varepsilon}{\mu^2} \right)^{-\epsilon} \end{aligned} \tag{2.23}$$

and we can again expand for $\epsilon \to 0$:

$$B_0(p_1; m_0, m_1) = \Delta - \int_0^1 \mathrm{d}\xi \ln \frac{\xi^2 p_1^2 - \xi(p_1^2 - m_1^2 + m_0^2) + m_0^2 - \mathrm{i}\varepsilon}{\mu^2} + \mathcal{O}(\epsilon)\,. \tag{2.24}$$

Observations:

- $B_0(p_1; m_0, m_1)$ depends on p_1 only through p_1^2 and we could write
$$B_0(p_1; m_0, m_1) = \tilde{B}_0(p_1^2; m_0, m_1) \tag{2.25}$$
- we have
$$B_0(p_1; m_0, m_1) = B_0(p_1; m_1, m_2) \tag{2.26}$$
because we could have shifted the loop momentum $q \to q - p_1$.

2.3 Tensor Reduction

Observation: since

$$\underbrace{p_i^\mu q_\mu}_{T_M^{(N)}} = \underbrace{\underbrace{\frac{1}{2}\left[(q+p_i)^2 - m_i^2\right]}_{i\text{th denominator}}}_{T_M^{(N-1)}} - \underbrace{\underbrace{\frac{1}{2}\left[q^2 - m_0^2\right]}_{0\text{th denominator}}}_{T_M^{(N-1)}} - \underbrace{\underbrace{\frac{1}{2}\left[p_i^2 - m_i^2 + m_0^2\right]}_{\text{lower rank tensor}}}_{T_{M-1}^{(N)}} \tag{2.27a}$$

$$\underbrace{g^{\mu\nu} q_\mu q_\nu}_{T_M^{(N)}} = \underbrace{\underbrace{q^2 - m_0^2}_{0\text{th denominator}}}_{T_M^{(N-1)}} + \underbrace{\underbrace{m_0^2}_{\text{lower rank tensor}}}_{T_{M-2}^{(N)}} \tag{2.27b}$$

all contractions of tensor integrals can be expressed by tensor integrals of strictly lower rank and/or strictly lower number of denominators. Likewise, contracting the expansion in covariants (2.6) results in linear combinations of the coefficient functions. Therefore, we obtain a hierarchy of systems of linear equations that can be solved recursively (provided we can avoid singularities).

2.3.1 B_μ

Notational shorthand

$$\langle f(q;\ldots)\rangle_q = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D q\, f(q;\ldots) \tag{2.28}$$

and all $+\mathrm{i}\epsilon$ in the denominators implied.

$$B^\mu(p_1;m_0,m_1) = p_1^\mu B_1(p_1;m_0,m_1) = \left\langle \frac{q^\mu}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)}\right\rangle_q . \tag{2.29}$$

Since there is only one invariant B_1, a single contraction suffices. Contracting both sides with $p_{1,\mu}$

$$\begin{aligned}
p_1^2 B_1(p_1;m_0,m_1) &= \left\langle \frac{p_1 q}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)}\right\rangle_q \\
&= \frac{1}{2}\left\langle \frac{\left((q+p_1)^2-m_1^2\right)-(q^2-m_0^2)-(p_1^2-m_1^2+m_0^2)}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)}\right\rangle_q \\
&= \frac{1}{2}\left\langle \frac{1}{q^2-m_0^2}\right\rangle_q - \frac{1}{2}\left\langle \frac{1}{(q+p_1)^2-m_1^2}\right\rangle_q \\
&\quad - \frac{p_1^2-m_1^2+m_0^2}{2}\left\langle \frac{1}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)}\right\rangle_q \\
&= \frac{1}{2}A_0(m_0^2) - \frac{1}{2}A_0(m_1^2) - \frac{p_1^2-m_1^2+m_0^2}{2}B_0(p_1;m_0,m_1)
\end{aligned} \tag{2.30}$$

i. e.

$$B_1(p_1;m_0,m_1) = \frac{1}{2p_1^2}\left(A_0(m_0^2) - A_0(m_1^2) - (p_1^2-m_1^2+m_0^2)B_0(p_1;m_0,m_1)\right) . \tag{2.31}$$

2.3.2 $B_{\mu\nu}$

Expand in available tensors with new scalar coefficient functions:

$$\begin{aligned}
B^{\mu\nu}(p_1;m_0,m_1) &= p_1^\mu p_1^\nu B_{11}(p_1;m_0,m_1) + g^{\mu\nu}B_{00}(p_1;m_0,m_1) \\
&= \left\langle \frac{q^\mu q^\nu}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)}\right\rangle_q .
\end{aligned} \tag{2.32}$$

In the following

$$B_{\ldots} = B_{\ldots}(p_1;m_0,m_1) \tag{2.33}$$

will be implied. We need two contractions for two invariants: first with $g^{\mu\nu}$

$$p_1^2 B_{11} + D B_{00} = \left\langle \frac{(q^2 - m_0^2) + m_0^2}{(q^2 - m_0^2)\left((q+p_1)^2 - m_1^2\right)} \right\rangle_q = A_0(m_1) + m_0^2 B_0 \tag{2.34a}$$

(note that $g^{\mu\nu} g_{\mu\nu} = D$) and

$$\begin{aligned} p_{1,\nu} p_1^2 B_{11} + p_{1,\nu} B_{00} = & \\ & \frac{1}{2} \left\langle \frac{q_\nu}{(q^2 - m_0^2)} \right\rangle_q - \frac{1}{2} \left\langle \frac{q_\nu}{((q+p_1)^2 - m_1^2)} \right\rangle_q \\ & - \frac{p_1^2 - m_1^2 + m_0^2}{2} \left\langle \frac{q_\nu}{(q^2 - m_0^2)\left((q+p_1)^2 - m_1^2\right)} \right\rangle_q \\ & = 0 - \frac{1}{2} \left\langle \frac{q'_\nu - p_{1,\nu}}{(q')^2 - m_1^2} \right\rangle_{q'} - \frac{p_1^2 - m_1^2 + m_0^2}{2} B_\nu \\ & = \frac{1}{2} p_{1,\nu} A_0(m_1) - \frac{(p_1^2 - m_1^2 + m_0^2}{2} p_{1,\nu} B_1 \end{aligned} \tag{2.34b}$$

where we have made use of *symmetric integration*

$$\left\langle q_\mu f(q^2) \right\rangle_q = 0\,. \tag{2.35}$$

Thus we obtain a linear equation for B_{00} and B_{11}:

$$\begin{pmatrix} D & p_1^2 \\ 1 & p_1^2 \end{pmatrix} \begin{pmatrix} B_{00} \\ B_{11} \end{pmatrix} = \begin{pmatrix} A_0(m_1) + m_0^2 B_0 \\ \frac{1}{2} A_0(m_1) - \frac{p_1^2 - m_1^2 + m_0^2}{2} B_1 \end{pmatrix}, \tag{2.36}$$

Lecture 09: Thu, 12. 05. 2016

with solution

$$B_{00} = \frac{A_0(m_1) + 2m_0^2 B_0 + (p_1^2 - m_1^2 + m_0^2) B_1}{2(D-1)} \tag{2.37a}$$

$$B_{11} = \frac{(D-2) A_0(m_1) - 2m_0^2 B_0 - D(p_1^2 - m_1^2 + m_0^2) B_1}{2(D-1) p_1^2} \tag{2.37b}$$

and divergent pieces

$$B_{00} = -\frac{1}{12}\left(p_1^2 - 3(m_0^2 + m_1^2)\right)\Delta + \text{finite} \tag{2.38a}$$

$$B_{11} = \frac{1}{3}\Delta + \text{finite} \tag{2.38b}$$

2.3.3 C_μ

$$\begin{aligned} C^\mu(p_1,p_2;m_0,m_1,m_2) &= p_1^\mu C_1(p_1,p_2;m_0,m_1,m_2)+p_2^\mu C_2(p_1,p_2;m_0,m_1,m_2) \\ &= \left\langle \frac{q^\mu}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)\left((q+p_2)^2-m_2^2\right)} \right\rangle_q . \end{aligned} \tag{2.39}$$

A simple exercise yields

$$\begin{aligned} &\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix}^{-1} \times \\ &\begin{pmatrix} B_0(p_2^2;m_0,m_2) - B_0((p_1-p_2)^2;m_1,m_2) - (p_1^2-m_1^2+m_0^2)C_0 \\ B_0(p_1^2;m_0,m_1) - B_0((p_1-p_2)^2;m_1,m_2) - (p_2^2-m_2^2+m_0^2)C_0 \end{pmatrix} . \end{aligned} \tag{2.40}$$

The divergent part of B_0 is independent of masses and momenta

$$B_0 = \Delta + \text{finite}\,, \tag{2.41}$$

therefore the divergencies cancel in $C_{1,2}$.

2.3.4 Gram Determinants

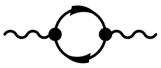
However, whenever the Gram determinant

$$G(p_1,p_2,\ldots,p_n) = \begin{vmatrix} p_1^2 & p_1p_2 & \ldots & p_1p_n \\ p_2p_1 & p_2^2 & \ldots & p_2p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_np_1 & p_np_2 & \ldots & p_n^2 \end{vmatrix} \tag{2.42}$$

vanishes, the expressions for the invariants become ill defined. This is easily understood geometrically, because it means that the momenta are not linearly independent. Fundamentally, this is no problem, because the values on the singular submanifolds can be obtained by continuity. Unfortunately, this complicates the numerical evaluation significantly and other, potentially better behaved, methods are being studied.

2.3.5 Example

Again, QED:

$\mathrm{i}\Sigma_{\mu\nu}(p) = -\ A_\mu(p)$  $A_\nu(-p)$ (2.2')

$$= e^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma_\mu \left(\not{q}+m\right)\gamma_\nu\left(\not{q}+\not{p}+m\right)\right]}{(q^2-m^2+\mathrm{i}\epsilon)\left((q+p)^2-m^2+\mathrm{i}\epsilon\right)}\,.$$

Traces

$$\begin{gathered}
\mathrm{tr}\left[\gamma_\mu \left(\not{q}+m\right)\gamma_\nu\left(\not{q}+\not{p}+m\right)\right] \\
= \mathrm{tr}\left[\gamma_\mu\not{q}\gamma_\nu\not{q}\right] + \mathrm{tr}\left[\gamma_\mu\not{q}\gamma_\nu\not{p}\right] + m^2\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\right] \\
= \left(2q_\mu q_\nu - q^2 g_{\mu\nu}\right)\mathrm{tr}\,\mathbf{1} + \left(q_\mu p_\nu + p_\mu q_\nu - qp g_{\mu\nu}\right)\mathrm{tr}\,\mathbf{1} + m^2 g_{\mu\nu}\,\mathrm{tr}\,\mathbf{1} \\
= \left(2q_\mu q_\nu + q_\mu p_\nu + p_\mu q_\nu - \left(q^2+qp-m^2\right)g_{\mu\nu}\right)\mathrm{tr}\,\mathbf{1} \\
= \left(2q_\mu q_\nu + q_\mu p_\nu + p_\mu q_\nu\right)\mathrm{tr}\,\mathbf{1} - \left((q^2-m^2) + ((q+p)^2-m^2) - p^2\right) g_{\mu\nu}\frac{\mathrm{tr}\,\mathbf{1}}{2}
\end{gathered} \tag{2.43}$$

hence ($+\mathrm{i}\epsilon$ implied, again):

$$\begin{aligned}
\Sigma_{\mu\nu}(p) &= \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{4}\left\langle \frac{2q_\mu q_\nu + q_\mu p_\nu + p_\mu q_\nu}{(q^2-m^2)\left((q+p)^2-m^2\right)}\right\rangle_q \\
&\quad + \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{8} g_{\mu\nu}\Bigg(\left\langle\frac{-1}{\left((q+p)^2-m^2\right)}\right\rangle_q + \left\langle\frac{-1}{(q^2-m^2)}\right\rangle_q \\
&\quad\quad + \left\langle \frac{p^2}{(q^2-m^2)\left((q+p)^2-m^2\right)}\right\rangle_q\Bigg) \\
&= \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{4}\left(2B_{\mu\nu}(p;m,m) + p_\mu B_\nu(p;m,m) + p_\nu B_\mu(p;m,m)\right) \\
&\quad - \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{8} g_{\mu\nu}\left(A_0(m) + A_0(m) - p^2 B_0(p;m,m)\right) \\
&= \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{4}\left(2g_{\mu\nu}B_{00}(p;m,m) + 2p_\mu p_\nu B_{11}(p;m,m) + 2p_\mu p_\nu B_{00}(p;m,m)\right) \\
&\quad - \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{4} g_{\mu\nu}\left(A_0(m) - \frac{p^2}{2}B_0(p;m,m)\right) \\
&= \frac{\alpha}{\pi}\frac{\mathrm{tr}\,\mathbf{1}}{4}\Bigg(p_\mu p_\nu\left(2B_{11}(p;m,m) + 2B_1(p;m,m)\right) \\
&\quad + g_{\mu\nu}\left(2B_{00}(p;m,m) - A_0(m) + \frac{p^2}{2}B_0(p;m,m)\right)\Bigg)
\end{aligned} \tag{2.44}$$

Useful decomposition

$$\Sigma_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Sigma_T(p^2) + \frac{p_\mu p_\nu}{p^2}\Sigma_L(p^2) \tag{2.45}$$

with

$$\Sigma_T(p^2) = \frac{\alpha}{\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\left(2B_{00}(p;m,m) + \frac{p^2}{2}B_0(p;m,m) - A_0(m)\right) \tag{2.46a}$$

$$\begin{aligned}\Sigma_L(p^2) = \frac{\alpha}{\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\Bigl(&2B_{00}(p;m,m) + 2p^2(B_{11}(p;m,m) + B_1(p;m,m)) \\ &+ \frac{p^2}{2}B_0(p;m,m) - A_0(m)\Bigr)\end{aligned} \tag{2.46b}$$

and ultimately

$$\Sigma_T(p^2) = \frac{\alpha}{3\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\left((p^2 + 2m^2)B_0(p;m,m) - \frac{p^2}{3} - 2m^2B_0(0;m,m)\right) \tag{2.47a}$$

$$\Sigma_L(p^2) = 0 \tag{2.47b}$$

(sse exercise) using $A_0(m) = m^2B_0(0;m,m) + m^2$ etc..

This is *not* an accident, as will be shown in the next chapter.

Remark #1

What is $\operatorname{tr}\mathbf{1}$?

- in fourdimensional Dirac algebra, the smallest faithful representation is also fourdimensional, thus $\operatorname{tr}\mathbf{1}\big|_{D=4} = 4$.
- in D-dimensional Dirac algebra, the smallest faithful representation is $2^{\lfloor D/2\rfloor}$-dimensional, thus $\operatorname{tr}\mathbf{1} = 2^{\lfloor D/2\rfloor}$.

In any case,

$$\operatorname{tr}\mathbf{1} = 4 + \mathcal{O}(D-4) \tag{2.48}$$

and any difference can be absorbed in the definition of

$$\Delta = \frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \ln(4\pi)\ . \tag{2.49}$$

Remark #2

What is the Feynman rule corresponding to

$$\mathcal{L}_I = -\frac{c}{4}F_{\mu\nu}F^{\mu\nu}\,? \tag{2.50}$$

Up to boundary terms

$$\begin{aligned}\mathcal{L}_I = -\frac{c}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{c}{2}\partial_\mu A_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= \frac{c}{2}A_\mu \left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu \to \frac{c}{2}A_\mu \left(p^\mu p^\nu - p^2 g^{\mu\nu}\right) A_\nu \end{aligned} \tag{2.51}$$

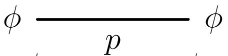
it is proportional to the *transversal* part that is divergent!

2.4 Renormalization Constants

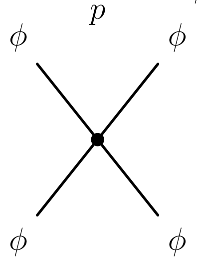
Consider the ϕ^4-theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.52}$$

with Feynman rules

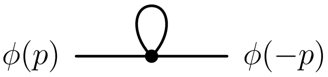


$$= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} \tag{2.53a}$$

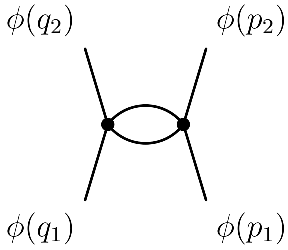


$$= -\mathrm{i}\lambda\,. \tag{2.53b}$$

There are only two divergent one loop diagrams



$$= \mathrm{i}\Gamma^{(2)}(p) \tag{2.54a}$$



$$= \mathrm{i}\tilde{\Gamma}^{(4)}(p_1 + p_2) \tag{2.54b}$$

with

$$\begin{aligned}\mathrm{i}\Gamma^{(2)}(p) = \frac{-\mathrm{i}\lambda}{2}\int\frac{\mathrm{d}^D q}{(2\pi)^D}\,\frac{\mathrm{i}}{q^2 - m^2 + \mathrm{i}\epsilon} \\ = \mathrm{i}\frac{\lambda}{32\pi^2}A_0(m) = \mathrm{i}\frac{\lambda}{32\pi^2}m^2\Delta + \text{finite}\end{aligned} \tag{2.55a}$$

and

$$\begin{aligned}\mathrm{i}\tilde\Gamma^{(4)}(p) &= \frac{(-\mathrm{i}\lambda)^2}{2!}\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\epsilon}\,\frac{\mathrm{i}}{(q+p)^2-m^2+\mathrm{i}\epsilon}\\ &= \mathrm{i}\frac{\lambda^2}{32\pi^2}B_0(p;m,m) = \mathrm{i}\frac{\lambda^2}{32\pi^2}\Delta + \text{finite}\,. \end{aligned}\tag{2.55b}$$

The complete one loop greensfunction is the sum of tree diagrams

$$\begin{aligned}\Gamma^{(4)}(p_1,p_2,p_3,p_4) &= \tilde\Gamma^{(4)}(p_1+p_2) + \tilde\Gamma^{(4)}(p_1+p_3) + \tilde\Gamma^{(4)}(p_1+p_4)\\ &= \frac{\lambda^2}{32\pi^2}\left(B_0(p_1+p_2;m,m) + B_0(p_1+p_3;m,m) + B_0(p_1+p_4;m,m)\right)\\ &= \frac{3\lambda^2}{32\pi^2}\Delta + \text{finite}\,. \end{aligned}\tag{2.56}$$

Allow renormalizations

$$\phi \to \sqrt{Z}\phi = \phi + \frac{1}{2}\delta Z\phi \tag{2.57a}$$

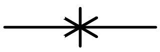
$$\lambda \to Z_\lambda\lambda = \lambda + \delta\lambda\lambda \tag{2.57b}$$

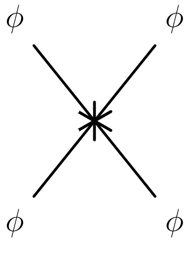
$$m^2 \to Z_m m^2 = m^2 + \delta m^2 \tag{2.57c}$$

i. e.

$$\begin{aligned}\mathcal{L} \to \mathcal{L} &= \frac{Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{ZZ_m}{2}m^2\phi^2 - \frac{Z^2Z_\lambda\lambda}{4!}\phi^4\\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\\ &+ \frac{\delta Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\delta Z}{2}m^2\phi^2 - 2\delta Z\frac{\lambda}{4!}\phi^4 - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda\lambda}{4!}\phi^4 + \mathcal{O}((\delta\ldots)^2)\end{aligned}\tag{2.58}$$

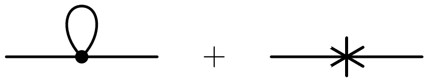
with new interactions

$\phi(p)$  $\phi(-p)$ $= -\mathrm{i}\delta Z(-p^2+m^2) - \mathrm{i}\delta m^2$ (2.59a)



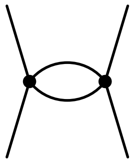
$$= -\mathrm{i}2\delta Z\lambda - \mathrm{i}\delta\lambda\lambda\,. \tag{2.59b}$$

Demand

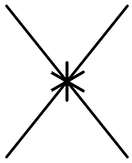


$$= \mathrm{i}\left(\Gamma^{(2)}(p) - \delta Z(-p^2+m^2) - \delta m^2\right) = \text{finite} \tag{2.60a}$$

and



+ crossed +



$$= \mathrm{i}\left(\Gamma^{(4)}(p_1,p_2,p_3,p_4) - 2\delta Z\lambda - \delta\lambda\lambda\right) = \text{finite}\,. \tag{2.60b}$$

Therefore

$$\delta Z(-p^2+m^2) + \delta m^2 = \Gamma^{(2)}(p) = \frac{\lambda}{32\pi^2}A_0(m) + \text{finite} \tag{2.61a}$$

$$2\delta Z\lambda + \delta\lambda\lambda = \Gamma^{(4)}(p_1,p_2,p_3,p_4) + \text{finite} = \frac{3\lambda}{32\pi^2}B_0(0;m,m) + \text{finite} \tag{2.61b}$$

from which we find

$$\delta Z = 0 + \text{finite} \tag{2.62a}$$

$$\delta m^2 = \frac{\lambda}{32\pi^2}A_0(m) + \text{finite} = \frac{\lambda}{32\pi^2}m^2\Delta + \text{finite} \tag{2.62b}$$

$$\delta\lambda = \frac{3\lambda}{32\pi^2}B_0(0;m,m) + \text{finite} = \frac{3\lambda}{32\pi^2}\Delta + \text{finite} \tag{2.62c}$$

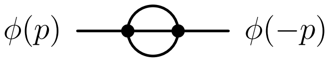
Lecture 10: Thu, 19. 05. 2016

Remarks:

- we can choose $\delta Z = 0$, because $\Gamma^{(2)}(p)$ is momentum independent at one-loop,

  $$\Gamma^{(2)}(p) = \frac{\lambda}{32\pi^2}A_0(m) \tag{2.63}$$

  but this no longer true at two loops, because the external momentum goes *through* the diagram:

  (2.64)

- in QED, $\delta Z \neq 0$ already at one loop, because Σ_T is divergent and corresponds to a momentum depended counterterm.

2.5 Power Counting

2.5.1 Dimensional Analysis

Free fields for scalars ϕ, spin-1/2 fermions ψ and vectors A_μ

$$S_0^\phi = \int \mathrm{d}^4x \left(\frac{1}{2} \frac{\partial\phi(x)}{\partial x_\mu} \frac{\partial\phi(x)}{\partial x^\mu} - \frac{m_\phi^2}{2}\phi^2(x) \right) \tag{2.65a}$$

$$S_0^\psi = \int \mathrm{d}^4x \left(\bar\psi(x)\mathrm{i}\gamma_\mu \frac{\partial}{\partial x_\mu}\psi(x) - m_\psi\bar\psi(x)\psi(x) \right) \tag{2.65b}$$

$$S_0^A = \int \mathrm{d}^4x \, \frac{-1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x) \,, \tag{2.65c}$$

$$F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu} \,. \tag{2.65d}$$

actions are dimensionsless. The *mass dimension*

$$\dim(m) = 1 \tag{2.66}$$

of the fields follows with

$$\dim\left(\mathrm{d}^4x\right) = -4 \tag{2.67}$$

$$\dim\left(\frac{\partial}{\partial x_\mu}\right) = 1 \tag{2.68}$$

as

$$\dim\left(\phi(x)\right) = 1 \tag{2.69a}$$

$$\dim\left(\psi(x)\right) = \frac{3}{2} \tag{2.69b}$$

$$\dim\left(A_\mu(x)\right) = 1 \,. \tag{2.69c}$$

As a result, the high energy asymptotics of the propagators is $p^{2\dim-4}$

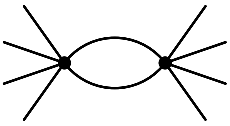
$$\int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}px} \left\langle 0 \middle| \mathrm{T}\phi(x)\phi(0) \middle| 0 \right\rangle = \frac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon} \tag{2.70a}$$

$$\int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}px} \left\langle 0 \middle| \mathrm{T}\psi(x)\bar\psi(0) \middle| 0 \right\rangle = \mathrm{i}\frac{\not p + m_\psi}{p^2 - m_\psi^2 + \mathrm{i}\epsilon} \tag{2.70b}$$

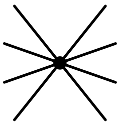
$$\int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}px} \left\langle 0 \middle| \mathrm{T}A_\mu(x)A_\mu(0) \middle| 0 \right\rangle = \frac{-ig_{\mu\nu}}{p^2 + \mathrm{i}\epsilon} \,. \tag{2.70c}$$

Therefore, the high energy asymptotics of integrands in Feynman loop diagrams is determined by dimensional analysis.

One loop diagrams with two ϕ^6-insertions require a ϕ^8-counterterm



$+$



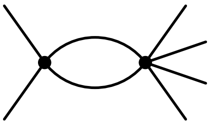
$$= \text{finite}\,. \tag{2.71}$$

because

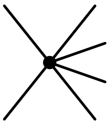
$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{1}{k^2(p-k)^2} \tag{2.72}$$

is logarithmically divergent.

For the same reason, a ϕ^4-operator and a ϕ^6-operator

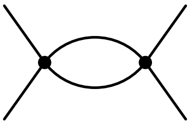


$+$

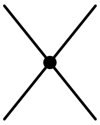


$$= \text{finite} \tag{2.73}$$

require a only another ϕ^6-counterterm and two ϕ^4-operators only another ϕ^4-counterterm



$+$



$$= \text{finite}\,. \tag{2.74}$$

These examples suggest the following conjecture:

- multiple insertions of operators of dimension *higher than 4* require (infinitely many) new counterterms of growing dimension, while
- multiple insertions of operators of dimension 4 or less require no new counterterms,

which has been proven rigorously by Weinberg's power counting theorem and later heroic combinatorical work.

If an operator has dimension higher than 4, the corresponding coupling constant has a *negative* dimension:

$$\frac{1}{\Lambda^2}\frac{1}{6!}\phi^6(x)\,,\quad \dim(\Lambda) = 1\,. \tag{2.75}$$

Since the loop integral does not depend on the coupling constant, the product has a prefactor of *more negative* dimension and the corresponding counterterm has higher dimension, e. g.

$$\frac{1}{\Lambda^2}\frac{1}{6!}\phi^6(x)\frac{1}{\Lambda^2}\frac{1}{6!}\phi^6(y) \to \frac{1}{16\pi^2}\frac{1}{\Lambda^4}\frac{1}{8!}\phi^8(x)\,. \tag{2.76}$$

For heuristic investigations, dimensional analysis can therefore replace the analysis of Feynman diagrams.

Important observation: *all* building blocks (fields and derivatives) have *positive* dimension and therefore

- in each order, there is only a *finite* set of possible counterterms
- interactions of dimension 4 or less can be renormalized by a finite set of counterterms *to all orders!*

2.5.2 Momentum Space

A diagram G with L loops, I_F fermion propagators, I_B boson propagators and δ_v derivatives at the vertex v scales like

$$\int \frac{\mathrm{d}k}{k}\, k^{\omega(G)} \tag{2.77}$$

with the *superficial degree of divergence*

$$\omega(G) = DL + \sum_v \delta_v - I_F - 2I_B\,. \tag{2.78}$$

On the other hand

$$L = I_F + I_B - \left(\sum_v 1 - 1\right)\,, \tag{2.79}$$

since there are $\sum_v 1$ momentum conserving vertices and one overall momentum conservation that can be factored. Therefore

$$\omega(G) = D + (D-1)I_F + (D-2)I_B + \sum_v (\delta_v - D)\,. \tag{2.80}$$

Also, each internal line ends at two vertices and if $\hat f_v$ and $\hat b_v$ denote the number of *internal* fermion and boson lines ending at v, we have

$$I_F = \frac{1}{2}\sum_v \hat f_v \tag{2.81a}$$

$$I_B = \frac{1}{2}\sum_v \hat b_v \tag{2.81b}$$

and thus

$$\omega(G) = D + \sum_v \left(\underbrace{\delta_v + \frac{D-1}{2}\hat{f}_v + \frac{D-2}{2}\hat{b}_v}_{\hat{\omega}_v} - D \right) . \tag{2.82}$$

Adding the external lines to the vertices

$$\omega_v = \delta_v + \frac{D-1}{2} f_v + \frac{D-2}{2} b_v \tag{2.83}$$

and subtracting them together with overall momenta factored from the vertex, we find

$$\omega(G) = D + \sum_v (\omega_v - D) - \frac{D-1}{2} E_F - \frac{D-2}{2} E_B - \delta\,. \tag{2.84}$$

(2.84) leads to an important classification:

- $\omega_v > D$: adding new vertices of this kind will make the diagram (superficially) more divergent: such vertices are called *nonrenormalizable*,
- $\omega_v = D$: adding new vertices of this kind will not change the diagram's (superficial) degree of divergence: such vertices are called *renormalizable*, and
- $\omega_v < D$: adding new vertices of this kind will make the diagram (superficially) more convergent: such vertices are called *superrenormalizable*.

It is of course no accident, that the renormalizable vertices have dimensionless couplings, whereas the nonrenormalizable couplings have negative mass dimension.

2.5.3 Renormalizability

Nonrenormalizable Theories

If there is at least one nonrenormalizable vertex, *all* Green's functions can become divergent by going to a sufficiently high order with enough insertions of nonrenormalizable vertices. Such theories require an *infinite* set of counterterms and have no predictive power as fundamental theories — but can be very useful as effective theories[2].

---

[2] There is a striking anology of rabbits and nonrenormalizable interactions: a single one is fine and will not cause additional trouble, but once you allow two (e. g. by not insisting on a certain symmetry breaking), you are in trouble, because they proliferate

Renormalizable Theories

If there is at least one renormalizable vertex, but no nonrenormalizable vertex, only a *finite* number of Green's functions with

$$\frac{D-1}{2}E_F + \frac{D-2}{2}E_B + \delta \leq D \tag{2.85}$$

can become divergent. Such theories require a *finite* set of counterterms and have predictive power as fundamental theories. NB: for $D = 2$ there is a loophole, since boson fields are dimensionless and an infinite number of counterterms is possible.

Superrenormalizable Theories

If there are *only* superrenormalizable vertices, only a *finite* number of *diagrams* can become divergent (with the same caveat for bosons in two dimensions).

2.5.4 Zoology

We can classify all models by computing

$$\frac{D-1}{2}E_F + \frac{D-2}{2}E_B + \delta - D = \frac{D}{2}\left(E_F + E_B - 2\right) - \frac{1}{2}E_F - E_B + \delta \tag{2.86}$$

which grows with D, since $E_F + E_B > 2$ for all interactions. Therefore a model can be renormalizable exactly for one value D_c of D and will be nonrenormalizable for $D > D_c$ and superrenormalizable for $D < D_c$.

- Scalar Models:
  - ϕ^4: $D_c = 4$
  - ϕ^3: $D_c = 6$
  - $\phi^\dagger \overleftrightarrow{\partial_\mu} \phi A^\mu$, $\phi^\dagger \phi A^\mu A_\mu$: $D_c = 4$
- Spinor Models:
  - $\bar\psi \not{A} \psi$: $D_c = 4$
  - $\bar\psi \Gamma \psi \bar\psi \Gamma' \psi$: $D_c = 2$

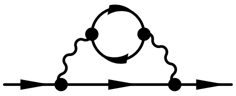
Note that in the presence of gauge models, additional counterterms could appear, e. g. $A_\mu A_\nu A^\mu A^\nu$, that would break gauge invariance. In this case, proving renormalizability requires to prove their absence.

2.5.5 Nitty-Gritty Details

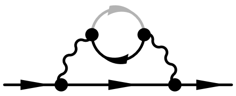
The rigorous analysis of higher orders is *much* more complicated than that.

Nested Divergencies

Nested divergencies

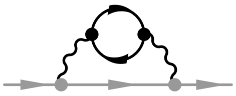


are simple, because the outer integration is often *finite*:



$$= \text{finite}\,. \tag{2.87}$$

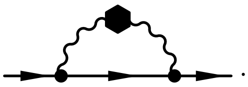
In this example, there is only one sub divergence:



$$= (p_\mu p_\nu - p^2 g_{\mu\nu})\frac{\alpha}{3\pi}\ln\frac{\Lambda^2}{-p^2} + \text{finite}\,. \tag{2.88}$$

Weinberg's Theorem: the convergence of the whole diagram is determined by the *power* of the outer loop momentum:

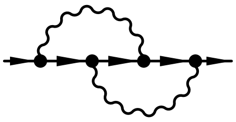
- logarithms of subdiagrams are only important for the finite pieces
- the outer loop looks like a self energy with local insertion:



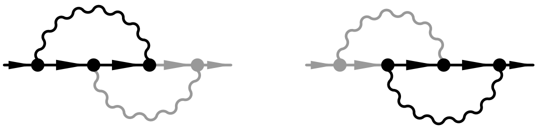
In the integration regions corresponding to nested divergencies, the inner momenta can be chosen to grow faster than the outer momenta and a simple recursive renormalization procedure can be applied.

Overlapping Divergencies

Unfortunately, in



there are two logarithmically divergente subdiagrams with *common* propagators



and there are several disjoint "dangerous" regions and in each a different momentum is growing faster that the others.

It is not intuitively clear, that a recursive addition of counter terms will cure overlapping divergencies correctly. It took almost half of a century and the smartest theorists to prove that it works.

2.6 Renormalization Procedure

Lecture 11: Tue, 24. 05. 2016

- Step 1: add *all* required counterterms
  - field renormalizations Z,
  - mass renormalizations δm, and
  - coupling constant renormalization δg.

  Note that there might be additional counterterms, that are not part of the initial model, but have dimension $\leq D$ and are allowed by symmetry.

- Step 2: compute *all* superficially divergent 1-loop Green's functions and show that they can be made finite by adjusting the value of the coefficients of the counterterms added in step 1.

- Step 2a: if divergencies remain, we have an *anomaly* and have to add counterterms that break a symmetry. This is harmless for non gauge symmetries, but a desaster for gauge symmetries (see below).

- Repeat this procedure to the desired number N of loops, taking into account the n-loop counterterms with $n < N$ in step 2.

- Step 3: the divergent pieces of the renormalization constants are now fixed. The μ-dependent finite pieces, remember

  $$A_0(m) = m^2 \left(\Delta + \ln\frac{\mu^2}{m^2} + 1\right) + \mathcal{O}(D-4) \tag{2.89}$$

  are determined by computing enough observables and comparing them with experiments done at the scale μ. Note that Z is not observable, but the ratio of Zs of fields related by a symmetry might exhibit deviations.

—3—

Gauge Theories

Experimental observation: the vector bosons

- photons: massless particles with spin-1 (triplet representation of SO(3) $\simeq$ SU(2)) and *two* degrees of freedom: left- and righthanded polarization,
- gluons: apparently massless strong force carriers,
- $W^{\pm}$ and Z bosons: massive spin 1 particles with three degrees of freedom transmitting weak interactions

are well established. Theoretical observation: the naive covariant quantization of spin-1 vector fields requires four components $\{A_\mu(y)\}_{\mu=0,1,2,3}$ with

$$[A_\mu(x), A_\nu(y)] = -\mathrm{i}g_{\mu\nu}D(x-y) \tag{3.1}$$

and is problematic:

- 4 degrees of freedom, not 3 or 2,
- $[A_0(x), A_0(y)]$ has the *wrong sign*.

Thus it *can not* be correct. Possible solutions

- gauge invariance of electrodynamics $A_\mu(x) \to A_\mu(x) + \partial_\mu\omega(x)$,
- cancellation of A_0 and A_L

require a more detailled and systematic investigation of local symmetries.

3.1 Global and Gauge Symmetries

3.1.1 Groups

Symmetries described as Groups $(G, \circ)$ with G a set and $\circ$ an inner operation

$$\begin{aligned} \circ : G \times G &\to G \\ (x, y) &\mapsto x \circ y \end{aligned} \tag{3.2}$$

with

1. closure: $\forall x, y \in G : x \circ y \in G$,
2. associativity: $x \circ (y \circ z) = (x \circ y) \circ z$,
3. identity element: $\exists e \in G : \forall x \in G : e \circ x = x \circ e = x$,
4. inverse elements: $\forall x \in G : \exists x^{-1} \in G : x \circ x^{-1} = x^{-1} \circ x = e\,.$

Many examples in physics

- permutations
- reflections
- parity
- translations
- rotations
- Lorentz boosts
- Runge-Lenz vector
- isospin
- …

3.1.2 Lie Groups

Particularly interesting are *Lie Groups*, i. e. groups, where the set is a *differentiable Manifold* and the composition is differentiable w. r. t. both operands.

Note that the choice of coordinates is not relevant:

$$\begin{aligned} B &= \left\{ b_1(\eta) = \exp\begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \middle| \, \eta \in \mathbf{R} \right\} \\ &= \left\{ b_2(\beta) = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \middle| \beta \in \left]-1,1\right[\right\} \end{aligned} \tag{3.3}$$

Both times we have the set of all real symmetric 2×2 matrices with unit determinant. The composition laws are given by matrix multiplication[1]:

$$b_1(\eta) \circ b_1(\eta') = b_1(\eta) b_1(\eta') = b_1(\eta + \eta') \tag{3.4a}$$

$$b_2(\beta) \circ b_2(\beta') = b_2(\beta) b_2(\beta') = b_2\left(\frac{\beta + \beta'}{1 + \beta\beta'}\right) . \tag{3.4b}$$

3.1.3 Lie Algebras

A Lie algebra $(A, [\cdot,\cdot])$ is a K-vector space[2] with a non-associative antisymmetric bilinear inner operation $[\cdot,\cdot]$:

$$\begin{aligned} [\cdot,\cdot] : A \times A &\to A \\ (a,b) &\mapsto [a,b] \end{aligned} \tag{3.5}$$

with

1. closure: $\forall a, b \in A : [a,b] \in A$,
2. antisymmetry: $[a,b] = -[b,a]$
3. bilinearity: $\forall \alpha, \beta \in K : [\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c]$
4. Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$

---

[1] NB:

$$|\beta| < 1 \land |\beta'| < 1 \Rightarrow \left|\frac{\beta + \beta'}{1 + \beta\beta'}\right| < 1$$

[2] K = R or C

Since A is a vector space, we can choose a basis and write

$$[a_i, a_j] = \sum_k C_{ijk} a_k \,. \tag{3.6}$$

A Lie algebra is called *simple*, if it has no ideals besides itself and $\{0\}$. Remarkably, *all* simple Lie algebras are known:

$$\mathrm{so}(N), \mathrm{su}(N), \mathrm{sp}(2N), g_2, f_4, e_6, e_7, e_8 \tag{3.7}$$

with $N \in \mathbf{N}$.

The infinitesimal generators of a Lie group form a Lie algebra. Vice versa, the elements of a Lie algebra can be exponentiated to obtain a Lie group (not necessarily the same, but a cover of the original group).

3.1.4 Homomorphisms

A *group homomorphism* f is a map

$$\begin{aligned} f : G &\to G' \\ x &\mapsto f(x) \end{aligned} \tag{3.8}$$

between two groups $(G, \circ)$ and $(G', \circ')$ that is compatible with the group structure

$$f(x) \circ' f(y) = f(x \circ y) \tag{3.9}$$

and therefore

$$f(e) = e' \tag{3.10a}$$

$$f(x^{-1}) = (f(x))^{-1} \,. \tag{3.10b}$$

A *Lie algebra homomorphism* ϕ is a map

$$\begin{aligned} \phi : A &\to A' \\ a &\mapsto \phi(a) \end{aligned} \tag{3.11}$$

between two Lie algebras $(A, [\cdot,\cdot])$ and $(A', [\cdot,\cdot]')$ that is compatible with the Lie algebra structure

$$[\phi(a), \phi(b)]' = \phi([a, b]) \,. \tag{3.12}$$

NB: these need *not* be isomorphisms: $f(x) = e', \forall x$ is a trivial, but well defined group homomorphism and $\phi(a) = 0, \forall a$ is a similarly trivial but also well defined Lie algebra homomorphism.

3.1.5 Representations

Lie groups and algebras are abstract objects, which can be made concrete by representations.

A *group representation*

$$R : G \to L \tag{3.13}$$

is a homomorphism from the group $(G, \circ)$ to a group of linear operators $(L, \cdot)$ with $(O_1 \cdot O_2)(v) = O_1(O_2(v))$. The representation is called *unitary* if the operators are unitary. The representation is called *faithful* if $\forall x \neq y : R(x) \neq R(y)$.

A *Lie algebra representation*

$$r : A \to L \tag{3.14}$$

is a homomorphism from the Lie algebra $(A, [\cdot,\cdot])$ to an associative algebra of linear operators $(L, [\cdot,\cdot]')$ with $[O_1, O_2]' = O_1 \cdot O_2 - O_2 \cdot O_1$ or $[O_1, O_2]'(v) = O_1(O_2(v)) - O_2(O_1(v))$, i. e. commutators for Lie brackets.

The Matrix groups $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(2N)$ and their Lie algebras have obvious defining representations.

Every Lie algebra has a *adjoint representation*, using the itself as the linear representation space $a \Leftrightarrow |a\rangle$:

$$r_{\text{adj.}}(a)\,|b\rangle = |[a,b]\rangle \tag{3.15}$$

using the Jacobi identity

$$\begin{aligned}(r_{\text{adj.}}(a) r_{\text{adj.}}(b) - r_{\text{adj.}}(b) r_{\text{adj.}}(a))\,|c\rangle &= |[a,[b,c]] - [b,[a,c]]\rangle \\ &= |[[a,b],c]\rangle = r_{\text{adj.}}([a,b])\,|c\rangle \end{aligned} \tag{3.16}$$

or, using a basis

$$r_{\text{adj.}}(a_i)\,|a_j\rangle = |[a_i,a_j]\rangle = |C_{ijk}a_k\rangle = C_{ijk}\,|a_k\rangle \tag{3.17}$$

we find the matrix elements

$$[r_{\text{adj.}}(a_i)]_{jk} = C_{ijk}\,. \tag{3.18}$$

Using Hausdorff's formula

$$\begin{aligned}\mathrm{e}^a b\,(\mathrm{e}^a)^{-1} = \mathrm{e}^a b \mathrm{e}^{-a} = \mathrm{e}^{\mathrm{ad}_a} b = \mathrm{e}^{[a,\cdot]} b \\ = b + [a,b] + \frac{1}{2!}[a,[a,b]] + \frac{1}{3!}[a,[a,[a,b]]] + \ldots \end{aligned} \tag{3.19}$$

we see that the map

$$\begin{aligned} f(x) : A &\to A \\ b &\mapsto xbx^{-1} \end{aligned} \tag{3.20}$$

is well defined and remains *inside* the Lie algebra. It's obviously linear and since

$$f(x)(f(y)(a)) = f(x)\left(yay^{-1}\right) = xyay^{-1}x^{-1} = (xy)a(xy)^{-1} = f(xy)(a) \tag{3.21}$$

it is also a representation, called the *adjoint representation of the group.*

3.1.6 Gauge Symmetries

A *global symmetry transformation* is constant through all of spacetime, while a *local symmetry transformation*, a. k. a. *gauge transformation*, may depend on the point in space and time. Obviously, the group of gauge transformations is much bigger than the group of global transformations and gauge invariance is much more demanding than global invariance.

3.2 Gauge Invariant Actions

Lecture 12: Tue, 31. 05. 2016

3.2.1 Global Transformations

Given a symmetry group G and a finite dimensional representation R, we can easily construct invariant actions for multiplets of fields transforming under this representation

$$U(\alpha) \in G : \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \cdots \\ \phi_n(x) \end{pmatrix} \mapsto \begin{pmatrix} \phi'_1(x) \\ \phi'_2(x) \\ \cdots \\ \phi'_n(x) \end{pmatrix} = R(U(\alpha)) \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \cdots \\ \phi_n(x) \end{pmatrix} \tag{3.22}$$

or, in components,

$$\phi_i(x) \mapsto \phi'_i(x) = \sum_{j=1}^{n} \left[R(U(\alpha))\right]_{ij} \phi_j(x) \tag{3.23}$$

or, combining the components to vectors,

$$\phi(x) \mapsto \phi'(x) = R(U(\alpha))\phi(x) \tag{3.24}$$

or, if there's no danger of mistaking the group for the representation,

$$\phi(x) \mapsto \phi'(x) = U(\alpha)\phi(x)\,. \tag{3.25}$$

However, while R is in many cases the defining representation, there are important examples for other representations in particle physics.

Parametrizing the group elements

$$U(\alpha) = \mathrm{e}^{\mathrm{i}t_a\alpha_a} = \mathrm{e}^{\mathrm{i}\alpha} \tag{3.26}$$

with $\{t_a\}$ a basis of the corresponding Lie algebra, we can often concentrate on infinitesimal transformations:

$$\phi(x) \mapsto \phi'(x) = \phi(x) + \delta\phi(x) \tag{3.27}$$

with

$$\delta\phi_i(x) = \mathrm{i}\sum_a \sum_{j=1}^n \alpha_a \left[r(t_a)\right]_{ij} \phi_j(x) = \mathrm{i}\sum_{j=1}^n \left[r(\alpha)\right]_{ij} \phi_j(x) \tag{3.28}$$

or

$$\delta\phi(x) = \mathrm{i}\sum_a \alpha_a r(t_a)\phi(x) = \mathrm{i}r(\alpha)\phi(x) \tag{3.29}$$

or

$$\delta\phi(x) = \mathrm{i}\sum_a \alpha_a t_a\phi(x) = \mathrm{i}\alpha\phi(x)\,. \tag{3.30}$$

Mass terms in a complex unitary representation,

$$\phi^\dagger(x)\phi(x) = \sum_{i=1}^n \phi_i^*(x)\phi_i(x) \tag{3.31}$$

and in a real orthogonal representation

$$\phi^T(x)\phi(x) = \sum_{i=1}^n \phi_i(x)\phi_i(x)\,, \tag{3.32}$$

are obviously invariant:

$$\phi^\dagger(x)\phi(x) \mapsto (\phi')^\dagger(x)\phi'(x) = \phi^\dagger(x)\phi(x) \tag{3.33}$$

and

$$\phi^T(x)\phi(x) \mapsto (\phi')^T(x)\phi'(x) = \phi^T(x)\phi(x)\,. \tag{3.34}$$

Since

$$\partial_\mu\phi'(x) = \partial_\mu\left(R(U(\alpha))\phi(x)\right) = R(U(\alpha))\partial_\mu\phi(x) \tag{3.35}$$

derivatives transform just like the fields and kinetic terms are invariant as well. Using this we can easily write invariant Lagrangians

$$\mathcal{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\phi^\dagger\phi - P(\phi^\dagger\phi) \tag{3.36}$$

and

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^T(\partial^\mu\phi) - \frac{m^2}{2}\phi^T\phi - P\left(\phi^T\phi\right) . \tag{3.37}$$

There are of course many more interaction terms, e. g.

$$\mathcal{L}_{333} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\phi_i\phi'_j\phi''_k \tag{3.38a}$$

$$\mathcal{L}_{\bar{2}23} = \sum_{i,j=1}^{2}\sum_{k=1}^{3} \sigma^k_{ij}\psi^*_i\psi'_j\phi_k \tag{3.38b}$$

$$\mathcal{L}_{333} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\chi_i\chi'_j\chi'_k \tag{3.38c}$$

$$\ldots \tag{3.38d}$$

for the ϕ in SU(2) triplets, the ψ in SU(2) dublets and the χ in SU(3) triplets.

3.2.2 Local Transformations

Considering local, x-dependent, transformations with

$$U(x) = \mathrm{e}^{\mathrm{i}t_a\alpha_a(x)} = \mathrm{e}^{\mathrm{i}\alpha(x)} \tag{3.39}$$

we find that derivatives no longer transform covariantly

$$\begin{aligned}\partial_\mu\phi'(x) = \partial_\mu\left(U(x)\phi(x)\right) &= U(x)\partial_\mu\phi(x) + \partial_\mu U(x)\phi(x) \\ &= U(x)\left[\partial_\mu + U^{-1}(x)\left(\partial_\mu U(x)\right)\right]\phi(x) .\end{aligned} \tag{3.40}$$

According to

$$\begin{aligned}U^{-1}(x)\partial_\mu U(x) = \mathrm{e}^{-\mathrm{i}\alpha(x)}\partial_\mu\mathrm{e}^{\mathrm{i}\alpha(x)} &= \mathrm{e}^{-\mathrm{i}[\alpha(x),\cdot]}\partial_\mu = \mathrm{e}^{-\mathrm{i}\mathrm{ad}_{\alpha(x)}}\partial_\mu \\ &= \partial_\mu - \mathrm{i}[\alpha(x),\partial_\mu] - \frac{1}{2!}[\alpha(x),[\alpha(x),\partial_\mu]] + \ldots \\ = \partial_\mu + \mathrm{i}\partial_\mu\alpha(x) &+ \frac{1}{2!}[\alpha(x),\partial_\mu\alpha(x)] - \frac{\mathrm{i}}{3!}[\alpha(x),[\alpha(x),\partial_\mu\alpha(x)]] + \ldots \\ &= \partial_\mu + U^{-1}(x)\left(\partial_\mu U(x)\right) ,\end{aligned} \tag{3.41}$$

the additional term is composed of multiple commutators of generators and their derivatives. Therefore it is defined *in* the *Lie algebra* representation and can be cancelled by a field in the same Lie algebra representation!

3.2.3 Covariant Derivative

Define a *covariant derivate*

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu(x) \tag{3.42}$$

such that

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu(x) \to D'_\mu = U(x) D_\mu U^{-1}(x) = \partial_\mu - \mathrm{i}A'_\mu(x) \tag{3.43}$$

and demand the transformation property of the Lie algebra valued *connection*

$$A_\mu(x) = t_a A^a_\mu(x) \tag{3.44}$$

accordingly

$$\begin{aligned} \partial_\mu - \mathrm{i}A'_\mu(x) = U(x)\left(\partial_\mu - \mathrm{i}A_\mu(x)\right)U^{-1}(x) &= U(x)\partial_\mu U(x) - \mathrm{i}U(x)A_\mu(x)U^{-1}(x) \\ &= \partial_\mu + U(x)\left(\partial_\mu U^{-1}(x)\right) - \mathrm{i}U(x)A_\mu(x)U^{-1}(x) \end{aligned} \tag{3.45}$$

i. e.

$$\begin{aligned} A_\mu(x) \to A'_\mu(x) &= U(x)A_\mu(x)U^{-1}(x) + \mathrm{i}U(x)\left(\partial_\mu U^{-1}(x)\right) \\ &= A_\mu(x) + \mathrm{i}[\alpha(x), A_\mu(x)] - \frac{1}{2!}[\alpha(x), [\alpha(x), A_\mu(x)]] + \ldots \\ &\quad + \partial_\mu\alpha(x) + \frac{\mathrm{i}}{2!}[\alpha(x), \partial_\mu\alpha(x)] - \frac{1}{3!}[\alpha(x), [\alpha(x), \partial_\mu\alpha(x)]] + \ldots \end{aligned} \tag{3.46}$$

NB: more precisely, D_μ depends on the representation

$$D^r_\mu = \partial_\mu - \mathrm{i}r(A_\mu(x)) \tag{3.47}$$

e. g.

$$D^{\text{adj.}}_\mu = \partial_\mu - \mathrm{i}[A_\mu(x), \cdot] = \partial_\mu - \mathrm{i}A^a_\mu(x)[t_a, \cdot] \tag{3.48}$$

and in

$$D^r_\mu = \partial_\mu - \mathrm{i}r(A_\mu(x)) \to D^{r\prime}_\mu = R(U(x)) D^r_\mu R(U^{-1}(x)) = \partial_\mu - \mathrm{i}r(A'_\mu(x)) \tag{3.49}$$

the representations r and R must match. However, by Hausdorff's formula,

$$A_\mu(x) \to A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + \mathrm{i}U(x)\left(\partial_\mu U^{-1}(x)\right) \tag{3.50}$$

is representation independent and we can use the *same* gauge connection for *all* representations.

NB: for the special case of abelian transformations

$$[\alpha(x), \alpha'(x)] = [\alpha(x), \partial_\mu \alpha'(x)] = 0 \tag{3.51}$$

we find

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{3.52}$$

to *all orders*, i. e. the gauge transformations of electrodynamics.

D_μ is called a *covariant derivative*, because it transforms as an adjoint

$$r(D_\mu) \to r(D'_\mu) = R(U(x)) r(D_\mu) R(U^{-1}(x)) \tag{3.53}$$

and we find

$$\begin{aligned} &r(D_\mu)\phi_R(x) \to r(D'_\mu)\phi'_R(x) \\ &\quad = R(U(x)) r(D_\mu) R(U^{-1}(x)) R(U(x)) \phi_R(x) = R(U(x)) r(D_\mu) \phi_R(x) \end{aligned} \tag{3.54}$$

iff the representations r and R match.

If we introduce the convention that *the appropriate representation is implied, depending on which field D_μ is acting*, we can drop r and R consistently in

$$D_\mu \to D'_\mu = U(x) D_\mu U^{-1}(x) \tag{3.55}$$

and

$$D_\mu \phi(x) \to D'_\mu \phi'(x) = U(x) D_\mu U^{-1}(x) U(x) \phi(x) = U(x) D_\mu \phi(x) \,. \tag{3.56}$$

We will adapt this convention from now on!

This way we can easily write invariant Lagrangians for matter fields

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - P(\phi^\dagger \phi) \,, \tag{3.57}$$

but the connection $A_\mu(x)$ is still an external field. We need dynamics for it.

3.2.4 Field Strength

The Ricci identity

$$F_{\mu\nu} = \mathrm{i}\,[D_\mu, D_\nu] = F^a_{\mu\nu} t_a \tag{3.58}$$

can be used to *define* a new object $F_{\mu\nu}$, *en detail*

$$\begin{aligned} F_{\mu\nu} = \mathrm{i}\,[\partial_\mu - \mathrm{i}A_\mu, \partial_\nu - \mathrm{i}A_\nu] = \mathrm{i}\,[\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] - \mathrm{i}\,[A_\mu, A_\nu] \\ = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\,[A_\mu, A_\nu] \;, \end{aligned} \tag{3.59}$$

that transforms like an adjoint

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \tag{3.60}$$

because

$$[D_\mu, D_\nu] \to \left[D'_\mu, D'_\nu\right] = \left[UD_\mu U^{-1}, UD_\nu U^{-1}\right] = U\,[D_\mu, D_\nu]\,U^{-1} \tag{3.61}$$

Finally

$$F_{\mu\nu}F^{\mu\nu} \to UF_{\mu\nu}F^{\mu\nu}U^{-1} \tag{3.62}$$

and by the cyclic invariance of the trace

$$\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) \to \mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) \tag{3.63}$$

we find a viable candidate for a Lagrangian for A_μ

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} \tag{3.64}$$

independent of the representation with normalization fixed by

$$\mathrm{tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab}\,. \tag{3.65}$$

3.2.5 Building Blocks

This way, gauge theory lagrangians are like Lego bricks: just plug matching blocks together so that pairs of $U^{-1}(x)$ and $U(x)$ cancel:

$$\phi, D_\mu, \psi, \not{D}, F_{\mu\nu}\,, \tag{3.66}$$

where the covariant derivative for fermions acts in the tensor product of Dirac spinors and gauge group representation

$$\not{D} = \mathbf{1}_R \otimes \gamma^\mu\partial_\mu - \mathrm{i}r(A_\mu(x)) \otimes \gamma^\mu = r(D_\mu) \otimes \gamma^\mu \tag{3.67}$$

Typical terms are for bosons

$$\phi^\dagger \cdots D_\mu \cdots F_{\rho\sigma} \cdots \phi\,, \tag{3.68a}$$

fermions

$$\bar\psi \cdots D_\mu \cdots F_{\rho\sigma} \cdots \gamma_\nu \cdots \psi \tag{3.68b}$$

and gauge bosons

$$\mathrm{tr}(F_{\mu\nu} \cdots D_\lambda \cdots F_{\rho\sigma}) \tag{3.68c}$$

but more complicated structures like

$$\sum_{abc} C^{abc} \left(\phi^T t_a D_\mu \phi\right) \left(\phi^T t_b D_\nu \phi\right) \left(\bar\psi t_c F^{\mu\nu} \not{D} \psi\right) \tag{3.69}$$

are also possible.

Note that due to (3.58), of the three combinations

$$F_{\mu\nu},\ D_\mu D_\nu,\ D_\nu D_\mu \tag{3.70}$$

only two are independent!

3.3 Constrained Dynamics

3.3.1 Hamiltonian Dynamics for Gauge Fields

Consider pure gauge theory

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) - 2\operatorname{tr}(A_\mu j^\mu) = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} - A_{a,\mu}j^\mu_a \tag{3.71}$$

and attempt canonical quantization of A_μ.

Three-Vector Notation

Convention for the gradient

$$\nabla^i = \frac{\partial}{\partial x^i} = \partial_i = -\partial^i \tag{3.72}$$

i. e.

$$\vec\nabla = -\vec\partial \tag{3.73}$$

and consequently for the corresponding covariant derivative

$$\vec D = \vec\nabla + \mathrm{i}\vec A\,, \tag{3.74}$$

where the representation is implied. Then with

$$E^i = F^{i0} \tag{3.75a}$$

$$B^i = \frac{1}{2}\epsilon^{ijk}F^{jk} \tag{3.75b}$$

we have[3]

$$\vec{E} = -\vec{D}A_0 - \dot{\vec{A}} \tag{3.76a}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} + \frac{\mathrm{i}}{2}\left[\vec{A} \overset{\times}{;} \vec{A}\right] \tag{3.76b}$$

with the notation

$$\left[\vec{V} \overset{\times}{;} \vec{W}\right] = \vec{V}_a \times \vec{W}_a [t_a, t_b] = \left[\vec{W} \overset{\times}{;} \vec{V}\right] . \tag{3.77}$$

This allows to write

$$\mathcal{L} = \mathrm{tr}(\vec{E}\vec{E} - \vec{B}\vec{B}) - 2\,\mathrm{tr}(A_\mu j^\mu) = \frac{1}{2}\vec{E}_a\vec{E}_a - \frac{1}{2}\vec{B}_a\vec{B}_a - A_{a,0}\rho_a + \vec{A}_a\vec{j}_a \tag{3.78}$$

just like in electrodynamics. However, $\vec{E}$ and $\vec{B}$ are just convenient shorthands, A^0 and A^i remain the dynamical variables.

Conjugate Momenta

$$\pi_a^i = \frac{\partial\mathcal{L}}{\partial \dot{A}_a^i} = \frac{\partial\mathcal{L}}{\partial E_b^j}\frac{\partial E_b^j}{\partial \dot{A}_a^i} = E_b^j \cdot (-\delta_{ab}\delta^{ij}) = -E_a^i = \dot{A}_a^i + (\vec{D}A^0)_a \tag{3.79a}$$

$$\pi_a^0 = 0 \tag{3.79b}$$

… oops!

Legendre Transform

Lecture 13: Thu, 02. 06. 2016

Nevertheless

$$\begin{aligned}\mathcal{H} = \pi_a^0\dot{A}_a^0 + \vec{\pi}_a\dot{\vec{A}}_a - \mathcal{L} &= \pi_a^0\dot{A}_a^0 + \vec{\pi}_a\dot{\vec{A}}_a - \frac{1}{2}\vec{E}_a\vec{E}_a + \frac{1}{2}\vec{B}_a\vec{B}_a + A_{a,0}\rho_a - \vec{A}_a\vec{j}_a \\ &= \frac{1}{2}\vec{\pi}_a\vec{\pi}_a - \vec{\pi}_a\left(\vec{D}A^0\right)_a + \frac{1}{2}\vec{B}_a\vec{B}_a + A_{a,0}\rho_a - \vec{A}_a\vec{j}_a \\ &= \frac{1}{2}\vec{\pi}_a\vec{\pi}_a + \frac{1}{2}\vec{B}_a\vec{B}_a + A_a^0\underbrace{(\vec{D}\vec{\pi}+\rho)_a}_{\text{Gauß' Law}} - \vec{A}_a\vec{j}_a + \text{surface terms}\end{aligned} \tag{3.80}$$

doesn't look too bad. If we enforce Gauß' Law, we obtain the Hamiltonian of classical electrodynamics.

---

[3] NB:

$$\vec{B} \neq \vec{D} \times \vec{A} = \vec{\nabla} \times \vec{A} + \mathrm{i}\left[\vec{A} \overset{\times}{;} \vec{A}\right]$$

Instead

$$2\mathrm{i}\vec{B} = \left[\vec{D} \overset{\times}{;} \vec{D}\right]$$

Poisson Brackets

We would like to have

$$\left\{A_a^i(\vec{x}), \pi_b^j(\vec{y})\right\} = \delta_{ab}\delta^{ij}\delta^3(\vec{x}-\vec{y}) \tag{3.81a}$$

$$\left\{A_a^0(\vec{x}), \pi_b^0(\vec{y})\right\} = \delta_{ab}\delta^3(\vec{x}-\vec{y}) \tag{3.81b}$$

but (3.81b) requires $\pi^0 \neq 0$ and is therefore not compatible with (3.79b).

Solutions

- *Gauge fixing:* remove π^0 from the theory as a dynamical variable and enforce (3.79b), $\pi^0 = 0$, while ignoring (3.81b). *Possible approach, but manifestly breaks Lorentz invariance.*
- *Gauge fixing redux:* add a term like $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ that vanishes unter classical gauge conditions to the Lagrangian, such that $\pi^0 \neq 0$ and (3.81b) becomes possible. *Works for QED, but fails subtly for nonabelian gauge theories.*
- *Constrained dynamics (Dirac):* enforce (3.79b) *only at the very end*, but calculate with (3.81b) and $\pi^0 \neq 0$ before.

3.3.2 Constraints

We shall say that a function χ on phase space (p,q) vanishes *weakly*

$$\chi \approx 0 \tag{3.82}$$

when we solve the dynamics without regard to the condition

$$\chi(p,q) = 0 \tag{3.83}$$

and only apply it at the very end to the solutions, before computing observables. Solutions of $\chi(p,q) = 0$ will be called the *constraint surface.*

Given a set of primary constraints

$$\chi_A \approx 0 \qquad (\text{for } A = 1,2,\ldots) \tag{3.84}$$

this approach only makes sense, if the dynamics doesn't leave the constraint surface, i. e.

$$\frac{\mathrm{d}\chi_A}{\mathrm{d}t} = \{\chi_A, H\} \approx 0 \qquad (\text{for } A = 1,2,\ldots) \tag{3.85}$$

which defines a set of secondary constraints

$$\chi'_A = \{\chi_A, H\} \approx 0 \tag{3.86}$$

some of which can be satisfied trivially. This process must be iterated until no new secondary constraints are generated.

In fact, since we know nothing about observables outside of the constraint surface, we can always add functions that vanish on the constraint surface to the Hamiltonian and other observables:

$$H(p,q) \to H'(p,q) = H(p,q) + \sum_A f_A(p,q)\chi_A(p,q) \tag{3.87}$$

We will find in our case that the Poisson-algebra of H and the *first class constraints* χ_A closes[4]:

$$\{H, \chi_A\} = V_{AB}\chi_B \tag{3.88a}$$

$$\{\chi_A, \chi_B\} = U_{ABC}\chi_C\,, \tag{3.88b}$$

where the U and V are not necessarily constant.

3.3.3 Gauß' Law Is Not An Equation Of Motion!

In our example

$$\mathcal{H} = \frac{1}{2}\vec\pi_a\vec\pi_a + \frac{1}{2}\vec B_a\vec B_a + A^0_a(\vec D\vec\pi + \rho)_a - \vec A_a\vec j_a + \text{surface terms} \tag{3.89}$$

we find the following constraints

$$\chi_0 = \pi_0 \approx 0 \tag{3.90a}$$

$$\chi_G = \vec D\vec\pi + j_0 \approx 0 \tag{3.90b}$$

that form with the Hamiltonian

$$H = \int \mathrm{d}^3x\, \mathcal{H} \tag{3.91}$$

a closed algebra

$$\{H, \chi_0(\vec x)\} = \chi_G(\vec x) \tag{3.92a}$$

---

[4] NB: this is not necessarily so and there are systems with additional constraints with

$$\{\chi_A, \chi_B\} \not\approx 0\,.$$

These are called *second class constraints* and are dealt with by a modified Poisson bracket, called *Dirac bracket*.

$$\{H, \chi_G(\vec{x})\} = -\mathrm{i}\left[A^0(\vec{x}), \chi_G(\vec{x})\right] \tag{3.92b}$$

$$\{\chi_{G,a}(\vec{x}), \chi_{G,b}(\vec{y})\} = \mathrm{i}f_{abc}\chi_{G,c}\delta^3(\vec{x}-\vec{y}) \tag{3.92c}$$

where (3.92b) requires that the current is *covariantly* conserved

$$D_\mu j^\mu = 0\,. \tag{3.93a}$$

and (3.92c) that the currents generate the gauge group

$$\left\{j^0_a(\vec{x}), j^0_b(\vec{y})\right\} = \mathrm{i}f_{abc}j^0_c\delta^3(\vec{x}-\vec{y})\,. \tag{3.93b}$$

Examples for such currents are

$$j^\mu_{F,a} = \bar\psi\gamma^\mu t_a\psi \tag{3.94a}$$

$$j^\mu_{B,a} = \mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}t_a\phi\,. \tag{3.94b}$$

3.3.4 Gauge Transformations

Another observation: consider the transformations generated by the constraints

$$\int \mathrm{d}^3y\, f_b(\vec{y})\left\{\chi_{0,b}(\vec{y}), A^0_a(\vec{x})\right\} = -\delta_{ab}f_b(\vec{x}) \tag{3.95a}$$

$$\int \mathrm{d}^3y\, f_b(\vec{y})\left\{\chi_{0,b}(\vec{y}), A^i_a(\vec{x})\right\} = 0 \tag{3.95b}$$

$$\int \mathrm{d}^3y\, g_b(\vec{y})\left\{\chi_{G,b}(\vec{y}), A^0_a(\vec{x})\right\} = 0 \tag{3.95c}$$

$$\int \mathrm{d}^3y\, g_b(\vec{y})\left\{\chi_{G,b}(\vec{y}), A^i_a(\vec{x})\right\} = \left(D^ig(\vec{x})\right)_a\,. \tag{3.95d}$$

With the choice

$$f = -D^0g \tag{3.96}$$

they are actually *gauge transformations*

$$A_\mu \to A_\mu + D_\mu g\,. \tag{3.97}$$

We will use this fact below.

3.3.5 Quantisation?

Just as in the classical version we can't enforce the constraints as operator identities in the quantum version of the theory:

$$\hat{\chi}_A \neq 0\,.$$

The next best option is to define a *physical* subspace $\mathcal{V}_{\text{phys.}}$ of the full *kinematical* Hilbert space $\mathcal{V}$

$$\mathcal{V}_{\text{phys.}} \subseteq \mathcal{V} \tag{3.98}$$

such that all matrix elements of the constraints vanish in $\mathcal{V}_{\text{phys.}}$:

$$\forall \Psi, \Phi \in \mathcal{V}_{\text{phys.}} : \langle \Psi | \hat{\chi}_A | \Phi \rangle = 0\,. \tag{3.99}$$

While this works (by accident) for QED, we have in general problems in perturbation theory

- should we sum over $\mathcal{V}_{\text{phys.}}$ or over $\mathcal{V}$ in intermediate states?
- since in addition to $[\hat{H}, \hat{\chi}_A] \neq 0$, we have $[\hat{H}_0, \hat{\chi}_A] \neq 0$, we can not diagonalize them simultaneously, so the former question doesn't even make a lot of sense!

We need a formalism, where we can use operator identities.

3.4 Classical BRST Formalism

Paradoxically, the solution involves enlarging the phase space even more, but with "negative" degrees of freedom.

3.4.1 Faddeev-Popov Ghosts

Introduce pairs of anticommuting degrees of freedom η_A, $\bar{\eta}_A$ with *symmetric* Poisson brackets $\{\cdot,\cdot\}_+$ amongst themselves

$$\{\bar{\eta}_A, \eta_B\}_+ = -\delta_{AB} \tag{3.100a}$$

$$\{\eta_A, \eta_B\}_+ = 0 \tag{3.100b}$$

$$\{\bar{\eta}_A, \bar{\eta}_B\}_+ = 0 \tag{3.100c}$$

and vanishing Poisson brackets with all other degrees of freedom. Note that the negative sign is just a convention and has nothing to do with "negative" degrees of freedom, the latter is in the "wrong" statistics. For complex conjugation, we choose

$$\eta_A^* = \eta_A \tag{3.101a}$$

$$\bar{\eta}_A^* = -\bar{\eta}_A\,. \tag{3.101b}$$

3.4.2 BRST Transformations

Then we can construct an anticommuting function Ω from the algebra (3.88) of the constraints

$$\Omega = \eta_A \chi_A - \frac{1}{2} \eta_B \eta_A U_{ABC} \bar\eta_C \tag{3.102}$$

with the remarkable property

$$\{\Omega, \Omega\}_+ = 0\,, \tag{3.103}$$

which is not trivial, because the bracket is symmetric. Our conventions also imply that Ω is real

$$\Omega^* = \Omega\,. \tag{3.104}$$

Note that (3.103) implies in concert with the super-Jacobi identity

$$\left\{\{\Omega, \Omega\}_+, F\right\} + \left\{\{F, \Omega\}, \Omega\right\}_+ - \left\{\{\Omega, F\}, \Omega\right\}_+ = 0 \tag{3.105}$$

imply that the transformation generated by $\delta_B = \{\cdot, \Omega\}$ is nilpotent:

$$\left\{\{F, \Omega\}, \Omega\right\}_+ = -\frac{1}{2} \left\{\{\Omega, \Omega\}_+, F\right\} = 0\,, \tag{3.106}$$

i. e. $\delta_B^2 = 0$. Ω encodes the algebra (3.88) of the constraints and observables. Since

$$\delta_B O = \{O, \Omega\} = \eta_A \{O, \chi_A\} = \eta_A V^O_{AB} \chi_B\,, \tag{3.107}$$

we have

$$\delta_B O = 0 \leftrightarrow \forall A : \{O, \chi_A\} = 0\,. \tag{3.108}$$

3.4.3 Observables

But we can do better and define a minimal extension of any observable

$$O_{\text{min.}} = O + \eta_A V^O_{AB} \bar\eta_B \tag{3.109}$$

with the properties

$$O^*_{\text{min.}} = O_{\text{min.}} \tag{3.110a}$$

$$O_{\text{min.}}\Big|_{\eta_A = \bar\eta_A = 0} = 0 \tag{3.110b}$$

$$\delta_B O_{\text{min.}} = \{O_{\text{min.}}, \Omega\} = 0 \tag{3.110c}$$

where the latter is valid everywhere, even *off the constraint surface.*

NB: due to the nilpotency of Ω, we can always add a term

$$O_B^\Psi = O_{\text{min.}} + \delta_{BRST}\Psi \tag{3.111}$$

with

$$\Psi = \psi_A \bar\eta_A + \text{higher ghost powers} \tag{3.112}$$

maintaining

$$\delta_B O_B^\Psi = 0\,. \tag{3.113}$$

3.4.4 Gauge Fields

Constraints and Ghosts

$$\chi_0 = t_a \chi_{0,a} \qquad \eta_0 = t_a \eta_{0,a} \qquad \bar\eta_0 = t_a \bar\eta_{0,a} \tag{3.114a}$$

$$\chi_G = t_a \chi_{G,a} \qquad \eta_G = t_a \eta_{G,a} \qquad \bar\eta_G = t_a \bar\eta_{G,a} \tag{3.114b}$$

BRST Charge and Hamiltonian

$$\Omega = \int \mathrm{d}^3x\, 2\,\mathrm{tr}\left(\eta_0\chi_0 + \eta_G\chi_G + \frac{\mathrm{i}}{2}\left[\eta_G,\eta_G\right]\bar\eta_G\right) \tag{3.115a}$$

$$H_{\text{min.}} = H + \int \mathrm{d}^3x\, 2\,\mathrm{tr}\left(\eta_0\bar\eta_G - \mathrm{i}\eta_G\left[A_0,\bar\eta_G\right]\right) \tag{3.115b}$$

BRST Transformations

Lecture 14: Tue, 07. 06. 2016

$$\delta_B\eta_0 = \{\eta_0,\Omega\}_+ = 0 \tag{3.116a}$$

$$\delta_B\bar\eta_0 = \{\bar\eta_0,\Omega\}_+ = -\pi^0 \tag{3.116b}$$

$$\delta_B\eta_G = \{\eta_G,\Omega\}_+ = -\frac{\mathrm{i}}{2}\left[\eta_G,\eta_G\right] \tag{3.116c}$$

$$\delta_B\bar\eta_G = \{\bar\eta_G,\Omega\}_+ = -\chi_G - \mathrm{i}\left[\eta_G,\bar\eta_G\right] \tag{3.116d}$$

$$\delta_B A^0 = \left\{A^0,\Omega\right\} = \eta_0 \tag{3.116e}$$

$$\delta_B\vec A = \left\{\vec A,\Omega\right\} = -\vec D\eta_G \tag{3.116f}$$

$$\delta_B\pi^0 = \left\{\pi^0,\Omega\right\} = 0 \tag{3.116g}$$

$$\delta_B\vec\pi = \{\vec\pi,\Omega\} = -\mathrm{i}\left[\vec\pi,\eta_G\right] \tag{3.116h}$$

Corollary

$$\delta_B\vec B = \left\{\vec B,\Omega\right\} = -\vec D\times\vec D\eta_G = -\mathrm{i}\left[\vec B,\eta_G\right] \tag{3.117}$$

BRST Hamiltonian

Educated guess

$$\Psi_\alpha = \int \mathrm{d}^3x \, 2 \operatorname{tr} \left(\left(\vec{\nabla}\vec{A} + \frac{\alpha}{2}\pi^0 \right) \bar{\eta}_0 \right) \tag{3.118}$$

Then

$$\begin{aligned} H_{\mathrm{BRST},\alpha} &= H_{\mathrm{min.}} + \{\Psi_\alpha, \Omega\}_+ \\ &= \int \mathrm{d}^3x \, 2 \operatorname{tr} \biggl(\frac{1}{2} (\vec{\pi})^2 - \frac{\alpha}{2} \left(\pi^0\right)^2 + \frac{1}{2} \left(\vec{B}\right)^2 + A^0 \vec{D}\vec{\pi} - \pi^0 \vec{\nabla}\vec{A} \\ &\qquad + \eta_0 \bar{\eta}_G - \vec{D}\eta_G \vec{\nabla}\bar{\eta}_0 - \mathrm{i}\eta_G \left[A^0, \bar{\eta}_G\right] + j^0 A^0 - \vec{j}\vec{A} \biggr) \end{aligned} \tag{3.119}$$

Hamiltonian Equations of Motion

$$\dot{\eta}_G = \{\eta_G, H_{\mathrm{BRST},\alpha}\} = \eta_0 + i\left[A^0, \eta_G\right] \tag{3.120a}$$

$$\dot{\bar{\eta}}_0 = \{\bar{\eta}_0, H_{\mathrm{BRST},\alpha}\} = -\bar{\eta}_G \tag{3.120b}$$

$$\dot{\vec{A}} = \left\{\vec{A}, H_{\mathrm{BRST},\alpha}\right\} = \vec{\pi} - \vec{D}A^0 \tag{3.120c}$$

$$\ldots \tag{3.120d}$$

These allow to eliminate $\vec{\pi}$, η_0 and $\bar{\eta}_G$

$$\vec{\pi} = \dot{\vec{A}} + \vec{D}A^0 \tag{3.121a}$$

$$\eta_0 = D^0\eta_G \tag{3.121b}$$

$$\bar{\eta}_G = -\dot{\bar{\eta}}_0 \tag{3.121c}$$

to find

$$\delta_B A_\mu = \{A_\mu, \Omega\} = D_\mu \eta_G \tag{3.122}$$

i. e. infinitesimal gauge transformation with η_G as parameter

Lorentz Covariant Equations of Motion

For the remaining fields, the suffixes G and 0 are redundant and we can use the abbreviations

$$\eta = \eta_G \tag{3.123a}$$

$$\bar{\eta} = \bar{\eta}_0 \tag{3.123b}$$

and find

$$\begin{aligned}
\partial_\mu D^\mu \eta &= 0 && (3.124\text{a})\\
D^\mu \partial_\mu \bar\eta &= 0 && (3.124\text{b})\\
D^\mu \partial_\mu \pi_0 &= \mathrm{i}\,[D_\mu\eta, \partial^\mu\bar\eta] && (3.124\text{c})\\
\partial_\mu A^\mu &= -\alpha\pi^0 && (3.124\text{d})\\
D_\mu F^{\mu\nu} &= \partial^\nu\pi^0 + j^\nu - \mathrm{i}\,[\eta, \partial^\nu\bar\eta]\ , && (3.124\text{e})
\end{aligned}$$

the BRST charge

$$\begin{aligned}
\Omega = \int \mathrm{d}^3x\, 2\,\mathrm{tr}\left(\pi^0 \overleftrightarrow{\partial}_0 \eta - \mathrm{i}\pi^0\left[A^0, \eta\right] + \frac{\mathrm{i}}{2}\,[\eta,\eta]\,\dot{\bar\eta}\right)& \\
= \int \mathrm{d}^3x\, 2\,\mathrm{tr}\left(\pi^0 D_0\eta - (\partial_0\pi^0)\eta + \frac{\mathrm{i}}{2}\,[\eta,\eta]\,\dot{\bar\eta}\right)& \quad (3.125)
\end{aligned}$$

and BRST transformations

$$\begin{aligned}
\delta_B A_\mu &= D_\mu\eta && (3.126\text{a})\\
\delta_B \eta &= -\frac{\mathrm{i}}{2}\,[\eta,\eta] && (3.126\text{b})\\
\delta_B \bar\eta &= -\pi^0 && (3.126\text{c})\\
\delta_B \pi^0 &= 0 && (3.126\text{d})
\end{aligned}$$

QED

$$\begin{aligned}
\Box\eta = \Box\bar\eta &= \Box\pi_0 = 0 && (3.127\text{a})\\
\partial_\mu A^\mu &= -\alpha\pi^0 && (3.127\text{b})\\
\partial_\mu F^{\mu\nu} &= \partial^\nu\pi^0 + j^\nu\,, && (3.127\text{c})
\end{aligned}$$

$$\Omega = \int \mathrm{d}^3x\, \pi^0 \overleftrightarrow{\partial}_0 \eta \quad (3.128)$$

and

$$\begin{aligned}
\delta_B A_\mu &= \partial_\mu\eta && (3.129\text{a})\\
\delta_B \eta &= 0 && (3.129\text{b})\\
\delta_B \bar\eta &= -\pi^0 && (3.129\text{c})\\
\delta_B \pi^0 &= 0\,. && (3.129\text{d})
\end{aligned}$$

Therefore all ghosts, η and $\bar\eta$, and $\pi = -\partial_\mu A^\mu/\alpha$ are free fields and decouple.

3.5 Quantum BRST Formalism

3.5.1 (Anti-)Commutation Relations

The Faddeev-Popov ghosts have canonical anti-commutation relations

$$[\bar\eta_A, \eta_B]_+ = -\mathrm{i}\delta_{AB} \tag{3.130a}$$
$$[\eta_A, \eta_B]_+ = 0 \tag{3.130b}$$
$$[\bar\eta_A, \bar\eta_B]_+ = 0 \tag{3.130c}$$

and commute with the other degrees of freedom. For hermition conjugation, we choose

$$\eta_A^\dagger = \eta_A \tag{3.131a}$$
$$\bar\eta_A^\dagger = -\bar\eta_A\,. \tag{3.131b}$$

Then, *iff the Poisson-bracket algebra of constraints can be represented as a commutator algebra of operators on Hilbert space*, we can use the same formulae to find extended observables O_B and a BRST charge Ω with

$$\Omega^2 = \frac{1}{2}\left[\Omega,\Omega\right]_+ = 0 \tag{3.132a}$$
$$\Omega^\dagger = \Omega \tag{3.132b}$$
$$[\Omega, H_B]_- = 0 \tag{3.132c}$$
$$H_B^\dagger = H_B \tag{3.132d}$$

3.5.2 The Cohomology of Ω

From (3.132c) we infer that Ω is constant in time. Therefore we can use it in equations defining the physical subspace. Let $\mathcal{V}$ be the *indefinite metric Hilbert space* in which we represent the canonical (anti-)commutation relations, including the unphysical degrees of freedom and ghosts. Then we demand that a physical state from $\mathcal{V}_{\text{phys.}}$ is annihilated by Ω:

$$\forall\Psi\in\mathcal{V}_{\text{phys.}} : \Omega\left|\Psi\right\rangle = 0 \tag{3.133}$$

or

$$\mathcal{V}_{\text{phys.}} \subset \mathrm{Ker}(\Omega)\,. \tag{3.134}$$

This is a very reasonable condition, because Ω often generates gauge transformations with ghosts as parameters.

Unfortunately, Ker(Ω) contains many zero-norm states. Since $\Omega^2 = 0$, we have

$$\forall \Psi \in \mathcal{V} : \Omega \left|\Psi\right\rangle \in \mathrm{Ker}(\Omega) \tag{3.135a}$$

as well as

$$\forall \Psi \in \mathcal{V} : \left|\left|\Omega \left|\Psi\right\rangle\right|\right|^2 = \left\langle\Psi\middle|\Omega^\dagger\Omega\middle|\Psi\right\rangle = \left\langle\Psi\middle|\Omega^2\middle|\Psi\right\rangle = 0\,. \tag{3.135b}$$

Fortunately, such states do not contribute to matrix elements of observables, for which we demand $[O, \Omega] = 0$:

$$\forall \Psi \in \mathrm{Ker}(\Omega), \Phi \in \mathrm{Im}(\Omega) : \langle\Psi|O|\Phi\rangle = \\ \langle\Psi|O\Omega|\Xi\rangle = \langle\Psi|\Omega O|\Xi\rangle = \left\langle\Psi\middle|\Omega^\dagger O\middle|\Xi\right\rangle = 0\,. \tag{3.136}$$

Therefore we can factor these states out without affecting predictions for observables

$$\mathcal{V}_{\text{phys.}} = \mathrm{Ker}(\Omega)/\mathrm{Im}(\Omega)\,, \tag{3.137}$$

a. k. a. the *cohomology* of the BRST-charge Ω in $\mathcal{V}$. Nevertheless, two crucial facts must still be shown for specific examples:

1. $\mathcal{V}_{\text{phys.}}$ is non-trivial, i. e. $\mathrm{Ker}(\Omega) \neq \mathrm{Im}(\Omega)$, and
2. all vectors in $\mathcal{V}_{\text{phys.}}$ have positive norm.

QED

Since $\pi^0 = -\partial_\mu A^\mu/\alpha$ and the ghosts are a free field, we can consistently split them in positive frequency (annihilation) and negative frequency (creation) parts. The BRST-Charge (3.125) assumes a very simple form

$$\Omega = \int \widetilde{\mathrm{d}k} \left(a_{\pi^0}(k)c_\eta^\dagger(k) + a_{\pi^0}^\dagger(k)c_\eta(k)\right) \tag{3.138}$$

and we recover the *Gupta-Bleuler* condition

$$\forall \Psi \in \mathcal{V}_{\text{phys.}} : (\partial_\mu A^\mu)^{(+)} \left|\Phi\right\rangle = 0\,. \tag{3.139}$$

3.5.3 Quartet Mechanism

Consider the six degrees of freedom

$$\{A_+, A_-, A_L, A_S, \eta, \bar\eta\} \tag{3.140}$$

where $A_\pm$ are the left- and righthanded polarization states, $A_S = \partial_\mu A^\mu$ the so called scalar polarization state and A_L the longitudinal polarization.

As can be shown [?], they can be decomposed into a pair of physical fields and a *quartet* of unphysical fields

$$P = \{A_+, A_-\} \;, \quad Q = \{A_L, A_S, \eta, \bar\eta\} \;, \tag{3.141}$$

where the operators from P generate a Hilbert space that can be identified with $\mathcal{V}_{\text{phys.}}$. Note that they are, while isomorphic, not identical, because the elements of $\mathcal{V}_{\text{phys.}}$ are equivalence classes of states in $\mathcal{V}$, not elements of $\mathcal{V}$ itself.

This proves that all elements in $\mathcal{V}_{\text{phys.}}$ have positive norm and since $[\Omega, H_B] = 0$ and $H_B^\dagger = H_B$ we a have unitary time evolution on $\mathcal{V}_{\text{phys.}}$ and, consequently, S-matrix that commutes with the BRST-Charge

$$[\Omega, S] = 0\,. \tag{3.142}$$

Quartet Mechanism w/Higgses

In the case of spontaneously broken gauge symmetries, there is a similar quartet mechanism, but the role of the longitudinally polarized gauge boson A_L is taken over by the would-be Goldstone boson so that it can become a physical degree of freedom.

3.6 Action

In practical applications, we calculate S-matrix elements and Green's functions using Feynman diagrams. There are two ways to obtain a useful action

1. perform the inverse Legendre transform of $H_{\text{BRST},\alpha}$, (3.119), or
2. construct one from scratch, demanding
   (a) gauge fixing, i. e. existence of a propagator,
   (b) hermiticity, and
   (c) BRST-invariance.

The latter can be easily achieved by adding a term of the form

$$\begin{aligned}\mathcal{L}_{\text{BRST}} &= -2\delta_B \operatorname{tr}\left(\bar\eta\left(\mathcal{G}(A) + \frac{\alpha}{2}\pi^0\right)\right) \\ &= 2\operatorname{tr}\left(\pi^0\mathcal{G}(A) + \frac{\alpha}{2}\left(\pi^0\right)^2 + \bar\eta\frac{\delta G(A)}{\delta A}\delta_B A\right)\end{aligned} \tag{3.143}$$

with an appropriately chosen $\mathcal{G}(A)$ to the classical Lagrangian. A popular choice is

$$\mathcal{G}(A) = \partial_\mu A^\mu\,, \tag{3.144}$$

resulting in

$$\frac{\delta G(A)}{\delta A}\delta_B A = \partial_\mu D^\mu \eta \tag{3.145}$$

and thus

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - 2\operatorname{tr}\left(j_\mu A^\mu\right) + 2\operatorname{tr}\left(\pi^0\partial_\mu A^\mu + \frac{\alpha}{2}\left(\pi^0\right)^2 - \partial_\mu\bar\eta D^\mu\eta\right)\,. \tag{3.146}$$

This has the useful property that

$$\frac{\partial\mathcal{L}}{\partial\partial_0 A_{a,0}} = \pi^0_a \tag{3.147}$$

and is therefore equivalent to our Hamiltonian construction.

3.6.1 Matter Fields

So far, we haven't discussed matter fields. For matter fields transforming like

$$\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi \tag{3.148}$$

with α in the appropriate representation, the infinitesimal gauge transformations are

$$\delta\psi = \mathrm{i}\alpha\psi \tag{3.149}$$

and the corresponding BRST-transformations obviously

$$\delta_B\psi = \mathrm{i}\eta\psi\,. \tag{3.150a}$$

Care must be taken with the sign of the BRST-transformation of the conjugate fermions

$$\delta_B\bar\psi = \mathrm{i}\bar\psi\eta\,, \tag{3.150b}$$

because

$$0 = \delta_B(\bar\psi\psi) = \delta_B\bar\psi\psi - \bar\psi\delta_B\psi \tag{3.151}$$

shows that the naive expectation

$$\delta_B^{\text{naive}}\bar\psi = -\mathrm{i}\bar\psi\eta^\dagger = -\mathrm{i}\bar\psi\eta$$

is inconsistent. This way, a gauge invariant matter lagragian can take the place of the external current in a BRST-invariant lagragian for fermions

$$\mathcal{L}_\psi = \bar\psi\left(\mathrm{i}\not{D} - m\right)\psi = \bar\psi\left(\mathrm{i}\not\partial - m\right)\psi + \bar\psi\not{A}\psi \tag{3.152}$$

i. e.

$$j_a^\mu = -\bar\psi\left(\gamma^\mu\otimes t_a\right)\psi \tag{3.153}$$

and bosons

$$\begin{aligned}\mathcal{L}_\phi &= \left(D_\mu\phi\right)^\dagger D^\mu\phi - m^2\phi^\dagger\phi \\ &= \left(\partial_\mu\phi\right)^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \mathrm{i}\phi^\dagger A_\mu\partial^\mu\phi + \mathrm{i}\phi^\dagger\overleftarrow{\partial_\mu}A^\mu\phi + \phi^\dagger A^\mu A_\mu\phi\end{aligned} \tag{3.154}$$

i. e.

$$j_a^\mu = -\mathrm{i}\phi^\dagger t_a\overleftrightarrow{\partial_\mu}\phi \tag{3.155}$$

and a quartic coupling, as in QED.

3.6.2 Perturbation Theory

So far, we have no small parameter in our action, that would allow a perturbative expansion. Therefore, we perform a *simultaneous* rescaling of our gauge connection, ghosts and gauge lagragian

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu \to \partial_\mu - \mathrm{i}gA_\mu \tag{3.156a}$$
$$F_{\mu\nu} \to gF_{\mu\nu} = g\left(\partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\left[A_\mu, A_\nu\right]\right) \tag{3.156b}$$
$$\eta \to g\eta \tag{3.156c}$$
$$\bar\eta \to g\bar\eta \tag{3.156d}$$
$$\pi^0 \to g\pi^0 \tag{3.156e}$$
$$\mathrm{tr}\left(\cdots\right) \to \frac{1}{g^2}\,\mathrm{tr}\left(\cdots\right) \tag{3.156f}$$

compatible with the gauge/BRST transformations

$$\delta_B\psi = \mathrm{i}g\eta\psi \tag{3.157a}$$
$$\delta_B\bar\psi = \mathrm{i}g\bar\psi\eta \tag{3.157b}$$
$$\delta_B A_\mu = D_\mu\eta \tag{3.157c}$$
$$\delta_B\eta = -\frac{\mathrm{i}g}{2}\left[\eta,\eta\right] \tag{3.157d}$$

$$\delta_B\bar\eta = -\pi^0 \tag{3.157e}$$

$$\delta_B\pi^0 = 0\,. \tag{3.157f}$$

Then the gauge fields A_μ decouple in the limit $g \to 0$.

In any case, the equation of motion for π^0 is algebraic

$$0 = \frac{\delta\mathcal{L}}{\delta\pi^0} = \alpha\pi^0 + \mathcal{G}(A) \tag{3.158}$$

and π^0 can be “integrated out” *exactly* by the substitution

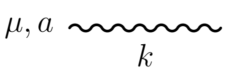
$$\pi^0 = -\frac{1}{\alpha}\mathcal{G}(A) = -\frac{1}{\alpha}\partial^\mu A_\mu\,. \tag{3.159}$$

This results in the following “free” lagragian

$$\begin{aligned}\mathcal{L}_{g=0} &= -\frac{1}{4}F_{a,\mu\nu}F_a^{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu A_a^\mu\right)^2 + \bar\eta^a\Box\eta^a \\ &= \frac{1}{2}A_\mu^a\left(\Box g^{\mu\nu} - \left(1-\frac{1}{\alpha}\right)\partial^\mu\partial^\nu\right)A_\nu^a + \bar\eta^a\Box\eta^a \end{aligned} \tag{3.160}$$

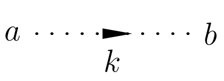
3.6.3 Feynman Rules

If $0 < |\alpha| < \infty$, we can construct a gauge propagator from (3.160):



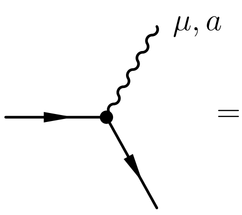
$$= \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\epsilon}\left(-g_{\mu\nu} + (1-\alpha)\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}\right) \tag{3.161a}$$

while the ghost propagator is simply

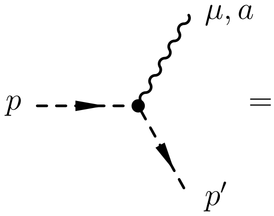


$$= -\frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\epsilon} \tag{3.161b}$$

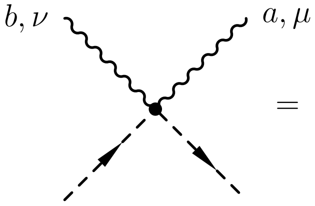
And vertices



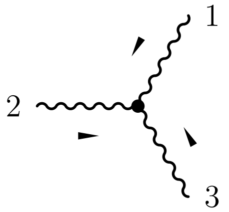
$$= \mathrm{i}g\gamma_\mu t_a \tag{3.161c}$$



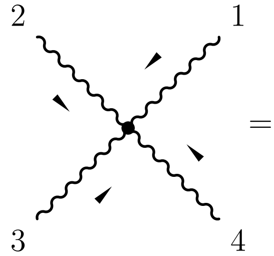
$$= \qquad \mathrm{i}g(p+p')_\mu t_a \tag{3.161d}$$



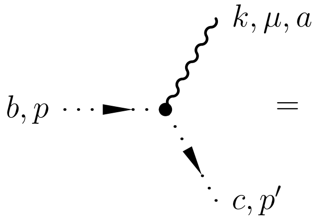
$$= \qquad \mathrm{i}g^2 g_{\mu\nu}\left(t_a t_b + t_b t_a\right) \tag{3.161e}$$



$$= \qquad \begin{aligned} & g f_{a_1a_2a_3} g_{\mu_1\mu_2}(k^1_{\mu_3} - k^2_{\mu_3}) \\ & + g f_{a_1a_2a_3} g_{\mu_2\mu_3}(k^2_{\mu_1} - k^3_{\mu_1}) \\ & + g f_{a_1a_2a_3} g_{\mu_3\mu_1}(k^3_{\mu_2} - k^1_{\mu_2}) \end{aligned} \tag{3.161f}$$



$$= \qquad \begin{aligned} & - \mathrm{i}g^2 f_{a_1a_2b} f_{a_3a_4b}(g_{\mu_1\mu_3} g_{\mu_4\mu_2} - g_{\mu_1\mu_4} g_{\mu_2\mu_3}) \\ & - \mathrm{i}g^2 f_{a_1a_3b} f_{a_4a_2b}(g_{\mu_1\mu_4} g_{\mu_2\mu_3} - g_{\mu_1\mu_2} g_{\mu_3\mu_4}) \\ & - \mathrm{i}g^2 f_{a_1a_4b} f_{a_2a_3b}(g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_4\mu_2}) \end{aligned} \tag{3.161g}$$



$$= \qquad g p'_\mu f_{abc}\,, \tag{3.161h}$$

where the ghost-gauge vertex is indeen *not* symmetric in the momenta.

3.6.4 Slavnov-Taylor Identities

Lecture 15: Tue, 14. 06. 2016

We can now use the BRST invariance of states in $\mathcal{V}_{\text{phys.}}$ together with the BRST transformation properties of the fields to derive non-trivial relations among Green's functions of the theory.

Since the vacuum state has to be in $\mathcal{V}_{\text{phys.}}$, we have

$$\Omega\,|0\rangle = 0 \tag{3.162}$$

and consequently

$$\langle 0|[\Omega, \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle = 0 \tag{3.163}$$

for *arbitrary* fields, including matter, gauge and ghost fields. Furthermore, Ω is constant in time and commutes with the time ordering operator. Therefore

$$\langle 0|\mathrm{T}\,[\Omega, \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)]|0\rangle = 0 \tag{3.164}$$

or

$$\begin{aligned}\langle 0|\mathrm{T}\,\delta_B\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)|0\rangle&\\ \pm\langle 0|\mathrm{T}\,\phi_1(x_1)\delta_B\phi_2(x_2)\cdots\phi_n(x_n)|0\rangle \pm \ldots&\\ \pm\langle 0|\mathrm{T}\,\phi_1(x_1)\phi_2(x_2)\cdots\delta_B\phi_n(x_n)|0\rangle = 0\,,&\end{aligned} \tag{3.165}$$

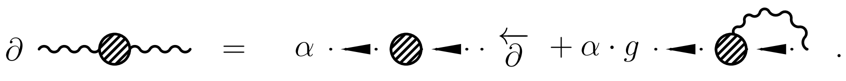
where minus signs appear for fermionic fields.

Examples

1. $\langle 0|\mathrm{T}\,[\Omega, \bar\eta(x)A_\nu(y)]|0\rangle = 0$ results in:

$$\begin{aligned}0 &= \langle 0|\mathrm{T}\,\delta_B\bar\eta(x)A_\nu(y)|0\rangle - \langle 0|\mathrm{T}\,\bar\eta(x)\delta_B A_\nu(y)|0\rangle\\ &= \frac{1}{\alpha}\langle 0|\mathrm{T}\,\partial^\mu A_\mu(x)A_\nu(y)|0\rangle - \langle 0|\mathrm{T}\,\bar\eta(x)D_\nu\eta(y)|0\rangle\end{aligned} \tag{3.166}$$

or graphically



(3.167)

In lowest order, we can immediately verify:

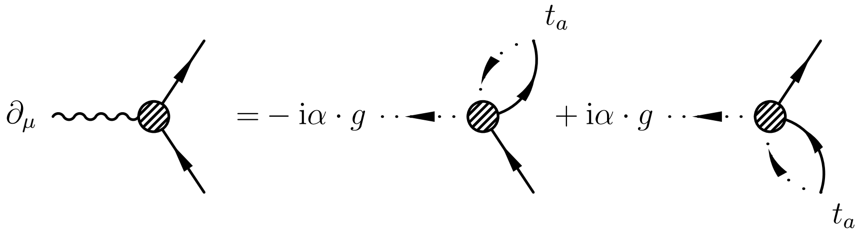
$$k^\mu \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\epsilon}\left(-g_{\mu\nu} + (1-\alpha)\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}\right) = \alpha\frac{-\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\epsilon}k_\nu \tag{3.168}$$

2. $\left\langle 0\middle|\mathrm{T}\left[\Omega, \bar\eta(x)\bar\psi(y)\psi(z)\right]\middle|0\right\rangle = 0$ results in:

$$\begin{aligned}0 &= \left\langle 0\middle|\mathrm{T}\,\delta_B\bar\eta(x)\bar\psi(y)\psi(z)\middle|0\right\rangle\\ &\quad - \left\langle 0\middle|\mathrm{T}\,\bar\eta(x)\delta_B\bar\psi(y)\psi(z)\middle|0\right\rangle + \left\langle 0\middle|\mathrm{T}\,\bar\eta(x)\bar\psi(y)\delta_B\psi(z)\middle|0\right\rangle\\ &= \frac{1}{\alpha}\left\langle 0\middle|\mathrm{T}\,\partial^\mu A_\mu(x)\bar\psi(y)\psi(z)\middle|0\right\rangle - \mathrm{i}g\left\langle 0\middle|\mathrm{T}\,\bar\eta(x)\bar\psi(y)\eta(y)\psi(z)\middle|0\right\rangle\end{aligned}$$

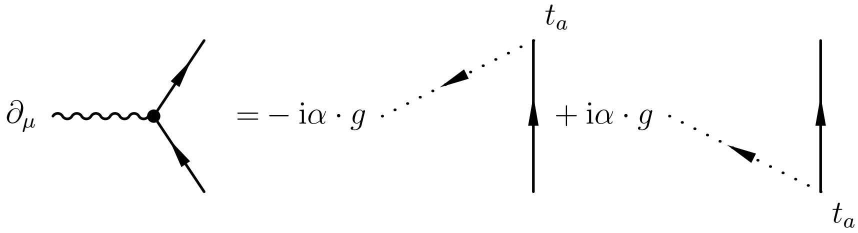
$$+\,\mathrm{i}g\left\langle 0\middle|\mathrm{T}\,\bar\eta(x)\bar\psi(y)\eta(z)\psi(z)\middle|0\right\rangle \quad (3.169)$$

Again, graphically:



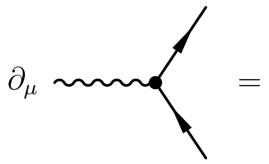
(3.170)

To lowest order

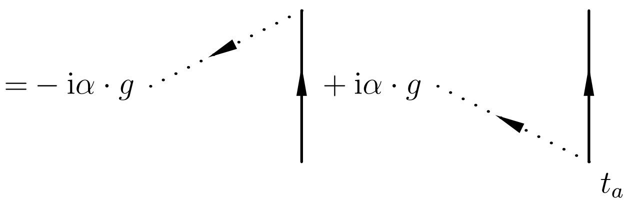


(3.171)

and indeed

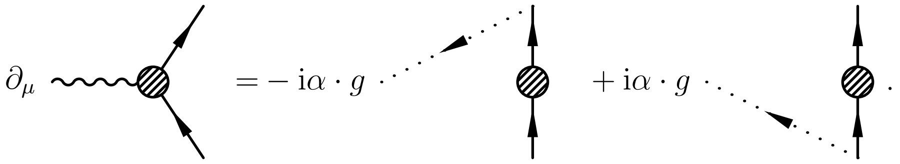


$$k^\mu\frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\epsilon}\left(-g_{\mu\nu}+(1-\alpha)\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}\right)\frac{\mathrm{i}}{\not p'-m+\mathrm{i}\epsilon}\mathrm{i}g\gamma^\nu t_b\frac{\mathrm{i}}{\not p-m+\mathrm{i}\epsilon}$$
$$=\frac{-\mathrm{i}\alpha}{k^2+\mathrm{i}\epsilon}\frac{\mathrm{i}}{\not p'-m+\mathrm{i}\epsilon}\mathrm{i}g\not k t_a\frac{\mathrm{i}}{\not p-m+\mathrm{i}\epsilon}$$
$$=\frac{-\mathrm{i}\alpha}{k^2+\mathrm{i}\epsilon}\frac{\mathrm{i}}{\not p'-m+\mathrm{i}\epsilon}\mathrm{i}g\left(\not p'-\not p\right)t_a\frac{\mathrm{i}}{\not p-m+\mathrm{i}\epsilon}$$
$$=\frac{-\mathrm{i}g\alpha}{k^2+\mathrm{i}\epsilon}\frac{\mathrm{i}}{\not p'-m+\mathrm{i}\epsilon}\mathrm{i}\left(\not p'-m-(\not p-m)\right)t_a\frac{\mathrm{i}}{\not p-m+\mathrm{i}\epsilon}$$
$$=g\alpha\frac{-\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(\frac{\mathrm{i}}{\not p'-m+\mathrm{i}\epsilon}t_a-t_a\frac{\mathrm{i}}{\not p-m+\mathrm{i}\epsilon}\right)$$



(3.172)

In QED, the ghost decouple and enter the equation only to ensure momentum conservation:



(3.173)

In higher orders of the perturbation series, these *Slavnov-Taylor Identities* form a powerful set of consistency relations among Green's functions with different numbers of ghosts and gauge bosons.

3.6.5 Ward Identities

So far, we have studied Green's functions, which contain vastly *more* information than is required for the calculation of S-matrix elements (nobody knows how to build a perturbation theory without that redundant information).

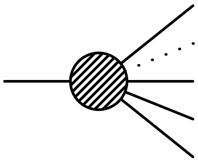
Reduction Formulae

From the LSZ-reduction formulae for scalars (for $Z = 1$)

$$\begin{aligned}\langle k|\mathrm{T}\,\phi(x_1)\ldots\phi(x_n)|0\rangle &= \langle 0|a(k)\,\mathrm{T}\,\phi(x_1)\ldots\phi(x_n)|0\rangle \\ &= \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{i}\left(\Box + m^2\right)\langle 0|\mathrm{T}\,\phi(x)\phi(x_1)\ldots\phi(x_n)|0\rangle\ , \end{aligned} \tag{3.174}$$

we see that external legs are always *amputated*

$$\lim_{k^2\to m^2} -\mathrm{i}(k^2 - m^2)$$



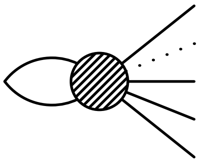
(3.175)

and we can only get a contribution, if there is a pole at the correct mass shell.

Contact Terms

Therefore

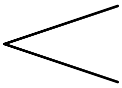
$$\lim_{k^2\to m^2} -\mathrm{i}(k^2-m^2)$$



$$= 0\,, \tag{3.176}$$

because there is no pole. E. g.

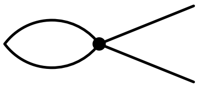
$$\lim_{k^2\to m^2} -\mathrm{i}(k^2-m^2)$$



$$= \lim_{k^2\to m^2} \mathrm{i}\frac{k^2-m^2}{(p_1^2-m^2)(p_2^2-m^2)} = 0\,, \tag{3.177}$$

because $k^2 \neq p_i^2$ (unless $p_j = 0$) and

$$\lim_{k^2\to m^2} -\mathrm{i}(k^2-m^2)$$



$$= 0\,, \tag{3.178}$$

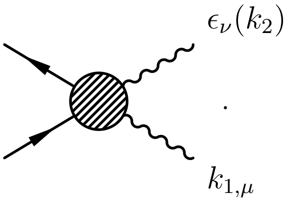
because the loop integral has no pole.

Example

Consequently, a dramatic simplification occurs in S-matrix elements, because the contact terms vanish on the mass shell and we can derive equations like

$$\left\langle 0\middle|\mathrm{T}\,\partial^\mu A_\mu(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\psi(y_2)\middle|0\right\rangle_{\text{amputated, on-shell}} = 0 \tag{3.179a}$$

for *physical* polarizations ϵ_ν or graphically



$$. \tag{3.179b}$$

In the derivation of (3.179a) from Slavnov-Taylor identities

$$0 = \left\langle 0\middle|\mathrm{T}\,\delta_B\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\psi(y_2)\middle|0\right\rangle - \left\langle 0\middle|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu\delta_B A_\nu(x_2)\bar\psi(y_1)\psi(y_2)\middle|0\right\rangle$$

$$-\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\delta_B\bar\psi(y_1)\psi(y_2)|0\rangle+\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\delta_B\psi(y_2)|0\rangle \tag{3.180}$$

most of the terms vanish on-shell:

- in

  $$\begin{aligned}\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu\delta_B A_\nu(x_2)\bar\psi(y_1)\psi(y_2)|0\rangle \\ = \langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu\partial_\nu\eta(x_2)\bar\psi(y_1)\psi(y_2)|0\rangle \\ -\mathrm{i}\,\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu\left[A_\nu(x_2),\eta(x_2)\right]\bar\psi(y_1)\psi(y_2)|0\rangle\end{aligned} \tag{3.181a}$$

  the first term vanishes, because the polarization is physical $\epsilon_\mu(k)k^\mu=0$ and the second is a contact term

- both of

  $$\begin{aligned}\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\delta_B\bar\psi(y_1)\psi(y_2)|0\rangle \\ = -\mathrm{i}\,\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\eta(y_1)\psi(y_2)|0\rangle\end{aligned} \tag{3.181b}$$

  and

  $$\begin{aligned}\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\delta_B\psi(y_2)|0\rangle \\ = -\mathrm{i}\,\langle 0|\mathrm{T}\,\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\eta(y_2)\psi(y_2)|0\rangle\end{aligned} \tag{3.181c}$$

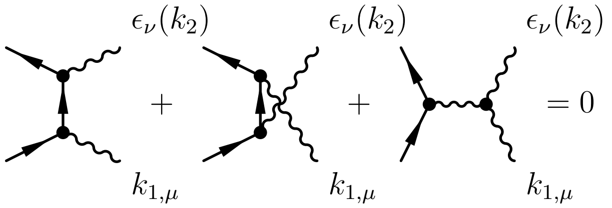
  are contact terms.

Thus only

$$\begin{aligned}0 = \langle 0|\mathrm{T}\,\delta_B\bar\eta(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\psi(y_2)|0\rangle \\ = \frac{1}{\alpha}\,\langle 0|\mathrm{T}\,\partial_\mu A^\mu(x_1)\epsilon^\nu A_\nu(x_2)\bar\psi(y_1)\psi(y_2)|0\rangle\end{aligned} \tag{3.181d}$$

remains, i. e. (3.179a).

At tree level, it is a simple exercise to verify



(3.182)

using the equations of motion

$$\frac{1}{\not p_1-\not k_1-m}\not k_1 u(p_1) = -u(p_1) \tag{3.183a}$$

$$\bar v(p_2)\not{k}_2\frac{1}{-\not{p}_2+\not{k}_2-m}=\bar v(p_1) \tag{3.183b}$$

etc. We also have to use $\epsilon_\mu(k_2)k_2^\mu=0$ to cancel unwanted terms in the triple gauge boson vertex.

In QED, the triple gauge boson vertex does not contribute and we can prove the stronger result

$$\left\langle 0\middle|\mathrm{T}\,\partial^\mu A_\mu(x_1)A_\nu(x_2)\bar\psi(y_1)\psi(y_2)\middle|0\right\rangle=0\,, \tag{3.184}$$

while in nonabelian gauge theories

$$\left\langle 0\middle|\mathrm{T}\,\partial^\mu A_\mu(x_1)A_\nu(x_2)\bar\psi(y_1)\psi(y_2)\middle|0\right\rangle\neq 0 \tag{3.185}$$

due to the triple gauge boson vertices.

3.6.6 Polarization Sums

We can write the polarization sum as

$$\sum_{\lambda=\pm}\epsilon_\lambda^\mu(k)\epsilon_\lambda^{*,\nu}(k)=-g^{\mu\nu}+\frac{c^\mu k^\nu+k^\mu c^\nu}{ck}\,, \tag{3.186}$$

with a suitable vector c. For example, with $k=(\omega,0,0,\omega)$ and $c=(\omega,0,0,-\omega)$

$$\frac{c^\mu k^\nu+k^\mu c^\nu}{ck}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix} \tag{3.187a}$$

and

$$-g^{\mu\nu}+\frac{c^\mu k^\nu+k^\mu c^\nu}{ck}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} \tag{3.187b}$$

Then in QED, where (3.184) holds, we can replace $\sum_{\lambda=\pm}\epsilon_\lambda^\mu(k)\epsilon_\lambda^{*,\nu}(k)$ by $-g^{\mu\nu}$ in all polarization sums, because the single k^μ in $(c^\mu k^\nu+k^\mu c^\nu)/ck$ suffices to make its contribution vanish (independently of c^μ).

However, in nonabelian gauge theories, where (3.184) does *not* hold, we have in general

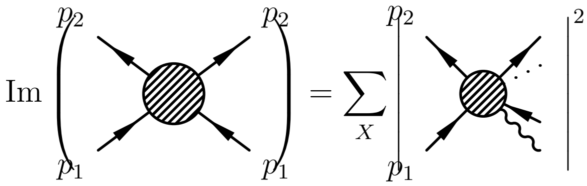
$$\langle 0 | \mathrm{T}\, \partial^\mu A_\mu(x_1) c^\nu A_\nu(x_2) \bar\psi(y_1)\psi(y_2) | 0 \rangle \neq 0\,, \tag{3.188}$$

where $c^\nu A_\nu(x_2)$ is a short hand for the corresponding momentum space expression and we *must not* make the replacement

$$\sum_{\lambda=\pm} \epsilon^\mu_\lambda(k)\epsilon^{*,\nu}_\lambda(k) \to -g^{\mu\nu}\,. \tag{3.189}$$

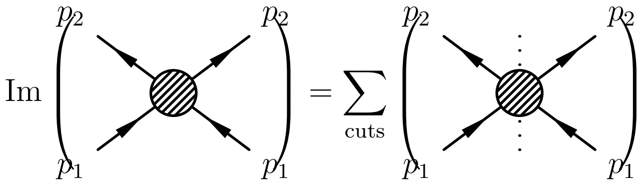
In fact, the corresponding differential cross sections were found to be *negative* in some regions of phase space.

The systematic solution is provided by the optical theorem



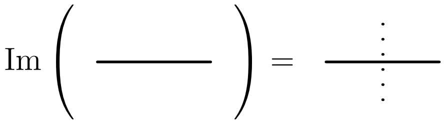
(3.190)

and the Cutkovsky cutting rules



(3.191)

with

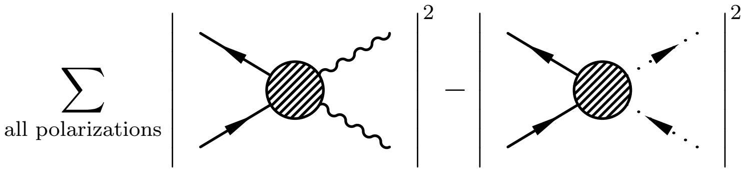


$= \pi\delta(p^2 - m^2)\,.$ (3.192)

This implies that we can use the simple polarization sum corresponding to Feynman gauge

$$\sum_{\lambda=\pm} \epsilon^\mu_\lambda(k)\epsilon^{*,\nu}_\lambda(k) = -g^{\mu\nu} \Longleftrightarrow \frac{-\mathrm{i}g^{\mu\nu}}{k^2+\mathrm{i}\epsilon} \tag{3.193}$$

if *and only if* we include external ghost states



(3.194)

where the sign comes from the fermi statistics of the ghosts

3.7 Path Integrals: Faddeev-Popov Procedure

Lecture 16: Thu, 16. 06. 2016

The ghosts were first guessed by Feynman and then derived systematically by Faddeev and Popov in the path integral formalism. BRST invariance and the canonical formulation came much later.

There's nothing wrong with the path integral

$$Z(j) = \int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A) - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \tag{3.195}$$

with the gauge invariant Yang-Mills action

$$S_{\mathrm{YM}}(A) = -\frac{1}{4}\int \mathrm{d}^4x\, F_{a,\mu\nu}F_a^{\mu\nu} \tag{3.196}$$

and it is used with great success in nonperturbative calculations on the lattice (to be precise an equivalent form that reduces to S_{YM} in the continuum limit).

However, we can not evaluate it in perturbation theory, because it has no propagator, unless we fix the gauge. We could obtain a propagator by fixing the gauge by brute force

$$Z_{\mathbf{BF}}(j,\chi) = \int \mathcal{D}A\, \delta\left(\mathcal{G}(A) - \chi\right) \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A) - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \tag{3.197}$$

but that would not guarantee that the physics remains unchanged. Instead, we should properly separate the gauge degrees of freedom in the functional integral and integrate once over each orbit, i. e. equivalence classes under

$$A_\mu \leftrightarrow UA_\mu U^{-1} + \mathrm{i}U\partial_\mu U^{-1}\,, \tag{3.198}$$

with the *same weight.* Just using the δ-distribution does *not* guarantee this:

$$\int \mathrm{d}x\, f(x)\delta(g(x)) = \sum_{x:g(x)=0} \frac{f(x)}{|\det g'(x)|}\,. \tag{3.199}$$

However

$$\int \mathrm{d}x\, f(x)\delta(g(x))|\det g'(x)| = \sum_{x:g(x)=0} f(x) \tag{3.200}$$

depends *only* on the zeros of g, *not* on any other property of g.

Thus we obtain a better gauge fixed path integral

$$Z_{\mathbf{FP}}(j,\chi) = \int \mathcal{D}A\, \delta\left(\mathcal{G}(A) - \chi\right) \det\left(\frac{\delta\mathcal{G}(A)}{\delta g}\right) \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A) - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \tag{3.201}$$

where $\delta\mathcal{G}(A)/\delta g$ is the functional derivative of the gauge fixing functional w.r.t. gauge transformations.

Since the generating functional does not depend on χ, we can get rid of the δ-distribution by integrating over χ with a suitable weight, e.g.

$$\begin{aligned} Z_{\mathbf{FP}}(j) &= \int \mathcal{D}\chi \mathrm{e}^{-\mathrm{i}\int \mathrm{d}^4x\, \frac{1}{2\alpha}\chi^2} Z_{\mathbf{FP}}(j,\chi) \\ &= \int \mathcal{D}A \,\det\left(\frac{\delta\mathcal{G}(A)}{\delta g}\right) \mathrm{e}^{\mathrm{i}S_{\mathrm{YM}}(A) - \mathrm{i}\int \mathrm{d}^4x\, \frac{1}{2\alpha}(\mathcal{G}(A))^2 - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \,. \end{aligned} \tag{3.202}$$

The functional determinant $\det \delta\mathcal{G}(A)/\delta g$ can be written as a *fermionic* gaussian path integral

$$\det\frac{\delta\mathcal{G}(A)}{\delta g} = \int \mathcal{D}\bar\eta\mathcal{D}\eta\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \bar\eta\frac{\delta\mathcal{G}(A)}{\delta g}\eta} \tag{3.203}$$

which turns out to be the generating functional for Faddeev-Popov ghosts, since

$$\bar\eta\frac{\delta\mathcal{G}(A)}{\delta g}\eta = \bar\eta\partial_\mu D^\mu\eta = \bar\eta\partial\delta_B A_\mu \,. \tag{3.204}$$

This is the same action as before, with π^0 integrated out:

$$Z_{\mathbf{FP}}(j) = \int \mathcal{D}A\mathcal{D}\bar\eta\mathcal{D}\eta\, \mathrm{e}^{\mathrm{i}S_{\mathrm{BRST}}(A) - \mathrm{i}\int \mathrm{d}^4x\, j_{a,\mu}A_a^\mu} \,. \tag{3.205}$$

In the Faddeev-Popov approach, the BRST symmetry is an afterthought, but the rôle is similar: allow a gauge fixing for perturbation theory that keeps the essential symmetry intact.

3.8 Role in Renormalization

We have $\delta_B^2 = 0$, irrespective of the gauge invariance of the action. Therefore we can use it to derive relations among counterterms (*Wess-Zumino Consistency Conditions*) that remain valid, even if we had to fix the gauge in perturbation theory. Using these relations among counterterms, one can prove as a theorem, that if the nth order is gauge invariant, then the $(n+1)$th order counterterms can be chosen such that the $(n+1)$th order is again gauge invariant.

—4—

Renormalization Group

4.1 Renormalizability

Typical expectations

- nature is described by a *renormalizable* QFT, i. e. by a theory consisting solely of interactions with dimension 4 or less,
- the predictive power derives from the fact that there is only a finite set of free parameters.

Open questions:

1. *why* is nature described by a renormalizable QFT? beschrieben?
2. *why* should a successful low energy theory remain valid up to arbitrarily high scales?
3. how do quantum gravity and string theory fit in?
4. the results are the difference of large terms: have the leading been included?

4.2 Pathintegrals à la Polchinski

4.2.1 Soft vs. Hard Modes

Consider the pathintegral for the generating functional

$$Z(j) = \int\mathcal{D}\varphi\, \mathrm{e}^{\mathrm{i}S(\varphi)+\mathrm{i}\int \mathrm{d}^4x\, \varphi(x)j(x)}\,. \tag{4.1}$$

and separate the high energy (*hard*, *fast*, short distance) from the low energy (*soft*, *slow*, long distance) degrees of freedom depending on a scale Λ:

$$\varphi = \varphi^{<}_{\Lambda} + \varphi^{>}_{\Lambda} \tag{4.2a}$$

$$\mathcal{D}\varphi = \mathcal{D}\varphi^{<}_{\Lambda}\mathcal{D}\varphi^{>}_{\Lambda} \tag{4.2b}$$

$$j = j^{<}_{\Lambda} + j^{>}_{\Lambda}\,, \tag{4.2c}$$

where the separation in momentum space is not necessarily sharp. With

$$\tilde\phi(k) = \int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}kx}\phi(x)\,, \tag{4.3}$$

we demand that, invariantly under $k \leftrightarrow -k$,

$$\tilde\phi(k) = \tilde\phi^{<}_{\Lambda}(k) + \tilde\phi^{>}_{\Lambda}(k) \tag{4.4a}$$

$$\tilde\phi^{<}_{\Lambda}(k) = \tilde\phi(k) \text{ for } k \ll \Lambda \tag{4.4b}$$

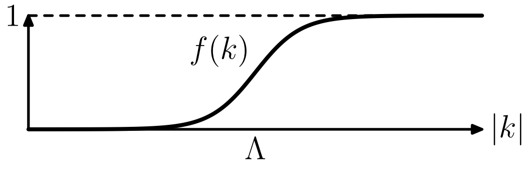
$$\tilde\phi^{>}_{\Lambda}(k) = \tilde\phi(k) \text{ for } k \gg \Lambda \tag{4.4c}$$

e. g.

$$\tilde\phi^{<}_{\Lambda}(k) = (1 - f(k)) \cdot \tilde\phi(k) \tag{4.5a}$$

$$\tilde\phi^{>}_{\Lambda}(k) = f(k) \cdot \tilde\phi(k) \tag{4.5b}$$

with a suitable cut-off function f



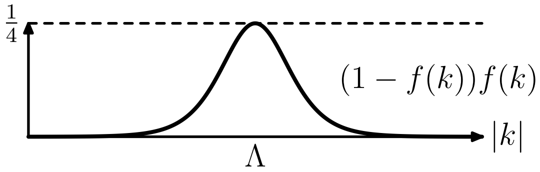
and similarly for the other fields and sources.

Due to momentum conservation, quadratic and bilinear terms in the action approximately separate without mixing soft and hard modes

$$\int \mathrm{d}^4x\,j(x)\varphi(x) = \int \mathrm{d}^4x\,j^{<}_{\Lambda}(x)\varphi^{<}_{\Lambda}(x) + \int \mathrm{d}^4x\,j^{>}_{\Lambda}(x)\varphi^{>}_{\Lambda}(x) \tag{4.6a}$$

$$S_0(\varphi) = S_0(\varphi^{<}_{\Lambda}) + S_0(\varphi^{>}_{\Lambda})\,, \tag{4.6b}$$

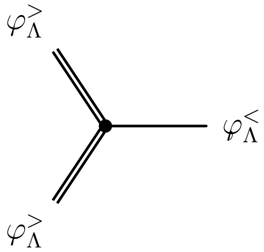
except for modes from the region where $f(k)(1 - f(k)) > 0$ for a cut-off function f with finite width.



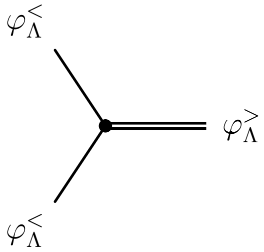
On the other hand, the interaction terms will couple very soft and very hard modes

$$S_I(\varphi) = S_I(\varphi_\Lambda^<) + S_{\text{mix}}(\varphi_\Lambda^<, \varphi_\Lambda^>) + S_I(\varphi_\Lambda^>)\,, \tag{4.7}$$

e. g. two fast modes coupling to a soft mode



making up their separation in momentum space. In a smaller region of momentum space, we can also have two soft modes joining forces to match a hard mode



4.2.2 "Integrating Out"

Then we can exactly rewrite the path integral

$$Z(j_\Lambda^<, j_\Lambda^>) = Z(j) = \int \mathcal{D}\varphi_\Lambda^< \,\mathcal{D}\varphi_\Lambda^>\, \mathrm{e}^{\mathrm{i}S(\varphi)+\mathrm{i}\int \mathrm{d}^4x\,\varphi(x)j(x)} \tag{4.8}$$

by separating

$$\begin{aligned} S(\varphi) + \int \mathrm{d}^4x\,\varphi(x)j(x) = S(\varphi_\Lambda^<) + \int \mathrm{d}^4x\,\varphi_\Lambda^<(x)j_\Lambda^<(x) \\ + S(\varphi_\Lambda^>) + S_{\text{mix}}(\varphi_\Lambda^<, \varphi_\Lambda^>) + \int \mathrm{d}^4x\,\varphi_\Lambda^>(x)j_\Lambda^>(x) \end{aligned} \tag{4.9}$$

as nested path integrals

$$
\begin{aligned}
Z(j^<_\Lambda, j^>_\Lambda) = \int\mathcal{D}\varphi^<_\Lambda\, \mathrm{e}^{\mathrm{i}S(\varphi^<_\Lambda)+\mathrm{i}\int\mathrm{d}^4x\,\varphi^<_\Lambda(x)j^<_\Lambda(x)} \times \\
\int\mathcal{D}\varphi^>_\Lambda\, \mathrm{e}^{\mathrm{i}S(\varphi^>_\Lambda)+\mathrm{i}S_{\mathrm{mix}}(\varphi^<_\Lambda,\varphi^>_\Lambda)+\mathrm{i}\int\mathrm{d}^4x\,\varphi^>_\Lambda(x)j^>_\Lambda(x)}\,. \qquad (4.10)
\end{aligned}
$$

Iff we don't want to study Green's functions and/or scattering matrixelements of particles with momenta $> \Lambda$, we can confine our interest to $Z(j^<_\Lambda, 0)$ and integrate $\varphi^>_\Lambda$ out:

$$
Z_\Lambda(j^<_\Lambda) = Z(j^<_\Lambda, 0) = \int\mathcal{D}\varphi^<_\Lambda\, \mathrm{e}^{\mathrm{i}S(\varphi^<_\Lambda)+\mathrm{i}\int\mathrm{d}^4x\,\varphi^<_\Lambda(x)j^<_\Lambda(x)}\mathrm{e}^{\mathrm{i}\delta S(\varphi^<_\Lambda)} \qquad (4.11)
$$

with

$$
\mathrm{e}^{\mathrm{i}\delta_\Lambda S(\varphi^<_\Lambda)} = \int\mathcal{D}\varphi^>_\Lambda\, \mathrm{e}^{\mathrm{i}S(\varphi^>_\Lambda)+\mathrm{i}S_{\mathrm{mix}}(\varphi^<_\Lambda,\varphi^>_\Lambda)}\,. \qquad (4.12)
$$

This suggests to introduce an *effective action* for the soft modes that contains all the effects on the hard modes

$$
S^{\mathrm{eff.}}_\Lambda(\varphi) = S(\varphi) + \delta_\Lambda S(\varphi) \qquad (4.13)
$$

and we can write the generating functional for soft modes as

$$
Z_\Lambda(j) = \int\mathcal{D}^<_\Lambda\varphi\, \mathrm{e}^{\mathrm{i}S^{\mathrm{eff.}}_\Lambda(\varphi)+\mathrm{i}\int\mathrm{d}^4x\,\varphi(x)j(x)}\,. \qquad (4.14)
$$

where we have written

$$
\mathcal{D}^<_\Lambda\varphi = \mathcal{D}\varphi^<_\Lambda \qquad (4.15a)
$$

to emphsize that $\varphi^<_\Lambda$ and φ are just integration variables and the restriction to soft modes should be considered as a property of the measure. Note that

$$
Z_\Lambda(j) = Z(j, 0) \qquad (4.15b)
$$

exactly, the index Λ only specifies, that only soft sources should be considered.

$$
\frac{\mathrm{d}}{\mathrm{d}\Lambda}Z_\Lambda(j) = 0 \text{ (for } j \text{ softer than } \Lambda) \qquad (4.15c)
$$

The physics interpretation of (4.15) is that we can perform the path integral over the hard modes, *without* effecting the generating functionals for the soft modes. We can choose whether we want to include effects of of the hard modes in the action or in the Feynman diagrams computed using this action. Restricting the integration to the soft modes ensures that there's no double counting.

Use results from Murayama et al. [arXiv:1604.01019] to link this section to the perturbative approach. Clarify the expansion of the non-local Wilsonian effective action into local operators.

4.3 Diagrammatic Approach

The careful reader will recognize the procedure in the previous section as formally equivalent to the Hubbard-Stratonovich transformation or the derivation of the Coleman-Weinberg potential described in section 1.3.4 and we can use the same diagrammatical interpretation.

4.3.1 Cut Off

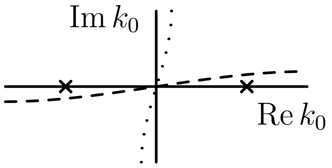
Propagator with *ultra violet* cut off in momentum space

$$\mathrm{i}D_\Lambda(k^2) = \frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\Theta(|k|\le\Lambda)\,, \tag{4.16}$$

where Θ could be replaced by a smooth function. However

- $|k| \le \Lambda$ is not Lorentz invariant!
- $|k^2| \le \Lambda^2$ would be Lorentz invariant, but is ineffective, because k can grow along $k^2 = 0$ without bounds.

As a result, $\Theta(|k|\le\Lambda)$ must be interpreted *symbolically* and a precise definition can be given only by *Wick rotation*: the k_0-integration contour in the loop integrals can be deformed from the dashed curves to the dotted curve



without crossing poles or cuts. With the subsequent substitution

$$(k^0,\vec{k}) \to (ik^0_E,\vec{k}_E)\,, \tag{4.17}$$

the Minkowski-“length” becomes a *euclidean* length

$$k^2 = (k^0)^2 - \vec{k}^2 = -(k^0_E)^2 - \vec{k}^2 = -k^2_E\,, \tag{4.18}$$

which makes the cut off $k^2_E \le \Lambda^2$ effective, because $k^2_E < \Lambda^2$ implies $|k^\mu_E| < \Lambda$.

Using this definition for the propagators, *all* integrals converge in the UV and *all* naive manupulations are allowed. But this comes at a price: all results depend on Λ and we will need the full machinery of the renormalization group to get rid of this dependence.

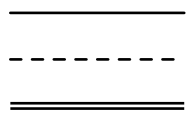
4.3.2 Sliding Cut Off

We can divide the cut off propagators in two pieces: a really soft part $|p| \leq \Lambda'$ and not quite as soft part $\Lambda' \leq |p| \leq \Lambda$:

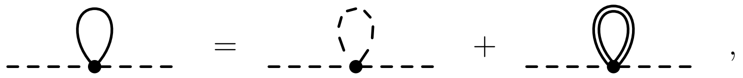
$$iD_\Lambda(k^2) = \tag{4.19a}$$

$$iD_{\Lambda'}(k^2) = \tag{4.19b}$$

$$iD_\Lambda(k^2) - iD_{\Lambda'}(k^2) = \tag{4.19c}$$

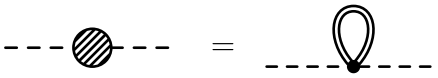


In case of the one loop self energy in ϕ^4-theory, we can write for $|p| \leq \Lambda'$



(4.20)

i. e. the theory with cut off $\Lambda' < \Lambda$ describes the same physics as the one with cut off Λ, as long as a new vertex



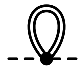
(4.21)

is added to the lagrangian.

Lecture 17: Tue, 21. 06. 2016

Loop Integrals

"tadpole":



$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{i}\Theta(\Lambda' \leq |k| \leq \Lambda)}{k^2 + \mathrm{i}\epsilon} \tag{4.22}$$

More general in D space time dimensions (we will need it later)

$$I_{n,m}^{\Lambda,\Lambda'}(D, M^2) = \int\limits_{\Lambda' \leq |k| \leq \Lambda} \frac{\mathrm{d}^Dk}{(2\pi)^D} \frac{(k^2)^n}{(k^2 - M^2 + \mathrm{i}\epsilon)^m} \,. \tag{4.23}$$

Wick rotation:

$$I_{n,m}^{\Lambda,\Lambda'}(D, M^2) = (-1)^{n+m}\mathrm{i} \int\limits_{\Lambda' \leq |k| \leq \Lambda} \frac{\mathrm{d}^Dk_E}{(2\pi)^D} \frac{(k_E^2)^n}{(k_E^2 + M^2)^m} \,. \tag{4.24}$$

surface of a D-dimensional sphere:

$$\int \mathrm{d}\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \,. \tag{4.25}$$

putting everything together:

$$I_{n,m}^{\Lambda,\Lambda'}(D,M^2) = \frac{(-1)^{n+m}\mathrm{i}}{(4\pi)^{D/2}\Gamma(D/2)} \int\limits_{(\Lambda')^2}^{\Lambda^2} \mathrm{d}k_E^2 \frac{(k_E^2)^{D/2+n-1}}{(k_E^2+M^2)^m} \tag{4.26}$$

This integral has a closed expression

$$\begin{aligned} I_{n,m}^{\Lambda,\Lambda'}(D,M^2) = \frac{(-1)^{n+m}\mathrm{i}}{(4\pi)^{D/2}\Gamma(D/2)} \left(M^2\right)^{D/2+n-m} \cdot \\ \cdot \Bigl(B_{\frac{1}{1+(\Lambda')^2/M^2}}(m-n-D/2, D/2+n) \\ - B_{\frac{1}{1+\Lambda^2/M^2}}(m-n-D/2, D/2+n) \Bigr) \end{aligned} \tag{4.27}$$

using the so-called incomplete beta funktion

$$B_z(x,y) = \int_0^z \mathrm{d}\xi\, \xi^{x-1}(1-\xi)^{y-1} \tag{4.28a}$$

$$B_1(x,y) = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{4.28b}$$

Non trivial limit $M \to 0$: logarithm

$$I_{k,k+D/2}^{\Lambda,\Lambda'}(D,0) = \frac{(-1)^{D/2}\mathrm{i}}{(4\pi)^{D/2}\Gamma(D/2)} \ln\frac{\Lambda^2}{(\Lambda')^2} \tag{4.29}$$

or power law

$$\begin{aligned} I_{n,m}^{\Lambda,\Lambda'}(D,0)\Big|_{m-n\neq D/2} = \frac{(-1)^{n+m}\mathrm{i}}{(4\pi)^{D/2}\Gamma(D/2)} \cdot \\ \cdot \frac{1}{D/2+n-m}\left(\Lambda^{D+2(n-m)} - (\Lambda')^{D+2(n-m)}\right)\,. \end{aligned} \tag{4.30}$$

Back to the tadpole

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\Theta(\Lambda' \le |k| \le \Lambda) = \mathrm{i}I_{0,1}^{\Lambda,\Lambda'}(4,0) = \frac{1}{(4\pi)^2}\left(\Lambda^2 - (\Lambda')^2\right)\,. \tag{4.31}$$

Later we will also need another limit

$$I_{n,m}(D,M^2) = \lim_{\Lambda'\to 0}\lim_{\Lambda\to\infty} I_{n,m}^{\Lambda,\Lambda'}(D,M^2)\,. \tag{4.32}$$

Here we can again find powers

$$I_{n,m}(D,M^2) = \frac{(-1)^{n+m}\mathrm{i}}{(4\pi)^{D/2}\Gamma(D/2)}\left(M^2\right)^{D/2+n-m}\frac{\Gamma(m-n-D/2)\Gamma(D/2+n)}{\Gamma(m)} \tag{4.33}$$

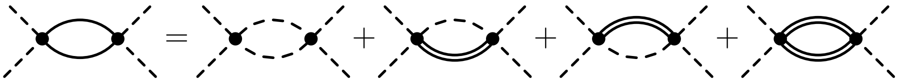
or logarithms

$$I_{0,2}(4-2\epsilon,M^2) = \frac{\mathrm{i}}{(4\pi)^{2-\epsilon}}\left(M^2\right)^{-\epsilon}\Gamma(\epsilon)\,. \tag{4.34}$$

4.3.3 Vertices

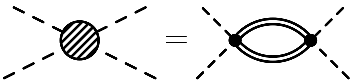
4-Vertex

Analogously for the vertex



, (4.35)

we need a new vertex



$$= -\frac{3}{2}\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{(\Lambda')^2}\frac{g^2}{4!}\phi^4(x)\,. \tag{4.36}$$

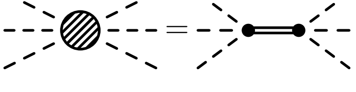
where the integral

$$\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}\Theta(\Lambda'\le|k|\le\Lambda)}{k^2+\mathrm{i}\epsilon}\frac{\mathrm{i}\Theta(\Lambda'\le|p-k|\le\Lambda)}{(p-k)^2+\mathrm{i}\epsilon} = -\frac{1}{2}\frac{\mathrm{i}}{16\pi^2}\ln\frac{\Lambda^2}{(\Lambda')^2}+\mathcal{O}\left(\frac{p^2}{\Lambda^2}\right)\,. \tag{4.37}$$

has been computed using $I_{0,2}^{\Lambda,\Lambda'}(4,|p|^2)$ under the assumption $|p|\ll\Lambda'<\Lambda$.

The computation for $|p|\approx\Lambda'<\Lambda$ is *much* more complicated, due to the non isotropic cut off. A more appropriate procedure will be developped below.

6-Vertex



$$= \frac{1}{(\Lambda')^2}\frac{g^2}{6!}\phi^6(x)+\mathcal{O}\left(\frac{(\Lambda')^2}{\Lambda^2}\right)\,. \tag{4.38}$$

General Procedure

- reduce the scale Λ' step by step
$$\Lambda > \Lambda' > \Lambda'' > \Lambda''' > \ldots \tag{4.39}$$
- change the lagrangian so, that the low energy physics is *unchanged*.

The resulting effective lagrangian will contain arbitrarily high powers of $\phi(x)$ and derivatives.

4.3.4 Renormalization Group Flow

The most general Lagrangian can be expanded in an (infinite) series of operators
$$\mathcal{L}(x) = \sum_i g_i \mathcal{O}_i(x)\,, \tag{4.40}$$
e. g. for a single scalar field
$$\mathcal{O}(x) = \left(\phi^2(x), \phi^4(x), (\partial\phi)^2(x), \phi^6(x), (\phi\partial\phi)^2(x), \ldots\right)\,. \tag{4.41}$$
The procedure of section 4.3.3 defines an infinite matrix $\Gamma_{ij}(\Lambda',\Lambda)$ (with finite coefficients) describing a finite *Renormalization Group* (RG) transformation acting on the couplings
$$g_\mathrm{i}(\Lambda) \to g_\mathrm{i}(\Lambda') = \sum_{j,n} \Gamma_{n,ij}(\Lambda',\Lambda) g_j^n(\Lambda) \tag{4.42}$$
such that both
$$\mathcal{L}(x;\Lambda) = \sum_i g_i(\Lambda)\mathcal{O}_i(x) \tag{4.43}$$
and $\mathcal{L}(x;\Lambda')$ give the same prediction for low energy physics, if the Feynman integrals are cut off at Λ and Λ' respectively.

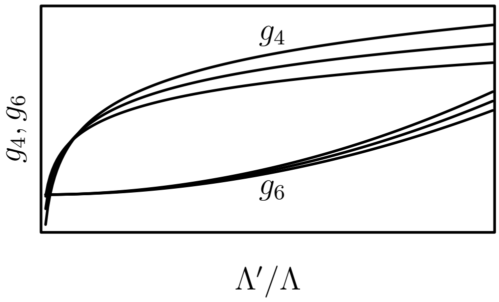
This is approach is not very useful in practice, because the finite transformations are hard to calculate. It is more convenient and transparent to study *continuous* transformations with *infinitesimal* generators. This produces the *Renormalization Group Equation* as a system of coupled ordinary differential equations:
$$(\Lambda')^2 \frac{\mathrm{d}g_\mathrm{i}(\Lambda')}{\mathrm{d}(\Lambda')^2} = \sum_j \gamma_{n,ij} g_j^n(\Lambda')\,. \tag{4.44}$$

Dimensional Transmutation

Consider the evolution of g_4 and g_6

$$\mathcal{L}_{\text{int}}(x) = \frac{g_4}{4!}\phi^4(x) + \frac{g_6}{6!}\phi^6(x) + \ldots \tag{4.45}$$

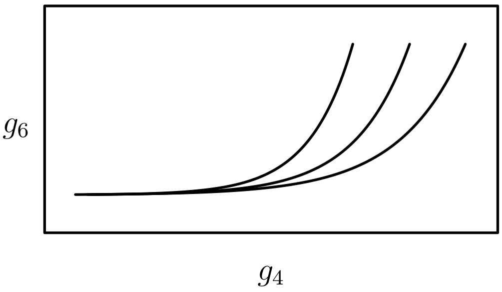
as a function of Λ':



By construction, the low energy physics remains *unchanged* along the trajectories. Λ' is just a *convention* for what is part of the Lagrangian and what should be computed by loop integration in Feynman diagrams. This has two important consequences:

1. Λ' is redundant und can be eliminated,
2. he physics is *not* determined by a point in parameter space, but by a trajectory.

An equivalent representation replaces the dimensionfull parameter Λ' by a dimensionless parameter, e. g. g_4:



This is called *dimensional transmutation* and should be familiar from QCD, where Λ_{QCD} can be traded for α_S and vice versa.

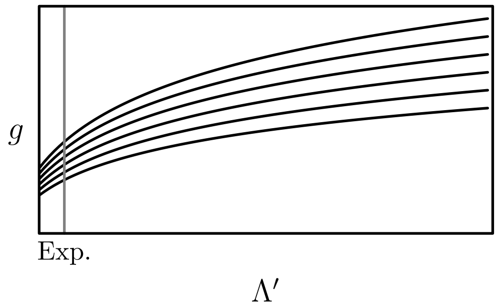
UV Limit

So far, we have ignored the UV cut off Λ that was required for making all integrals finite. Since it is required, we must not simply ignore it.

In the renormalization group picture, Λ plays the rôle of the starting point of the *renormalization group flow* and we can ask the question how the low energy (a. k. a. IR) physics depends on the value of this starting point.

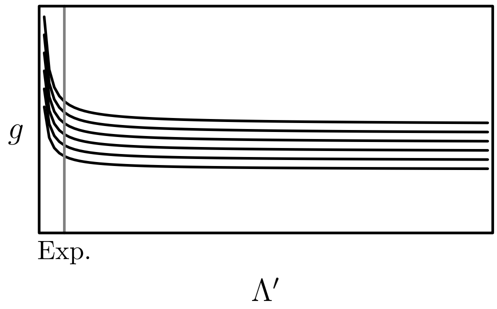
We can identify four basic scenarios:

1. all trajectories can be extended from the IR to $\Lambda' \to \infty$

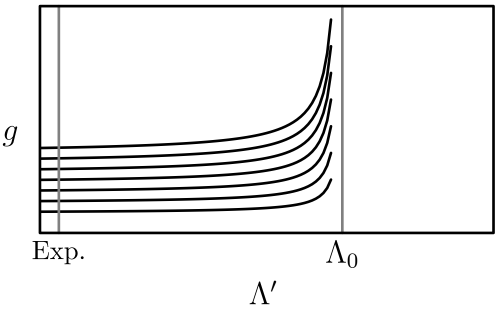
   In this case, we can take the *continuum limit* and the cut off can be removed, even if the couplings do not remain bounded. NB: the perturbative calculation can become unreliable, if the trajectory corresponding to the low energy measurements passes through a non-perturbative region $g \gg 1$.

2. all trajectories remain bounded for $\Lambda' \to \infty$:

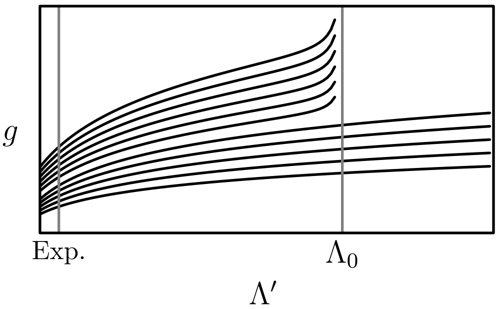
   perturbative calculations reliable if the trajectory corresponding to the low energy measurements remains in the region $g \ll 1$. If all couplings vanish for $\Lambda' \to \infty$, we find *asymptotic freedom*.

3. no trajectory corresponding to low energy measurement can be continued from $\Lambda' < \Lambda_0$ do $\Lambda' > \Lambda_0$ for some value Λ_0:



In this case, the quantum field theory that describes nature at scales $\Lambda' < \Lambda_0$, can *not* describe nature at higher scales. Something *very* interesting is bound to happen at Λ_0. Even if we have to account for a breakdown of perturbation theory below Λ_0, this behaviour is a very strong hint at *new physics*.

4. some trajectories are trapped in the region $\Lambda' < \Lambda_0$, others can be extended to $\Lambda' \to \infty$. Representative example: the trajectories with $g \leq g_0$ in the low energy limit can be extended, but not those with $g > g_0$:



In this case new physics can only be avoided, if the low energy couplings are not too large. In the standard model, this leads to upper limits for the Higgs self coupling and mass $m_H^2 = g/2 \cdot \langle\phi\rangle^2$.

4.3.5 Relevant, marginal & irrelevant

Lecture 18: Thu, 23. 06. 2016

The graphical representation of the renormalization group flow is intuitive for a few couplings, but can we extend it to a real calculational procedure in the *infinite*-dimensional space of all couplings?

In fact, higher dimensional operators induce even more divergent contributions to Feynman diagrams and require counterterms of increasing dimensions. Fortunately *But many that are first shall be last; and the last shall* *be first*, as can be seen again from dimensional analysis for the differential change in the couplings:

$$(\Lambda')^2\frac{\mathrm{d}g_2(\Lambda')}{\mathrm{d}(\Lambda')^2} = (\Lambda')^2\frac{1}{16\pi^2}g_4(\Lambda') + \ldots \tag{4.46a}$$

$$(\Lambda')^2\frac{\mathrm{d}g_4(\Lambda')}{\mathrm{d}(\Lambda')^2} = \frac{3}{2}\frac{1}{16\pi^2}g_4^2(\Lambda') + (\Lambda')^2\frac{1}{16\pi^2}g_6(\Lambda') + \ldots \tag{4.46b}$$

$$(\Lambda')^2\frac{\mathrm{d}g_6(\Lambda')}{\mathrm{d}(\Lambda')^2} \propto \frac{1}{(\Lambda')^2}g_4^2(\Lambda') + \frac{1}{16\pi^2}g_4(\Lambda')g_6(\Lambda') + \ldots\,. \tag{4.46c}$$

Let's write this more concisely as

$$\mu\frac{\mathrm{d}g_2(\mu)}{\mathrm{d}\mu} = \mu^2\beta_2 g_4(\mu) \tag{4.47a}$$

$$\mu\frac{\mathrm{d}g_4(\mu)}{\mathrm{d}\mu} = \beta_4 g_4^2(\mu) + \mu^2\beta'_4 g_6(\mu) \tag{4.47b}$$

$$\mu\frac{\mathrm{d}g_6(\mu)}{\mathrm{d}\mu} = \mu^{-2}\beta_6 g_4^2(\mu) + \beta'_6 g_4(\mu)g_6(\mu)\,, \tag{4.47c}$$

with the identifications

$$\mu = \Lambda' \tag{4.48a}$$

$$\beta_2 = \frac{2}{16\pi^2} \tag{4.48b}$$

$$\beta_4 = \frac{3}{16\pi^2} \tag{4.48c}$$

$$\beta'_4 = \frac{2}{16\pi^2} \tag{4.48d}$$

$$\beta_6 = 2 \tag{4.48e}$$

$$\beta'_6 = \frac{2}{16\pi^2}\,, \tag{4.48f}$$

and introduce dimensionless couplings λ_n with

$$g_n(\mu) = \mu^{4-n}\lambda_n(\mu)\,. \tag{4.49}$$

Then

$$\mu\frac{\mathrm{d}g_n(\mu)}{\mathrm{d}\mu} = \mu^{4-n}\left((4-n)\lambda_n(\mu) + \mu\frac{\mathrm{d}\lambda_n(\mu)}{\mathrm{d}\mu}\right) \tag{4.50}$$

and

$$\mu\frac{\mathrm{d}\lambda_2}{\mathrm{d}\mu} = -2\lambda_2 + \beta_2\lambda_4 \tag{4.51a}$$

$$\mu\frac{\mathrm{d}\lambda_4}{\mathrm{d}\mu} = \beta_4\lambda_4^2 + \beta_4'\lambda_6 \tag{4.51b}$$

$$\mu\frac{\mathrm{d}\lambda_6}{\mathrm{d}\mu} = 2\lambda_6 + \beta_6\lambda_4^2 + \beta_6'\lambda_4\lambda_6\,. \tag{4.51c}$$

Now consider a solution $\bar\lambda_n$ and it's neighborhood, parametrized by small deviations ϵ_n

$$\lambda_n(\mu) = \bar\lambda_n(\mu) + \epsilon_n(\mu)\,, \tag{4.52}$$

leading to the linearized equations

$$\mu\frac{\mathrm{d}\epsilon_2}{\mathrm{d}\mu} = -2\epsilon_2 + \beta_2\epsilon_4 \tag{4.53a}$$

$$\mu\frac{\mathrm{d}\epsilon_4}{\mathrm{d}\mu} = 2\beta_4\bar\lambda_4\epsilon_4 + \beta_4'\epsilon_6 \tag{4.53b}$$

$$\mu\frac{\mathrm{d}\epsilon_6}{\mathrm{d}\mu} = 2\epsilon_6 + 2\beta_6\bar\lambda_4\epsilon_4 + \beta_6'\bar\lambda_6\epsilon_4 + \beta_6'\bar\lambda_4\epsilon_6\,. \tag{4.53c}$$

In the perturbative regime, we know that

$$\beta_n\bar\lambda_{n'} \ll 1 \tag{4.54}$$

and we can use

$$\mu\frac{\mathrm{d}\epsilon_6}{\mathrm{d}\mu} = 2\epsilon_6 \tag{4.55}$$

as a good approximation with solution

$$\epsilon_6(\mu) = \text{const.}\cdot\mu^2 \xrightarrow{\mu\to 0} 0\,. \tag{4.56}$$

If we are in the regime, where ϵ_6 can already be neglected and where in addition $\bar\lambda_4$ is approximately constant, we can solve

$$\mu\frac{\mathrm{d}\epsilon_4}{\mathrm{d}\mu} = 2\beta_4\bar\lambda_4\epsilon_4 \tag{4.57}$$

by

$$\epsilon_4(\mu) = \text{const.}\cdot\mu^{2\beta_4\bar\lambda_4}\,. \tag{4.58}$$

This will tend to zero or blow up, depending on the sign of $\beta_4\bar\lambda_4$. The qualitative behaviour is the same, if $\bar\lambda_4$ is not constant. The region in which ϵ_6 can not yet be neglected will produce a "head-start" for the running of ϵ_4.

Similarly, if we stay close enough to $\bar\lambda_4$, so that $\beta_2\epsilon_4$ can be neglected w. r. t. 2, we can solve

$$\mu\frac{\mathrm{d}\epsilon_2}{\mathrm{d}\mu} = -2\epsilon_2 \tag{4.59}$$

to find

$$\epsilon_2(\mu) = \text{const.} \cdot \mu^{-2} \overset{\mu\to 0}{\longrightarrow} \infty\,. \tag{4.60}$$

This means that the trajectories diverge for $\mu \to 0$.

where the lowest dimensional operators receive the largest corrections!

In the absence of higher order contributions, all dimensionless matrix elements of operators with dimension d must carry a factor of

$$\left(\frac{|p|}{\Lambda}\right)^{d-4}\,. \tag{4.61}$$

Therefore, there are three cases for low energy physics:

- $d < 4$: the contributions of these, so-called *”‘relevant”’*, operators becomes *more important* at low energies,
- $d = 4$: such, so-called *”‘marginal”’*, operators are scale invariant until higher orders are switched on,
- $d > 4$: these, so-called *”‘irrelevant”’*, operators become *less important* at low energies.

Therefore, we recognize the previously “dangerous” nonrenormalizable operators as *irrelevant* and harmless at low energies. As a result, the low energy world can be described by a renormalizable quantum field theory.

Weak (as in “not strong”) interactions don’t change the classification *relevant* and *irrelevant* operators.

E. g.

- $d = 2$ (i. e. g_2 or m^2): the change in the coupling is of dimension 2, like the coupling itself: $\mathcal{O}((\Lambda')^2)$. Therefore without extensive *“fine-tuning”* of the initial conditions, we find $g_2 = \mathcal{O}(\Lambda^2) \gg |p|^2$ for the renormalized coupling.
- $d = 6$ (i.e. g_6): the change in the coupling is of dimension -2: $\mathcal{O}((\Lambda')^{-2})$. Therefore without strong interactions, it is impossible to compensate the factor $1/\Lambda^2$.

On the other hand, the qualitative behaviour of marginal operators will in general be sensitive be affected by weak interactions.

For every theory, i. e. a set of fields with given transformation properties under internal and space time symmetries, there can only be a *finite* number

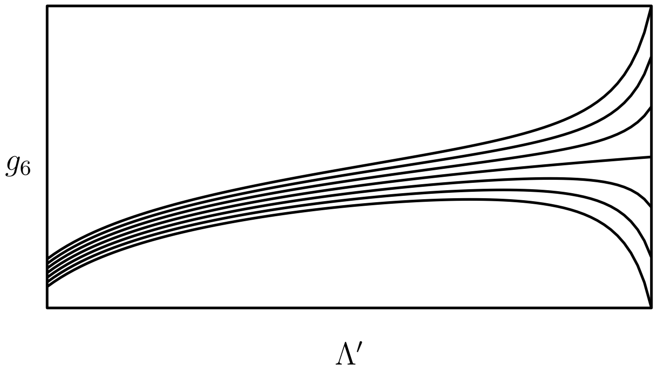


Figure 4.1: *Contributions of irrelevant operators do* not *necessarily vanish in the infrared, but the dependence of these contributions on their initial conditions at high energy vanishes in the infrared.*

of relevant operators. Thus, for practical purposes, we will only need few of the infinitely many possible operators.

To avoid a possible misunderstanding induced by the technical term "irrelevant operator", it is important to stress that the contributions of irrelevant operators do *not* necessarily vanish in the infrared. Instead, the dependence of these contributions on their initial conditions at high energy vanishes, because the RG trajectories flow together in the infrared, as shown in figure 4.1.

Thus the initial conditions for the irrelevant operators have no effect on the results of low energy experiments. In fact, the coefficients of the irrelevant operators can be set to zero at the high scale without changing the observable physics at lower energies. Therefore, the phenomena can be described by a renormalizable theory.

To avoid another possible misunderstanding, this observation does not "prove" that only a renormalizable QFT can describe the observed phenomena. It only states that there is always are renormalizable QFT that is indistinguishable from a non-renormalizable theory at low energies. Since renormalizable theories are technically more convenient and depend on less parameters, common sense and Occam's razor suggest to prefer the renormalizable QFT with only relevant or marginal interactions at the high scale over the others with the same infrared behavior.

German notes start here ...

Komplizierter ist der Fall einer dimensionslosen Kopplung, die ohne Wechselwirkung einen konstanten Beitrag bei allen Skalen liefert.

$\therefore$ die Wechselwirkung bestimmt, ob die Kopplung bei niedrigen Energien groß oder klein ist.

β-Function

Aus der Renormierungsgruppengleichung

$$(\Lambda')^2 \frac{\mathrm{d}g_4(\Lambda')}{\mathrm{d}(\Lambda')^2} = \beta(g_4(\Lambda')) \tag{4.62}$$

kann man ablesen, daß das Verhalten der Kopplung wird vom Vorzeichen der β-Funktion bestimmt wird bestimmt

- $\beta > 0$ (unser Fall $\beta = 3g_4^2/(32\pi^2)$): die Kopplung wird im Ultravioletten stärker und im Infraroten schwächer.
- $\beta < 0$ (z. B. QCD): die Kopplung wird im Ultravioletten schwächer und im Infraroten stärker.

Die Lösung ist in unserem Fall

$$g_4(\Lambda') = \frac{g_4(\Lambda)}{1 + \frac{3}{32\pi^2} g_4(\Lambda) \ln \frac{\Lambda^2}{(\Lambda')^2}} \tag{4.63}$$

Weil das Verhalten von dimensionslosen Kopplungen sensitiv von schwachen Wachenwechselwirkungen abhängt, werden sie als *marginal* bezeichnet.

Weitere Beiträge zur β-Funktion für g_4

- alle haben Faktoren $1/\Lambda^2$ und tragen deshalb *nicht* zur führenden Ordnung bei.

Falls die Wechselwirkung stark genug ist, kann das Skalenverhalten so beeinflußt werden, daß naiv relevante Operatoren irrelevant werden und umgekehrt.

- Kann im perturbativen Bereich *nicht* passieren.

Warum ist die Beschreibung durch Renormierungsgruppenflüsse "'besser"' als direkte Evaluation von Feynmandiagrammen?

- Koeffizienten der Renormierungsgruppentransformation enthalten keine unkontrollierten Integrationen über weit separierte Impulsbereiche

$\therefore$ große Logarithmen werden erst bei der Integration der Renormierungsgruppengleichung *langsam* aufgesammelt

- kontrollierte Rechnung

Im Prinzip auch *nichtperturbative* Berechnung von Pfadintegralen durch Lösung von Renormierungsgruppengleichungen möglich

- technisch schwierig, noch wenig Erfolge.

Fine Tuning

Massenparameter sind *relevant*: sofern die Masse eines Teilchens nicht durch eine Symmetrie vor Renormierung geschützt ist, erfordert die Auswahl einer Trajektorie mit $m \ll \Lambda$ ein unnatürliches *”‘fine tuning”’:*

> Because strait is the gate, and narrow is the way, which leadeth unto life, and few there be that find it.

- starkes Argument für Supersymmetrie
  - Fermionmassen können durch chirale Symmetrie geschützt werden.
  - Bosonmassen können durch eine Supersymmetrie von der chiralen Symmetrie profitieren.
- und/oder *dynamische* elektroschwache Symmetriebrechung
  - keine elementare Skalare, die unter *”‘fine tuning”’* leiden.

4.4 Callan-Symanzik Gleichung

Endliche Integrale mit mehreren Massenparametern sind technisch *schwierig* (vgl. unvollständige Beta-Funktion oben).

- gibt es einen Trick, um die Rechnungen zu vereinfachen?

Beobachtung:

- eine Absenkung des unteren Abschneideparameters Λ' entspricht einer Änderung der Lagrangefunktion

$\therefore$ wir können Λ' mit dem (Re)-Normierungspunkt μ in einer perturbativen Rechnung identifizieren.

☡ der *harte* Cut Off $|k| < \Lambda$ im Impulsraum wird durch einen *weichen* Cut Off durch Counter Terme ersetzt.

$\therefore$ Unterschied ist höherer Ordnung.

Betrachte beliebige n-Punkt Greensfunktion

$$G^{(n)}(x_1, x_2, \ldots, x_n; g, \mu) = \langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle \tag{4.64}$$

renormiert ”‘am Punkt μ”’, d. h. mit den Invarianten der Impulse in den Renormierungsbedingungen gleich μ.

Eine Änderung des Renormierungspunktes ändert die Physik nicht, wir wandern lediglich zu einem neuen Punkt auf der *gleichen* Trajektorie.

$\therefore$ wir können neue renormierte Kopplungen g' finden, sodaß sich nur die (unbeobachtbare) Normierung der Feldoperatoren ändert: $\phi(x) \to 1/\sqrt{Z}\phi(x)$.

Also

$$G^{(n)}(x_1, x_2, \ldots, x_n; g', \mu') = Z^{-n/2}(\mu, \mu')G^{(n)}(x_1, x_2, \ldots, x_n; g, \mu) \tag{4.65}$$

Kontinuierliche Transformationen mit infinitesimalen Erzeugenden sind wieder einfacher:

$$\mu' \frac{\mathrm{d}}{\mathrm{d}\mu'} \left(Z^{n/2}(\mu, \mu')G^{(n)}(x_1, \ldots, x_n; g', \mu')\right) = 0 \tag{4.66}$$

$\mathrm{d}/\mathrm{d}\mu$ ist eine *totale* Ableitung, die die Änderung der Kopplungen g berücksichtigt. Also

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma(g)\right) G^{(n)}(x_1, \ldots; g, \mu) = 0 \tag{4.67}$$

wobei

$$\beta(g) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu) \tag{4.68}$$

$$\gamma(g) = \frac{1}{2Z(\mu_0, \mu)}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z(\mu_0, \mu) \tag{4.69}$$

β ist dimensionslos und kann in einer masselosen Theorie aus Dimensionsgründen nicht von μ abhängen (in massiven Theorien gibt es immer Renormierungsvorschriften, die dies beibehalten).

Es ist *nicht* offensichtlich, daß γ nicht von μ_0 abhängt!

- Gilt für *renormierbare* Theorien, weil dort der Limes $\mu_0 \to \infty$ existiert.
- Es steht uns frei, eine Trajektorie zu wählen, die einer renormierbaren Theorie entspricht.

Offensichtlicher Verallgemeinerung für mehrere Kopplungen und/oder Felder:

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_{\mathrm{i}=1}^{k} \beta_{\mathrm{i}}(g)\frac{\partial}{\partial g_{\mathrm{i}}} + \sum_{j=1}^{m} n_j\gamma_j(g)\right) G^{(n_1,\ldots,n_m)}(x_1,\ldots;g_1,\ldots,g_k,\mu) = 0 \tag{4.70}$$

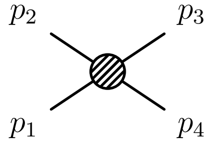
Z. B. QED-Vertex:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + 2\gamma_\psi(e) + \gamma_A(e)\right) \langle 0|\mathrm{T}\psi(x_1)\bar{\psi}(x_2)A_\nu(x_3)|0\rangle = 0 \tag{4.71}$$

4.4.1 Lösung der C-S Gleichung

Lecture 19: Tue, 28. 06. 2016

Betrachte Vierpunktfunktion



im Euklidischen (also keine physikalische Amplitude, aber möglicher Teil einer solchen)

$$p_{\mathrm{i}}^2 = -P^2 \tag{4.72}$$

$$p_i p_j = 0 \tag{4.73}$$

Niedrigste Ordnung Störungsrechnung:

$$G^{(4)}(P;g,\mu) = \text{F.T.}\,\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \tag{4.74}$$

$$= \left(\frac{-\mathrm{i}}{P^2}\right)^4 (-ig) + \mathcal{O}(g^2) \tag{4.75}$$

Dimensionsbetrachtung ohne Massenterme:

$$G^{(4)}(P;g,\mu) = \left(\frac{-\mathrm{i}}{P^2}\right)^4 \hat{G}^{(4)}(P/\mu;g) \tag{4.76}$$

also homogene Funktion

$$\mu\frac{\partial}{\partial\mu}G^{(4)}(P;g,\mu) = -\left(8+P\frac{\partial}{\partial P}\right)G^{(4)}(P;g,\mu) \tag{4.77}$$

Renormierungsgruppengleichung:

$$\left(P\frac{\partial}{\partial P}-\beta(g)\frac{\partial}{\partial g}+8-4\gamma(g)\right)G^{(4)}(P;g,\mu)=0 \tag{4.78}$$

Integration der partiellen Differentialgleichung: die *gleitende Kopplungskonstante* $\bar g(P;g)$:

$$P\frac{\mathrm{d}}{\mathrm{d}P}\bar g(P;g)=\beta(\bar g(P;g)) \tag{4.79a}$$

$$\bar g(\mu;g)=g \tag{4.79b}$$

"'absorbiert"' den Differentialoperator[1]

$$\left(P\frac{\partial}{\partial P}-\beta(g)\frac{\partial}{\partial g}\right)\bar g(P;g)=0 \tag{4.83}$$

also

$$\left(P\frac{\partial}{\partial P}-\beta(g)\frac{\partial}{\partial g}\right)\exp\left(4\int\limits_g^{\bar g(P;g)}\mathrm{d}g'\,\frac{\gamma(g')}{\beta(g')}\right) \\ = -\beta(g)\frac{\partial}{\partial g}\exp\left(4\int\limits_g \mathrm{d}g'\,\frac{\gamma(g')}{\beta(g')}\right) = 4\gamma(g)\exp\left(4\int\limits_g \mathrm{d}g'\,\frac{\gamma(g')}{\beta(g')}\right) \tag{4.84}$$

---

[1] The initial value problem (4.79) is equivalent to the implicit equation

$$\int_g^{\bar g(P;g)}\frac{\mathrm{d}g'}{\beta(g')}=\int_\mu^P\frac{\mathrm{d}P'}{P'} \tag{4.80}$$

Then

$$0=\frac{\mathrm{d}}{\mathrm{d}g}\mathrm{RHS}(4.80)=\frac{\mathrm{d}}{\mathrm{d}g}\mathrm{LHS}(4.80)=\frac{1}{\beta(\bar g(P,g))}\frac{\partial\bar g(P,g)}{\partial g}-\frac{1}{\beta(g)} \tag{4.81}$$

i. e.

$$\beta(g)\frac{\partial\bar g(P,g)}{\partial g}=\beta(\bar g(P,g))=P\frac{\partial\bar g(P;g)}{\partial P} \tag{4.82}$$

using (4.79).

Damit ist die allgemeine Lösung

$$G^{(4)}(P;g,\mu) = \left(\frac{-\mathrm{i}}{P^2}\right)^4 \mathcal{G}^{(4)}(\bar{g}(P;g)) \cdot \exp\left(4 \int\limits_{g}^{\bar{g}(P;g)} \mathrm{d}g' \, \frac{\gamma(g')}{\beta(g')}\right) \tag{4.85}$$

wobei $\mathcal{G}^{(4)}$ eine *beliebige* Funktion ist, durch Renormierungsgruppengleichung *nicht* festgelegt.

Vergleich mit Störungsrechnung (”‘*matching*”’) für $P = \mu$ liefert

$$\mathcal{G}^{(4)}(\bar{g}) = -\mathrm{i}\bar{g} + \mathcal{O}(\bar{g}^2) \tag{4.86}$$

Zwei Elemente der Lösung:

1. perturbatives Resultat $\mathcal{G}^{(4)}$ mit renormierter Kopplungskonstanten
2. Exponentialfaktor modifiziert Skalenverhalten für jedes Feld

$\therefore$ γ heißt *anomale Dimension*

Aus der β-Funktion für ϕ^4

$$\beta(g) = \frac{3}{16\pi^2} g^2 + \mathcal{O}(g^3) \tag{4.87}$$

folgt die gleitende Kopplungskonstante

$$\bar{g}(P;g) = \frac{g}{1 - \frac{3}{16\pi^2} g \ln \frac{P}{\mu}} \tag{4.88}$$

4.4.2 Führende Logarithmen

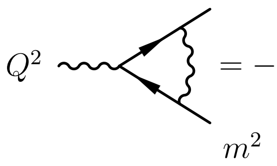
Große Logarithmen $\ln(P^2/\mu^2)$ kommen nur noch in der gleitenden Kopplungskonstanten $\bar{g}(P;g)$ und den Exponentialfaktoren vor

$\therefore$ erfolgreich resummiert

- Störungsrechnung ist zuverlässig, sofern $\bar{g}(P;g)$ klein

Verfahren funktioniert *nicht* für alle Greensfunktionen so einfach:

- Renormierungsgruppengleichung schwerer zu lösen, wenn nicht alle externen Impulse gleichförmig wachsen
- Wenn mehr als eine Massenskala im Spiel ist, können auch Koeffizientenfunktionen große Logarithmen entwickeln
- klassisches Beispiel: *Sudakov*-Logarithmen für exklusive Streuung



$$= -\frac{\alpha}{2\pi} \ln^2\left(\frac{Q^2}{m^2}\right) \tag{4.89}$$

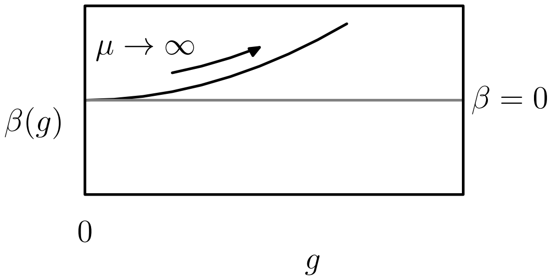
4.4.3 Asymptotische Freiheit

Das Verhalten der β-Funktion in der Nähe von $g = 0$ bestimmt die Eigenschaften der Störungstheorie.

Offensichtlich gilt $\beta(0) = 0$, weil ohne Wechselwirkung kein Mechanismus existiert, die die Kopplungskonstante gleiten läßt.

Es vier qualitativ verschiedene Szenarien:

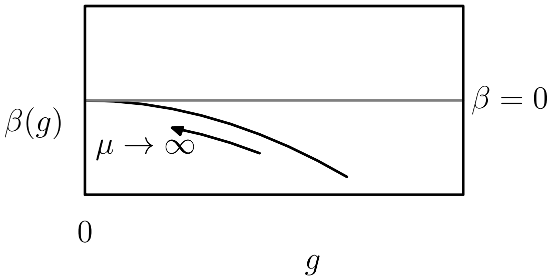
1. $\beta > 0$:



$\therefore$ Gleitende Kopplungskonstante wächst im Ultravioletten ohne Grenze

$\therefore$ Störungsrechnung im Infraroten zuverlässig

2. $\beta < 0$:



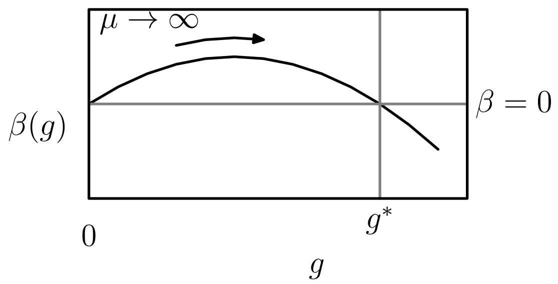
$\therefore$ Gleitende Kopplungskonstante wächst im Infraroten ohne Grenze

$\therefore$ Störungsrechnung im Ultravioletten zuverlässig

- *asympotische Freiheit* ist *sehr* interessante Alternative für die Hochenergiephysik
- Es gibt nur *eine* Klasse von Quantenfeldtheorien mit dieser Eigenschaft
- Niederenergiephysik *schwierig*
- Niederenergiephysik *interessant*

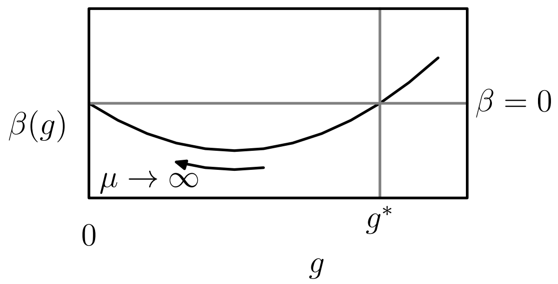
4.4.4 Fixpunkte

3. $\beta > 0$ mit ultraviolett stabilem Fixpunkt:



$\therefore$ Kopplung wächst im Ultravioletten bis g^* und bleibt dann wegen $\beta(g^*) = 0$ konstant.

4. $\beta < 0$ mit infrarot stabilem Fixpunkt:



$\therefore$ Kopplung wächst im Infraroten bis g^* und bleibt dann wegen $\beta(g^*) = 0$ konstant.

4.4.5 Dimensionale Regularisierung

Wie können wir die Renormierungsgruppenfunktionen $\beta(g)$ und $\gamma(g)$ *effizient* berechnen?

- $\beta(g)$ und $\gamma(g)$ sind universell, d. h. unabhängig von der betrachteten Greensfunktion

$\therefore$ berechne in Störungstheorie $\mu\partial G^{(n)}/\partial\mu$ für einen hinreichenden Satz von $G^{(n)}$, sodaß die Callan-Symanzik Gleichungen nach $\beta(g)$ und $\gamma(g)$ aufgelöst werden können.

Zurück zu

$$\hat{I}_{n,m}(D,M^2) = \mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{(k^2)^n}{(k^2-M^2+\mathrm{i}\epsilon)^m} \tag{4.90}$$

wobei diesmal μ^{4-D} die Dimension von $\hat{I}$ unabhängig von der Raumzeitdimension macht

$$\begin{aligned}\hat{I}_{n,m}(D,M^2) = \frac{(-1)^{n+m}\mathrm{i}}{16\pi^2}\left(M^2\right)^{2+n-m}\cdot \\ \cdot\left(\frac{M^2}{16\pi^2\mu^2}\right)^{D/2-2}\frac{\Gamma(m-n-D/2)\Gamma(D/2+n)}{\Gamma(m)\Gamma(D/2)}\end{aligned} \tag{4.91}$$

$\hat{I}_{n,m}(D,M^2)$ ist wohldefiniert, solange die Argumente $m-n-D/2$ und $D/2+n$ der Γ-Funktionen im Zähler keine negativen ganzen Zahlen oder 0 sind.

- $\therefore$ die logarithmische Divergenz von $\hat{I}_{0,2}(4,M^2)$ findet sich im Pol der Γ-Funktion wieder.
- $\therefore$ wenn wir etwas von $D=4$ weggehen, ist die Divergenz regularisiert:

$$\hat{I}_{0,2}(4-2\epsilon,M^2) = \frac{\mathrm{i}}{16\pi^2}\left(\frac{M^2}{16\pi^2\mu^2}\right)^{-\epsilon}\Gamma(\epsilon) \tag{4.92}$$

Entwicklung vom $\Gamma(\epsilon)$:

$$\hat{I}_{0,2}(4-2\epsilon,M^2) = \frac{\mathrm{i}}{16\pi^2}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{M^2}+2\ln 4\pi-\gamma_E\right) \tag{4.93}$$

In einer masselosen Theorie muß die Abhängigkeit von μ identisch zu Abhängigkeit vom Renormierungspunkt sein.

- es genügt, die Koeffizienten der Pole in ϵ zu bestimmen

4.4.6 Eichtheorien

Lecture 20: Thu, 30. 06. 2016

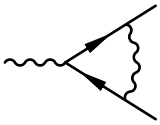
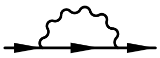
Quantenelektrodynamik:

$$\beta(e) = \frac{e^3}{12\pi^2} \tag{4.94}$$

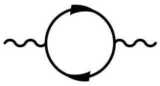
$$\gamma_\psi(e) = \frac{e^2}{16\pi^2} \tag{4.95}$$

$$\gamma_A(e) = \frac{e^2}{12\pi^2} \tag{4.96}$$

Besonderheit: Ward-Identität (Eichinvarianz, Stromerhaltung) erzwingt, daß sich die Beiträge von

zu $\beta(e)$ aufheben. Nur



trägt bei.

$\therefore$ QED ist *nicht* asympotisch frei

- es scheint auch in höheren Ordnungen keinen Fixpunkt zu geben

Nicht-Abelsche SU(N_C) Eichtheorie mit N_f Quarks

$$\beta(g) = \frac{g^3}{16\pi^2}\left(C_F\frac{N_f}{2} - \frac{11}{3}N_C\right) \tag{4.97}$$

mit

$$C_F = \frac{N_C^2 - 1}{2N_C} \tag{4.98}$$

Quantenchromodynamik ($N_C = 3$):

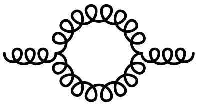
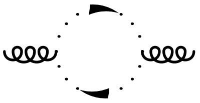
$$\beta_{\text{QCD}}(g) = \frac{g^3}{16\pi^2}\left(\frac{2}{3}N_f - 11\right) \tag{4.99}$$

- ist für $N_f < 33/2$ *asympotisch frei*
- einzige Klasse von Theorien, die in vier Raumzeitdimension *asympotisch frei* sind!

$\therefore$ QCD führender Kandidat für Theorie der starken Wechselwirkung

  - perturbative QCD funktioniert bei hohen Energien
  - Wechselwirkung stark bei niedrigen Energien

Offensichtlicher Unterschied zur Quantenelektrodynamik:

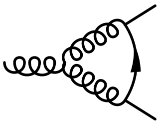
ergibt negativen Beitrag zur β-Funktion.

intuitiv richtig und erklärt das Wechselspiel von positiven Quark-Beiträgen und negativen Gluon-Beiträgen … aber technisch falsch!

- Wardidentitäten der Chromodynamik sind komplizierter

∴ es gibt nicht triviale Beiträge von Selbstenergien und Vertices

Z. B.



4.4.7 Aschenbrödel

1971: Wilson's Renormierungsgruppe für Phasenübergänge (”‘kritische Phänomene”’) and starke Wechselwirkungen zog *Nutzen* aus der Renormierung.

1972: Renormierbarkeitsbeweis für Eichtheorien und spontan gebrochene Eichtheorien etablierte Kandidaten für realistische Quantenfeldtheorien.

1973: ”‘Asymptotische Freiheit”’ legte die Grundlage für Quantenchromodynamik.

1979: Weinberg's Arbeit über effektive Feldtheorien faßte die Folklore über systematische Entwicklungen bei niedrigen Energien zusammen

Heute: Auch nicht-renormierbare Quantenfeldtheorien sind unter dem Namen ”‘effektive Feldtheorien”’ als angesehene Mitglieder der Gesellschaft eingebürgert worden.

4.5 Massen & Schwellen

Bislang alle Massen vernachlässigt

- *oft* eine gute Approximation: z. B. in der vier Fermionen Produktion bei LEP2 kann im größten Teil des Phasenraums mit masselossen Fermionen gerechnet werden: $m_b \ll \sqrt{s} < m_t$.
- *nicht immer* eine gute Approximation

Caveat emptor:

☡ Renormierungsgruppenmethoden *nicht* optimal für die präzise Beschreibung von Schwelleneffekten

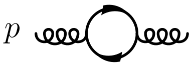
Dennoch

- viele Masseneffekte können systematisch in einer Renormierungsgruppenrechung berücksichtigt werden, am einfachsten und übersichtlichsten mit einer *effektiven Feltheorie*

Erinnern wir uns an die β-Funktion einer $\mathrm{SU}(N_C)$-Eichtheorie

$$\beta(g) = \frac{g^3}{16\pi^2}\left(C_F\frac{N_f}{2} - \frac{11}{3}N_C\right) \tag{4.100}$$

Sie ist durch das Diagram



von der Anzahl der Quark-Flavors abhängig:

☡ wie groß ist N_f?

- so lange alle Quarks masselos sind, ist $N_f = 6$

Betrachte den Fall $m_b^2 \ll -p^2 \ll m_t$

$$\Pi^R(p^2) = \mathcal{O}(\frac{p^2}{m_t^2}, \frac{\mu^2}{m_t^2}) \tag{4.101}$$

sehr klein, solange Renormierungspunkt $\mu \ll m_t$

- Renormierungsgruppengleichung soll (unter anderen) die Logarithmen in der Vakuumpolarisation aufsummieren.

- unterhalb der Top-Schwelle keine Logarithmen

$\therefore$ unterhalb der Top-Schwelle: $N_f = 5$.

4.6 Matching & Running

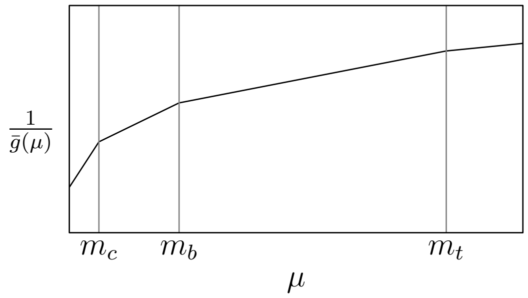
Lösung der Renormierungsgruppengleichung mit Massen:

1. starte mit $\bar{g}(\mu_0) = g_0$ bei $\mu_0 \gg m_t$
2. löse masselose RGG für $\bar{g}(\mu)$ mit $\beta_{N_f=6}(g)$ im Intervall $[\mu_0, m_t]$
3. starte erneut mit $\bar{g}(m_t)$ bei $\mu = m_t$
4. löse masselose RGG für $\bar{g}(\mu)$ mit $\beta_{N_f=5}(g)$ im Intervall $[m_t, m_b]$
5. starte erneut mit $\bar{g}(m_b)$ bei $\mu = m_b$
6. usw.

Verfahren im Jargon als *"‘matching, running, matching, running"’* bekannt.

- der Anschluß (das "‘matching"’) erfolgt an jeder Schwelle stetig, aber nicht differenzierbar, weil sich die β-Funktion ändert.

  Skizze:



- Approximation in der Nähe der Schwelle schlecht
- falls Logarithmen wichtig, dann Schwellenregion klein im Vergleich zur Strecke zwischen den Schwellen

$\therefore$ korrekte Resummation der großen Logarithmen

- systematische Verbesserung *"‘next-to-leading"’* order (NLA, NNLA, usw.) möglich: Zwei-Schleifen β-Funktion mit Ein-Schleifen Matching, usw.

Berühmtes Beispiel:

- die gleitenden Kopplungskonstanten der drei Eichgruppen des Standardmodells treffen sich *nicht* in einem Punkt

$\therefore$ Problem für *Grand Unified Theories*

- Matching der Evolution vom Standardmodell zum supersymmetrischen Standardmodell bei $\mu \approx 1\,\text{TeV}$ bewirkt, daß sich die Kopplungen doch treffen

$\therefore$ stärkster (indirekter) experimenteller Hinweis auf Supersymmetrie

Allgemeines Ergebnis:

- schwere Teilchen (schwerer als die betrachtete Energieskala) können aus der Theorie entfernt werden
- hinterlassen Renormierung von Kopplungskonstanten und Feldern

4.7 Effektive Theorien

Irrelevante Kopplungen "‘sterben aus"’ sofern der Renormierungsgruppenfluß ausreichend "‘Zeit"’ hat.

Komplizierter:

1. irrelevante Wechselwirkung kann eine Symmetrie der marginalen und relevanten Wechselwirkungen verletzen

$\therefore$ Auswirkungen irrelevanter Wechselwirkungen können beobachtbar sein.

2. Hierarchie der Skalen nicht groß genug

$\therefore$ irrelevante Wechselwirkungen überleben

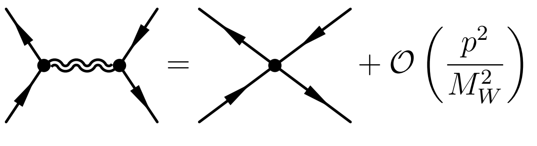
Typisches Beispiel für den ersten Fall:

- schwache Wechselwirkung

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}}\,\bar\psi(1-\gamma_5)\gamma_\mu\psi\,\bar\psi(1-\gamma_5)\gamma^\mu\psi \tag{4.102}$$

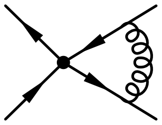
- irrelevant (Dimension 6), aber nicht zu vernachlässigen, weil Flavoränderung

Matching der Standardmodellwechselwirkung an der W-Masse:



$$+\,\mathcal{O}\left(\frac{p^2}{M_W^2}\right)$$

Strahlungskorrekturen unterhalb M_W können durch die anomale Dimension der Fermi-Wechselwirkung aufsummiert werden:

 $+ \ldots$

Theorie im herkömmlichen Sinne nicht renormierbar

- kein Problem, weil die Anschlußbedingung bei $\mu = M_W$ für *alle* denkbaren Operatoren Renomierungsbedingungen bereitstellt

Mögliche Sichtweise

- die Schleifenimpulse sind bei M_W abgeschnitten

$\therefore$ alles ist endlich

$\therefore$ kein Problem mit der Renomierbarkeit

Harter Cut-off ist aber technisch unpraktisch und sogar gefährlich

- schwierige Integrale
- Verletzung der Eichinvarianz bei naivem Vorgehen

Besser

- Renormierungsgruppenfluß in dimensionaler Regularisierung ausrechnen
- Eichinvarianz bleibt erhalten
- Trajektorie gemäß der Anschlußbedingung wählen und in den physikalischen Bereich verfolgen

—5—

SPONTANEOUS SYMMETRY BREAKING

Lecture 21: Tue, 05. 07. 2016

5.1 Wigner-Weyl vs. Nambu-Goldstone

5.1.1 Unbroken Symmetry: Wigner-Weyl

So far, we have identified unbroken symmetries of a quantum mechanical system with the existence of unitary operators in a Hilbert space representation $(\mathcal{H}, \pi)$. A priori, this is a too strong requirement, because a state of a physical system is *not* described by a single normalized vector $\Phi \in \mathcal{H}$, but by a *ray*

$$\hat{\Psi} = \{\mathrm{e}^{\mathrm{i}\lambda}\Psi : \lambda \in [0, 2\pi)\} \in P(\mathcal{H}) \tag{5.1}$$

and a physical symmetry is only required to preserve probabilies, i. e. the moduli of matrix elements, which obviously don't depend on the representative chosen for each ray

$$\forall \Psi \in \hat{\Psi}, \Phi \in \hat{\Phi}, \lambda, \mu \in [0, 2\pi) : \left|\left(\mathrm{e}^{\mathrm{i}\lambda}\Psi, \mathrm{e}^{\mathrm{i}\mu}\Phi\right)\right| = |(\Psi, \Phi)| =: |(\hat{\Psi}, \hat{\Phi})| . \tag{5.2}$$

Definition 5.1 (Wigner Symmetry)**.** A *Wigner symmetry* of a quantum mechanical system with states described by rays in a Hilbert space $\mathcal{H}$ is a mapping $g : P(\mathcal{H}) \to P(\mathcal{H})$ of the projective Hilbert space of rays to itself, which preserves all transition probabilies

$$\forall \hat{\Psi}, \hat{\Phi} \in P(\mathcal{H}) : \left|\left(g\hat{\Psi}, g\hat{\Phi}\right)\right| = \left|\left(\hat{\Psi}, \hat{\Phi}\right)\right| . \tag{5.3}$$

However, there is a famous theorem by Wigner in Hilbert space quantum mechanics for a finite number of degrees of freedom:

Theorem 5.2 (Wigner)**.** *Given a quantum mechanical system, that is described by the rays in a Hilbert space $\mathcal{H}$, all Wigner symmetries are realized by unitary or anti-unitary operators $U(g)$ with*

$$\forall\Psi \in \mathcal{H} : g\hat{\Psi} = \widehat{U(g)\Psi}\,, \tag{5.4}$$

where the $U(g)$ are determined upto a common phase.

An anti-unitary operator $U : \mathcal{H} \to \mathcal{H}$ satisfies

$$\forall\Psi, \Phi \in \mathcal{H} : (U\Psi, U\Phi) = \overline{(\Psi, \Phi)} = (\Phi, \Psi) \tag{5.5}$$

and is required for the realization of symmetries involving time-reversal $t \to -t$, since they switch the rôles of initial and final state in transition matrix elements.

The non-obvious aspect of Wigner's theorem is that the phases of the operators can be consistently chosen in the whole Hilbert space to obtain unitary or anti-unitary operators. In particular, it must be possible to compose symmetries without additional phases

$$U(g)U(g') = \underbrace{\mathrm{e}^{\mathrm{i}\phi(g,g')}}_{=1} U(g \circ g')\,. \tag{5.6}$$

Proof. See the textbook [2] and the article [6]. □

5.1.2 Broken Symmetry: Nambu-Goldstone

In the case of an infinite number of degrees of freedom (d.o.f.), we must distinguish between symmetries realized algebraically and represented as unitary operators on Hilbert space.

Definition 5.3. An *algebraic symmetry* of a physical system is a $*$-automorphism or $*$-anti-automorphism β of the C^*-algebra $\mathcal{A}$ generated by the observables of the system.

Definition 5.4. An anti-automorphism $\beta : \mathcal{A} \to \mathcal{A}$ of a C^*-algebra $\mathcal{A}$ is an anti-linear map β preserving the structure of $\mathcal{A}$

$$\forall\lambda, \mu \in \mathbf{C}, A, B \in \mathcal{A} : \beta(\lambda A + \mu B) = \bar{\lambda}\beta(A) + \bar{\mu}\beta(B) \tag{5.7a}$$

$$\forall A, B \in \mathcal{A} : \beta(AB) = \beta(B)\beta(A) \tag{5.7b}$$

$$\forall A \in \mathcal{A} : \beta(A^*) = (\beta(A))^*\,. \tag{5.7c}$$

Definition 5.5. An *internal algebraic symmetry* is an algebraic symmetry that commutes with all time and space translations α_t, α_x

$$\forall t \in \mathbf{R}, x \in \mathbf{R}^n : [\alpha_t, \beta] = [\alpha_x, \beta] = 0\,. \tag{5.8}$$

Theorem 5.6. *In a "physically relevant" representation $(\mathcal{H}, \pi, \Omega)$, i. e. a representation satisfying the conditions*[1] *described in section* **??**, *an internal algebraic symmetry β is unbroken, i. e. represented unitarily, if and only if (iff) the ground state correlation functions are invariant under β:*

$$\forall A \in \mathcal{A} : (\beta^*\omega)(A) := \omega(\beta(A)) := (\Omega, \beta(A)\Omega) = (\Omega, A\Omega) = \omega(A) \tag{5.9}$$

Proof. If β is unbroken, it is realized by a unitary operator $U(\beta)$. Then $U(\beta)\Omega$ is also translation invariant, since β is an internal symmetry. However, since Ω is by assumption the unique translation invariant state, we must have $\beta^*\omega = \omega$. The reverse direction is corollary **??**. ☐

This theorem allows a simple characterization of symmetry breaking by

Definition 5.7 (order parameter)**.** A ground state expectation value of an observable $A \in \mathcal{A}$ that is not invariant under an internal algebraic symmetry β

$$\omega(\beta(A)) \neq \omega(A) \tag{5.10}$$

is called an *order parameter*.

5.2 Charges

Conserved currents

$$\partial^\mu j_\mu(x) = 0 \tag{5.11}$$

and associated charges

$$Q(t) = \int_{x_0=t} \mathrm{d}^3x\, j_0(x) \tag{5.12}$$

that are also conserved

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \mathrm{i}[H, Q] = 0 \tag{5.13}$$

and act on the fields

$$[Q, \phi_n(x)] = \mathrm{i} \sum_m T_{nm}\phi_m(x)\,, \tag{5.14}$$

---

[1] Existence of energy and momentum, stability, and existence of a ground state.

where the fields on the right hand side can be composite.

Acting with Q on a translation invariant state $|\Psi\rangle$ (i. e. $\vec{P}\,|\Psi\rangle = 0$)

$$\begin{aligned}\|Q\,|\Psi\rangle\|^2 &= \langle\Psi|Q^\dagger Q|\Psi\rangle = \langle\Psi|QQ|\Psi\rangle = \int\mathrm{d}^3x\ \langle\Psi|j_0(x)Q|\Psi\rangle \\ &= \int\mathrm{d}^3x\ \langle\Psi|\mathrm{e}^{-\mathrm{i}\vec{P}\vec{x}}j_0(0)\mathrm{e}^{\mathrm{i}\vec{P}\vec{x}}Q|\Psi\rangle = \int\mathrm{d}^3x\ \langle\Psi|\mathrm{e}^{-\mathrm{i}\vec{P}\vec{x}}j_0(0)Q\mathrm{e}^{\mathrm{i}\vec{P}\vec{x}}|\Psi\rangle \\ &= \int\mathrm{d}^3x\ \langle\Psi|j_0(0)Q|\Psi\rangle = \langle\Psi|j_0(0)Q|\Psi\rangle\underbrace{\int\mathrm{d}^3x}_{\to\infty}, \end{aligned} \tag{5.15}$$

we see that either $Q\,|\Psi\rangle = 0$ or $Q\,|\Psi\rangle$ is not a normalizable state in the Hilbert space.

5.3 Goldstone's Theorem

$$\begin{aligned}&\langle 0|\,[j_\mu(x),\phi_n(y)]\,|0\rangle = \\ &\sum_\alpha\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\langle 0|j_\mu(x)|p,\alpha\rangle\,\langle p,\alpha|\phi_n(y)|0\rangle - \langle 0|\phi_n(y)|p,\alpha\rangle\,\langle p,\alpha|j_\mu(x)|0\rangle\right) \\ &= \sum_\alpha\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\mathrm{e}^{-\mathrm{i}p(x-y)}\,\langle 0|j_\mu(0)|p,\alpha\rangle\,\langle p,\alpha|\phi_n(0)|0\rangle - \mathrm{e}^{\mathrm{i}p(x-y)}\,\langle 0|\phi_n(0)|p,\alpha\rangle\,\langle p,\alpha|j_\mu(0)|0\rangle\right) \\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\mathrm{e}^{-\mathrm{i}p(x-y)}\rho^n_\mu(p) - \mathrm{e}^{\mathrm{i}p(x-y)}\tilde\rho^n_\mu(p)\right)\end{aligned} \tag{5.16}$$

with

$$\rho^n_\mu(p) = \sum_\alpha\langle 0|j_\mu(0)|p,\alpha\rangle\,\langle p,\alpha|\phi_n(0)|0\rangle = p_\mu\Theta(p_0)\rho^n(p^2) \tag{5.17a}$$

$$\tilde\rho^n_\mu(p) = \sum_\alpha\langle 0|\phi_n(0)|p,\alpha\rangle\,\langle p,\alpha|j_\mu(0)|0\rangle = p_\mu\Theta(p_0)\tilde\rho^n(p^2) \tag{5.17b}$$

then

$$\begin{aligned}\langle 0|\,[j_\mu(x),\phi_n(y)]\,|0\rangle &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}p_\mu\left(\Theta(p_0)\mathrm{e}^{-\mathrm{i}p(x-y)}\rho^n(p^2) - \Theta(p_0)\mathrm{e}^{\mathrm{i}p(x-y)}\tilde\rho^n(p^2)\right) \\ &= \mathrm{i}\partial_\mu\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\Theta(p_0)\mathrm{e}^{-\mathrm{i}p(x-y)}\rho^n(p^2) + \Theta(p_0)\mathrm{e}^{\mathrm{i}p(x-y)}\tilde\rho^n(p^2)\right)\end{aligned} \tag{5.18}$$

introducing

$$\Delta_+(x;m^2) = \int\widetilde{\mathrm{d}k}_m\,\mathrm{e}^{-\mathrm{i}kx} = \int\frac{\mathrm{d}^4k}{(2\pi)^4}2\pi\Theta(k_0)\delta(k^2-m^2)\mathrm{e}^{-\mathrm{i}kx} \tag{5.19}$$

we can write

$$\begin{aligned}&\langle 0|[j_\mu(x),\phi_n(y)]|0\rangle\\&\quad = 2\pi\mathrm{i}\partial_\mu\int_0^\infty \mathrm{d}m^2\left(\rho^n(m^2)\Delta_+(x-y;m^2)+\tilde\rho^n(m^2)\Delta_+(y-x;m^2)\right)\end{aligned} \tag{5.20}$$

$\Delta_+(x;m^2)$ depends on x only through x^2 and x_0, but the latter *only* for time- or lightlike x. Therefore we have for spacelike x

$$\Delta_+(x;m^2)=\Delta_+(-x;m^2)\qquad(\text{for } x^2<0)\,. \tag{5.21}$$

Also, by causality, all commutators must vanish for spacelike $x-y$

$$\begin{aligned}0 \overset{(x-y)^2<0}{=} &\langle 0|[j_\mu(x),\phi_n(y)]|0\rangle\\&= 2\pi\mathrm{i}\partial_\mu\int_0^\infty \mathrm{d}m^2\left(\rho^n(m^2)+\tilde\rho^n(m^2)\right)\Delta_+(x-y;m^2)\end{aligned} \tag{5.22}$$

i. e.

$$\rho^n(m^2)=-\tilde\rho^n(m^2) \tag{5.23}$$

and therefore

$$\begin{aligned}&\langle 0|[j_\mu(x),\phi_n(y)]|0\rangle\\&\quad = 2\pi\mathrm{i}\partial_\mu\int_0^\infty \mathrm{d}m^2\rho^n(m^2)\left(\Delta_+(x-y;m^2)-\Delta_+(y-x;m^2)\right)\\&\quad = 2\pi\mathrm{i}\partial_\mu\int_0^\infty \mathrm{d}m^2\rho^n(m^2)\Delta(x-y;m^2)\end{aligned} \tag{5.24}$$

If j_μ is conserved

$$\begin{aligned}0=\partial^\mu\langle 0|[j_\mu(x),\phi_n(y)]|0\rangle &= 2\pi\mathrm{i}\partial^\mu\partial_\mu\int_0^\infty \mathrm{d}m^2\rho^n(m^2)\Delta(x-y;m^2)\\&= -2\pi\mathrm{i}\int_0^\infty \mathrm{d}m^2 m^2\rho^n(m^2)\Delta(x-y;m^2)\end{aligned} \tag{5.25}$$

and we find

$$m^2\rho^n(m^2)=0\,. \tag{5.26}$$

On the other hand, in the case

$$\langle 0|[j_\mu(x),\phi_n(y)]|0\rangle\neq 0\,, \tag{5.27}$$

we must have

$$\rho^n(m^2)\propto\delta(m^2) \tag{5.28}$$

i. e. there is a massless state created by j_μ out of the vacuum, since (5.17) implies

$$\langle 0|j_0(x)|p,\alpha\rangle = 0\to\rho(p^2)=0\,. \tag{5.29}$$

—A—

FORMULAE

"Today's students can no longer calculate": such is the grievance frequently directed against current teaching of mathematics by physicists and engineers, and it must be admitted that this criticism is often justified. When one has seen a second or third year undergraduate toil over a change of variable or an integration by parts, one can scarcely be other than alarmed, particularly (as is sometimes the case) when the same student seasons his ignorance and clumsiness with a pretentious and useless jargon which he has also failed to understand.

It must be continually repeated that there is no "modern mathematics" as opposed to "classical mathematics" but simply the mathematics of today, which continues that of yesterday without any deep rupture, and which above all is dedicated to solving the great problems left by our predecessors. To do this, mathematics has gradually developed a profusion of new abstract concepts, which, by concentrating on the heart of a given problem and by eliminating trivial details, have made possible a steady advance in areas still considered inaccessible scarcely fifty years ago. Those mathematicians who create abstraction for the sake of abstraction are mostly mediocrities.

A by no means negligible consequence of this tendency to abstraction has been a "tidying up" which these new concepts have helped to create in the teaching of the fundamentals of mathematics (particularly in algebra and geometry). Prior to this, ridiculous traditions had encumbered teaching with trivialities and with useless and even harmful developments. Nevertheless the substance of so-called "classical mathematics" has remained intact, and the basis of modern analysis is still the wonderful tool wrought by the mathematicians of the last three centuries, the Infinitesimal Calculus. To pretend to neglect it in order to plunge immediately into the most recent functional analysis is to build on sand and can produce nothing but sterility and verbiage.

Until this year this stumbling block was hardly avoidable. Trapped on the one hand by a secondary teaching in the hands of a mandarinate cut off from living mathematics for 80 years and exclusively devoted to the con-

templation of its navel, and on the other hand by the teaching of modern analysis given in the Faculties, which "stick" to research in order to prepare for it efficiently, the unfortunate student had just one year to initiate himself into the classical Infinitesimnl Calculus and to learn how to handle its techniques fluently. Experience soon showed that this wns insufficient, and ***the palliative introduced under the title of "Mathematical Techniques of Physics" given by mathematicians more concerned with rigor than with efficiency, achieved in many Faculties the teaching of a painless version of abstract analysis, stressing principles rather than calculation****.*

The new syllabuses, by stretching the "first cycle" over two years, should re-establish the equilibrium and give the conscientious student the solid technical basis which will enable him later to assimilate more abstract concepts without falling into psittacism. Essential parts of classical analysis, which can and should be approached without too much abstract preparation, like the theory of analytic functions and of differential equations, have fortunately been included in these syllabuses, particularly in the second year. This book is above all devoted to the development of these fundamental techniques assuming known the fundamentals of the differential and integral Calculus taught in the first year of the first cycle.

We must therefore "know how to calculate" before claiming access to modern analysis. But what does "to calculate" mean? There are in fact two types of "calculus" which there is a tendency to confuse. On the one hand, there is the "algebraic calculus" which (oversimplifying the issue) can be characterized as the establishing of equalities *the prototype is given by* ***the formulae for the solution of equations (the "closed formulae" of the Anglo-Saxons) which wield a strange kind or fascination on the users of mathematics: how many times have I met an engineer or a physicist who wants mathematics to be a kind of automatic machine producing formulae for the solution of problems!***

This kind of relation also exists in analysis and can often be of great importance—Cauchy's formula and the development into Fourier series are typical examples of this. But in my opinion the essence of the Infinitesimal Calculus does not lie here. Physicists insist, with good reason, that for them a theorem is without interest if it does not entail at least the possibility of calculating numerically the numbers or functions under consideration. They will have nothing to do with those "existence theorems" of the pure mathematicians which do not fulfil these conditions. But to speak of numerical calculation is to speak of approximation, a real number being "known" only when a method to approximate it has been given (with an approximation which the mathematician want: to be arbitrarily small, whereas the user of *mathematics is content with much less). If it is remembered that the teaching of mathematics, in the first cycle, is addressed at least as much to the future physicists and chemists as to the mathematicians, it will be understood why this side of analysis is particularly insisted upon in this work. I have not tried to write a treatise on the Numerical Calculus proper, which should be the object of specialized teaching, but no concept has been introduced which is not susceptible to numerical evaluation. At each stage the theoretical means of obtaining such calculations has been indicated, if required.*

The pure mathematicians would in fact be wrong to despise this "down to earth" side of the Infinitesimal Calculus. To acquire a "feeling for analysis" indispensable even in the most abstract speculations, one must have learnt to distinguish between what is "large" and what is "small", what is "dominant" and what is "negligible". In other words, Infinitesimal Calculus, as it is presented in this book, is an apprenticeship in the handling of inequalities far more than of equalities and can be summed up in three words:

MAJORIZE, MINIMIZE, APPROXIMATE.

The adoption of this point of view by no means implies that I have sacrificed rigor to convenience, or reduced the Infinitesimal Calculus to a series of recipes. We have to shape thinking beings, not robots, to induce the student to understand what he is doing, not to teach him mechanical methods. To have a "feeling for analysis" is to have acquired an "intuitive" idea of the operations of the Infinitesimal Calculus and this is obtained only through use and numerous concrete examples. But the test which proves that one has really reached this stage is to know how to give precise definitions of the notions used and to employ these to build correct proofs, for these last are no more, in the end, than a "pulling into shape" of intuition.

On this point, the physicists often jeer at the pure mathematician for always wanting to prove everything and for "splitting hairs" to establish "self-evident" results. They are not always wrong, and a beginner would do well to accept plausible results without encumbering his mind with subtle proofs,[1] *so that he can reserve his efforts for the assimilation of new and not "self-evident" ideas. I have therefore had no hesitation in admitting a certain number of basic theorems of analysis nor in pointing out to students that they may, at first reading, dispense with knowing certain long or slightly delicate proofs, by printing the latter in small print.*

The physicists venture onto dangerous ground where they have a tendency to accept as "evident" that which is not so at all

---

[1] In the end this simply means increasing the number of axioms, an inflation against which only the logccians protest.

and to forget that our intuition is but a rudimentary instrument, which at times leads us into gross errors. *Contrary to what many of them believe, it is not necessary to look for functions as "monstrous" as continuous functions without derivatives in order to fault them in results which they accept without discussion. The "Runge phenomenon" (Chapter IX, Appendix) shows that the classical method of polynomial interpolation can diverge for analytic functions as "nice" as we could wish; and there are functions analytic for* $|z| < 1$, *continuous in the whole disc* $|z| \leq 1$, *which however transform the circle* $|z| = 1$ *onto a Peano curve filling a square.*

Implicit faith therefore has its perils. In any case ***one cannot meet serious experimentalists without being struck by the extreme care which they take in making sure of the correctness of their measurements and in avoiding fallacious interpretations. To handle mathematics correctly requires an equal care, and I do not think it is good teaching practice to try to inculcate strict habits of work in some spheres, while allowing (or even encouraging) slackness and vagueness in others***.

I have not adhered slavishly to the official syllabuses, and I have stressed particularly that which seemed to me most important for the student who completes his first cycle with a view to going on to his License or Maitrise in Physics or Mathematics (pure or applied). Thus I have omitted everything concerning multiple integrals and differential forms. I have said elsewhere what I have thought of the "Stokes mania" of some of my colleagues, and the coverage of the subject in the first year of the first cycle seems quite sufficient to me, without trying to enter into refinements which at this level can only be sterile. On the other hand I have included a number of topics of the Infinitesimal Calculus which do not expressly appear in the syllabus, or which, like the serious study of differential equations, are in my opinion left too late, at the level of the Maitrise. Roughly speaking, it can be said that the analysis expounded in this book is essentially analysis "of one variable", real or complex. All mathematicians know that the passage from one to several variables is a brutal "jump" which gives rise to great difficulties, and necessitates quite new methods, On the other hand, analysis of one variable is an essential tool for working towards more general questions, I have thought it wholly appropriate to put this "mutation" at the junction of the two cycles.

The present timetables do not therefore permit the teaching of the whole of this book in the second year of the first cycle, and the teacher or student who uses it will make his own choice. Nevertheless one may be forgiven for hoping that one day secondary teaching will place in the lumber room of histery the fossilized mathematics at present taught and that the time thus gained will be usefully employed in teaching in the last three years at high school what is *now taught in the first year of the first cycle. The first four chapters of this book, which are only complementary to the syllabus of the present first year (and usually omitted), could then be advantageously incorporated into the first year, and all of the remaining chapters into the second year. A student who had properly assimilated them would, in my opinion, be well prepared either to apply his mathematical knowledge to concrete problems, or to move to a higher level of abstraction and begin the present syllabus of the Maitrise in pure mathematics.*

Jean Dieudonné, preface of *Calcul Infinitésimal*, 1968 [3]
bold face selections by T. O.

BIBLIOGRAPHY

[1] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, Reading, Mass., 1995)

[2] S. Weinberg, *The Quantum Theory of Fields. Volume I: Foundations* (Cambridge University Press, Cambridge — New York — Melbourne, 1995)

[3] J. Dieudonné, *Infinitesimal Calculus* (Kershaw, London, 1973)

[4] J. Dieudonné, *Calcul Infinitésimal*, *Collection Méthodes*, 2ème ed. (Hermann, Paris, 1980)

[5] R. Jackiw, Phys. Rev. **D9**, 1686 (1974)

[6] V. Bargmann, Ann. Math. **59**, 1 (1954).

Alphabet Soup

AF	Asymptotic Freedom
d.o.f.	degrees of freedom
DR	Dimensional Regularization
EFT	Effective Field Theory
EQFT	Effective Quantum Field Theory
EW	Electro Weak
iff	if and only if
MS	Minimal Subtraction
OPE	Operator Product Expansion

PDF Parton Distribution Function

PT Perturbation Theory

QCD Quantum Chromodynamics

QED Quantum Electrodynamics

QFT Quantum Field Theory

QM Quantum Mechanics

RG Renormalization Group

RGE Renormalization Group Equation

SM Standard Model

SSB Spontaneous Symmetry Breaking

1PI One Particle Irreducible