Relativistic Quantum Field Theory

(Winter 2017/18)

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Vorbemerkung

Dieses Manuskript ist mein persönliches Vorlesungsmanuskript, an vielen Stellen nicht ausformuliert und kann jede Menge Fehler enthalten. Es handelt sich hoffentlich um weniger Denk- als Tippfehler, trotzdem kann ich deshalb ich keine Verantwortung für Fehler übernehmen. Zeittranslationsinvarianz ist natürlich auch nicht gegeben ...

Dennoch, oder gerade deshalb, bin ich für alle Korrekturen und Vorschläge dankbar!

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Aktuelle Informationen

Vorlesungs-URL zu lang, einfach zu

- `http://physik.uni-wuerzburg.de/ohl/`

gehen und der "'Kachel"' *QM3 / QFT1* folgen.

Übungszettel

- Mittwochs online, Besprechung am Mittwoch der folgenden Woche.

—1— Introduction

Lecture 01: Tue, 17. 10. 2017

QFT[1, 2, 3] plays a dual role: the

- "QM" of classical field theory, e. g. quantized radiation field in Quantum Electrodynamics (QED) and the
- quantum mechanics for (infinitely) many particles with possible creation and annihilation

are described by the *same* formalism[1]. Historically, the emphasis has been on the former aspect, but more recently, the latter aspect has become more prominent in the form of "Effective (Quantum) Field Theory (EFT)".

1.1 Limitations of QM

- The non-relativistic Schrödingerequation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(\vec x,t) = H\psi(\vec x,t) = \left(-\frac{1}{2m}\Delta + V(\vec x,t)\right)\psi(\vec x,t) \tag{1.1}$$

  is *not* covariant under Lorentz transformation, i. e. its interpretation depends on the observer, unless relative velocities are *much* smaller than the speed of light c.

- If the differences of energies among states accessible to the system are larger than the mass of particles, we must allow for particle creation and annihilation and move from a one particle Hilbert space to a much larger many particle Hilbert space.

[1] The second interpretation requires the notion of "particle", however, which is not always available on curved background geometries.

The non-covariance of the Schrödinger equation is certainly a nuisance, but not necessarily a problem: classical Hamiltonian mechanics can be make relativistic after all[2]. The creation and annihilation must however always be taken into account, if some particles are light enough — even in non-relativistic many body physics and solid state physics with light quasi-particles.

1.2 (Special) Relativistic QFT

This lecture is concerned with *relativistic* QFT, i. e. systems with velocities close to the speed of light

$$v \lesssim c \tag{1.2a}$$

and typical actions close to Planck's constant

$$S \gtrsim \hbar\,. \tag{1.2b}$$

Therefore, we will use units with

$$c = \hbar = 1 \tag{1.3}$$

and the following quantities will have the same units

mass, momentum, energy, inverse length .

In order to obtain the physical values, in the end the proper powers of c and $\hbar$ have to added. In our applications, the most important combinations are

$$\hbar c = 197.327\,053(59)\,\mathrm{MeV\,fm} \tag{1.4a}$$

$$(\hbar c)^2 = 0.389\,379\,66(23)\,\mathrm{GeV}^2\,\mathrm{mb} = 0.389\,379\,66(23)\,\mathrm{TeV}^2\,\mathrm{nb}\,. \tag{1.4b}$$

The goal will be a formalism that is compatible with special relativity

- *covariance* of observables unter Lorentz Transformations (LTs) so that observers travelling a different speeds can agree on observations
- *causality*, i. e. all observables at spacelike distances must commute

The latter requirement will lead us to *local* theories, with pointlike objects and interactions at one point in space, that are propagated by particles (field quanta) along timelike trajectories.

[2]See, e. g.[4]

1.3 Limitations of QFT

Our treatment of QFT will be limited to Perturbation Theory (PT). Indeed, while there exist mathematical rigorous treatments of QM for non-trivial, i. e. interacting, systems, this is *not* the case for QFTs with realistic particle content in four space-time dimensions. Nevertheless perturbation theory has been *very* successful in Quantum Electrodynamics, e. g. the theoretical prediction for the anomalous magnetic moment of the electron has been tested to $0.7 \cdot 10^{-9}$. This is still considered to be the most precise prediction in theoretical physics[3].

[3] You will compute the first non-trivial term in this expansion in the exercises near the end of this lecture.

—2— Symmetries

Our most important guiding principles will be symmetries

- of space-time (a. k. a. Lorentz invariance)
- among particles

(it is a very deep result, that these two kinds of symmetries can be mixed only in a few very special ways). These symmetries will restrict the allowed interactions and make calculations possible.

2.1 Principles of QM

QFT does *not* invalidate the principles of QM, the notions of *states* and *observables* remain intact.

States

All pure states of a quantum mechanical system is given by unit *rays* in a Hilbert space $\mathcal{H}$:

$$\widehat{|\psi\rangle} = \left\{ |\psi\rangle \in \mathcal{H} : |\psi\rangle = \mathrm{e}^{\mathrm{i}\alpha} |\psi_0\rangle \,,\; \alpha \in [0, 2\pi),\; \langle\psi|\psi\rangle = 1 \right\} . \tag{2.1}$$

Mixed states are represented by density matrices, but will not be considered in this lecture. Note, however, that a ray corresponds exactly to a density matrix that is a projection operator

$$|\psi\rangle \langle\psi| \tag{2.2}$$

because the phases cancel in the projection operator.

Superposition Principle

For any two state vectors $|\phi_1\rangle, |\phi_1\rangle \in \mathcal{H}$ the superposition

$$|\psi\rangle = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle \in \mathcal{H} \tag{2.3}$$

is again a valid state vector, unless prohibited by one of a handful *superselection rules* (e. g. the fermion number superselection rule).

Observables

Observables correspond to self-adjoint linear operators $A^\dagger = A : \mathcal{H} \to \mathcal{H}$ with

$$\forall |\psi\rangle, |\phi\rangle \in \mathcal{H} : \langle\psi|A^\dagger|\phi\rangle = \langle\phi|A|\psi\rangle^* \tag{2.4}$$

Possible results of measurements of the observable A are given by the eigenvalues a_n of the corresponding operator A:

$$A |\psi_n\rangle = a_n |\psi_n\rangle \tag{2.5}$$

(with the appropriate generalization for possible continuous parts of the spectrum). An the probability P_n of obtaining the eigenvalue a_n in a measurement of a system in thre state $\widehat{|\psi\rangle}$ is given by the *Born rule*

$$P_n = |\langle\psi_n|\psi\rangle|^2 . \tag{2.6}$$

Obviously, P_n does not depend on the state vector chosen to represent the state, i. e. the representative $|\psi\rangle \in \widehat{|\psi\rangle}$. The eigenvalues of a self-adjoint operator form a complete set

$$\sum_n |\psi_n\rangle \langle\psi_n| = \mathbf{1} : \mathcal{H} \to \mathcal{H} . \tag{2.7}$$

Dynamics

The time evolution of the system is governed by a Schrödingerequation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H |\psi(t)\rangle . \tag{2.8}$$

This will remain true in the relativistic theory, despite the fact that the time coordinate t plays a special role.

2.2 Symmetries in QM

A symmetry is a transformation of states

$$\widehat{|\psi\rangle} \to \widehat{|\psi\rangle}' \tag{2.9}$$

such that all probabilities are preserved:

$$\forall \widehat{|\psi\rangle}, \widehat{|\phi\rangle} : |\langle\psi'|\phi'\rangle| = |\langle\psi|\phi\rangle| \,. \tag{2.10}$$

There is a celebrated theorem by Wigner that shows that each symmetry can be realized either by a unitary operator $U : \mathcal{H} \to \mathcal{H}$

$$|\psi\rangle \to |\psi'\rangle = U\,|\psi\rangle \tag{2.11}$$

with U invertible and

$$\forall\,|\psi\rangle\,, |\phi\rangle \in \mathcal{H} : \langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle \tag{2.12a}$$

$$U\left(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle\right) = c_1 U\,|\psi_1\rangle + c_2 U\,|\psi_2\rangle \tag{2.12b}$$

or by a unitary operator $T : \mathcal{H} \to \mathcal{H}$

$$|\psi\rangle \to |\psi'\rangle = T\,|\psi\rangle \tag{2.13}$$

with T invertible and

$$\forall\,|\psi\rangle\,, |\phi\rangle \in \mathcal{H} : \langle T\psi|T\phi\rangle = \langle\phi|\psi\rangle = \langle\psi|\phi\rangle^* \tag{2.14a}$$

$$T\left(c_1\,|\psi_1\rangle + c_2\,|\psi_2\rangle\right) = c_1^* T\,|\psi_1\rangle + c_2^* T\,|\psi_2\rangle \tag{2.14b}$$

This theorem frees us from the complicated projective geometry of the space of states and we can work directly in Hilbert space.

2.3 Groups

Lecture 02: Wed, 18. 10. 2017

Symmetries are described mathematically by groups $(G, \circ)$ with G a set and $\circ$ an inner operation

$$\begin{aligned} \circ : G \times G &\to G \\ (x, y) &\mapsto x \circ y \end{aligned} \tag{2.15}$$

with

1. closure: $\forall x, y \in G : x \circ y \in G$,
2. associativity: $x \circ (y \circ z) = (x \circ y) \circ z$,
3. identity element: $\exists e \in G : \forall x \in G : e \circ x = x \circ e = x$,
4. inverse elements: $\forall x \in G : \exists x^{-1} \in G : x \circ x^{-1} = x^{-1} \circ x = e$.

Many examples in physics

- permutations
- reflections
- parity
- translations
- rotations
- Lorentz boosts
- isospin
- ...

2.3.1 Lie Groups

Particularly interesting are *Lie Groups*, i. e. groups, where the set is a *differentiable manifold* and the composition is differentiable w. r. t. both operands.

Note that the choice of coordinates is not relevant:

$$B = \left\{ b_1(\eta) = \exp\begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \middle| \eta \in \mathbf{R} \right\}$$
$$= \left\{ b_2(\beta) = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \middle| \beta \in]-1,1[\right\} \quad (2.16)$$

Both times we have the set of all real symmetric 2×2 matrices with unit determinant. The composition laws are given by matrix multiplication[1]:

$$b_1(\eta) \circ b_1(\eta') = b_1(\eta) b_1(\eta') = b_1(\eta + \eta') \quad (2.17a)$$
$$b_2(\beta) \circ b_2(\beta') = b_2(\beta) b_2(\beta') = b_2\left(\frac{\beta + \beta'}{1 + \beta\beta'}\right). \quad (2.17b)$$

[1]NB:

$$|\beta| < 1 \wedge |\beta'| < 1 \Rightarrow \left| \frac{\beta + \beta'}{1 + \beta\beta'} \right| < 1$$

2.3.2 Lie Algebras

A Lie algebra $(A, [\cdot,\cdot])$ is a K-vector space[2] with a non-associative antisymmetric bilinear inner operation $[\cdot,\cdot]$:

$$\begin{aligned} [\cdot,\cdot] : A \times A &\to A \\ (a,b) &\mapsto [a,b] \end{aligned} \tag{2.18}$$

with

1. closure: $\forall a,b \in A : [a,b] \in A$,
2. antisymmetry: $[a,b] = -[b,a]$
3. bilinearity: $\forall \alpha,\beta \in \mathrm{K} : [\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c]$
4. Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$

Since A is a vector space, we can choose a basis and write

$$[a_i, a_j] = \sum_k C_{ijk} a_k \,. \tag{2.19}$$

A Lie algebra is called *simple*, if it has no ideals besides itself and $\{0\}$. Remarkably, *all* simple Lie algebras are known[5]:

$$\mathrm{so}(N), \mathrm{su}(N), \mathrm{sp}(2N), g_2, f_4, e_6, e_7, e_8 \tag{2.20}$$

with $N \in \mathbf{N}$.

As we will see below, the infinitesimal generators of a Lie group form a Lie algebra. Vice versa, the elements of a Lie algebra can be exponentiated to obtain a Lie group (not necessarily the same, but a cover of the original group).

2.3.3 Homomorphisms

A *group homomorphism* f is a map

$$\begin{aligned} f : G &\to G' \\ x &\mapsto f(x) \end{aligned} \tag{2.21}$$

between two groups $(G, \circ)$ and $(G', \circ')$ that is compatible with the group structure

$$f(x) \circ' f(y) = f(x \circ y) \tag{2.22}$$

[2] $\mathrm{K} = \mathrm{R}$ or C

and therefore

$$f(e) = e' \tag{2.23a}$$
$$f(x^{-1}) = (f(x))^{-1} . \tag{2.23b}$$

A *Lie algebra homomorphism* ϕ is a map

$$\begin{aligned} \phi : A &\to A' \\ a &\mapsto \phi(a) \end{aligned} \tag{2.24}$$

between two Lie algebras $(A, [\cdot,\cdot])$ and $(A', [\cdot,\cdot]')$ that is compatible with the Lie algebra structure

$$[\phi(a), \phi(b)]' = \phi([a, b]) . \tag{2.25}$$

NB: these need *not* be isomorphisms: $f(x) = e', \forall x$ is a trivial, but well defined group homomorphism and $\phi(a) = 0, \forall a$ is a similarly trivial but also well defined Lie algebra homomorphism.

2.3.4 Representations

Lie groups and algebras are abstract objects, which can be made concrete by representations.

A *group representation*

$$R : G \to L \tag{2.26}$$

is a homomorphism from the group $(G, \circ)$ to a group of linear operators $(L, \cdot)$ with $(O_1 \cdot O_2)(v) = O_1(O_2(v))$. The representation is called *unitary* if the operators are unitary. The representation is called *faithful* if $\forall x \neq y : R(x) \neq R(y)$.

A *Lie algebra representation*

$$r : A \to L \tag{2.27}$$

is a homomorphism from the Lie algebra $(A, [\cdot,\cdot])$ to an associative algebra of linear operators $(L, [\cdot,\cdot]')$ with $[O_1, O_2]' = O_1 \cdot O_2 - O_2 \cdot O_1$ or $[O_1, O_2]'(v) = O_1(O_2(v)) - O_2(O_1(v))$, i. e. commutators for Lie brackets.

The Matrix groups $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(2N)$ and their Lie algebras have obvious defining representations.

Every Lie algebra has a *adjoint representation*, using the itself as the linear representation space $a \Leftrightarrow |a\rangle$:

$$r_{\text{adj.}}(a)\, |b\rangle = |[a, b]\rangle \tag{2.28}$$

using the Jacobi identity

$$(r_{\text{adj.}}(a)r_{\text{adj.}}(b) - r_{\text{adj.}}(b)r_{\text{adj.}}(a))\,|c\rangle = |[a,[b,c]] - [b,[a,c]]\rangle \\ = |[[a,b],c]\rangle = r_{\text{adj.}}([a,b])\,|c\rangle \quad (2.29)$$

or, using a basis

$$r_{\text{adj.}}(a_i)\,|a_j\rangle = |[a_i,a_j]\rangle = |C_{ijk}a_k\rangle = C_{ijk}\,|a_k\rangle \quad (2.30)$$

we find the matrix elements

$$\left[r_{\text{adj.}}(a_i)\right]_{jk} = C_{ijk}\,. \quad (2.31)$$

Using Hausdorff's formula[3]

$$\mathrm{e}^a b\,(\mathrm{e}^a)^{-1} = \mathrm{e}^a b \mathrm{e}^{-a} = \mathrm{e}^{\mathrm{ad}_a} b = \mathrm{e}^{[a,\cdot]} b \\ = b + [a,b] + \frac{1}{2!}[a,[a,b]] + \frac{1}{3!}[a,[a,[a,b]]] + \ldots \quad (2.32)$$

we see that the map

$$\begin{aligned} f(x) : A &\to A \\ b &\mapsto xbx^{-1} \end{aligned} \quad (2.33)$$

is well defined and remains *inside* the Lie algebra. It's obviously linear and since

$$f(x)(f(y)(a)) = f(x)\left(yay^{-1}\right) = xyay^{-1}x^{-1} = (xy)a(xy)^{-1} = f(xy)(a) \quad (2.34)$$

it is also a representation, called the *adjoint representation of the group*.

2.4 Infinitesimal Generators

If we have a representation a Lie group and parametrize the neighborhood of the unit element by

$$\begin{aligned} x : \mathbf{R}^n &\to G \\ \alpha &\mapsto x(\alpha) \end{aligned} \quad (2.35)$$

such that

$$x(0) = e\,, \quad (2.36)$$

we find

$$R(\alpha) \equiv R(x(\alpha)) = \mathbf{1} + \mathrm{i}\sum_a \alpha_a T_a + \mathcal{O}(\alpha^2) \quad (2.37)$$

[3] The proof is left as an exercise.

with

$$T_a = -\mathrm{i}\,\frac{\partial}{\partial\alpha_a} R(\alpha)\bigg|_{\alpha=0} \tag{2.38}$$

the (matrix representation of the) *generators* of the group. Vice versa, we can obtain (a matrix representation of) group elements for finite α

$$R(\alpha) = \mathrm{e}^{\mathrm{i}\sum_a \alpha_a T_a}\,. \tag{2.39}$$

We must have

$$\forall\alpha,\beta\in\mathbf{R}^n : \exists\gamma\in\mathbf{R}^n : R(\alpha)R(\beta) = R(\gamma) \tag{2.40a}$$

i. e.

$$\mathrm{e}^{\mathrm{i}\sum_a \alpha_a T_a}\mathrm{e}^{\mathrm{i}\sum_a \beta_a T_a} = \mathrm{e}^{\mathrm{i}\sum_a \gamma_a T_a}\,. \tag{2.40b}$$

This is not a trivial condition, as can seen from expanding the logarithm[4]

$$\mathrm{i}\gamma_a T_a = \ln\left(\mathbf{1} + \left(\mathrm{e}^{\mathrm{i}\alpha_a T_a}\mathrm{e}^{\mathrm{i}\beta_a T_a} - \mathbf{1}\right)\right) \tag{2.42}$$

to second order[5]

$$\mathrm{i}\gamma_a T_a = \mathrm{i}\alpha_a T_a + \mathrm{i}\beta_a T_a - \frac{1}{2}\alpha_a\beta_b[T_a,T_b] + \mathcal{O}((\alpha,\beta)^3) \tag{2.43}$$

which can only be true if the commutator of two generators can be written as a linear combination of the generators

$$[T_a,T_b] = \mathrm{i}f_{abc}T_c$$

(summation over c is implied). Thus, the representation of the generators T_a must form a Lie algebra and it can be shown more abstractly, that the same is true for the generators of an abstract Lie-Group, where a Lie-bracket of two generators is well defined, but not the product. The Jacobi-Identity is trivial for matrix representations, because it follows from expanding the commutators. In the abstract case it is required by consistency.

[4] We use a summation convention

$$\alpha_a T_a = \sum_a \alpha_a T_a \tag{2.41}$$

etc.

[5] The details are left as an exercise.

2.4.1 Unitary and Conjugate Representations

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We may use any faithful representation of a Lie algebra to compute the structure constants

$$f_{abc} = \frac{1}{2\mathrm{i}} \operatorname{tr} \left([T_a, T_b] \, T_c \right) \tag{2.44a}$$

assuming the customary normalization

$$\operatorname{tr} \left(T_a T_b \right) = \frac{1}{2} \,. \tag{2.44b}$$

If the Lie algebra allows at least one faithful unitary representation, we can use the hermitian generators in this one

$$\begin{aligned} f^*_{abc} &= -\frac{1}{2\mathrm{i}} \operatorname{tr} \left([T^*_a, T^*_b] \, T^*_c \right) = -\frac{1}{2\mathrm{i}} \operatorname{tr} \left(\left[T^T_a, T^T_b \right] T^T_c \right) \\ &= -\frac{1}{2\mathrm{i}} \operatorname{tr} \left(T_c \, [T_b, T_a] \right) = \frac{1}{2\mathrm{i}} \operatorname{tr} \left([T_a, T_b] \, T_c \right) = f_{abc} \end{aligned} \tag{2.45}$$

to find that the structure constants are *real*.

There's always the *conjugate representation* of a Lie algebra

$$\overline{T_a} = -T^T_a \tag{2.46}$$

since

$$\left[\overline{T_a}, \overline{T_b} \right] = \left[T^T_a, T^T_b \right] = [T_b, T_a]^T = \mathrm{i} f_{bac} T^T_c = -\mathrm{i} f_{abc} T^T_c = \mathrm{i} f_{abc} \overline{T_c} \,. \tag{2.47}$$

Note that there are cases in which this representation is equivalent to the original one. If the representation is unitary representations, the conjugate representation is actually be the complex conjugate

$$\overline{T_a} = -T^*_a \,. \tag{2.48}$$

Also note that the complex conjugates of a Lie group representation matrix are also a representation

$$R^*(g) R^*(g') = R^*(g \circ g') \,, \tag{2.49}$$

which may be or not be equivalent to the original representation.

2.5 SO(3) *and* SU(2)

The Lie groups SO(3) and SU(2) have been discussed extensively in the QM lecture(s). Nevertheless, we should recall the most important elements, because we will need them again in the discussion of the Lorentz- and the Poincaré-Group and their representations.

2.5.1 O(3) *and* SO(3)

The group O(3) of real orthogonal 3×3-matrices R corresponds to the group of transformations that leave the eucledian inner product

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle \mathbf{R}^3 \times \mathbf{R}^3 &\to \mathbf{R}\\ (\vec x,\vec y) &\mapsto \langle\vec x,\vec y\rangle = \vec x\vec y = \sum_{i=1}^{3} x_i y_i\end{aligned} \tag{2.50}$$

invariant with

$$R^T R = \mathbf{1} = RR^T\,. \tag{2.51}$$

One immediately sees that

$$\det R = \pm 1 \tag{2.52}$$

and since the determinant is a continuous map, the group consists of (at least) two disconnected components, one with $\det R = 1$ the other with $\det R = -1$. The former is again a Lie-group, the group SO(3) of all unimodular, orthogonal 3×3-matrices. The second component is *not*[6].

A prominent element of the second component is the *parity* operation

$$\begin{aligned}P : \mathbf{R}^3 &\to \mathbf{R}^3\\ \vec x &\mapsto -\vec x\end{aligned} \tag{2.53}$$

with matrix representation

$$P = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\,. \tag{2.54}$$

Indeed, every element of the second component can be written as the product of P with an element of SO(3).

Since the orthogonality condition is non-linear, it is not obvious how to parametrize the elements of SO(3) uniquely. We start with noting that we can write

$$R(\vec\alpha) = \mathrm{e}^{-\mathrm{i}\vec\alpha\vec L} \tag{2.55}$$

[6] $\det P^2 = +1$, even if $\det P = -1$.

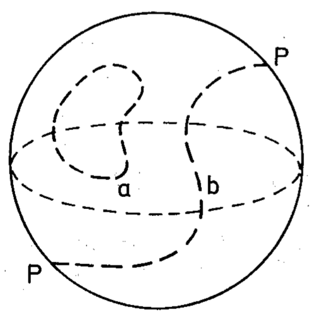

Figure 2.1: *Both curves* a *and* b *are closed on the* SO(3) *group manifold, but only* a *can be shrunk to a point.*

with the generators

$$-\mathrm{i}L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.56a}$$

$$-\mathrm{i}L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{2.56b}$$

$$-\mathrm{i}L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.56c}$$

Indeed, since $\mathrm{tr}\, L_i = 0$, we have $\det R(\vec{\alpha}) = 1$ and since $L_i = L_i^\dagger$, we have $R^T(\vec{\alpha})R(\vec{\alpha}) = \mathbf{1}$. Geometrically, one can visualize $R(\vec{\alpha})$ as a rotation around the axis $\vec{\alpha}$, with angle $|\vec{\alpha}|$. Therefore $R(\vec{\alpha}) = \mathbf{1}$, if $|\vec{\alpha}|/2\pi \in \mathbf{N}$. Furthermore, also

$$\forall \vec{\alpha} \in \mathbf{R}^3, |\vec{\alpha}| = \pi : R(\vec{\alpha}) = R(-\vec{\alpha}) \tag{2.57}$$

and we see that the group manifold of SO(3) is the sphere in $\mathbf{R}^3$ with radius π and opposing boundary points identified. Thus not all closed loops can be contracted to a point, cf. figure 2.1.

It is straightforward to compute the structure constants of the corresponding Lie algebra so(3) that is spanned by the generators $\{L_i\}_{i=1,2,3}$:

$$[L_i, L_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} L_k \tag{2.58}$$

where ϵ_{ijk} is the totally antisymmetric rank-3 tensor with $\epsilon_{123} = 1$.

2.5.2 SU(2)

The group SU(2) of unimodular unitary 2×2-matrices can be parametrized

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.59}$$

with conditions

$$U^\dagger U = \mathbf{1} = UU^\dagger \tag{2.60a}$$

$$\det U = 1 \tag{2.60b}$$

Therefore, the group SU(2) can be parametrized as

$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \text{with } |a|^2 + |b|^2 = 1 \tag{2.61}$$

or

$$U(\theta,\zeta,\eta) = \begin{pmatrix} \cos\theta\,\mathrm{e}^{\mathrm{i}\zeta} & -\sin\theta\,\mathrm{e}^{\mathrm{i}\eta} \\ \sin\theta\,\mathrm{e}^{-\mathrm{i}\eta} & \cos\theta\,\mathrm{e}^{-\mathrm{i}\zeta} \end{pmatrix} \tag{2.62}$$

with $\theta \in [0,\pi)$ and $\eta,\zeta \in [0,2\pi)$. Note that the range $\theta \in [\pi,2\pi]$ is redundant, because it is just a change in the signs of $\sin\theta$, which is already covered by $\mathrm{e}^{\mathrm{i}\eta}$.

A third option uses the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.63}$$

and four real dependent parameters $a = \beta_0 + \mathrm{i}\beta_3$, $b = \beta_2 + \mathrm{i}\beta_1$

$$U(\beta_0,\vec\beta) = \begin{pmatrix} \beta_0 + \mathrm{i}\beta_3 & \beta_2 + \mathrm{i}\beta_1 \\ -\beta_2 + \mathrm{i}\beta_1 & \beta_0 - \mathrm{i}\beta_3 \end{pmatrix} = \beta_0\mathbf{1} + \mathrm{i}\vec\beta\vec\sigma \qquad \text{with } \sum_{i=0}^{3} \beta_i^2 = 1\,. \tag{2.64}$$

Thus the group manifold of SU(2) is S^3, the unit sphere in $\mathbf{R}^4$, where all closed loops can be shrunk to a point.

The exponential parametrization with three real parameters α

$$U(\vec\alpha) = \mathrm{e}^{-\mathrm{i}\vec\alpha\vec\sigma/2} \tag{2.65}$$

is more convenient in a neighborhood of the identity. Unitarity and unimodularity are immmediate consequences of the properties of the Pauli matrices

$$\sigma_i^\dagger = \sigma_i \tag{2.66a}$$

$$\operatorname{tr}\sigma_i = 0 \tag{2.66b}$$

and the exponential function

$$\left(\mathrm{e}^A\right)^\dagger = \mathrm{e}^{A^\dagger} \tag{2.67a}$$
$$\left(\mathrm{e}^A\right)^{-1} = \mathrm{e}^{-A} \tag{2.67b}$$
$$\det\left(\mathrm{e}^A\right) = \mathrm{e}^{\operatorname{tr}A} \tag{2.67c}$$

We can read off the generators of SU(2)

$$\left\{S_i = \frac{1}{2}\sigma_i\right\}_{i=1,2,3} \tag{2.68}$$

and find again the commutation relation

$$[S_i, S_j] = \mathrm{i}\sum_{k=1}^{3}\epsilon_{ijk}S_k\,. \tag{2.69}$$

Therefore

$$\mathrm{su}(2) \cong \mathrm{so}(3)\,. \tag{2.70}$$

2.5.3 SU(2) → SO(3)

Thus we have found that the Lie algebras su(2) ≅ so(3) are isomorphic, but the corresponding Lie Groups SU(2) and SO(3) are not.

We can use the Pauli matrices to construct an isomorphism between the space H_2 of hermitian, traceless 2×2-matrices and $\mathbf{R}^3$

$$\begin{aligned}\phi : \mathbf{R}^3 &\to H_2\\ \vec x &\mapsto X = \vec x\vec\sigma = \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3\end{pmatrix}\end{aligned} \tag{2.71a}$$

with the inverse map

$$\begin{aligned}\phi^{-1} : H_2 &\to \mathbf{R}^3\\ X &\mapsto \vec x = \frac{1}{2}\operatorname{tr}(\vec\sigma X)\end{aligned} \tag{2.71b}$$

from

$$\operatorname{tr}(\sigma_i\sigma_j) = 2\delta_{ij}\,. \tag{2.72}$$

Any transformation

$$X \to AXA^\dagger \tag{2.73}$$

preserves hermiticity of X. Furthermore, any similarity transformation

$$X \to AXA^{-1} \tag{2.74}$$

preserves all products of traces

$$\mathrm{tr}(XY\cdots Z) \to \mathrm{tr}(AXA^{-1}AYA^{-1}\cdots AZA^{-1}) = \mathrm{tr}(XY\cdots Z) \tag{2.75}$$

and in particular tracelessness, inner product and volume element

$$0 = \mathrm{tr}\,(X) \tag{2.76a}$$

$$\vec{x}\vec{y} = \frac{1}{2}\,\mathrm{tr}\,(XY) \tag{2.76b}$$

$$(\vec{x}\times\vec{y})\vec{z} = \frac{1}{4\mathrm{i}}\,\mathrm{tr}\,([X,Y]Z)\ . \tag{2.76c}$$

Therefore, any similarity transformation with a unitary 2×2-matrix in H_2 corresponds to a SO(3) transformation of $\mathbf{R}^3$

$$\begin{array}{ccc} \mathbf{R}^3 & \xrightarrow{\ \phi\ } & H_2 \\ {\scriptstyle \vec{x}\to R\vec{x}}\downarrow & & \downarrow{\scriptstyle X\to UXU^\dagger} \\ \mathbf{R}^3 & \xleftarrow[\ \phi^{-1}\ ]{} & H_2 \end{array} \tag{2.77}$$

and since the phase cancels in the similarity transformation, we may choose the unitary matrix from SU(2).

This way we have defined a map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Since the sign of the unitary matrix cancels in the similarity transformation, the kernel of this map contains $\mathrm{Z}_2 = \{\mathbf{1}, -\mathbf{1}\}$. By explicit calculation with a concrete parameterization of SO(3), e. g. Euler angles, one can show the that the kernel is indeed $\mathbf{Z}_2$ and also that the map is surjective. Therefore

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbf{Z}_2 \tag{2.78}$$

and SU(2) is a double cover of SO(3).

2.5.4 SO(3) *and* SU(2) *Representations*

The representations of a Lie algebra are direct sums of *irreducible representations*, i. e. representations that have to non-trivial invariant subspaces. The irreducible representations of the Lie algebra $\mathrm{su}(2) \cong \mathrm{so}(3)$ should be familiar from the QM lecture.

An operator, that commutes with all operators in the Lie algebra is called a *Casimir Operator*. By *Schur's Lemma*, such an operator is proportional to the unit matrix in an irreducible representation. The eigenvalues of Casimir operators can be used to identify an irreducible representation. For so(3) such an operator is

$$\vec{L}^2 = L_1 L_1 + L_2 L_2 + L_3 L_3 \,, \tag{2.79}$$

and its eigenvalues will turn out to be $\{l(l+1)\}_{l\in 0,1/2,1,3/2,\ldots}$.

The construction proceeds by switching from $\{L_1, L_2, L_3\}$ to

$$\{L_+ = L_1 + \mathrm{i}L_2, L_0 = L_3, L_- = L_1 - \mathrm{i}L_2\} \tag{2.80}$$

and to use the *shift operators* $L_\pm$ to change the eigenvalues of the *Cartan generators* L_0 that labels the states within the representation. In order to have a finite dimensional representation, the repeated application of one of the shift operators must yield the zero vector. From this one obtains conditions on the eigenvalues of the Casimir operator(s) and Cartan generator(s):

$$\vec{L}^2 \, |l,m\rangle = l(l+1) \, |l,m\rangle \tag{2.81a}$$
$$L_0 \, |l,m\rangle = m \, |l,m\rangle \tag{2.81b}$$
$$L_\pm \, |l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)} \, |l,m\pm 1\rangle \tag{2.81c}$$

where

$$l \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right\} \tag{2.82a}$$
$$m \in \{-l, -l+1, \ldots, l-1, l\} \,. \tag{2.82b}$$

It turns out that all representations with $l \geq 1$ can be constructed as the symmetrical tensor product of $2l$ copies of the $l = 1/2$ representation. The latter is supplied by the Pauli matrices, of course.

Note that

$$\overline{\sigma_i} = -\sigma_i^T = -(-1)^{\delta_{i2}} \sigma_i = \sigma_2 \sigma_i \sigma_2 = (\mathrm{i}\sigma_2) \sigma_i (\mathrm{i}\sigma_2)^\dagger \tag{2.83}$$

since σ_2 is antisymmetric and the other Pauli matrices are symmetric. Furthermore $\mathrm{i}\sigma_2$ is unitary

$$(\mathrm{i}\sigma_2)^\dagger (\mathrm{i}\sigma_2) = \sigma_2 \sigma_2 = \mathbf{1} \tag{2.84}$$

and the (complex) conjugate representation turns out to be unitarily equivalent and adds nothing new.

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We can compute non-infinitesimal rotations in the exponential representation

$$\mathrm{e}^{-\mathrm{i}\vec\alpha\vec\sigma/2} = \mathbf{1}\cos\frac{|\vec\alpha|}{2} - \mathrm{i}\frac{\vec\alpha\vec\sigma}{|\vec\alpha|}\sin\frac{|\vec\alpha|}{2} = \begin{pmatrix} \cos\frac{|\vec\alpha|}{2} - \mathrm{i}\frac{\alpha_3}{|\vec\alpha|}\sin\frac{|\vec\alpha|}{2} & -\frac{\alpha_2+\mathrm{i}\alpha_1}{|\vec\alpha|}\sin\frac{|\vec\alpha|}{2} \\ \frac{\alpha_2-\mathrm{i}\alpha_1}{|\vec\alpha|}\sin\frac{|\vec\alpha|}{2} & \cos\frac{|\vec\alpha|}{2} + \mathrm{i}\frac{\alpha_3}{|\vec\alpha|}\sin\frac{|\vec\alpha|}{2} \end{pmatrix}, \tag{2.85}$$

in particular along the coordinate axes

$$\mathrm{e}^{-\mathrm{i}\alpha\sigma_1/2} = \begin{pmatrix} \cos\frac{\alpha}{2} & -\mathrm{i}\sin\frac{\alpha}{2} \\ -\mathrm{i}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \tag{2.86a}$$

$$\mathrm{e}^{-\mathrm{i}\alpha\sigma_2/2} = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \tag{2.86b}$$

$$\mathrm{e}^{-\mathrm{i}\alpha\sigma_3/2} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha/2} \end{pmatrix}. \tag{2.86c}$$

The simplest case is the rotation in x_1-x_2 plane

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\alpha\sigma_3/2}X\mathrm{e}^{\mathrm{i}\alpha\sigma_3/2} &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha/2} \end{pmatrix} \begin{pmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha/2} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha/2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2}x_3 & \mathrm{e}^{-\mathrm{i}\alpha/2}(x_1 - \mathrm{i}x_2) \\ \mathrm{e}^{\mathrm{i}\alpha/2}(x_1 + \mathrm{i}x_2) & -\mathrm{e}^{-\mathrm{i}\alpha/2}x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_3 & \mathrm{e}^{-\mathrm{i}\alpha}(x_1 - \mathrm{i}x_2) \\ \mathrm{e}^{\mathrm{i}\alpha}(x_1 + \mathrm{i}x_2) & -x_3 \end{pmatrix} \end{aligned} \tag{2.87}$$

which turns out to be a rotation by the angle α:

$$x_1' \pm \mathrm{i}x_2' = \mathrm{e}^{\pm\mathrm{i}\alpha}(x_1 \pm \mathrm{i}x_2) \tag{2.88}$$

i. e.

$$x_1' = \frac{\mathrm{e}^{\mathrm{i}\alpha}(x_1 + \mathrm{i}x_2) + \mathrm{e}^{-\mathrm{i}\alpha}(x_1 - \mathrm{i}x_2)}{2} = \cos\alpha\, x_1 - \sin\alpha\, x_2 \tag{2.89a}$$

$$x_2' = \frac{\mathrm{e}^{\mathrm{i}\alpha}(x_1 + \mathrm{i}x_2) - \mathrm{e}^{-\mathrm{i}\alpha}(x_1 - \mathrm{i}x_2)}{2\mathrm{i}} = \cos\alpha\, x_2 + \sin\alpha\, x_1 . \tag{2.89b}$$

This exercise can be repeated for the other axis and we find that

$$U(\vec\alpha) = \mathrm{e}^{-\mathrm{i}\vec\alpha\vec\sigma/2} \tag{2.90}$$

corresponds to a rotation by $|\alpha|$ around the axis α. Note that rotations of $l = 1/2$ states by 2π change the sign

$$\mathrm{e}^{-\mathrm{i}\alpha\sigma_1/2}\Big|_{\alpha=2\pi} = \mathrm{e}^{-\mathrm{i}\alpha\sigma_2/2}\Big|_{\alpha=2\pi} = \mathrm{e}^{-\mathrm{i}\alpha\sigma_3/2}\Big|_{\alpha=2\pi} = -\mathbf{1}, \tag{2.91}$$

but this sign cancels in the rotation of $l = 1$ states, because two such matrices appear.

2.6 Lorentz- and Poincaré-Group

We know from experimental observations with high precission, that the speed of light c is the same in all inertial frames. The wave fronts emerging from $\vec{x} = 0$ at $t = 0$ lie on the sphere

$$|\vec{x}| = ct = x^0 \tag{2.92}$$

in all inertial systems. It is helpful to express the time t by an equivalent length x^0

$$t = \frac{x^0}{c} \tag{2.93}$$

and to set $c = 1$. We denote *events* by their point in time and space and assign to them a *four vector*

$$\mathbf{R}^4 \ni x = (x^0, \vec{x}) = (x^0, x^1, x^2, x^3) = (ct, \vec{x})\,. \tag{2.94}$$

A ray of light can connect two events x and y if and only if

$$0 = (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 = c^2(t_x - t_y)^2 - (\vec{x} - \vec{y})^2\,. \tag{2.95}$$

This defines a relation that is the same for a all inertial observers.

Since

$$xy = \frac{1}{2}\left((x+y)^2 - x^2 - y^2\right) \tag{2.96}$$

we are led to define an invariant inner product[7]

$$\begin{aligned} g : \mathbf{M} \times \mathbf{M} \cong \mathbf{R}^4 \times \mathbf{R}^4 &\to \mathbf{R} \\ (x,y) &\mapsto xy = g(x,y) = g(y,x) = x^0 y^0 - \vec{x}\vec{y} \end{aligned} \tag{2.97}$$

on *Minkowski space* $\mathbf{M}$. This inner product can be realized by a symmetric *metric tensor* of rank 2

$$g(x,y) = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} x^\mu y^\nu\,, \tag{2.98}$$

with

$$g_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.99}$$

We will streamline our notation with the *Einstein summation convention*:

[7] This is the "west coast" or "particle physics" choice of sign. The opposite "east coast" of "general relativity" sign is equally valid, but one must be consistent. We will use the "west coast" sign throughout this lecture.

Every pair of greek indices, with one upper and one lower, shall be interpreted as a sum from 0 *to* 3*:*

$$X^{\cdots\mu\cdots\cdots}_{\cdots\cdots\mu\cdots} := \sum_{\mu=0}^{3} X^{\cdots\mu\cdots\cdots}_{\cdots\cdots\mu\cdots} \tag{2.100}$$

e. g.

$$g(x,y) = g_{\mu\nu}x^\mu y^\nu \,. \tag{2.101}$$

Furthermore, we will use the metric tensor to raise and lower indices

$$x_\mu = g_{\mu\nu}x^\nu \tag{2.102a}$$
$$g_\mu{}^\nu = g_{\mu\rho}g^{\rho\nu} = \delta_\mu{}^\nu \tag{2.102b}$$
$$g_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma} \tag{2.102c}$$
$$x^\mu = g^{\mu\nu}x_\nu \tag{2.102d}$$

etc. *Numerically*, $g^{\mu\nu}$ and $g_{\mu\nu}$ have the same entries, but

$$x^\mu = (x_0, \vec{x}) \tag{2.103a}$$
$$x_\mu = (x_0, -\vec{x}) \,. \tag{2.103b}$$

Nevertheless, we are free to exchange the position of pairs of indices and to remove redundant metric tensors

$$g(x,y) = g_{\mu\nu}x^\mu y^\nu = g^{\mu\nu}x_\mu y_\nu = x_\mu y^\mu = x^\mu y_\mu = xy \,, \tag{2.104}$$

where the latter is only used, if no ambiguities can arise.

A *Lorentz transformation* among inertial systems

$$\begin{aligned} \Lambda : \mathbf{M} \cong \mathbf{R}^4 &\to \mathbf{M} \cong \mathbf{R}^4 \\ x^\mu &\mapsto (x')^\mu = \Lambda^\mu{}_\nu x^\nu \end{aligned} \tag{2.105}$$

leaves the inner product invariant:

$$x'y' = xy \tag{2.106}$$

or

$$g_{\mu\nu}x^\mu y^\nu = g_{\mu\nu}(x')^\mu(y')^\nu = g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma x^\rho y^\sigma = g_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu x^\mu y^\nu \,. \tag{2.107}$$

Since this must hold for *all* $x, y \in \mathbf{M}$, we find the condition

$$g_{\mu\nu} = g_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu \tag{2.108}$$

or in matrix notation

$$g = \Lambda^T g \Lambda\,. \tag{2.109}$$

From

$$1 = g_{00} = g_{\rho\sigma}\Lambda^\rho{}_0\Lambda^\sigma{}_0 = \left(\Lambda_0{}^0\right)^2 - \sum_{i=1}^{3}\left(\Lambda^i{}_0\right)^2 \tag{2.110a}$$

$$\det g = \det\Lambda^T \det g \det\Lambda\,, \tag{2.110b}$$

we find

$$|\Lambda^0{}_0| \geq 1 \tag{2.111a}$$

$$|\det\Lambda| = 1 \tag{2.111b}$$

and that the Lorentz group consists of four disconnected components

$$\mathcal{L}_+^\uparrow = \left\{\Lambda\in\mathcal{L} : \det\Lambda = +1 \wedge \Lambda^0{}_0 \geq 1\right\} \ni \mathbf{1} \tag{2.112a}$$

$$\mathcal{L}_-^\uparrow = \left\{\Lambda\in\mathcal{L} : \det\Lambda = -1 \wedge \Lambda^0{}_0 \geq 1\right\} \ni P \tag{2.112b}$$

$$\mathcal{L}_-^\downarrow = \left\{\Lambda\in\mathcal{L} : \det\Lambda = -1 \wedge \Lambda^0{}_0 \leq -1\right\} \ni T \tag{2.112c}$$

$$\mathcal{L}_+^\downarrow = \left\{\Lambda\in\mathcal{L} : \det\Lambda = +1 \wedge \Lambda^0{}_0 \leq -1\right\} \ni PT = TP\,, \tag{2.112d}$$

with parity, time reversal and their combination

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.113a}$$

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.113b}$$

$$PT = TP = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{2.113c}$$

Of those, only the proper, orthochronous Lorentz transformations $\mathcal{L}_+^\uparrow$ form a subgroup.

There are 16 conditions in (2.108), but since (2.108) is symmetrical, only 10 are independent. This leaves us with 6 independent parameters for the

Lorentz group. Indeed, expanding the Lorentz transformation matrices to first order in a small parameter

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} + \mathcal{O}(\omega^2) \tag{2.114}$$

we find the conditions

$$\begin{aligned} g_{\mu\nu} = g_{\rho\sigma}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu} = g_{\rho\sigma}(\delta^{\rho}{}_{\mu} + \omega^{\rho}{}_{\mu} + \ldots)(\delta^{\sigma}{}_{\nu} + \omega^{\sigma}{}_{\nu} + \ldots) \\ = g_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} + \ldots \end{aligned} \tag{2.115}$$

i. e. the ω are antisymmetric

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \,. \tag{2.116}$$

Thus there are 6 parameters and 6 generators. Three of these generators must correspond to rotations $\vec{L}$ and the other 3 to Lorentz boosts $\vec{K}$, i. e. transformations to a moving coordinate system.

The commutation relations among these generators turn out to be

$$[L_i, L_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} L_k \tag{2.117a}$$

$$[L_i, K_j] = \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} K_k \tag{2.117b}$$

$$[K_i, K_j] = -\mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} L_k \,. \tag{2.117c}$$

Here, (2.117a) has been computed before and (2.117b) simply states that $\vec{K}$ is a vector under rotations. Finally, (2.117c) follows from a simple calculation.

It is convenient for the following discussion, to combine the generators $\vec{L}$ and $\vec{K}$ into an antisymmetric 4×4-matrix that transforms like a rank two tensor under Lorentz transformations ($i, j, k = 1, 2, 3$):

$$M^{ij} = \sum_{k} \epsilon_{ijk} L_k \tag{2.118a}$$

$$M^{0i} = -M^{i0} = -K_i \,. \tag{2.118b}$$

The commutation relations are then

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\,(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho}) \\ = \mathrm{i} g^{\mu\rho}M^{\nu\sigma} + \text{antisymmetric} \,. \end{aligned} \tag{2.119}$$

2.6.1 Lorentz-Group and SL(2, **C**)

As in the case of SU(2), we can introduce a map from the four vectors to the hermitean 2×2 matrices

$$\begin{aligned}\phi : \mathbf{M} &\to H_2 \\ x \mapsto X = x_\mu \sigma^\mu &= x_0 \mathbf{1} - \vec{x}\vec{\sigma} = \begin{pmatrix} x_0 - x_3 & -x_1 + \mathrm{i}x_2 \\ -x_1 - \mathrm{i}x_2 & x_0 + x_3 \end{pmatrix}\end{aligned} \tag{2.120a}$$

with the inverse map

$$\begin{aligned}\phi^{-1} : H_2 &\to \mathbf{M} \\ X \mapsto x_\mu &= \frac{1}{2}\,\mathrm{tr}(\sigma_\mu X)\,,\end{aligned} \tag{2.120b}$$

where we have introduced the notation

$$\sigma^\mu = (\sigma^0, \vec{\sigma}) = (\mathbf{1}, \vec{\sigma})\,. \tag{2.121}$$

This map is useful, because

$$x^2 = x_\mu x^\mu = \det X \tag{2.122}$$

and we can look for maps that preserve hermiticity and the determinant. As noted above, any transformation

$$X \to AXA^\dagger\,, \tag{2.123}$$

including complex matrices, the preserves hermiticity of X. As long as $|\det A| = 1$, it also preserves the determinant. Since an overall phase cancels in the transformation, we can restrict ourselves to the case

$$\det A = 1 \tag{2.124}$$

and arrive at SL(2, **C**), the group of unimodular 2×2 matrices with complex entries. There are 8 real parameters and two real conditions from the complex determinant, leaving us with 6 independent real parameters for SL(2, **C**).

Since this is the same number of parameters as the Lorentz group, and each element of SL(2, **C**) corresponds to an element of the Lorentz group, the Lie algebras of $\mathcal{L}_+^\uparrow$ and SL(2, **C**) must agree.

There is again an exponential representation

$$\mathrm{e}^{-\mathrm{i}\vec{\alpha}\vec{\sigma}/2} = \mathbf{1}\cos\frac{\sqrt{\vec{\alpha}^2}}{2} - \mathrm{i}\frac{\vec{\alpha}\vec{\sigma}}{\sqrt{\vec{\alpha}^2}}\sin\frac{\sqrt{\vec{\alpha}^2}}{2}$$

$$= \begin{pmatrix} \cos\frac{\sqrt{\vec\alpha^2}}{2} - \mathrm{i}\frac{\alpha_3}{\sqrt{\vec\alpha^2}}\sin\frac{\sqrt{\vec\alpha^2}}{2} & -\frac{\alpha_2+\mathrm{i}\alpha_1}{\sqrt{\vec\alpha^2}}\sin\frac{\sqrt{\vec\alpha^2}}{2} \\ \frac{\alpha_2-\mathrm{i}\alpha_1}{\sqrt{\vec\alpha^2}}\sin\frac{\sqrt{\vec\alpha^2}}{2} & \cos\frac{\sqrt{\vec\alpha^2}}{2} + \mathrm{i}\frac{\alpha_3}{\sqrt{\vec\alpha^2}}\sin\frac{\sqrt{\vec\alpha^2}}{2} \end{pmatrix}, \quad (2.125)$$

but this time with three complex parameters and care has been taken not to confuse $|\vec\alpha| = \sqrt{\vec\alpha^*\vec\alpha}$ with $\sqrt{\vec\alpha^2}$. For purely imaginary values of α, we can replace the trigonometric functions of $\sqrt{\vec\alpha^2}$ by hyperbolic functions of the imaginary part. These functions are not periodic and we see that the group $\mathrm{SL}(2,\mathbf{C})$ is not compact. There is a theorem, that there are no finite dimensional unitary representations of non-compact groups [6]. Thus it is not a surprise that the matrices $\mathrm{e}^{-\mathrm{i}\vec\alpha\vec\sigma/2}$ are *not* unitary.

2.6.2 Lorentz-Group Representations

Lecture 05: Tue, 07. 11. 2017

The representations of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ are constructed from $\mathrm{SL}(2,\mathbf{C})$ matrices

$$\begin{aligned} A : \mathcal{L}_+^\uparrow &\to \mathrm{SL}(2,\mathbf{C}) \\ \Lambda &\mapsto A(\Lambda) \end{aligned} \quad (2.126)$$

and their complex conjugates. The rotations correspond to the compact $\mathrm{SU}(2)$ subgroup

$$\begin{aligned} A : \mathrm{SO}(3) \in \mathcal{L}_+^\uparrow &\to \mathrm{SU}(2) \in \mathrm{SL}(2,\mathbf{C}) \\ \Lambda &\mapsto A(\Lambda) \end{aligned} \quad (2.127)$$

generated by $\vec L$, while the Lorentz boosts correspond to the non-compact directions generated by $\vec K$.

The most general irreducible representation $\mathcal{D}^{(k/2,l/2)}$ is given by the tensor product of the k-fold symmetrical tensor product with the l-fold symmetrical tensor product of two-component spinors

$$\xi_{\alpha_1\alpha_2\ldots\alpha_k,\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_l} \xrightarrow{\Lambda} \sum_{\beta,\dot\beta=1}^{2} A_{\alpha_1\beta_1}(\Lambda)\cdots A_{\alpha_k\beta_k}(\Lambda)A^*_{\dot\alpha_1\dot\beta_1}(\Lambda)\cdots A^*_{\dot\alpha_l\dot\beta_l}(\Lambda)\xi_{\beta_1\beta_2\ldots\beta_k,\dot\beta_1\dot\beta_2\ldots\dot\beta_l}\,. \quad (2.128)$$

The dotted indices $\dot\alpha_i$ have no relation to the undotted indices α_i. The dot is used to distinguish indices that transform according to the $\mathrm{SL}(2,\mathbf{C})$-matrix A from those transforming according to its complex conjugate A. In the case of $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ there was no difference between dotted and undotted indices, but in case of the Lorentz group we have to make this distinction.

By looking at the SO(3) subgroup of the Lorentz group, we see that $\mathcal{D}^{(1/2,0)}$ and $\mathcal{D}^{(0,1/2)}$ correspond to spinors for spin 1/2 particles. They transform identically under rotations, but differently under boosts. In general, $\mathcal{D}^{(k,l)}$ describes particles of spins in the Clebsh-Gordan decomposition of the tensor product $(2k+1)\otimes(2l+1)$.

Note that under parity

$$P : \begin{pmatrix} \vec{L} \\ \vec{K} \end{pmatrix} \mapsto \begin{pmatrix} \vec{L} \\ -\vec{K} \end{pmatrix} \tag{2.129}$$

and therefore

$$P : \mathcal{D}^{(k,l)} \to \mathcal{D}^{(l,k)} . \tag{2.130}$$

Thsi means that if we want to represent tho whole Lorentz group and not just $\mathcal{L}^{\uparrow}_{+}$, the representations are

$$\mathcal{D}^{(k,l)} \oplus \mathcal{D}^{(l,k)} . \tag{2.131}$$

2.6.3 Poincaré-Group

Our argument that transformations among inertial systems should leave the inner product $g(x,y) = x_\mu y^\mu$ invariant has led us to the *linear* LTs. However, the argument does not apply to points in Minkowski space $\mathbf{M}$, but to differences bewteen two points. Therefore, we are led to consider the *affine Poincaré Transformations*

$$\begin{aligned} (\Lambda, a) : \mathbf{M} \cong \mathbf{R}^4 &\to \mathbf{M} \cong \mathbf{R}^4 \\ x^\mu &\mapsto (x')^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \end{aligned} \tag{2.132}$$

that include LTs, space-time translations and their combinations. Sometimes the Poincaré-group is also called the inhomogeneous Lorentz-group. Obviously, the matrices Λ must satisfy the same constraints as in the case of the homogeneous Lorentz-group and the translations $a \in \mathbf{R}^4$ are arbitrary. Thus we have 10 independent parameters and 10 generators: $M^{\mu\nu} = -M^{\nu\mu}$ and P^μ.

Note that the composition of Poincaré transformations mixes LTs and translations

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1) . \tag{2.133}$$

Thus we can not study their representations independently. Indeed the inverse is

$$(\Lambda, a)^{-1} = \left(\Lambda^{-1}, -\Lambda^{-1} a\right) . \tag{2.134}$$

Not that we can obtain the inverse matrix by rewriting (2.108)

$$g_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = g_{\mu\nu} \tag{2.135a}$$

$$\Lambda_\sigma{}^\mu\Lambda^\sigma{}_\nu = \delta^\mu_\nu = \left(\Lambda^{-1}\right)^\mu{}_\sigma\Lambda^\sigma{}_\nu \tag{2.135b}$$

to find

$$\left(\Lambda^{-1}\right)^\mu{}_\sigma = \Lambda_\sigma{}^\mu = g_{\sigma\rho}g^{\mu\nu}\Lambda^\rho{}_\nu\,. \tag{2.136}$$

If we use an exponential parametrization of (representations of) Poincaré transformations in terms of (representations of) the generators $M^{\mu\nu}$ and P^ν

$$U(\Lambda,a) = \mathrm{e}^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}+\mathrm{i}a_\mu P^\mu} \tag{2.137}$$

we can write infinitesimal transformations as

$$U^{(1)}(\omega,\epsilon) = \mathbf{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu} + \mathrm{i}\epsilon_\mu P^\mu\,. \tag{2.138}$$

Using the composition law (2.133) we find

$$\begin{aligned}U(\Lambda,a)U^{(1)}(\omega,\epsilon)U^{-1}(\Lambda,a) &= U(\Lambda,a)U^{(1)}(\omega,\epsilon)U\left(\Lambda^{-1},-\Lambda^{-1}a\right)\\ &= U(\Lambda,a)U\left((\mathbf{1}+\omega)\Lambda^{-1},-(\mathbf{1}+\omega)\Lambda^{-1}a+\epsilon\right)\\ &= U\left(\Lambda(\mathbf{1}+\omega)\Lambda^{-1},-\Lambda(\mathbf{1}+\omega)\Lambda^{-1}a+\Lambda\epsilon+a\right)\\ &= U\left(\Lambda(\mathbf{1}+\omega)\Lambda^{-1},\Lambda\epsilon-\Lambda\omega\Lambda^{-1}a\right)\end{aligned} \tag{2.139}$$

and subtracting the unit from both sides of the equation

$$\begin{aligned}U(\Lambda,a)&\left(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}+\epsilon_\mu P^\mu\right)U^{-1}(\Lambda,a)\\ &= \frac{1}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}M^{\mu\nu} + \left(\Lambda\epsilon-\Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu\end{aligned} \tag{2.140}$$

we can compare coefficients of $\omega_{\mu\nu}$ and ϵ_μ:

$$\begin{aligned}U(\Lambda,a)&M^{\mu\nu}U^{-1}(\Lambda,a)\\ &= \Lambda_\rho{}^\mu\left(\Lambda^{-1}\right)^\nu{}_\sigma M^{\rho\sigma} - \Lambda_\rho{}^\mu\left(\Lambda^{-1}\right)^\nu{}_\sigma a^\sigma P^\rho + \Lambda_\rho{}^\nu\left(\Lambda^{-1}\right)^\mu{}_\sigma a^\sigma P^\rho\\ &= \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu M^{\rho\sigma} - \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu a^\sigma P^\rho + \Lambda_\rho{}^\nu\Lambda_\sigma{}^\mu a^\sigma P^\rho\\ &= \Lambda_\rho{}^\mu\Lambda_\sigma{}^\nu\left(M^{\rho\sigma}-a^\sigma P^\rho+a^\rho P^\sigma\right)\end{aligned} \tag{2.141a}$$

where we have taken into account the antisymmetry of $\omega_{\mu\nu}$. We also see immediately

$$U(\Lambda,a)P^\mu U^{-1}(\Lambda,a) = \Lambda_\nu{}^\mu P^\nu\,. \tag{2.141b}$$

If we expand $U(\Lambda,a)$ itself as $U^{(1)}(\omega,\epsilon)$

$$U^{(1)}(\omega,\epsilon)A\left(U^{(1)}(\omega,\epsilon)\right)^{-1} = A + \mathrm{i}\left[\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + \epsilon_\rho P^\rho, A\right] + \ldots \tag{2.142}$$

we can read off commutators by comparing the coefficients of the first order using $\Lambda_\mu{}^\nu = \delta_\mu^\nu + \omega_\mu{}^\nu + \ldots$

$$\begin{aligned}\mathrm{i}\left[\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + \epsilon_\rho P^\rho, M^{\mu\nu}\right] &= \left(\omega_\rho{}^\mu\delta_\sigma^\nu + \delta_\rho^\mu\omega_\sigma{}^\nu\right)M^{\rho\sigma} - \epsilon^\nu P^\mu + \epsilon^\mu P^\nu \\ &= \omega_{\rho\sigma}g^{\sigma\mu}M^{\rho\nu} + \omega_{\sigma\rho}g^{\rho\nu}M^{\mu\sigma} - \epsilon^\nu P^\mu + \epsilon^\mu P^\nu \\ &= \frac{1}{2}\omega_{\rho\sigma}\left(g^{\sigma\mu}M^{\rho\nu} - g^{\rho\nu}M^{\mu\sigma} - g^{\sigma\nu}M^{\rho\mu} + g^{\rho\mu}M^{\nu\sigma}\right) - \epsilon^\nu P^\mu + \epsilon^\mu P^\nu \\ &= \frac{1}{2}\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right) - \epsilon^\nu P^\mu + \epsilon^\mu P^\nu\end{aligned} \tag{2.143a}$$

and

$$\begin{aligned}\mathrm{i}\left[\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma} + \epsilon_\rho P^\rho, P^\mu\right] &= \omega_\rho{}^\mu P^\rho \\ &= \omega_{\rho\sigma}g^{\sigma\mu}P^\rho = \frac{1}{2}\omega_{\rho\sigma}\left(g^{\sigma\mu}P^\rho - g^{\rho\mu}P^\sigma\right)\end{aligned} \tag{2.143b}$$

to find

$$\mathrm{i}\left[M^{\rho\sigma}, M^{\mu\nu}\right] = g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho} \tag{2.144a}$$
$$\mathrm{i}\left[P^\rho, M^{\mu\nu}\right] = -g^{\rho\nu}P^\mu + g^{\rho\mu}P^\nu \tag{2.144b}$$
$$\mathrm{i}\left[M^{\rho\sigma}, P^\mu\right] = g^{\sigma\mu}P^\rho - g^{\rho\mu}P^\sigma \tag{2.144c}$$
$$\left[P^\mu, P^\nu\right] = 0\,. \tag{2.144d}$$

Rearranging the indices we see that (2.144b) and (2.144c) are equivalent and we can write the *Poincaré Algebra*

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = \mathrm{i}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma}\right) \tag{2.145a}$$
$$\left[M^{\mu\nu}, P^\rho\right] = \mathrm{i}\left(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu\right) \tag{2.145b}$$
$$\left[P^\mu, P^\nu\right] = 0\,. \tag{2.145c}$$

2.6.4 Poincaré-Group Representations

It turns our that there are two Casimir operators for the representations of the Poincaré group

$$P^2 = P_\mu P^\mu \tag{2.146a}$$

$$W^2 = W_\mu W^\mu \tag{2.146b}$$

with the *Pauli-Lubanski vector*

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma\,, \tag{2.147}$$

that commute with all generators $M^{\mu\nu}$ and P^μ. Therefore one can use the eigenvalues

$$P^2 \left|m,s\right\rangle = m^2 \left|m,s\right\rangle \tag{2.148a}$$
$$W^2 \left|m,s\right\rangle = -m^2 s(s+1) \left|m,s\right\rangle \tag{2.148b}$$

with $m \geq 0$ and $s \in 0, 1/2, 1, 3/2, \ldots$ to label the representations[8]

Since P^μ generates space-time translations, we interpret is as the energy-momentum operator $(H, \vec{P})$ and expect that eigenvalues $p^\mu = (E, \vec{p})$ in plane wave states satisfy Einstein's dispersion relation

$$H = \sqrt{m^2 + \vec{p}^2} \tag{2.149}$$

or

$$p^2 = m^2\,. \tag{2.150}$$

We must treat the cases $m > 0$ and $m = 0$ separately:

1. For $m > 0$, we can find a LT into a *rest frame* with $p = (m, \vec{0})$. In this frame, the Pauli-Lubanski vector turns out to be proportional to the angular momentum
$$W^\mu = (0, m\vec{L})\,. \tag{2.151}$$
Then
$$W^2 = -m^2 \vec{L}^2 \tag{2.152}$$
and we can obtain the eigenvalues from the discussion of the representations of the *Little Group* SU(2) that leaves the vector $(m, \vec{0})$ invariant.

2. For $m = 0$, there is no rest frame. Instead we can choose a frame with $p = (E, 0, 0, E)$. In this frame, the little group is the euclidean group in the (x_1, x_2)-plane. The representations are again labelled by a half-integer number s, but this time, all irreducible representations are one-dimensional and s is called the *helicity* and corresponds to the projection of the spin on the momentum vector.

[8]Note that there are also representation with $P^2 < 0$, but these appear not to be realized in nature and violate causality. Therefore we will not discuss them here.

2.6.5 Poincaré-Group Action on Functions

Lecture 06: Wed, 08. 11. 2017

Consider the partial derivatives of functions on $\mathbf{M} \cong \mathbf{R}^4$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{2.153a}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} \tag{2.153b}$$

and their commutation relations with space time coordinates

$$\begin{aligned} [x^\mu, \partial_\nu]\, \phi(x) &= x^\mu \frac{\partial}{\partial x^\nu}\phi(x) - \frac{\partial}{\partial x^\nu}\left(x^\mu \phi(x)\right) \\ &= x^\mu \frac{\partial}{\partial x^\nu}\phi(x) - \left(\frac{\partial}{\partial x^\nu} x^\mu\right)\phi(x) - x^\mu \frac{\partial}{\partial x^\nu}\phi(x) = -\delta^\mu_\nu \end{aligned} \tag{2.154}$$

or

$$[x^\mu, \partial^\nu] = -g^{\mu\nu}\,. \tag{2.155}$$

This suggests to identify

$$\mathrm{i}\partial_\mu = P_\mu \tag{2.156}$$

so that plane waves are (improper) eigenfunctions

$$P_\mu \mathrm{e}^{-\mathrm{i}xp} = \mathrm{i}\partial_\mu \mathrm{e}^{-\mathrm{i}xp} = p_\mu \mathrm{e}^{-\mathrm{i}xp} \tag{2.157}$$

and we find

$$[x^\mu, P^\nu] = -\mathrm{i}g^{\mu\nu}\,. \tag{2.158}$$

Now consider

$$M_{\mu\nu} = -x_\mu P_\nu + x_\nu P_\mu = -\mathrm{i}\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \tag{2.159}$$

and with

$$[M_{\mu\nu}, P_\rho] = [-x_\mu P_\nu + x_\nu P_\mu, P_\rho] = \mathrm{i}\left(g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu\right) \tag{2.160}$$

and

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= [-x_\mu P_\nu + x_\nu P_\mu, -x_\rho P_\sigma - x_\sigma P_\rho] \\ &= \mathrm{i}\,(g_{\nu\rho} x_\mu P_\sigma - g_{\mu\sigma} x_\rho P_\nu - g_{\nu\sigma} x_\mu P_\rho + g_{\mu\rho} x_\sigma P_\nu \\ &\qquad - g_{\mu\rho} x_\nu P_\sigma + g_{\nu\sigma} x_\mu P_\rho + g_{\mu\sigma} x_\nu P_\rho - g_{\nu\rho} x_\sigma P_\mu) \\ &= \mathrm{i}\left(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} - g_{\nu\rho} M_{\mu\sigma} + g_{\nu\sigma} M_{\mu\rho}\right) \end{aligned} \tag{2.161}$$

we find the Poincaré algebra (2.145).

Scalar Functions

In the simplest case, a function does not change under Poincaré transformations

$$\begin{aligned} U(\Lambda,a) : C^\infty(\mathbf{M}) &\to C^\infty(\mathbf{M}) \\ \phi &\mapsto \phi' = U(\Lambda,a)\phi \,. \end{aligned} \tag{2.162}$$

However we must be careful about “does not change”: do we mean

$$\phi' = U(\Lambda,a)\phi = \phi \,,$$

i. e. that the *function* does not change, or

$$\phi'(x') = (U(\Lambda,a)\phi)((\Lambda,a)x) = (U(\Lambda,a)\phi)(\Lambda x + a) = \phi(x) \,, \tag{2.163}$$

i. e. that the *value* at a fixed point in space, that must also be transformed, does not change?

From a physics perspective, that relies on functions to describe quantities at a given space-time point, only the latter interpretation makes sense. We thus require for a scalar function

$$\phi'(x) = (U(\Lambda,a)\phi)(x) = \phi\left((\Lambda,a)^{-1}x\right) = \phi\left(\Lambda^{-1}(x-a)\right) \,. \tag{2.164}$$

Note that

$$\begin{aligned} (U(\Lambda_1,a_1)U(\Lambda_2,a_2)\phi)(x) &= (U(\Lambda_2,a_2)\phi)\left((\Lambda_1,a_1)^{-1}x\right) \\ = \phi\left((\Lambda_2,a_2)^{-1}((\Lambda_1,a_1)^{-1}x)\right) &= \phi\left(((\Lambda_1,a_1)\circ(\Lambda_2,a_2))^{-1}x)\right) \\ &= (U((\Lambda_1,a_1)\circ(\Lambda_2,a_2))\phi)(x) \end{aligned} \tag{2.165}$$

as required by the group axioms. This can also be expressed as

$$\phi' = \phi \circ (\Lambda,a)^{-1} \tag{2.166}$$

and

$$\begin{aligned} U(\Lambda_1,a_1)U(\Lambda_2,a_2)\phi &= U(\Lambda_1,a_1)\left(\phi\circ(\Lambda_2,a_2)^{-1}\right) = \phi\circ(\Lambda_2,a_2)^{-1})\circ(\Lambda_1,a_1)^{-1} \\ &= \phi\circ\left((\Lambda_1,a_1)\circ(\Lambda_2,a_2)^{-1}\right) = U((\Lambda_1,a_1)\circ(\Lambda_2,a_2))\phi \,. \end{aligned} \tag{2.167}$$

We can Taylor expand (2.164) to first order

$$\begin{aligned} \phi'(x) &= \phi\left((\mathbf{1}+\omega)^{-1}(x-\epsilon)\right) + \ldots = \phi\left((\mathbf{1}-\omega)(x-\epsilon)\right) + \ldots \\ &= \phi(x - \omega x - \epsilon) + \ldots = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu\phi(x) - \epsilon^\mu\partial_\mu\phi(x) + \ldots \\ &= \phi(x) + \frac{1}{2}\omega_{\mu\nu}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\phi(x) - \epsilon^\mu\partial_\mu\phi(x) + \ldots \\ &= \phi(x) + \frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu}\phi(x) + \mathrm{i}\epsilon_\mu P^\mu\phi(x) + \ldots \end{aligned} \tag{2.168}$$

consistent with (2.156) and (2.159).

Functions with Spin

In the general case, we will have fields with several components that transform under a non-trivial representation of the Lorentz group

$$\phi(x) \xrightarrow{(\Lambda,a)} \phi'(x) = (R(\Lambda)\phi)(\big(\Lambda^{-1}(x-a)\big) \,. \tag{2.169}$$

The most important cases will be

- $\mathcal{D}^{(0,0)}$: scalar particles: Higgs bosons
- $\mathcal{D}^{(1/2,0)} \oplus \mathcal{D}^{(0,1/2)}$: Dirac spinors: electrons, quarks etc.
- $\mathcal{D}^{(1/2,1/2)}$: vector bosons: photons, etc.

In fact, it can be shown that the maximum spin for interacting fundamental fields that can be realized in Minkowski space is 1. Including gravity pushes this limit to 2.

—3—
Asymptotic States

In addition to the Casimir operators P^2 and W^2 that commute with every generator of the Poincaré group, we can simultaneously diagonalize

$$\left\{P^2, \vec{P}, W^2, W_3\right\}$$

with the eigenvalues corresponding to mass squared, three momentum, total angular momentum and projection of angular momentum on an axis (choosen to be $\vec{e}_3$ here).

For a quantum theory, we need to find irreducible unitary representations of the Poincaré group and algebra. Since there are no finite dimensional unitary representations of the non-compact Poincaré group, we immediately have to go to a suitable Hilbert space.

We will start with the simplest possible states corresponding to single non-interacting particles, from which we can construct non-interacting multi particle states.

3.1 Relativistic One Particle States

We can represent vectors in the Hilbert space for non-interacting scalar particles of mass m by (the limit points of) square integrable functions

$$\mathcal{H}_1 \ni \Psi \cong \psi \in L^2(\mathbf{R}^4, \mathbf{C}, \widetilde{\mathrm{d}p}) \tag{3.1}$$

where the integration measure[1]

$$\widetilde{\mathrm{d}p} = \left.\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p_0}\right|_{p_0=\sqrt{\vec{p}^2+m^2}} = \frac{\mathrm{d}^4p}{(2\pi)^4} 2\pi\Theta(p^0)\delta^4(p^2-m^2) \tag{3.2}$$

[1] The integration measure $\widetilde{\mathrm{d}p}$ depends on the mass, but this is usually left implicit.

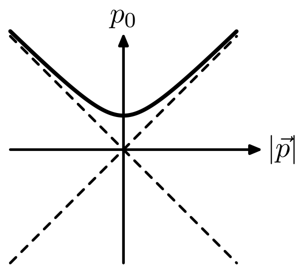


Figure 3.1: *The positive mass shell* $p_0 = +\sqrt{\vec p^2 + m^2}$ *or* $p^2 = m^2$ *and* $p_0 \geq 0$.

is invariant under LTs, since $|\det\Lambda| = 1$ and $(\Lambda p)^2 = p^2$. The inner product is then

$$\langle\Psi|\Phi\rangle = \int \widetilde{\mathrm{d}p}\, \psi^*(p)\phi(p)\,. \tag{3.3}$$

Note that only the values on the three dimensional *positive mass shell* (cf. fig. 3.1)

$$\left\{p \in \mathbf{R}^4 : p^2 = m^2\right\} \subset \mathbf{R}^4 \tag{3.4}$$

are relevant. Poincaré transformations are represented as

$$(U(\Lambda, a)\psi)(p) = \mathrm{e}^{\mathrm{i}pa}\psi(\Lambda^{-1}p)\,. \tag{3.5}$$

Introducing (improper) momentum eigenstates $|p\rangle$ with

$$P_\mu\,|p\rangle = p_\mu\,|p\rangle \tag{3.6}$$

and normalization

$$\langle p|q\rangle = (2\pi)^3 2p_0\delta(\vec p - \vec q)\,, \tag{3.7}$$

we can also write

$$|\Psi\rangle = \int \widetilde{\mathrm{d}p}\,\psi(p)\,|p\rangle \tag{3.8}$$

and

$$\psi(p) = \langle p|\Psi\rangle \tag{3.9}$$

since

$$\begin{aligned}\langle\Psi|\Phi\rangle &= \int \widetilde{\mathrm{d}p}\,\widetilde{\mathrm{d}q}\,\psi^*(p)\phi(q)\,\langle p|q\rangle \\ &= \int \widetilde{\mathrm{d}p}\,\widetilde{\mathrm{d}q}\,\psi^*(p)\phi(q)\,(2\pi)^3 2p_0\delta(\vec p - \vec q) = \int \widetilde{\mathrm{d}p}\,\psi^*(p)\phi(p)\,.\end{aligned} \tag{3.10}$$

Note that

$$\Pi_m = \int \widetilde{\mathrm{d}p}\, |p\rangle \langle p| \tag{3.11}$$

is the projection operator on one-particle states of mass m.

For the general case, it is more convenient to use the momentum eigenstates instead of the normalizable states in Hilbert space. We must allow for Quantum Numbers (QNs) α in addition to the momentum p and group them into multiplets

$$\{|p,\alpha\rangle\}_\alpha \,. \tag{3.12}$$

The action of pure translations is fixed by the exponential representation

$$U(\mathbf{1},a)\,|p,\alpha\rangle = \mathrm{e}^{\mathrm{i}a_\mu P^\mu}\,|p,\alpha\rangle = \mathrm{e}^{\mathrm{i}a_\mu p^\mu}\,|p,\alpha\rangle \,. \tag{3.13}$$

The action of LTs is more complicated. Using (2.141b), we can determine the transformation of the eigenvalues of the momentum operator

$$\begin{aligned} P^\mu U(\Lambda,0)\,|p,\alpha\rangle &= U(\Lambda,0)U^{-1}(\Lambda,0)P^\mu U(\Lambda,0)\,|p,\alpha\rangle \\ &= U(\Lambda,0)U(\Lambda^{-1},0)P^\mu U^{-1}(\Lambda^{-1},0)\,|p,\alpha\rangle \\ &= U(\Lambda,0)\left(\Lambda^{-1}\right)_\nu{}^\mu P^\nu\,|p,\alpha\rangle = \Lambda^\mu{}_\nu p^\nu U(\Lambda,0)\,|p,\alpha\rangle \end{aligned} \tag{3.14}$$

i. e. $U(\Lambda,0)\,|p,\alpha\rangle$ is an (improper) eigenvector of Λp. Therefore we must be able to expand it in $\{|\Lambda p,\alpha\rangle\}_\alpha$, i. e.

$$U(\Lambda,0)\,|p,\alpha\rangle = \sum_\beta C_{\beta\alpha}(\Lambda,p)\,|\Lambda p,\beta\rangle \tag{3.15}$$

with coefficients $C_{\beta\alpha}(\Lambda,p)$ to be determined. In general, the matrices $C_{\beta\alpha}(\Lambda,p)$ can be decomposed by a basis transformation of the $|p,\alpha\rangle$ of into a block diagonal form, corresponding to irreducible representations of the Poincaré group.

3.1.1 Little Group

Lecture 07: Tue, 14. 11. 2017

The eigenvalue m^2 of P^2 is invariant in each irreducible representation. In addition, as long as $m^2 \geq 0$, also the sign of p_0 is invariant under proper isochronous transformations[2]. In each irreducible representation, may choose

[2]For $m^2 < 0$, the sign of p_0 depends on the reference frame.

a reference momentum k and express all other momenta as a LT[3] of this momentum

$$p^\mu = \overline{\Lambda}^\mu{}_\nu(p) k^\nu \,. \tag{3.16}$$

This allows us to fix the QNs α by *defining*

$$|p,\alpha\rangle := N(p) U(\overline{\Lambda}(p),0) \,|k,\alpha\rangle \,, \tag{3.17}$$

with $N(p)$ a normalization factor to be determined later. Note that $N(p)$ and $\overline{\Lambda}(p)$ depend on k, but since the latter is supposed to be fixed for each irreducible representation, we don't write it explicitely.

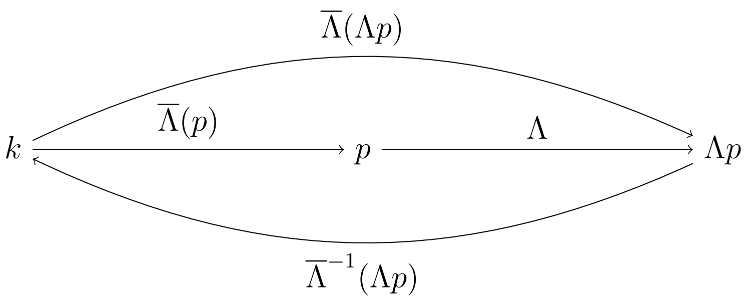
Applying an arbitrary LT Λ to this state

$$\begin{aligned} U(\Lambda,0)\,|p,\alpha\rangle &= N(p)U(\Lambda,0)U(\overline{\Lambda}(p),0)\,|k,\alpha\rangle = N(p)U(\Lambda\overline{\Lambda}(p),0)\,|k,\alpha\rangle \\ &= N(p)U(\overline{\Lambda}(\Lambda p),0)U(\overline{\Lambda}^{-1}(\Lambda p)\Lambda\overline{\Lambda}(p),0)\,|k,\alpha\rangle \,, \end{aligned} \tag{3.18}$$

we are lead to study the action of a special LT, the *Wigner rotation*

$$W(\Lambda,p) = \overline{\Lambda}^{-1}(\Lambda p)\Lambda\overline{\Lambda}(p) \tag{3.19}$$

on k:



(3.20)

Thus W is an element of the *little group*, the subgroup of LTs that leave the reference vector k invariant. Obviously

$$U(W,0)\,|k,\alpha\rangle = \sum_\beta D_{\beta\alpha}(W)\,|k,\beta\rangle \tag{3.21}$$

for each W in the little group. Note that the positioning of the indices of D is chosen to make the matrices a representation of the little group

$$\sum_\beta D_{\beta\alpha}(W'W)\,|k,\beta\rangle = U(W'W)\,|k,\alpha\rangle = U(W')U(W)\,|k,\alpha\rangle$$

[3]Note, however, that the explicit form of this LT will depend on m^2.

$$= U(W') \sum_\gamma D_{\gamma\alpha}(W) |k,\gamma\rangle = \sum_\gamma D_{\gamma\alpha}(W) U(W') |k,\gamma\rangle$$
$$= \sum_\gamma D_{\gamma\alpha}(W) \sum_\beta D_{\beta\gamma}(W') |k,\beta\rangle$$
$$= \sum_\beta \sum_\gamma D_{\beta\gamma}(W') D_{\gamma\alpha}(W) |k,\beta\rangle = \sum_\beta [D(W')D(W)]_{\beta\alpha} |k,\beta\rangle \quad (3.22)$$

i. e.

$$D(W')D(W) = D(W'W)\,. \quad (3.23)$$

Returning to (3.18), we find

$$U(\Lambda,0)|p,\alpha\rangle = N(p) U(\overline{\Lambda}(\Lambda p),0) U(W(\Lambda,p)|k,\alpha\rangle$$
$$= N(p) \sum_\beta D_{\beta\alpha}(W(\Lambda,p)) U(\overline{\Lambda}(\Lambda p),0) |k,\beta\rangle$$
$$= \frac{N(p)}{N(\Lambda p)} \sum_\beta D_{\beta\alpha}(W(\Lambda,p)) |\Lambda p,\beta\rangle\,. \quad (3.24)$$

Thus we have reduced the problem of finding the $C(\Lambda,p)$ to fixing the normalization factor $N(p)$ and to the problem of finding representations of the little group. If we choose latter D to be unitary

$$D^\dagger(W) = D^{-1}(W) = D(W^{-1})\,, \quad (3.25)$$

and use the covariant inner product

$$\langle p,\alpha|q,\beta\rangle = (2\pi)^3 2p_0 \delta_{\alpha\beta}\delta(\vec{p}-\vec{q}) = \langle \Lambda p,\alpha|\Lambda q,\beta\rangle \quad (3.26)$$

we can choose

$$N(p) = 1 \quad (3.27)$$

to guarantee unitarity of $U(\Lambda,a)$.

3.1.2 Wigner Classification

Since the little group is the invariance group of the reference momentum k, the nature of the little group depends on the Lorentz invariant properties of p and k. There are six different cases, that have been collected in table 3.1. Of these, only three appear to be realized in nature

1) massive particles,

3) massless particles,

	p		k	little group
1)	$p^2 = m^2 > 0$	$p_0 > 0$	$(m, 0, 0, 0)$	SO(3)
2)	$p^2 = m^2 > 0$	$p_0 < 0$	$(-m, 0, 0, 0)$	SO(3)
3)	$p^2 = m^2 = 0$	$p_0 = \omega > 0$	$(\omega, 0, 0, \omega)$	ISO(2)
4)	$p^2 = m^2 = 0$	$p_0 = -\omega < 0$	$(-\omega, 0, 0, \omega)$	ISO(2)
5)	$p^2 = -m^2 < 0$		$(0, 0, 0, m)$	SO(1, 2)
6)	$p = 0$		$(0, 0, 0, 0)$	SO(1, 3)

Table 3.1: *Little groups for different momenta.* SO$(1, N)$ *is the Lorentz group in* N *space and* 1 *time dimensions.* ISO(2) *is the euclidean group in two dimensions, consisting of rotations and translations in plane.*

6) the translation invariant *vacuum* state.

We do not have to consider the remaining ones

2) massive particle with unphysical negative energy,

4) massless particle with unphysical negative energy,

5) *tachyonic* particle traveling faster than the speed of light, violating causality.

Vacuum

The vacuum state is usually the unique state of lowest energy. It is space and time translation invariant, i. e. its four momentum vanishes. Note that there are situations in which there are more than one of such states, this will cause Spontaneous Symmetry Breaking (SSB), after which the vacuum state will be unique.

Massive Particles

The little group for a massive particle is SO(3), the rotation group in three dimensions. We know its representations, which are specified by a half integer l and are $2l + 1$ dimensional. As above, we can then associate a Wigner rotation to every LT

$$W(\Lambda, p) = \overline{\Lambda}^{-1}(\Lambda p)\Lambda\overline{\Lambda}(p) \qquad ((3.19))$$

and find the representation $U^{(l)}$ of the Lorentz group from the representation $D^{(l)}$ of the rotation group:

$$U^{(l)}(\Lambda, 0) |p, \sigma\rangle = \sum_{\sigma'=-l}^{l} D^{(l)}_{\sigma'\sigma}(W(\Lambda, p)) |\Lambda p, \sigma'\rangle \,. \tag{3.28}$$

One can show that the Wigner rotation of a LT that is a rotation, is in fact the *same* rotation, independent of the momentum p. This fact allows us to take over the whole angular momentum formalism from non-relativistic QM to the description of relativistic states of single massive particles.

One can compute the Wigner rotation explicitely by noting that, the LT

$$\overline{\Lambda}^{\mu}{}_{\nu}(p) = \delta^{\mu}_{\nu} - \frac{(p+k)^{\mu}(p+k)_{\nu}}{kp + m^2} + 2\frac{p^{\mu}k_{\nu}}{m^2} \tag{3.29}$$

is well defined for $k^2 = p^2 = m^2 > 0$ and satifies the condition (3.16)

$$\overline{\Lambda}^{\mu}{}_{\nu}(p)k^{\nu} = k^{\mu} - (p+k)^{\mu} + 2p^{\mu} = p^{\mu} \,. \tag{3.30}$$

Massless Particles

In the massless case, we choose $k = (\omega, 0, 0, \omega)$ as the reference vector. One can show (see the exercises), that

$$S^{\mu}{}_{\nu}(\alpha, \beta) = \begin{pmatrix} 1+\zeta & \alpha & \beta & -\zeta \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \zeta & \alpha & \beta & 1-\zeta \end{pmatrix} \quad \text{with } 2\zeta = \alpha^2 + \beta^2 \tag{3.31}$$

is a LT that transforms the unit vector $q = (1, 0, 0, 0)$ such that the inner product with k is unchanged.

$$(Sq)^{\mu}k_{\mu} = q^{\mu}k_{\mu} = \omega \,. \tag{3.32}$$

On the other hand, a Wigner rotation with $Wk = k$ must also satisfy

$$\omega = q^{\mu}k_{\mu} = (Wq)^{\mu}(Wk)_{\mu} = (Wq)^{\mu}k_{\mu} \tag{3.33a}$$

$$1 = q^2 = (Wq)^2 \,. \tag{3.33b}$$

Therefore one can find a $S(\alpha, \beta)$ that acts on q just as W acts on q. Consequently, even if $S(\alpha, \beta) \neq W$, the combined LT

$$S^{-1}(\alpha, \beta)W \tag{3.34}$$

must leave q invariant. It's easy to see that

$$S(\alpha,\beta)k = k \tag{3.35}$$

and therefore

$$S^{-1}(\alpha,\beta)Wk = k\,. \tag{3.36}$$

Any transformation that leaves both k and q invariant, must be a rotation around the $\vec e_3$-axis:

$$\left(S^{-1}(\alpha,\beta)W\right)^{\mu}{}_{\nu} = R^{\mu}{}_{\nu}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.37}$$

and we have identified the most general element of the little group

$$W(\theta,\alpha,\beta) = S(\alpha,\beta)R(\theta)\,. \tag{3.38}$$

It is another straightforward exercise to show that we have abelian subgroups

$$W(\theta,0,0)W(\theta',0,0) = W(\theta+\theta',0,0) \tag{3.39a}$$

$$W(0,\alpha,\beta)W(0,\alpha',\beta') = W(0,\alpha+\alpha',\beta+\beta')\,, \tag{3.39b}$$

one of which is even invariant

$$W(\theta,0,0)W(0,\alpha,\beta)W^{-1}(\theta,0,0) = W(0,\alpha\cos\theta-\beta\sin\theta,\alpha\sin\theta+\beta\cos\theta)\,. \tag{3.39c}$$

It's easy see that the multiplication laws (3.39) correspond to ISO(2), the Euclidean group of translations and rotations in two dimensions.

Expanding $W(\theta,\alpha,\beta)$ to foirst order, one can derive the generators of ISO(2) and their representation in this parametrization

$$U(W(\theta,\alpha,\beta)) = \mathbf{1} + \mathrm{i}\alpha A + \mathrm{i}\beta B + \mathrm{i}\theta L_3 + \ldots \tag{3.40}$$

with commutation relations

$$[A,B] = 0 \tag{3.41a}$$

$$[L_3,A] = \mathrm{i}B \tag{3.41b}$$

$$[L_3,B] = -\mathrm{i}A\,, \tag{3.41c}$$

as was to be expected from the geometrical interpretation. Since A and B commute, we can diagonalize them simultaneously.

$$A\,|k,a,b,h\rangle = a\,|k,a,b,h\rangle \tag{3.42a}$$

$$B\,|k,a,b,h\rangle = b\,|k,a,b,h\rangle \ , \tag{3.42b}$$

with h standing for the remaining QNs. However, we can use $U(W(\theta,0,0)) = \mathrm{e}^{\mathrm{i}\theta J_3}$ to rotate these eigenvalues, leading to an unphysical continuum. We are thus led to *require*

$$A\,|k,h\rangle = 0 \tag{3.43a}$$
$$B\,|k,h\rangle = 0\,, \tag{3.43b}$$

and

$$L_3\,|k,h\rangle = h\,|k,h\rangle \tag{3.43c}$$

where we have already suppressed the QNs $a = b = 0$ and written only the *helicity* h.

This way, only the angle θ, as determined in the decomposition of the Wigner rotation in (3.38) appears in the representation of the Wigner rotation, which is diagonal in the helicity

$$D_{\beta\alpha}(W(\theta,\alpha,\beta)) = \mathrm{e}^{\mathrm{i}h\theta}\delta_{\beta\alpha} \tag{3.44}$$

resulting in the representation of LTs

$$U(\Lambda)\,|p,h\rangle = \mathrm{e}^{\mathrm{i}h\theta(\Lambda,p)}\,|\Lambda p,h\rangle \ . \tag{3.45}$$

It follows, that the helicity of an helicity eigenstate is invariant under LTs. However, since the LT adds a phase, superpositions of states with different helicities are *not* invariant.

From the algebraic treatment alone, the helicity could be any real number. However, since rotations by 4π must leave the state invariant, it is restricted to half integer values.

Lecture 08: Wed, 15. 11. 2017

The physical interpretation of the helicity h is, of course, the projection of the angular momentum onto the momentum $\vec{p}$. This is trivially invariant rotations. It is also invariant under Lorentz boosts for massless particles, but can change for massive particles, because an observer can move faster than a massive particle.

3.1.3 Parity and Time Reversal

The parity, a. k. a. space inversion, and time reversal operations (2.113b) and (2.113b) are *not* exact symmetries of nature. Experiments have demonstrated small violations in certain processes involving *weak interactions*, while strong and electromagnetic interactions appear to be invariant.

In order to be able to study these effects systematically, we start we assuming that parity and time reversal are symmetries. Later we can add interactions that violate them in a controlled fashion. Then we must have

$$U(P,0)U(\Lambda,a)U^{-1}(P,0) = U(P\Lambda P^{-1},Pa) \tag{3.46a}$$
$$U(T,0)U(\Lambda,a)U^{-1}(T,0) = U(T\Lambda T^{-1},Ta)\,, \tag{3.46b}$$

but any of $U(P,0)$ and $U(T,0)$ could be unitary or anti-unitary. Note that operators continuously connected with unity must not be anti-unitary.

As in section 2.6.3 above, we can derive the transformation properties of the Lorentz generators[4]

$$U(P,0)\mathrm{i}M^{\mu\nu}U^{-1}(P,0) = \mathrm{i}P_\rho{}^\mu P_\sigma{}^\nu M^{\rho\sigma} \tag{3.47a}$$
$$U(P,0)\mathrm{i}P^\mu U^{-1}(P,0) = \mathrm{i}P_\rho{}^\mu P^\rho \tag{3.47b}$$
$$U(T,0)\mathrm{i}M^{\mu\nu}U^{-1}(T,0) = \mathrm{i}T_\rho{}^\mu T_\sigma{}^\nu M^{\rho\sigma} \tag{3.47c}$$
$$U(T,0)\mathrm{i}P^\mu U^{-1}(T,0) = \mathrm{i}T_\rho{}^\mu P^\rho\,, \tag{3.47d}$$

but we must not cancel the factors of i, because $U(P,0)$ and $U(T,0)$ might be anti-unitary. If we were to assume that $U(P,0)$ is anti-unitary, we find for the Hamiltonian $H = P^0$

$$\begin{aligned} H &= -\mathrm{i}U(P,0)\mathrm{i}HU^{-1}(P,0) \\ &= (-\mathrm{i})^2U(P,0)HU^{-1}(P,0) = -U(P,0)HU^{-1}(P,0)\,. \end{aligned} \tag{3.48}$$

This would imply that an energy eigenstate $|E\rangle$

$$H\,|E\rangle = E\,|E\rangle \tag{3.49}$$

must have an associated state

$$|-E\rangle = U(P,0)\,|E\rangle \tag{3.50}$$

with negative energy

$$H\,|-E\rangle = -E\,|-E\rangle\ . \tag{3.51}$$

[4]Hopefully, there is no chance to confuse the momentum P^μ with the parity transformation $P^\mu{}_\nu$.

Since the energy can not be bounded from above, this would imply that, energy would also not be bounded from below. This unphysical choice must be rejected and $U(P,0)$ must be unitary.

Similarly, assuming that $U(T,0)$ is unitary

$$\begin{aligned} H &= +\mathrm{i}U(T,0)\mathrm{i}HU^{-1}(T,0) \\ &= \mathrm{i}^2U(T,0)HU^{-1}(T,0) = -U(T,0)HU^{-1}(T,0)\,, \end{aligned} \tag{3.52}$$

leads to the same unboundedness from below and we conclude that $U(T,0)$ is anti-unitary.

Massive Particles

A simultaneous eigenstate of Hamiltonian, momentum and one component of angular momentum corresponding to the reference momentum k

$$H\,|k,\sigma\rangle = m\,|k,\sigma\rangle \tag{3.53a}$$

$$\vec{P}\,|k,\sigma\rangle = 0 \tag{3.53b}$$

$$L_3\,|k,\sigma\rangle = \sigma\,|k,\sigma\rangle \tag{3.53c}$$

is transformed by $U(P,0)$ into another eigenstate with the same eigenvalues. Since $U(P,0)$ commutes with all three operators,

$$U(P,0)\,|k,\sigma\rangle = \eta_\sigma\,|k,\sigma\rangle \tag{3.54}$$

with phases $|\eta_\sigma| = 1$ that could depend on σ. However also the shift operators $L_\pm = L_1 \pm \mathrm{i}L_2$ commute with $U(P,0)$ and the phase η_σ must not depend on σ

$$U(P,0)\,|k,\sigma\rangle = \eta\,|k,\sigma\rangle \tag{3.55}$$

and the phase $|\eta| = 1$ can only depend on the representation or additional internal QNs. Finally, for general momenta p

$$U(P,0)\,|p,\sigma\rangle = \eta\,|Pp,\sigma\rangle\ . \tag{3.56}$$

From

$$U(T,0)L_3U^{-1}(T,0) = -L_3 \tag{3.57}$$

we conclude

$$HU(T,0)\,|k,\sigma\rangle = mU(T,0)\,|k,\sigma\rangle \tag{3.58a}$$

$$\vec{P}U(T,0)\,|k,\sigma\rangle = 0 \tag{3.58b}$$

$$L_3U(T,0)\,|k,\sigma\rangle = -\sigma U(T,0)\,|k,\sigma\rangle\ , \tag{3.58c}$$

i. e.

$$U(T,0)\,|k,\sigma\rangle = \zeta_\sigma\,|k,-\sigma\rangle \ , \tag{3.59}$$

with another phase $|\zeta_\sigma| = 1$. One can again use the shift operators[5] to show that the phase can be absorbed into the state and obtain

$$U(T,0)\,|p,\sigma\rangle = (-1)^{l-\sigma}\,|Pk,-\sigma\rangle \ . \tag{3.60}$$

Massless Particles

One can show [6] that

$$U(P,0)\,|p,\sigma\rangle = \eta_\sigma \mathrm{e}^{-\mathrm{i}\pi\sigma\Theta(p^2)}\,|Pp,-\sigma\rangle \tag{3.61a}$$

$$U(T,0)\,|p,\sigma\rangle = \zeta_\sigma \mathrm{e}^{\mathrm{i}\pi\sigma\Theta(p^2)}\,|Pp,\sigma\rangle \ , \tag{3.61b}$$

but it should be intuitively clear that the angular momentum changes sign, but the helicity not.

3.2 Relativistic Many-Particle States

Multi particle states can be constructed as tensor products, e. g. for two particles

$$|p_1,\alpha_1;p_2,\alpha_2\rangle = |p_1,\alpha_1\rangle \otimes |p_2,\alpha_2\rangle \ . \tag{3.62}$$

Normalizable states are represented as square integrable functions on the product of two positive mass shells

$$\mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_1 \ni \Psi \cong \psi \in L^2(\mathbf{R}^4 \times \mathbf{R}^4, \mathbf{C}, \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2})\,, \tag{3.63}$$

where we must allow for mass shells with different masses. The generalization to N-particle states is obvious.

3.2.1 Fermions and Bosons

In the case of identical particles however, observations tell us that not all tensor products are allowed physical states. Instead, there are only *totally* symmetric states of identical (up to Poincaré group QNs) *bosons* and *totally* antisymmetric states of identical (up to Poincaré group QNs) *fermions*. Furthermore, one can show[7] that particles with integer spins *must* be bosons and particles with half-integer spin *must* be fermions to avoid contradictions.

[5] Cf., e. g., [3], p. 77f.
[6] Cf., e. g., [3], p. 78f.
[7] Cf., e. g., [7] for rigorous proof relying only on well motivated general axioms, not PT.

One can easily define totally symmetric and anti-symmetric tensor products for two

$$\mathcal{H}_{+,2} \ni |\psi_1\rangle \vee |\psi_2\rangle = |\psi_1\rangle \otimes_S |\psi_2\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle \right) \tag{3.64a}$$

$$\mathcal{H}_{-,2} \ni |\psi_1\rangle \wedge |\psi_2\rangle = |\psi_1\rangle \otimes_A |\psi_2\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle \otimes |\psi_2\rangle - |\psi_2\rangle \otimes |\psi_1\rangle \right) \tag{3.64b}$$

and n identical (up to Poincaré group QNs) particles

$$\mathcal{H}_{+,n} = \mathcal{H}_1^{\vee n} \ni \bigvee_{i=1}^{n} |\psi_i\rangle = \frac{1}{\sqrt{n!}} \sum_\pi \bigotimes_{i=1}^{n} |\psi_{\pi(i)}\rangle \tag{3.65a}$$

$$\mathcal{H}_{-,n} = \mathcal{H}_1^{\wedge n} \ni \bigwedge_{i=1}^{n} |\psi_i\rangle = \frac{1}{\sqrt{n!}} \sum_\pi \epsilon(\pi) \bigotimes_{i=1}^{n} |\psi_{\pi(i)}\rangle \,, \tag{3.65b}$$

where

$$\epsilon(\pi) = \begin{cases} +1 & \pi \text{ is an even permutation} \\ -1 & \pi \text{ is an odd permutation} \end{cases} . \tag{3.66}$$

While these are mathematically well defined, computations using these states can be very tedious. Instead it is more efficient to move to the direct sum of all n-particle spaces.

3.2.2 Fock Space

Introducing a unique Poincaré invariant vacuum state $|0\rangle$

$$M^{\mu\nu} |0\rangle = 0 \tag{3.67a}$$

$$P^{\mu\nu} |0\rangle = 0 \tag{3.67b}$$

$$\langle 0|0\rangle = 1 \tag{3.67c}$$

and the one dimensionsal Hilbert space $\mathcal{H}_0 = \mathcal{H}_{+,0} = \mathcal{H}_{-,0} \ni |0\rangle$, we can form the direct sum of all n particle Hilbert spaces

$$\mathcal{F}_\pm = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\pm,n} \,, \tag{3.68}$$

the so called *Fock space*. If there is more than one species of particles, the n-particle spaces will be formed from a combination of unsymmetrical,

symmetrical and antisymmetrical tensor products. This can become very complicated and it is more convenient to give a recursive description, using operators

$$a_\alpha^\dagger : \mathcal{H}_{\pm,n} \to \mathcal{H}_{\pm,n+1} \tag{3.69a}$$
$$a_\alpha : \mathcal{H}_{\pm,n} \to \mathcal{H}_{\pm,n-1} \tag{3.69b}$$

to construct $\mathcal{F}_\pm$ from $\mathcal{H}_{\pm,0}$.

Bosonic

Starting with the bosonic case, we demand Canonical Commutation Relations (CCRs) among the *creation* and *annihilation* operators

$$\left[a_\alpha(k), a_\beta(p)\right] = 0 \tag{3.70a}$$
$$\left[a_\alpha^\dagger(k), a_\beta^\dagger(p)\right] = 0 \tag{3.70b}$$
$$\left[a_\alpha(k), a_\beta^\dagger(p)\right] = (2\pi)^3 2k_0 \delta_{\alpha\beta}\delta^3(\vec{k}-\vec{p}) \tag{3.70c}$$

with the *commutator*

$$[A,B] = [A,B]_- = AB - BA \tag{3.71}$$

and since $a_\alpha \mathcal{H}_{\pm,0}$ lies outside of Fock space

$$\forall p,\alpha : a_\alpha(p)\left|0\right\rangle = 0\,. \tag{3.72}$$

We can obtain all of $\mathcal{H}_{+,1}$ by acting with $a_\alpha^\dagger(p)$ on the vacuum state

$$\left|p,\alpha\right\rangle = a_\alpha^\dagger(p)\left|0\right\rangle\ . \tag{3.73}$$

Note that the normalization of these states is already determined by the CCRs

$$\begin{aligned}\langle p,\alpha|q,\beta\rangle &= \langle 0|a_\alpha(p)a_\beta^\dagger(q)|0\rangle = \langle 0|a_\beta^\dagger(q)a_\alpha(p)|0\rangle + \left\langle 0\middle|\left[a_\alpha(p),a_\beta^\dagger(q)\right]\middle|0\right\rangle \\ &= 0 + (2\pi)^3 2p_0\delta_{\alpha\beta}\delta^3(\vec{p}-\vec{q})\left\langle 0|0\right\rangle = (2\pi)^3 2p_0\delta_{\alpha\beta}\delta^3(\vec{p}-\vec{q})\end{aligned} \tag{3.74}$$

and turns out to be Lorentz invariant. The states in $\mathcal{H}_{+,2}$ are constructed by one more application of $a_\alpha^\dagger(p)$

$$\begin{aligned}\left|p_1,\alpha_1;p_2,\alpha_2\right\rangle &= a_{\alpha_1}^\dagger(p_1)\left|p_2,\alpha_2\right\rangle = a_{\alpha_1}^\dagger(p_1)a_{\alpha_2}^\dagger(p_2)\left|0\right\rangle \\ &= a_{\alpha_2}^\dagger(p_2)a_{\alpha_1}^\dagger(p_1)\left|0\right\rangle = a_{\alpha_2}^\dagger(p_2)\left|p_1,\alpha_1\right\rangle \\ &= \left|p_2,\alpha_2;p_1,\alpha_1\right\rangle = \left|p_1,\alpha_1\right\rangle \vee \left|p_2,\alpha_2\right\rangle\end{aligned} \tag{3.75}$$

and the normalization can again be verified from the CCRs. At this point, one might worry that the operators $a^\dagger_\alpha(p)$ do not create normalizable states, but this can be solved by using CCRs

$$\left[a_\alpha(f), a_\beta(g)\right] = 0 \tag{3.76a}$$

$$\left[a^\dagger_\alpha(f), a^\dagger_\beta(g)\right] = 0 \tag{3.76b}$$

$$\left[a_\alpha(f), a^\dagger_\beta(g)\right] = \delta_{\alpha\beta} \int \widetilde{\mathrm{d}p}\, f^*(p) g(p) \tag{3.76c}$$

with functions f square integrable on the positive mass shell and creation and annihilation operators formally defined as

$$a_\alpha(f) = \int \widetilde{\mathrm{d}p}\, f^*(p) a_\alpha(p) \tag{3.77a}$$

$$a^\dagger_\alpha(f) = \int \widetilde{\mathrm{d}p}\, f(p) a_\alpha(p)\,. \tag{3.77b}$$

This way, everything can be made rigorous, but at the price or serious tedium in applications. We will therefore use the δ-function normalized states for convenience.

The action of $a_\alpha : \mathcal{H}_{\pm,1} \to \mathcal{H}_{\pm,0}$ is already determined by the CCRs

$$a_\alpha(p) \left|q,\beta\right\rangle = a_\alpha(p) a^\dagger_\beta(q) \left|0\right\rangle = \left[a_\alpha(p), a^\dagger_\beta(q)\right] \left|0\right\rangle \\ = (2\pi)^3 2p_0 \delta_{\alpha\beta} \delta^3(\vec p - \vec q) \left|0\right\rangle \tag{3.78}$$

also $a_\alpha : \mathcal{H}_{\pm,2} \to \mathcal{H}_{\pm,1}$

$$a_\alpha(p) \left|q_1,\beta_1; q_2,\beta_2\right\rangle \\ = (2\pi)^3 2p_0 \left(\delta_{\alpha\beta_1} \delta^3(\vec p - \vec q_1) \left|q_2,\beta_2\right\rangle - \delta_{\alpha\beta_2} \delta^3(\vec p - \vec q_2) \left|q_1,\beta_1\right\rangle\right) \tag{3.79}$$

and the general case $a_\alpha : \mathcal{H}_{\pm,n} \to \mathcal{H}_{\pm,n-1}$ should be obvious. It's easy to see that $a^\dagger_\alpha(k)$ is indeed the operator adjoint to $a_\alpha(k)$. The δ-distributions could be avoided by using (3.76) instead of (3.70).

An interesting operator $N : \mathcal{F}_+ \to \mathcal{F}_+$ is the *number operator*

$$N = \sum_\alpha \int \widetilde{\mathrm{d}p}\, a^\dagger_\alpha(p) a_\alpha(p) \tag{3.80}$$

with

$$\forall \psi \in \mathcal{H}_{+,n} : N \left|\psi\right\rangle = n \left|\psi\right\rangle\,. \tag{3.81}$$

Fermionic

The symmetry of the states in the Fock space constructed with $a^\dagger_\alpha(p)$ is an unavoidable consequence of the CCRs (3.70). In order to describe fermions, we therefore have to switch to Canonical Anticommutation Relations (CARs)

$$\left[c_\alpha(k), c_\beta(p)\right]_+ = 0 \tag{3.82a}$$
$$\left[c^\dagger_\alpha(k), c^\dagger_\beta(p)\right]_+ = 0 \tag{3.82b}$$
$$\left[c_\alpha(k), c^\dagger_\beta(p)\right]_+ = (2\pi)^3 2k_0 \delta_{\alpha\beta}\delta^3(\vec{k}-\vec{p}) \tag{3.82c}$$

with the *anti commutator*

$$[A,B]_+ = [B,A]_+ = AB + BA \tag{3.83}$$

to construct fermionic creation and annihilation operators

$$c^\dagger_\alpha : \mathcal{H}_{-,n} \to \mathcal{H}_{-,n+1} \tag{3.84a}$$
$$c_\alpha : \mathcal{H}_{-,n} \to \mathcal{H}_{-,n-1} \,. \tag{3.84b}$$

Everything goes through as in the case of bosons, except for additional signs that have to be taken care of.

3.2.3 Poincaré Transformations

Lecture 09: Tue, 21. 11. 2017

We can now use the transformation properties of the states $|p,\alpha\rangle$ under Poincaré transformations to derive the transformation properties of the creation and annihilation operators.

In general, the creation operators transform under similarity transformations just like the states. The transformation properties of the annihilation operators are different unless the coefficients are real, because we have to take the adjoint. In particular

$$\begin{aligned}\sum_\beta C_{\beta\alpha}(\Lambda,p) a^\dagger_\beta(\Lambda p)\,|0\rangle \\ &= \sum_\beta C_{\beta\alpha}(\Lambda,p)\,|\Lambda p,\beta\rangle = U(\Lambda,0)\,|p,\alpha\rangle = U(\Lambda,0)a^\dagger_\alpha(p)\,|0\rangle \\ &= U(\Lambda,0)a^\dagger_\alpha(p)U^{-1}(\Lambda,0)U(\Lambda,0)\,|0\rangle = U(\Lambda,0)a^\dagger_\alpha(p)U^{-1}(\Lambda,0)\,|0\rangle\end{aligned} \tag{3.85}$$

i. e.

$$U(\Lambda,0)a^\dagger_\alpha(p)U^{-1}(\Lambda,0) = \sum_\beta C_{\beta\alpha}(\Lambda,p)a^\dagger_\beta(\Lambda p) = \sum_\beta C^*_{\alpha\beta}(\Lambda^{-1},p)a^\dagger_\beta(\Lambda p)\,, \tag{3.86}$$

using

$$C_{\beta\alpha}(\Lambda,p) = \left(C^{-1}\right)_{\beta\alpha}(\Lambda^{-1},p) = \left(C^\dagger\right)_{\beta\alpha}(\Lambda^{-1},p) = C^*_{\alpha\beta}(\Lambda^{-1},p)\,. \tag{3.87}$$

Taking the adjoint, we find

$$U(\Lambda,0)a_\alpha(p)U^{-1}(\Lambda,0) = \sum_\beta C^*_{\beta\alpha}(\Lambda,p)a_\beta(\Lambda p) = \sum_\beta C_{\alpha\beta}(\Lambda^{-1},p)a_\beta(\Lambda p)\,. \tag{3.88}$$

Also

$$\begin{aligned} \mathrm{e}^{\mathrm{i}xp}a^\dagger_\alpha(p)\,|0\rangle &= \mathrm{e}^{\mathrm{i}xp}\,|p,\alpha\rangle = U(0,x)\,|p,\alpha\rangle = U(0,x)a^\dagger_\alpha(p)\,|0\rangle \\ &= U(0,x)a^\dagger_\alpha(p)U^{-1}(0,x)U(0,x)\,|0\rangle = U(0,x)a^\dagger_\alpha(p)U^{-1}(0,x)\,|0\rangle \end{aligned} \tag{3.89}$$

and thus

$$U(0,x)a^\dagger_\alpha(p)U^{-1}(0,x) = \mathrm{e}^{\mathrm{i}xp}a^\dagger_\alpha(p) \tag{3.90a}$$

$$U(0,x)a_\alpha(p)U^{-1}(0,x) = \mathrm{e}^{-\mathrm{i}xp}a_\alpha(p)\,. \tag{3.90b}$$

Furthermore[8]

$$U(\mathbf{1},x)U(\Lambda,0) = U(\Lambda,x) \tag{3.91}$$

and thus finally

$$U(\Lambda,x)a_\alpha(p)U^{-1}(\Lambda,x) = \mathrm{e}^{-\mathrm{i}x(\Lambda p)}\sum_\beta C_{\alpha\beta}(\Lambda^{-1},p)a_\beta(\Lambda p) \tag{3.92a}$$

$$U(\Lambda,x)a^\dagger_\alpha(p)U^{-1}(\Lambda,x) = \mathrm{e}^{\mathrm{i}x(\Lambda p)}\sum_\beta C^*_{\alpha\beta}(\Lambda^{-1},p)a^\dagger_\beta(\Lambda p) \tag{3.92b}$$

[8] But $U(\Lambda,0)U(\mathbf{1},x) = U(\Lambda,\Lambda x)$!

—4— Free Quantum Fields

In chapter 6, we will construct the S-matrix, the unitary operator that transforms non-interacting incoming states into outgoing states. The quantum mechanical transition amplitude is then

$$A(\text{incoming} \to \text{outgoing}) = \langle \text{outgoing}|S|\text{incoming}\rangle \tag{4.1}$$

and the transition probability is its modulus squared

$$P(\text{incoming} \to \text{outgoing}) \propto |A(\text{incoming} \to \text{outgoing})|^2 , \tag{4.2}$$

up to normalization factors that are independent of the interaction.

We will be able to show that the S-matrix is Poincaré invariant if the interaction is described by an interaction density

$$H(t) = H_0 - \int_{x_0=t} \mathrm{d}^3\vec{x}\, \mathcal{L}_I(x) \tag{4.3}$$

that is both local

$$[\mathcal{L}_I(x), \mathcal{L}_I(x')]_- = 0 \qquad \text{if } (x-x')^2 < 0 \tag{4.4a}$$

and a scalar under Poincaré transformations

$$U(\Lambda, a)\mathcal{L}_I(x)U^{-1}(\Lambda, a) = \mathcal{L}_I(\Lambda x + a) . \tag{4.4b}$$

The *free hamiltonian* H_0 in (4.3) is absorbed into the time depence of the creation and annihilation operators in the interaction picture of QM (cf. chapter **??**).

We should be able to construct the interaction density $\mathcal{L}_I(x)$ as an operator in Fock space out of creation and annihilation operators. Doing so will require us to switch from the momenta p that label the creation and annihilation operators to positions x via some kind of Fourier transform.

Unfortunately, the transformation properties (3.92) of the creation and annihilation operators under LTs depend on the momentum and the LTs of the Fourier transforms can be very complicated.

Fortunately, we will be able to construct local quantum fields that have simple transformation properties and are already local for bosons and fermions

$$U(\Lambda,a)\phi_\alpha(x)U^{-1}(\Lambda,a) = \sum_\beta D_{\alpha\beta}(\Lambda^{-1})\phi_\beta(\Lambda x + a) \tag{4.5a}$$

$$[\phi_\alpha(x),\phi_\beta(x')]_\mp = 0 \qquad \text{if } (x-x')^2 < 0\,, \tag{4.5b}$$

where the upper sign in the commutator applies to bosons and the lower to fermions. For the U to form a representation of the Poincaré group

$$\begin{aligned}
U(\Lambda_1\Lambda_2,\Lambda_1 a_2 + a_1)&\phi_\alpha(x)U^{-1}(\Lambda_1\Lambda_2,\Lambda_1 a_2 + a_1) \\
&= U(\Lambda_1,a_1)U(\Lambda_2,a_2)\phi_\alpha(x)U^{-1}(\Lambda_2,a_2)U^{-1}(\Lambda_1,a_1) \\
&= \sum_\beta D_{\alpha\beta}(\Lambda_2^{-1})U(\Lambda_1,a_1)\phi_\beta(\Lambda_2 x + a_2)U^{-1}(\Lambda_1,a_1) \\
&= \sum_{\beta,\gamma} D_{\alpha\beta}(\Lambda_2^{-1})D_{\beta\gamma}(\Lambda_1^{-1})\phi_\gamma(\Lambda_1(\Lambda_2 x + a_2) + a_1) \\
&\stackrel{!}{=} \sum_\gamma D_{\alpha\gamma}((\Lambda_1\Lambda_2)^{-1})\phi_\gamma(\Lambda_1\Lambda_2 x + \Lambda_1 a_2 + a_1)\,,
\end{aligned} \tag{4.6}$$

the $D_{\alpha\beta}$ must also form a representation of the Lorentz group

$$\sum_\beta D_{\alpha\beta}(\Lambda_2^{-1})D_{\beta\gamma}(\Lambda_1^{-1}) \stackrel{!}{=} D_{\alpha\gamma}((\Lambda_1\Lambda_2)^{-1}) = D_{\alpha\gamma}(\Lambda_2^{-1}\Lambda_1^{-1})\,. \tag{4.7}$$

If the interactions are local polynomials in these fields with appropriately contracted indices

$$\mathcal{L}_I(x) = \sum_{\alpha,\beta,\gamma,\ldots} C_{\alpha,\beta,\gamma,\ldots}\phi_\alpha(x)\psi_\beta(x)\psi_\gamma(x)\cdots \tag{4.8}$$

locality and Lorentz invariance will be guaranteed by construction (cf. exercise).

As we will see later, real scalar fields with these properties and $D^\phi = 1$ can be written as

$$\phi(x) = \int\widetilde{\mathrm{d}p}\,\left(a(p)\mathrm{e}^{-\mathrm{i}xp} + a^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right) : \mathcal{F}_+ \to \mathcal{F}_+ \tag{4.9}$$

and we will also see that locality requires the fields to always have both a creation and an annihilation part, but not necessarily corresponding to the same QNs.

4.1 Massive Particles with Arbitrary Spin

Before we construct the local fields, we treat the "positive energy" annihilation and "negative energy" creation parts separately[1]

$$\phi^{(+)}_\alpha(x) = \sum_\sigma \int \widetilde{\mathrm{d}p}\, u^\sigma_\alpha(x,p) a_\sigma(p) : \mathcal{H}_{\pm,n} \to \mathcal{H}_{\pm,n-1} \tag{4.10a}$$

$$\phi^{(-)}_\alpha(x) = \sum_\sigma \int \widetilde{\mathrm{d}p}\, v^\sigma_\alpha(x,p) a^\dagger_\sigma(p) : \mathcal{H}_{\pm,n} \to \mathcal{H}_{\pm,n+1}\,, \tag{4.10b}$$

but it should be obvious, that we need *both* in the interaction in order to obtain a self-adjoint Hamiltonian.

All formulae will be written as if for bosons, but they can be taken over unchanged for fermions. The indices σ enumerate states in the Poincaré group representations discussed in section 3.1, while the indices α enumerate members in multiplets forming Lorentz group representations (4.7). The fields $\phi^\pm$, functions u, v and operators a, $a^\dagger$ carry additional QNs that distinguish different particles of different mass, spin and charges. These will not be spelled out, but it is obvious that the u and v must depend on the mass and spin of the Poincaré group representation.

Thus we need to find functions

$$u^\sigma_\alpha(x,p)\,,\ v^\sigma_\alpha(x,p) \tag{4.11}$$

that yield the desired transformation properties

$$U(\Lambda,b)\phi^{(\pm)}_\alpha(x)U^{-1}(\Lambda,b) = \sum_\beta D_{\alpha\beta}(\Lambda^{-1})\phi^{(\pm)}_\beta(\Lambda x + b) \tag{4.12}$$

with D *independent* of $x \in \mathbf{M}$. It will be shown below, that we can always choose the u and v such that the same D appears in the positive and negative energy parts. We will assume that $p^2 = m^2 > 0$ and treat the massless case later.

Obviously,

$$\left[\phi^{(+)}_\alpha(x), \phi^{(+)}_\beta(y)\right]_\pm = 0 = \left[\phi^{(-)}_\alpha(x), \phi^{(-)}_\beta(y)\right]_\pm \tag{4.13}$$

from the CCRs or CARs. On the other hand,

$$\left[\phi^{(+)}_\alpha(x), \phi^{(-)}_\beta(y)\right]_\pm = \sum_\sigma \int \widetilde{\mathrm{d}p}\, u^\sigma_\alpha(x,p) v^\sigma_\beta(y,p) \neq 0 \tag{4.14}$$

[1] The rationale for the postive/negative energy terminology and notation will become clear below.

and we have to choose the u and v appropriately to get combinations that (anti-)commute for space-like distances.

For massive particles, we can use the Wigner rotations to rewrite (3.92) as

$$U(\Lambda,b)a_\sigma(p)U^{-1}(\Lambda,b) = \mathrm{e}^{-\mathrm{i}b(\Lambda p)}\sum_{\sigma'} R_{\sigma\sigma'}(W^{-1}(\Lambda,p))a_{\sigma'}(\Lambda p) \tag{4.15a}$$

$$U(\Lambda,b)a^\dagger_\alpha(p)U^{-1}(\Lambda,b) = \mathrm{e}^{\mathrm{i}b(\Lambda p)}\sum_{\sigma'} R^*_{\sigma\sigma'}(W^{-1}(\Lambda,p))a^\dagger_{\sigma'}(\Lambda p)\,, \tag{4.15b}$$

where R is the representation of the rotation group corresponding to the spin of the particles created by $a^\dagger$. From this, we obtain

$$\begin{aligned}
U(\Lambda,b)&\phi^{(+)}_\alpha(x)U^{-1}(\Lambda,b)\\
&= \sum_{\sigma\sigma'}\int\widetilde{\mathrm{d}p}\, u^\sigma_\alpha(x,p)\mathrm{e}^{-\mathrm{i}b(\Lambda p)}R_{\sigma\sigma'}(W^{-1}(\Lambda,p))a_{\sigma'}(\Lambda p)\\
&\stackrel{!}{=} \sum_{\beta\sigma'}\int\widetilde{\mathrm{d}p}\, D_{\alpha\beta}(\Lambda^{-1})u^{\sigma'}_\beta(\Lambda x+b,p)a_{\sigma'}(p)\\
&= \sum_{\beta\sigma'}\int\widetilde{\mathrm{d}p}\, D_{\alpha\beta}(\Lambda^{-1})u^{\sigma'}_\beta(\Lambda x+b,\Lambda p)a_{\sigma'}(\Lambda p)
\end{aligned} \tag{4.16a}$$

and

$$\begin{aligned}
U(\Lambda,b)&\phi^{(-)}_\alpha(x)U^{-1}(\Lambda,b)\\
&= \sum_{\sigma\sigma'}\int\widetilde{\mathrm{d}p}\, v^\sigma_\alpha(x,p)\mathrm{e}^{\mathrm{i}b(\Lambda p)}R^*_{\sigma\sigma'}(W^{-1}(\Lambda,p))a^\dagger_{\sigma'}(\Lambda p)\\
&\stackrel{!}{=} \sum_{\beta\sigma'}\int\widetilde{\mathrm{d}p}\, D_{\alpha\beta}(\Lambda^{-1})v^{\sigma'}_\beta(\Lambda x+b,p)a^\dagger_{\sigma'}(p)\\
&= \sum_{\beta\sigma'}\int\widetilde{\mathrm{d}p}\, D_{\alpha\beta}(\Lambda^{-1})v^{\sigma'}_\beta(\Lambda x+b,\Lambda p)a^\dagger_{\sigma'}(\Lambda p)\,,
\end{aligned} \tag{4.16b}$$

where we have used

$$\widetilde{\mathrm{d}p} = \widetilde{\mathrm{d}(\Lambda p)}\,. \tag{4.17}$$

Comparing coefficients, we find

$$\sum_\sigma u^\sigma_\alpha(x,p)\mathrm{e}^{-\mathrm{i}b(\Lambda p)}R_{\sigma\sigma'}(W^{-1}(\Lambda,p)) = \sum_\beta D_{\alpha\beta}(\Lambda^{-1})u^{\sigma'}_\beta(\Lambda x+b,\Lambda p) \tag{4.18a}$$

$$\sum_\sigma v^\sigma_\alpha(x,p)\mathrm{e}^{\mathrm{i}b(\Lambda p)}R^*_{\sigma\sigma'}(W^{-1}(\Lambda,p)) = \sum_\beta D_{\alpha\beta}(\Lambda^{-1})v^{\sigma'}_\beta(\Lambda x+b,\Lambda p) \tag{4.18b}$$

or getting rid of the inverse transformations

$$\sum_\beta D_{\alpha\beta}(\Lambda)u^\sigma_\beta(x,p)\mathrm{e}^{-\mathrm{i}b(\Lambda p)} = \sum_{\sigma'} u^{\sigma'}_\alpha(\Lambda x+b,\Lambda p)R_{\sigma'\sigma}(W(\Lambda,p)) \tag{4.19a}$$

$$\sum_\beta D_{\alpha\beta}(\Lambda)v^\sigma_\beta(x,p)\mathrm{e}^{\mathrm{i}b(\Lambda p)} = \sum_{\sigma'} v^{\sigma'}_\beta(\Lambda x+b,\Lambda p)R^*_{\sigma'\sigma}(W(\Lambda,p))\,. \tag{4.19b}$$

We can now proceed to solve (4.19). First the special case $\Lambda = \mathbf{1}$:

$$u^\sigma_\alpha(x,p)\mathrm{e}^{-\mathrm{i}bp} = u^\sigma_\alpha(x+b,p) \tag{4.20a}$$

$$v^\sigma_\alpha(x,p)\mathrm{e}^{\mathrm{i}bp} = v^\sigma_\alpha(x+b,p)\,, \tag{4.20b}$$

i. e. u and v must be plane waves with momentum and spin dependent coefficients

$$u^\sigma_\alpha(x,p) = \mathrm{e}^{-\mathrm{i}xp}u^\sigma_\alpha(p) \tag{4.21a}$$

$$v^\sigma_\alpha(x,p) = \mathrm{e}^{\mathrm{i}xp}v^\sigma_\alpha(p)\,, \tag{4.21b}$$

where we reuse the symbols u and v, since there is no risk of confusion. The conditions (4.19) now read

$$\sum_\beta D_{\alpha\beta}(\Lambda)u^\sigma_\beta(p)\mathrm{e}^{-\mathrm{i}xp-\mathrm{i}b(\Lambda p)} = \sum_{\sigma'} u^{\sigma'}_\alpha(\Lambda p)\mathrm{e}^{-\mathrm{i}(\Lambda x+b)(\Lambda p)}R_{\sigma'\sigma}(W(\Lambda,p)) \tag{4.22a}$$

$$\sum_\beta D_{\alpha\beta}(\Lambda)v^\sigma_\beta(x,p)\mathrm{e}^{\mathrm{i}xp+\mathrm{i}b(\Lambda p)} = \sum_{\sigma'} v^{\sigma'}_\beta(\Lambda p)\mathrm{e}^{\mathrm{i}(\Lambda x+b)(\Lambda p)}R^*_{\sigma'\sigma}(W(\Lambda,p))\,. \tag{4.22b}$$

and using

$$(\Lambda x+b)(\Lambda p) = xp + b(\Lambda p)\,, \tag{4.23}$$

we see that the exponentials cancel

$$\sum_\beta D_{\alpha\beta}(\Lambda)u^\sigma_\beta(p) = \sum_{\sigma'} u^{\sigma'}_\alpha(\Lambda p)R_{\sigma'\sigma}(W(\Lambda,p)) \tag{4.24a}$$

$$\sum_\beta D_{\alpha\beta}(\Lambda)v^\sigma_\beta(p) = \sum_{\sigma'} v^{\sigma'}_\beta(\Lambda p)R^*_{\sigma'\sigma}(W(\Lambda,p))\,. \tag{4.24b}$$

Using boosts without rotations, i. e. $W(\Lambda,p) = \mathbf{1}$, we can compute the momentum dependence of u and v from u and v at the reference momentum in the Wigner classification for any representation D of the Lorentz group

$$u^\sigma_\alpha(p) = \sum_\beta D_{\alpha\beta}(\overline{\Lambda}(p))u^\sigma_\beta(k) \tag{4.25a}$$

$$v^\sigma_\beta(p) = \sum_\beta D_{\alpha\beta}(\overline{\Lambda}(p))v^\sigma_\beta(k)\,. \tag{4.25b}$$

4.1.1 (Anti-)Commutators

Lecture 10: Wed, 22. 11. 2017

$$\left[\phi_\alpha^{(+)}(x), \phi_\beta^{(-)}(y)\right]_\pm = \sum_\sigma \int \widetilde{\mathrm{d}p}\, \mathrm{e}^{-\mathrm{i}p(x-y)} u_\alpha^\sigma(p) v_\beta^\sigma(p)\,. \tag{4.26}$$

If we choose $x - y = (0, \vec{x} - \vec{y})$ purely spacelike, the commutators are the fourier transform of

$$f(\vec{p}) = \sum_\sigma \frac{u_\alpha^\sigma(p) v_\beta^\sigma(p)}{2\sqrt{\vec{p}^2 + m^2}}\,. \tag{4.27}$$

In order for the (anti-)commutators to vanish for spacelike distances, f would thus have to vanish.

However, if we introduce combinations

$$\phi_\alpha(x) = \kappa_\alpha \phi_\alpha^{(+)}(x) + \lambda_\alpha \phi_\alpha^{(-)}(x)\,, \tag{4.28}$$

where $\phi^{(+)}$ and $\phi^{(-)}$ can be made from annihilation and creation for *different* particles that are distiguished by QNs that have not been spelled out, the (anti-)commutators are

$$[\phi_\alpha(x), \phi_\beta(y)]_\pm = \sum_\sigma \int \widetilde{\mathrm{d}p} \left(\mathrm{e}^{-\mathrm{i}p(x-y)} \kappa_\alpha \lambda_\beta u_\alpha^\sigma(p) v_\beta^\sigma(p) - \mathrm{e}^{\mathrm{i}p(x-y)} \kappa_\beta \lambda_\alpha u_\beta^\sigma(p) v_\alpha^\sigma(p)\right)\,, \tag{4.29}$$

which can easily be made to vanish for spacelike separations, e. g. already for $\kappa = \lambda = u = v = 1$.

4.2 Massive Scalar Fields

In the case of the trivial representation of the Lorentz group, $D(\Lambda) = \mathbf{1}$, i. e. scalar particles, we can choose

$$\forall \Lambda \in \mathcal{L} : u(\Lambda p) = v(\Lambda p) = 1 \tag{4.30}$$

together with $\kappa = \lambda = 1$ and find

$$\phi^{(+)}(x) = \int \widetilde{\mathrm{d}p}\, a(p) \mathrm{e}^{-\mathrm{i}xp} \tag{4.31a}$$

$$\phi^{(-)}(x) = \int \widetilde{\mathrm{d}p}\, a^\dagger(p) \mathrm{e}^{\mathrm{i}xp} \tag{4.31b}$$

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) \tag{4.31c}$$

with

$$\phi^{(\pm)}(x) = \left(\phi^{(\mp)}(x)\right)^\dagger \tag{4.32a}$$
$$\phi(x) = (\phi(x))^\dagger \tag{4.32b}$$

and

$$\left[\phi^{(+)}(x), \phi^{(-)}(y)\right]_- = \int \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}p(x-y)} = \mathrm{i}\Delta^{(+)}(x-y) \tag{4.33a}$$
$$\left[\phi^{(-)}(x), \phi^{(+)}(y)\right]_- = -\mathrm{i}\Delta^{(+)}(y-x) \tag{4.33b}$$
$$[\phi(x), \phi(y)]_- = \mathrm{i}\Delta^{(+)}(x-y) - \mathrm{i}\Delta^{(+)}(y-x) = \mathrm{i}\Delta(x-y)\,. \tag{4.33c}$$

Here we only consider the commutator and not the anti-commutator, because it can be shown, that for integer spins only the commutator and for half-integer spins only the anti-commutator leads to a consistent theory.

From the fact that the measure $\widetilde{\mathrm{d}p}$ vanishes outside of the mass shell, we immediately conclude that

$$\left(\Box + m^2\right)\phi^{(\pm)}(x) = \left(\partial_\mu\partial^\mu + m^2\right)\phi^{(\pm)}(x) = 0 \tag{4.34a}$$
$$\left(\Box + m^2\right)\phi(x) = 0 \tag{4.34b}$$
$$\left(\Box + m^2\right)\Delta^{(+)}(x) = 0 \tag{4.34c}$$
$$\left(\Box + m^2\right)\Delta(x) = 0\,. \tag{4.34d}$$

Using

$$\phi^{(+)}(x)\,|0\rangle = 0\,, \tag{4.35}$$

one can compute matrix elements like

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|\phi^{(+)}(y)\phi^{(-)}(x)|0\rangle \\ &= \left\langle 0\middle|\left[\phi^{(+)}(x), \phi^{(-)}(y)\right]_-\middle|0\right\rangle = \mathrm{i}\Delta^{(+)}(x-y)\,.\end{aligned} \tag{4.36}$$

4.2.1 Commutator Function

Note that the distribution Δ, the so-called *commutator function*, defined in (4.33) depends on the mass m in the Lorentz invariant integration measure. Sometimes, this dependence should be made explicit

$$\mathrm{i}\Delta^{(+)}(x;m) = \int \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}xp} = \int \frac{\mathrm{d}^3\vec p}{(2\pi)^3 2p_0}\mathrm{e}^{-\mathrm{i}xp}\bigg|_{p_0=\sqrt{\vec p^2+m^2}}\,. \tag{4.37}$$

Note that $\Delta^{(+)}$ is invariant under orthochronous LTs

$$\mathrm{i}\Delta^{(+)}(\Lambda x;m) = \int\widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}(\Lambda x)p} = \int\widetilde{\mathrm{d}(\Lambda p)}\,\mathrm{e}^{-\mathrm{i}(\Lambda x)(\Lambda p)} = \mathrm{i}\Delta^{(+)}(x;m) \tag{4.38}$$

and can therefore depend only on x^2 and on the sign of x_0 timelike x. For spacelike x, we can choose a coordinate system with $x = (0,\vec{x})$ and find

$$\mathrm{i}\Delta^{(+)}((0,\vec{x});m) = \int\widetilde{\mathrm{d}p}\,\mathrm{e}^{\mathrm{i}\vec{x}\vec{p}} = \frac{m}{4\pi^2}\frac{K_1(m|\vec{x}|)}{|\vec{x}|} \tag{4.39}$$

or, from Lorentz invariance,

$$\mathrm{i}\Delta^{(+)}(x;m)\Big|_{x^2<0} = \frac{m}{4\pi^2}\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}}\,, \tag{4.40}$$

where K_1 is a Hankel function and we see that $\Delta^{(+)}(x;m)$ does *not* depend on the sign of x as long as $x^2 < 0$. Hence

$$\Delta(x;m)\Big|_{x^2<0} = \Delta^{(+)}(x;m)\Big|_{x^2<0} - \Delta^{(+)}(-x;m)\Big|_{x^2<0} = 0\,. \tag{4.41}$$

Another fun fact, that we will use later extensively, is

$$\frac{\partial}{\partial x_0}\Delta^{(+)}(x;m)\Big|_{x_0=0} = -\int\widetilde{\mathrm{d}p}\,p_0\mathrm{e}^{\mathrm{i}\vec{x}\vec{p}} = -\frac{1}{2}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{x}\vec{p}} = -\frac{1}{2}\delta^3(\vec{x}) \tag{4.42}$$

or

$$\frac{\partial}{\partial x_0}\Delta(x;m)\Big|_{x_0=0} = -\delta^3(\vec{x}) \tag{4.43}$$

and

$$\left[\phi(x),\frac{\partial\phi}{\partial y_0}(y)\right]_{-,x_0=y_0} = \mathrm{i}\delta^3(\vec{x}-\vec{y})\,. \tag{4.44}$$

suggesting that

$$\pi(x) = \frac{\partial\phi}{\partial x_0}(x) \tag{4.45}$$

could play the role of a momentum conjugate to ϕ in the canonical formalism to be introduced in chapter 5.

4.2.2 Charged Scalar Fields

Assume that there is a conserved self adjoint charge $Q = Q^\dagger$ with associated QN q

$$Q\,|q\rangle = q\,|q\rangle \tag{4.46}$$

and that $\phi^{(+)}(x)$ and $\phi^{(-)}(x)$ annihilates and create particles with charge q respectively

$$\left[Q, \phi^{(\pm)}(x)\right] = \mp q\phi^{(\pm)}(x)\,. \tag{4.47}$$

Then

$$\left[Q, \phi^{(+)}(x) + \phi^{(-)}(x)\right] = -q\left(\phi^{(+)}(x) - \phi^{(-)}(x)\right) \tag{4.48}$$

and we can *not* use $\phi(x)$ to simply construct interactions that commute with Q and ensure that Q is conserved.

We can solve this problem by introducing a *charge conjugate* states with annihilation and creation operators a^c and $a^{c\dagger}$ for *anti particles*

$$[Q, a(p)] = -qa(p) \tag{4.49a}$$
$$\left[Q, a^\dagger(p)\right] = qa^\dagger(p) \tag{4.49b}$$
$$[Q, a^c(p)] = qa^c(p) \tag{4.49c}$$
$$\left[Q, a^{c\dagger}(p)\right] = -qa^{c\dagger}(p) \tag{4.49d}$$

and corresponding non-hermitian fields

$$\phi(x) = \int \widetilde{\mathrm{d}p}\,\left(a(p)\mathrm{e}^{-\mathrm{i}xp} + a^{c\dagger}(p)\mathrm{e}^{\mathrm{i}xp}\right) \tag{4.50a}$$
$$\phi^\dagger(x) = \int \widetilde{\mathrm{d}p}\,\left(a^c(p)\mathrm{e}^{-\mathrm{i}xp} + a^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right)\,. \tag{4.50b}$$

For those, we have now simply

$$[Q, \phi(x)] = -q\phi(x) \tag{4.51a}$$
$$\left[Q, \phi^\dagger(x)\right] = q\phi^\dagger(x)\,. \tag{4.51b}$$

The charge conjugate states are independent

$$\left[a(p), a^{c\dagger}(q)\right] = 0 \tag{4.52}$$

and thus

$$[\phi(x), \phi(y)]_- = 0 \tag{4.53a}$$
$$\left[\phi^\dagger(x), \phi^\dagger(y)\right]_- = 0 \tag{4.53b}$$

$$\left[\phi(x), \phi^\dagger(y)\right]_- = \mathrm{i}\Delta(x-y)\,. \tag{4.53c}$$

Note that (4.49b) also follows from (4.49a) without making any physical assumptions

$$\left[Q, a^\dagger(p)\right] = \left[a(p), Q^\dagger\right]^\dagger = -\left[Q, a(p)\right]^\dagger = q a^\dagger(p)\,. \tag{4.54}$$

Therefore, we are always forced to introduce anti-particles, when there is a non-vanishing charge.

4.2.3 Parity, Charge Conjugation and Time Reversal

If there is no danger of confusion, we can write

$$P = U(P,0) \tag{4.55a}$$
$$T = U(T,0) \tag{4.55b}$$

for the representations of parity and time reversal. Then

$$Pa(p)P^{-1} = \eta^* a(Pp) \tag{4.56a}$$
$$Pa^{c\dagger}(p)P^{-1} = \eta^c a^{c\dagger}(Pp)\,, \tag{4.56b}$$

with $|\eta| = |\eta^c| = 1$ the *intrinsic parity* of particle and anti-particle. For the field to have definite transformation properties

$$P\phi(x)P^{-1} = \eta^*\phi(Px) \tag{4.57}$$

we must require $\eta^c = \eta^*$.

Introducing a charge conjugation operator C

$$Ca(p)C^{-1} = \xi^* a^c(p) \tag{4.58a}$$
$$Ca^{c\dagger}(p)C^{-1} = \xi^c a^\dagger(p)\,, \tag{4.58b}$$

with $|\xi| = |\xi^c| = 1$ the *intrinsic charge conjugation parity* of particle and anti-particle. For the field to have definite transformation properties

$$C\phi(x)C^{-1} = \xi^*\phi^\dagger(x) \tag{4.59}$$

we must again require $\xi^c = \xi^*$.

Finally for time reversal

$$Ta(p)T^{-1} = \zeta^* a(Pp) \tag{4.60a}$$
$$Ta^{c\dagger}(p)T^{-1} = \zeta^c a^{c\dagger}(Pp)\,, \tag{4.60b}$$

with $|\zeta| = |\zeta^c| = 1$.

$$T\phi(x)T^{-1} = \int \widetilde{\mathrm{d}p}\, T\left(a(p)\mathrm{e}^{-\mathrm{i}xp} + a^{c\dagger}(p)\mathrm{e}^{\mathrm{i}xp}\right)T^{-1}$$
$$= \int \widetilde{\mathrm{d}p}\, \left(Ta(p)T^{-1}\mathrm{e}^{\mathrm{i}xp} + Ta^{c\dagger}(p)T^{-1}\mathrm{e}^{-\mathrm{i}xp}\right)$$
$$= \int \widetilde{\mathrm{d}p}\, \left(\zeta^* a(Pp)\mathrm{e}^{\mathrm{i}xp} + \zeta^c a^{c\dagger}(Pp)\mathrm{e}^{-\mathrm{i}xp}\right)$$
$$= \int \widetilde{\mathrm{d}p}\, \left(\zeta^* a(p)\mathrm{e}^{-\mathrm{i}(-Px)p} + \zeta^c a^{c\dagger}(p)\mathrm{e}^{\mathrm{i}(-Px)p}\right) = \zeta^*\phi(-Px) \quad (4.61)$$

using the anti-unitarity of T and requiring $\zeta^* = \zeta^c$.

4.3 Massive Vector Fields

Lecture 11: Tue, 28. 11. 2017

In the case of vector particles, we simply have

$$D^\mu{}_\nu(\Lambda) = \Lambda^\mu{}_\nu \quad (4.62)$$

as the representation of the Lorentz group and the field carry one Lorentz index

$$\phi^{(+),\mu}(x) = \sum_\sigma \int \widetilde{\mathrm{d}p}\, u^\mu_\sigma(p)a_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} \quad (4.63a)$$
$$\phi^{(-),\mu}(x) = \sum_\sigma \int \widetilde{\mathrm{d}p}\, v^\mu_\sigma(p)a^\dagger_\sigma(p)\mathrm{e}^{\mathrm{i}xp}\,. \quad (4.63b)$$

However, in (4.24) and (4.25) there are two different little group representations for $u^\mu_\sigma(k)$ and $v^\mu_\sigma(k)$ at the reference momentum $k = (m, \vec{0})$

1. spin 0, where the reference vectors

$$u(k) = (-\mathrm{i}m, \vec{0}) \quad (4.64a)$$
$$v(k) = (\mathrm{i}m, \vec{0}) \quad (4.64b)$$

are invariant under little group transformations, and

2. spin 1, where the reference vectors

$$u_i(k) = (0, \vec{u}_i) \quad (4.65a)$$
$$v_i(k) = (0, \vec{v}_i) \quad (4.65b)$$

transform like vectors under little group transformations.

Spin 0

In the first case, we have

$$u^\mu(p) = D^\mu{}_\nu(\overline{\Lambda}(p))u^\nu(k) = -\mathrm{i}p^\mu \tag{4.66a}$$
$$u^\mu(p) = D^\mu{}_\nu(\overline{\Lambda}(p))v^\nu(k) = \mathrm{i}p^\mu \tag{4.66b}$$

and we find nothing new, because

$$\phi^\mu(x) = \int \widetilde{\mathrm{d}p} \left(-\mathrm{i}p^\mu a(p)\mathrm{e}^{-\mathrm{i}xp} + \mathrm{i}p^\mu a^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right) = \partial^\mu\phi(x) \tag{4.67}$$

with a scalar field ϕ. For describing a charged particle, we can again introduce anti particles, as above.

Spin 1

In the spin one case, the choices for the reference vectors are usually called polarizations

$$u_0(k) = v_0(k) = \epsilon_0(k) = \epsilon_0^*(k) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.68a}$$

$$u_1(k) = -v_{-1}(k) = \epsilon_1(k) = -\epsilon_{-1}^*(k) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \mathrm{i} \\ 0 \end{pmatrix} \tag{4.68b}$$

$$u_{-1}(k) = -v_1(k) = \epsilon_{-1}(k) = -\epsilon_1^*(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \\ 0 \end{pmatrix} \tag{4.68c}$$

and we define again

$$\epsilon_\sigma^\mu(p) = D^\mu{}_\nu(\overline{\Lambda}(p))\epsilon_\sigma^\nu(k) = \overline{\Lambda}^\mu{}_\nu(p)\epsilon_\sigma^\nu(k) \tag{4.69}$$

with the properties

$$p_\mu\epsilon_\sigma^\mu(p) = 0 \tag{4.70a}$$
$$\epsilon_\sigma^{*,\mu}(p) = -(-1)^{\delta_{\sigma 0}}\epsilon_{-\sigma}^\mu(p) \tag{4.70b}$$
$$\epsilon_\sigma^\mu(p)\epsilon_{\sigma',\mu}^*(p) = -\delta_{\sigma\sigma'} \tag{4.70c}$$

$$\sum_{\sigma=-1,0,1} \epsilon^{\mu}_{\sigma}(k)\epsilon^{*,\nu}_{\sigma}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.70d}$$

Then a spin 1 massive neutral vector field is

$$\phi^{\mu}(x) = \sum_{\sigma=-1,0,1} \int \widetilde{\mathrm{d}p} \left(\epsilon^{\mu}_{\sigma}(p) a_{\sigma}(p) \mathrm{e}^{-\mathrm{i}xp} + \epsilon^{*,\mu}_{\sigma}(p) a^{\dagger}_{\sigma}(p) \mathrm{e}^{\mathrm{i}xp} \right) \tag{4.71}$$

and we can compute the commutator

$$\left[\phi^{(+),\mu}(x), \phi^{(-),\nu}(y)\right]_{-} = \sum_{\sigma=-1,0,1} \int \widetilde{\mathrm{d}p}\, \epsilon^{\mu}_{\sigma}(p)\epsilon^{*,\nu}_{\sigma}(p) \mathrm{e}^{-\mathrm{i}xp} = \int \widetilde{\mathrm{d}p}\, \Pi^{\mu\nu}(p) \mathrm{e}^{-\mathrm{i}xp} \tag{4.72}$$

with the projection

$$\Pi^{\mu\nu}(p) = \sum_{\sigma=-1,0,1} \epsilon^{\mu}_{\sigma}(p)\epsilon^{*,\nu}_{\sigma}(p) = -g^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{m^2} \tag{4.73}$$

orthogonal to p

$$p_{\mu}\Pi^{\mu\nu}(p) = p_{\nu}\Pi^{\mu\nu}(p) = 0\,. \tag{4.74}$$

Using $\mathrm{i}\partial^{\mu}\mathrm{e}^{\mp\mathrm{i}px} = \pm p^{\mu}\mathrm{e}^{-\mathrm{i}px}$, the commutator can be written

$$\left[\phi^{(+),\mu}(x), \phi^{(-),\nu}(y)\right]_{-} = -\mathrm{i} \left(g^{\mu\nu} + \frac{\partial^{\mu}\partial^{\nu}}{m^2} \right) \Delta^{(+)}(x-y)\,. \tag{4.75}$$

We also see that the vector field is *transversal*

$$\partial_{\mu}\phi^{\mu}(x) = 0 \tag{4.76}$$

and each component satisfies the Equation of Motion (EOM)

$$\left(\Box + m^2\right) \phi^{\mu}(x) = 0\,. \tag{4.77}$$

Unsurprisingly, a spin 1 massive charged vector field is again constructed by introducing the charge conjugate creation operators in the negative energy part

$$\phi^{\mu}(x) = \sum_{\sigma=-1,0,1} \int \widetilde{\mathrm{d}p} \left(\epsilon^{\mu}_{\sigma}(p) a_{\sigma}(p) \mathrm{e}^{-\mathrm{i}xp} + \epsilon^{*,\mu}_{\sigma}(p) a^{c\dagger}_{\sigma}(p) \mathrm{e}^{\mathrm{i}xp} \right) \tag{4.78}$$

and we find

$$\left[\phi^{\mu}(x), \phi^{\nu\dagger}(y)\right]_{-} = -\mathrm{i} \left(g^{\mu\nu} + \frac{\partial^{\mu}\partial^{\nu}}{m^2} \right) \Delta(x-y)\,. \tag{4.79}$$

4.3.1 Parity, Charge Conjugation and Time Reversal

In the case of charge conjugation, there is no change from the scalar fields

$$C\phi^\mu(x)C^{-1} = \xi^* \phi^{\mu\dagger}(x) . \tag{4.80}$$

Howver, in order to understand the behaviour of vector fields under parity and time reversal, we need to determine the behaviour of the polarization vectors. From the definition

$$\epsilon^\mu_\sigma(Pp) = \overline{\Lambda}^\mu{}_\nu(Pp)\epsilon^\nu_\sigma(k) = P^\mu{}_\nu \overline{\Lambda}^\nu{}_\kappa(p) \underbrace{P^\kappa{}_\lambda \epsilon^\lambda_\sigma(k)}_{= -\epsilon^\kappa_\sigma(k)} = -P^\mu{}_\nu \epsilon^\nu_\sigma(p) \tag{4.81}$$

and therefore

$$P\phi^\mu(x)P^{-1} = -\eta^* P^\mu{}_\nu \phi^\nu(Px) \tag{4.82}$$

where we had again to chose the same phase for particle and anti-particle. From the definition, we also have

$$(-1)^{1-\sigma}\epsilon^\mu_{-\sigma}(Pp) = P^\mu{}_\nu \epsilon^\nu_\sigma(p) , \tag{4.83}$$

which nicely cancels the phases in the creation and annihilation operators required by (3.60):

$$T\phi^\mu(x)T^{-1} = \zeta^* P^\mu{}_\nu \phi^\nu(-Px) . \tag{4.84}$$

4.4 Massive Spinor Fields

In the caseof spin-1/2 particles, there is a very elegant method for constructing Lorentz group representations.

4.4.1 Dirac Algebra

The *Clifford algebra* of *Dirac matrices* or γ *matrices*

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \cdot \mathbf{1} \tag{4.85}$$

can be used to construct a representation of the Lorentz group, independent of the representation of the Clifford algebra. Indeed, define the six elements

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2} [\gamma^\mu, \gamma^\nu]_- , \tag{4.86}$$

then (cf. exercise)

$$[\sigma^{\mu\nu},\gamma^{\rho}]_{-} = 2\mathrm{i}\left(\gamma^{\mu}g^{\nu\rho} - \gamma^{\nu}g^{\mu\rho}\right) \tag{4.87}$$

and (cf. exercise)

$$[\sigma^{\mu\nu},\sigma^{\rho\sigma}]_{-} = 2\mathrm{i}\left(g^{\mu\rho}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\rho} - g^{\nu\rho}\sigma^{\mu\sigma} + g^{\nu\sigma}\sigma^{\mu\rho}\right)\,, \tag{4.88}$$

i. e. we have found a representation of the Lorentz algebra up to a normalization factor

$$D(M^{\mu\nu}) = -\frac{1}{2}\sigma^{\mu\nu}\,. \tag{4.89}$$

We can perform the computations following (2.141b) and (2.141a) backwards to find

$$D(\Lambda)\gamma^{\mu}D^{-1}(\Lambda) = {\Lambda_{\nu}}^{\mu}\gamma^{\nu} \tag{4.90a}$$

$$D(\Lambda)\frac{1}{2}\sigma^{\mu\nu}D^{-1}(\Lambda) = {\Lambda_{\rho}}^{\mu}{\Lambda_{\sigma}}^{\nu}\frac{1}{2}\sigma^{\rho\sigma}\,. \tag{4.90b}$$

We can also write

$$D(\Lambda)\mathbf{1}D^{-1}(\Lambda) = \mathbf{1} \tag{4.90c}$$

and conclude that $\mathbf{1}$ transforms like a scalar, γ^{μ} transforms like a vector and $\sigma^{\mu\nu}$ transforms like a rank-2 tensor. We will therefore use the metric to raise and lower indices

$$\gamma_{\mu} = g_{\mu\nu}\gamma^{\nu} \tag{4.91}$$

and the summation convention. Note that (4.85) implies

$$\left(\gamma^{0}\right)^{2} = \mathbf{1} = -\left(\gamma^{i}\right)^{2}\,. \tag{4.92}$$

Thus we can define

$$\beta = \gamma^{0} = \beta^{-1} \tag{4.93}$$

to get

$$\beta\gamma^{0}\beta^{-1} = \gamma^{0}\gamma^{0}\gamma^{0} = \gamma^{0} \tag{4.94a}$$

$$\beta\gamma^{i}\beta^{-1} = \gamma^{0}\gamma^{i}\gamma^{0} = -\gamma^{i} \tag{4.94b}$$

as well as

$$\beta\sigma^{0i}\beta^{-1} = -\sigma^{0i} \tag{4.95a}$$

$$\beta\sigma^{ij}\beta^{-1} = \sigma^{ij} \tag{4.95b}$$

i. e. β represents parity:

$$D(P) = \beta = \gamma^0 . \tag{4.96}$$

Defining the object

$$\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{i}\gamma^3\gamma^2\gamma^1\gamma^0 = -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{4.97}$$

we can use (4.85) to prove

$$[\gamma_5, \gamma^\mu]_+ = 0 \tag{4.98}$$

and γ_5 transforms as a pseudo scalar

$$[\sigma^{\mu\nu}, \gamma_5]_- = 0 \tag{4.99a}$$

$$\beta\gamma_5\beta^{-1} = -\gamma_5 , \tag{4.99b}$$

as was to be expected from the definition using the ϵ-tensor. Note that

$$(\gamma_5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0 = \mathbf{1} . \tag{4.100}$$

The axial vector $\gamma^\mu\gamma_5$

$$\beta\left(\gamma^0\gamma_5\right)\beta^{-1} = -\gamma^0\gamma_5 \tag{4.101a}$$

$$\beta\left(\gamma^i\gamma_5\right)\beta^{-1} = \gamma^i\gamma_5 \tag{4.101b}$$

completes the sex of 16 Dirac matrices

$\mathbf{1}$	1	"skalar"	(4.102a)
γ^μ	4	"vector"	(4.102b)
$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_-$	6	"tensor"	(4.102c)
$\gamma^\mu\gamma_5$	4	"axial vector"	(4.102d)
$\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$	1	"pseudo scalar"	(4.102e)

and it will not come as a surprise that the smallest faithful representations of (4.85) use (anti-)hermitian 4×4-matrices.

4.4.2 Feynman Slash Notation

Below, we will use components of four vectors as coefficients in linear combinations

$$p_\mu\gamma^\mu = p^0\gamma^0 - p^1\gamma^1 - p^2\gamma^2 - p^3\gamma^3 \tag{4.103}$$

so often, that it warrants to introduce a shorthand, the *Feynman slash*

$$\not{p} = p_\mu \gamma^\mu = p^\mu \gamma_\mu \,. \tag{4.104}$$

In this notation, the Dirac algebra (4.85) can be specified equivalently as

$$\forall a, b \in \mathbf{M} : [\not{a}, \not{b}]_+ = 2ab = 2a_\mu b^\mu \,. \tag{4.105}$$

As a consequence, we can write

$$\not{p}^2 = \frac{1}{2} \left(\not{p}\not{p} + \not{p}\not{p}\right) = p^2 \,. \tag{4.106}$$

4.4.3 Dirac Matrices

Lecture 12: Wed, 29. 11. 2017

From (4.85), we see that any matrix realization of γ^0 has real eigenvalues $\sqrt{1}$ and of γ^i imaginary eigenvalues $\sqrt{-1}$. Therefore

$$\gamma^{0\dagger} = \gamma^0 \tag{4.107a}$$
$$\gamma^{i\dagger} = -\gamma^i \tag{4.107b}$$
$$\gamma_5{}^\dagger = \gamma_5 \,. \tag{4.107c}$$

Introducing the *Dirac adjoint*

$$\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 \tag{4.108}$$

for any sum and product Γ of Dirac matrices, this can be written compactly

$$\overline{\mathbf{1}} = \mathbf{1} \tag{4.109a}$$
$$\overline{\gamma^\mu} = \gamma^\mu \tag{4.109b}$$
$$\overline{\sigma^{\mu\nu}} = \sigma^{\mu\nu} \tag{4.109c}$$
$$\overline{\gamma^\mu \gamma_5} = \gamma^\mu \gamma_5 \tag{4.109d}$$
$$\overline{\gamma_5} = -\gamma_5 \,. \tag{4.109e}$$

Since the Lorentz group is not compact in the direction of the boosts, their representations are not unitary. However, from (4.109c) we have

$$\gamma^0 \left(\frac{\mathrm{i}}{2}\sigma^{\mu\nu}\right)^\dagger \gamma^0 = -\frac{\mathrm{i}}{2}\sigma^{\mu\nu} \tag{4.110}$$

and therefore

$$\gamma^0 D^\dagger(\Lambda)\gamma^0 = D(\Lambda^{-1}) = D^{-1}(\Lambda)\,. \tag{4.111}$$

Therefore if a spinor ψ transforms according to

$$\psi \to D(\Lambda)\psi \tag{4.112}$$

the adjoint spinor $\psi^\dagger$ transforms according to

$$\psi^\dagger \to \psi^\dagger D^\dagger(\Lambda) = \psi^\dagger\gamma^0\gamma^0 D^\dagger(\Lambda) = \psi^\dagger\gamma^0 D^{-1}(\Lambda)\gamma^0 \neq \psi^\dagger D^{-1}(\Lambda)\,. \tag{4.113}$$

This suggests to introduce a Dirac adjoint also for spinors

$$\bar\psi = \psi^\dagger\gamma^0 \tag{4.114}$$

because it transforms as

$$\bar\psi \to \bar\psi D^{-1}(\Lambda)\,. \tag{4.115}$$

For the realization of the charge conjugation operation below, we will also need an operator that relates γ-matrices with their complex conjugates. Since we know the behaviour under hermitian conjugation, we can use transposed matrices instead of complex conjugates. Thus we will look for a *charge conjugation matrix* C with the properties

$$C\mathbf{1}C^{-1} = \mathbf{1} \tag{4.116a}$$

$$C\gamma^\mu C^{-1} = -\gamma^{\mu T} \tag{4.116b}$$

$$C\sigma^{\mu\nu} C^{-1} = -\sigma^{\mu\nu T} \tag{4.116c}$$

$$C\gamma_5 C^{-1} = {\gamma_5}^T \tag{4.116d}$$

$$C\gamma^\mu\gamma_5 C^{-1} = \left(\gamma^\mu\gamma_5\right)^T\,. \tag{4.116e}$$

We can *not* expect the form of the matrix to be independent of the representation.

Dirac Representation

In the *Dirac representation*, the matrices γ^0 and σ^{ij} are (block-)diagonal

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{4.117a}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.117b}$$

$$\gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{4.117c}$$

$$\sigma^{0i} = \mathrm{i} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{4.117d}$$

$$\sigma^{ij} = \sum_{k=1}^{3} \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{4.117e}$$

and it is best suited for small momenta. A suitable charge conjugation matrix is

$$C = -\mathrm{i} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.118}$$

with

$$C^{-1} = C^\dagger = C^T = -C\,. \tag{4.119}$$

Note that we can write $C = \mathrm{i}\gamma^2\gamma^0$, but this will *not* always be the case in other realizations.

Chiral Representation

In the *chiral representation*, the matrices γ^5 and $\sigma^{\mu\nu}$ are (block-)diagonal

$$\gamma^0 = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \tag{4.120a}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{4.120b}$$

$$\gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{4.120c}$$

$$\sigma^{0i} = \mathrm{i} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \tag{4.120d}$$

$$\sigma^{ij} = \sum_{k=1}^{3} \epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{4.120e}$$

and it will turn out to be best suited for light particles. Note in particular, that the boost generators σ^{0i} are block-diagonal, which is not the case in the Dirac representation. Also note that other variants of the chiral representation are used, where the signs of γ^0, γ_5 and σ^{0i} are flipped.

A suitable charge conjugation matrix is

$$C = \mathrm{i} \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\,, \tag{4.121}$$

which again satisfies

$$C^{-1} = C^{\dagger} = C^T = -C\,, \tag{4.122}$$

and can be written $C = \mathrm{i}\gamma^2\gamma^0$.

Majorana Representation

In the *Majorana representation*, we will find $C \neq \mathrm{i}\gamma^2\gamma^0$ (cf. exercise).

4.4.4 Dirac Fields

A charged *Dirac field*

$$\psi^\alpha(x) = \sum_\sigma \int \widetilde{\mathrm{d}p}\, \left(u^\alpha_\sigma(p) c_\sigma(p) \mathrm{e}^{-\mathrm{i}xp} + v^\alpha_\sigma(p) d^\dagger_\sigma(p) \mathrm{e}^{\mathrm{i}xp}\right) \tag{4.123}$$

with the customary notation $d_\sigma(p) = c^c_\sigma(p)$ for the charge conjugate creation and annihilation operators. The anticommutator is then

$$\left[\psi^\alpha(x), \psi^{\beta\dagger}(y)\right]_+ = \sum_\sigma \int \widetilde{\mathrm{d}p}\, \left(u^\alpha_\sigma(p) {u^{\beta}}^*_\sigma(p) \mathrm{e}^{-\mathrm{i}p(x-y)} + v^\alpha_\sigma(p) {v^{\beta}}^*_\sigma(p) \mathrm{e}^{\mathrm{i}p(x-y)}\right)\,, \tag{4.124}$$

but we still have to choose the coefficients $u^\alpha_\sigma(p)$ and $v^\alpha_\sigma(p)$.

In order to do so sensibly, we need to commit to a representation of the Dirac matrices. We will use the chiral representation (4.120), since all Lorentz generators are block diagonal there. The block diagonal form of the rotation generators σ^{ij} makes it clear that we are dealing with the direct sum of two spin-1/2 representations. For particles, we can write

$$u_\uparrow(k) = \begin{pmatrix} u_+ \\ 0 \\ u_- \\ 0 \end{pmatrix} \tag{4.125a}$$

$$u_\downarrow(k) = -\begin{pmatrix} 0 \\ u_+ \\ 0 \\ u_- \end{pmatrix} \tag{4.125b}$$

and for anti-particles

$$v_\uparrow(k) = \begin{pmatrix} 0 \\ v_+ \\ 0 \\ v_- \end{pmatrix} \tag{4.126a}$$

$$v_\downarrow(k) = \begin{pmatrix} v_+ \\ 0 \\ v_- \\ 0 \end{pmatrix}, \tag{4.126b}$$

such that

$$\frac{1}{2}\sigma^{12} u_\uparrow(k) = \frac{1}{2} u_\uparrow(k) \tag{4.127a}$$
$$\frac{1}{2}\sigma^{12} u_\downarrow(k) = -\frac{1}{2} u_\downarrow(k) \tag{4.127b}$$
$$\frac{1}{2}\sigma^{12} v_\uparrow(k) = -\frac{1}{2} v_\uparrow(k) \tag{4.127c}$$
$$\frac{1}{2}\sigma^{12} v_\downarrow(k) = \frac{1}{2} v_\downarrow(k)\,. \tag{4.127d}$$

Note that opposite spins for anti-particles are consistent, because $v^\alpha_\sigma(p)$ appears in front of annihilation operators, while $v^\alpha_\sigma(p)$ appears in front of creation operators.

Remember that $\beta = \gamma^0$ realizes the parity operation. In the chiral representation, it exchanges the upper and lower pairs of indices. Thus, in order to have simple transformation rules under parity, we can choose $u_\pm$ and $v_\pm$ as eigenstates of β

$$u_\uparrow(k) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{4.128a}$$

$$u_\downarrow(k) = \sqrt{m}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{4.128b}$$

$$v_\uparrow(k) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.128c}$$

$$v_\downarrow(k) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{4.128d}$$

where the signs of the eigenvalues and the normalization have been chosen to obtain a simple EOM and to make the anti-commutator vanish for spacelike distances. Indeed,

$$(\mathrm{i}\partial\!\!\!/ - m)\,\psi(x) = \sum_{\sigma=\uparrow,\downarrow}\int\widetilde{\mathrm{d}p}\,\left((p\!\!\!/ - m)\,u_\sigma(p)c_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} - (p\!\!\!/ + m)\,v_\sigma(p)d^\dagger_\sigma(p)\mathrm{e}^{\mathrm{i}xp}\right) \tag{4.129}$$

and we obtain the *Dirac equation*

$$(\mathrm{i}\partial\!\!\!/ - m)\,\psi(x) = 0 \tag{4.130}$$

iff

$$(p\!\!\!/ - m)\,u_\sigma(p) = 0 \tag{4.131a}$$
$$(p\!\!\!/ + m)\,v_\sigma(p) = 0\,. \tag{4.131b}$$

For the reference momentum $k = (m, \vec{0})$, this means

$$m\left(\gamma^0 - \mathbf{1}\right)u_\sigma(k) = 0 \tag{4.132a}$$
$$m\left(\gamma^0 + \mathbf{1}\right)v_\sigma(k) = 0 \tag{4.132b}$$

with

$$\left(\mathbf{1} \mp \gamma^0\right) = \begin{pmatrix} \mathbf{1} & \pm\mathbf{1} \\ \pm\mathbf{1} & \mathbf{1} \end{pmatrix}, \tag{4.133}$$

justifying our choice for $u(k)$ and $v(k)$. Furthermore

$$\sum_{\sigma=\uparrow,\downarrow} u_\sigma(k)\otimes\bar u_\sigma(k) = m\left(\mathbf{1}+\gamma^0\right)\gamma^0 = m\left(\gamma^0+\mathbf{1}\right) = k\!\!\!/ + m\mathbf{1} \tag{4.134a}$$
$$\sum_{\sigma=\uparrow,\downarrow} v_\sigma(k)\otimes v^\dagger_\sigma(k) = m\left(\mathbf{1}-\gamma^0\right)\gamma^0 = m\left(\gamma^0-\mathbf{1}\right) = k\!\!\!/ - m\mathbf{1}\,. \tag{4.134b}$$

Since we have formed the dyadic products as $\psi\otimes\bar\psi$ instead of $\psi\otimes\psi^\dagger$, we may use the standard transformation properties

$$D(\overline{\Lambda}(p))k\!\!\!/ D^{-1}(\overline{\Lambda}(p)) = p\!\!\!/ \tag{4.135}$$

for the boost from the reference momentum to obtain

$$\sum_{\sigma=\uparrow,\downarrow} u_\sigma(p)\otimes\bar u_\sigma(p) = p\!\!\!/ + m \tag{4.136a}$$
$$\sum_{\sigma=\uparrow,\downarrow} v_\sigma(p)\otimes\bar v_\sigma(p) = p\!\!\!/ - m\,. \tag{4.136b}$$

Note that on the mass shell $p^2 = m^2$, the matrices

$$\Pi_\pm(p) = \frac{\pm p\!\!\!/ + m}{2m} \tag{4.137}$$

are orthogonal projections

$$(\Pi_\pm(p))^2 = \Pi_\pm(p) \tag{4.138a}$$
$$\Pi_\pm(p)\Pi_\mp(p) = 0 \tag{4.138b}$$

as can be seen using $\not{p}^2 = p^2 = m^2$. Returning to the anticommutator (4.124) and replacing $\psi^\dagger$ by $\bar\psi$, we find

$$\left[\psi_\alpha(x), \bar\psi_\beta(y)\right]_+ = \int \widetilde{\mathrm{d}p}\, \left((\not{p}+m)_{\alpha\beta}\mathrm{e}^{-\mathrm{i}p(x-y)} + (\not{p}-m)_{\alpha\beta}\mathrm{e}^{\mathrm{i}p(x-y)}\right)$$
$$= (\mathrm{i}\not{\partial}+m)_{\alpha\beta}\,\mathrm{i}\Delta(x-y)\,, \tag{4.139}$$

which vanishes for spacelike distances. Going to equal times

$$\left[\psi_\alpha(x), \bar\psi_\beta(y)\right]_{+,x_0=y_0} = \mathrm{i}\gamma^0_{\alpha\beta}\partial_0 \mathrm{i}\Delta(x-y)\Big|_{x_0=y_0} = \gamma^0_{\alpha\beta}\delta^3(\vec x - \vec y) \tag{4.140}$$

we find again canonical commutation relations

$$\left[\psi_\alpha(x), \psi^\dagger_\beta(y)\right]_{+,x_0=y_0} = \delta_{\alpha\beta}\delta^3(\vec x - \vec y)\,. \tag{4.141}$$

The right-hand side must be real here because of the anti-commutator.

4.4.5 Parity, Charge Conjugation and Time Reversal

Lecture 13: Tue, 05. 12. 2017

Parity

From the definition of the coefficient function as LTs

$$u_\sigma(p) = D(\overline\Lambda(p))u_\sigma(k) \tag{4.142a}$$
$$v_\sigma(p) = D(\overline\Lambda(p))v_\sigma(k) \tag{4.142b}$$

and using the realization of parity the Dirac algebra

$$D(\Lambda(Pp)) = \beta D(\Lambda(p))\beta^{-1} = \gamma^0 D(\Lambda(p))\gamma^0 \tag{4.143}$$

we find

$$u_\sigma(Pp) = \beta D(\overline\Lambda(p))\beta u_\sigma(k) \tag{4.144a}$$
$$v_\sigma(Pp) = \beta D(\overline\Lambda(p))\beta v_\sigma(k) \tag{4.144b}$$

In our chiral representation

$$\gamma^0 u_\sigma(k) = u_\sigma(k) \tag{4.145a}$$
$$\gamma^0 v_\sigma(k) = -v_\sigma(k) \tag{4.145b}$$

and therefore

$$u_\sigma(Pp) = \gamma^0 u_\sigma(p) \tag{4.146a}$$
$$v_\sigma(Pp) = -\gamma^0 v_\sigma(p) \tag{4.146b}$$

This time, assigning *opposite* intrinsic parity to particle and antiparticle $\eta^c = -\eta^*$

$$Pc_\sigma(p)P^{-1} = \eta^* c_\sigma(Pp) \tag{4.147a}$$
$$Pd_\sigma^\dagger(p)P^{-1} = -\eta^* d_\sigma^\dagger(Pp)\,, \tag{4.147b}$$

we obtain nice transformation properties under parity

$$\begin{aligned} P\psi(x)P^{-1} &= \sum_\sigma \int \widetilde{\mathrm{d}p}\, \left(\eta^* u_\sigma(p)c_\sigma(Pp)\mathrm{e}^{-\mathrm{i}xp} - \eta^* v_\sigma(p)d_\sigma^\dagger(Pp)\mathrm{e}^{\mathrm{i}xp}\right) \\ &= \sum_\sigma \int \widetilde{\mathrm{d}p}\, \left(\eta^* u_\sigma(Pp)c_\sigma(p)\mathrm{e}^{-\mathrm{i}p(Px)} - \eta^* v_\sigma(Pp)d_\sigma^\dagger(p)\mathrm{e}^{\mathrm{i}p(Px)}\right) \\ &= \sum_\sigma \int \widetilde{\mathrm{d}p}\, \left(\eta^* \gamma^0 u_\sigma(p)c_\sigma(p)\mathrm{e}^{-\mathrm{i}p(Px)} + \eta^* \gamma^0 v_\sigma(p)d_\sigma^\dagger(p)\mathrm{e}^{\mathrm{i}p(Px)}\right)\,, \end{aligned} \tag{4.148}$$

i. e.

$$P\psi(x)P^{-1} = \eta^* \gamma^0 \psi(Px)\,. \tag{4.149}$$

Charge Conjugation

Moving on to charge conjugation[2]

$$\mathcal{C}c_\sigma(p)\mathcal{C}^{-1} = \xi^* d_\sigma(p) \tag{4.150a}$$
$$\mathcal{C}d_\sigma^\dagger(p)\mathcal{C}^{-1} = \xi^c c_\sigma^\dagger(p) \tag{4.150b}$$

and choosing $\xi^c = \xi^*$, we obtain

$$\mathcal{C}\psi^\alpha(x)\mathcal{C}^{-1} = \sum_\sigma \int \widetilde{\mathrm{d}p}\, \left(\xi^* u_\sigma^\alpha(p)d_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} + \xi^* v_\sigma^\alpha(p)c_\sigma^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right)$$

[2] We're temporarily using $\mathcal{C}$ for the operator in Hilbert space to avoid confusion with the charge conjugation matrix C.

$$\stackrel{?}{=} \sum_\beta \Gamma_{\alpha\beta} {\psi^\beta}^\dagger(x)\,, \tag{4.151}$$

with a suitable Dirac matrix Γ, provided the complex conjugation of $u(p)$ and $v(p)$ gives a suitable result. Using ${\gamma^0}^T = \gamma^0$ in our representation, we find

$$\sigma^{\mu\nu *} = \left(\sigma^{\mu\nu\dagger}\right)^T = \left(\gamma^0\sigma^{\mu\nu}\gamma^0\right)^T = {\gamma^0}^T {\sigma^{\mu\nu}}^T {\gamma^0}^T = -\gamma^0 C \sigma^{\mu\nu} C^{-1}\gamma^0\,. \tag{4.152}$$

Therefore

$$\left(\frac{\mathrm{i}}{2}\sigma^{\mu\nu}\right)^* = \gamma^0 C \frac{\mathrm{i}}{2}\sigma^{\mu\nu} C^{-1}\gamma^0 \tag{4.153}$$

and with

$$D^*(\Lambda) = \gamma^0 C D(\Lambda) C^{-1}\gamma^0 \tag{4.154}$$

we can indeed compute $u^*(p)$ and $v^*(p)$. From (4.121) and (4.128), we confirm by explicit calculation, that we have chosen $u(k)$ and $v(k)$ real and satisfying

$$C^{-1}u_\sigma(k) = -v_\sigma(k) \tag{4.155a}$$
$$C^{-1}v_\sigma(k) = u_\sigma(k)\,. \tag{4.155b}$$

Thus

$$u^*_\sigma(p) = \gamma^0 C D(\overline{\Lambda}(p)) C^{-1}\gamma^0 u_\sigma(k) = -\gamma^0 C v_\sigma(p) \tag{4.156a}$$
$$v^*_\sigma(p) = \gamma^0 C D(\overline{\Lambda}(p)) C^{-1}\gamma^0 v_\sigma(k) = -\gamma^0 C u_\sigma(p) \tag{4.156b}$$

and consequently

$$\mathcal{C}\psi(x)\mathcal{C}^{-1} = -\xi^*\gamma^0 C {\psi^\dagger}^T(x)\,. \tag{4.157}$$

Time Reversal

Since time reversal is represented by an anti-unitary operator, we need again the complex conjugate coefficient functions. Using the charge conjugation matrix, one can show[3]

$$T\psi(x)T^{-1} = -\zeta^*\gamma_5 C\psi(-Px)\,. \tag{4.158}$$

[3] Cf., e. g., [3], p. 227f.

4.4.6 Transformation of Bilinears

Given a 4×4-matrix M, we can construct bilinears in the Dirac fields

$$\bar\psi(x)M\psi(x) = \psi^\dagger(x)\gamma^0 M\psi(x) = \sum_{\alpha,\beta,\gamma} \psi^{\alpha\dagger}(x)\gamma^0_{\alpha\beta}M_{\beta\gamma}\psi^\gamma(x)\,. \tag{4.159}$$

Since the fields transform as

$$U(\Lambda,a)\psi(x)U^{-1}(\Lambda,a) = D(\Lambda^{-1})\psi(\Lambda x+a) = D^{-1}(\Lambda)\psi(\Lambda x+a) \tag{4.160a}$$
$$U(\Lambda,a)\bar\psi(x)U^{-1}(\Lambda,a) = \bar\psi(\Lambda x+a)D^{-1}(\Lambda^{-1}) = \bar\psi(\Lambda x+a)D(\Lambda) \tag{4.160b}$$

the bilinears transform as

$$U(\Lambda,a)\bar\psi(x)M\psi(x)U^{-1}(\Lambda,a) = \bar\psi(\Lambda x+a)D(\Lambda)MD^{-1}(\Lambda)\psi(\Lambda x+a)\,. \tag{4.161}$$

Obviously, the composition of M in terms of products of γ-matrices determines the transformation properties of such bilinears.

For charge conjugation

$$\mathcal{C}\psi\mathcal{C}^{-1} = -\xi^*\gamma^0 C\psi^{\dagger T} \tag{4.162a}$$
$$\mathcal{C}\bar\psi\mathcal{C}^{-1} = \left(\mathcal{C}\psi\mathcal{C}^{-1}\right)^\dagger\gamma^0 = \left(-\xi^*\gamma^0 C\psi^{\dagger T}\right)^\dagger\gamma^0$$
$$= -\xi\psi^T C^\dagger = -\xi\psi^T C^T \tag{4.162b}$$

we find

$$\begin{aligned}\mathcal{C}\bar\psi M\psi\mathcal{C}^{-1} &= \psi^T C^T M\gamma^0 C\psi^{\dagger T} \overset{\text{fermions!}}{=} -\psi^\dagger C^T\gamma^{0T}M^T C\psi \\ &= -\psi^\dagger \underbrace{C^{-1}\gamma^{0T}C}_{-\gamma^0} C^{-1}M^T C\psi = \psi^\dagger\gamma^0 C^{-1}M^T C\psi = \bar\psi C^{-1}M^T C\psi\,,\end{aligned} \tag{4.163}$$

and conclude from (4.116) that vector and tensor bilinears are odd, while scalar, pseudo-scalar and pseudo-vector bilinears are even under charge conjugation.

4.5 Massless Fields

In the case of scalar particles, nothing changes in the limit $m \to 0$, because the representation of the little group is trivial. In the case of spin-1/2 Dirac particles, there is also no problem in taking the limit $m \to 0$, it will only turn out that there appears new conserved QN, *chirality*, corresponding to the handedness of the particles.

However, in the case of spin 1 vector particles, there must be terms proportional to $1/m$ in the polarization vectors $\epsilon^\mu_\sigma(p)$, because they appear in the commutators of two fields via the polarization sum

$$\Pi^{\mu\nu}(p) = \sum_{\sigma=-1,0,1} \epsilon^\mu_\sigma(p)\epsilon^{*,\nu}_\sigma(p) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\,. \tag{4.73'}$$

Thus the limit $m \to 0$ is not well defined in this case.

It turns out[4], that the 'bad' polarization vector can be decoupled consistently, but only if the transformation property of a massless vector field $A^\mu(x)$ is generalized to

$$U(\Lambda,a)A^\mu(x)U^{-1}(\Lambda,a) = \Lambda_\nu{}^\mu A^\nu(\Lambda x + a) + \partial^\mu\omega(x,\Lambda) \tag{4.164}$$

and physical observables can be shown to be unaffected by *gauge transformations*

$$A^\mu(x) \to A^\mu(x) + \partial^\mu\omega(x)\,. \tag{4.165}$$

How to achieve this will be discussed in chapter 5.4.

[4] Cf., e. g., [3], p. 246ff.

—5— Lagrange Formalism for Fields

So far, we have managed to construct non-interacting local quantum fields, that are covariant under LTs and causal in the sense that they (anti-)commute for spacelike separations. Interactions can be added as integrals of local polynomials in the fields and their derivatives. In order to develop some intuition for sensible interactions, we can start with the classical limit $\hbar \to 0$.

5.1 Classical Field Theory

Configuration space: linear space of all functions ϕ

$$\begin{aligned} \phi : M &\to \mathbf{C}^n \\ x &\mapsto \phi(x) \end{aligned} \tag{5.1}$$

or rather of all distributions, since we often encounter singularities, e. g. in the Coulomb potential of point charges. Mathematically, the space of all (tempered) distributions is the dual of the space of smooth testfunctions, that (fall off faster than any power for $|x| \to \infty$) have compact support:

$$\begin{aligned} \phi : C^\infty(M)^{\times n} &\to \mathbf{C} \\ f &\mapsto \phi(f) = \sum_{i=1}^{n} \int \mathrm{d}^4x\, f_i(x)\phi_i(x)\,. \end{aligned} \tag{5.2}$$

The dynamics of the fields ϕ is governed by first or second order Partial Differential Equations (PDEs), e. g. the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0 \tag{5.3}$$

or the Dirac equation

$$(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0 \tag{5.4}$$

with appropriate Cauchy data for $\psi(x)$ or $\phi(x)$ and $\partial_0\phi(x)$ on a spacelike hypersurface, e. g. $x_0 = 0$.

5.1.1 Action Principle, Euler-Lagrange-Equations

Since the study of coupled nonlinear PDEs is complicated and in particular symmetries are not manifest for multi-component fields, it helps to derive the EOM from an action principle:

$$\delta S(\phi_1,\ldots,\phi_n) = \sum_{i=1}^n \int \mathrm{d}^4x \, \frac{\delta S}{\delta\phi_i}(\phi_1,\ldots\phi_n,x)\delta\phi_i(x) = 0 \tag{5.5}$$

for all variations $\{\delta\phi_i\}_{i=1,\ldots,n}$ that vanish together with their partial derivatives sufficiently fast at spatial and temporal infinity. The resulting Euler-Lagrange equations are

$$\frac{\delta S}{\delta\phi_i}(\phi_1,\ldots\phi_n,x) = 0\,. \tag{5.6}$$

In order to not run into problems with causality, we shall assume that the action can be written as the integral of a *local Lagrangian density*

$$S(\phi_1,\ldots,\phi_n) = \int \mathrm{d}^4x \, \mathcal{L}\left(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu}, \{\partial_\mu\partial_\nu\phi_j(x)\}_{j,\mu,\nu},\ldots\right)\,, \tag{5.7}$$

where $\mathcal{L}$ is a *polynomial* of the fields and their partial derivatives of finite order[1]. Then the variations can be computed

$$\begin{aligned}\delta S(\phi_1,\ldots,\phi_n) = \\ \sum_{i=1}^n \int \mathrm{d}^4x \Biggl(&\frac{\partial\mathcal{L}}{\partial\phi_i(x)}\left(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots\right)\delta\phi_i(x) \\ &+ \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i(x)}\left(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots\right)\delta\partial_\mu\phi_i(x) + \ldots\Biggr)\,.\end{aligned} \tag{5.8}$$

Using

$$\delta\partial_\mu\phi_i(x) = \partial_\mu\delta\phi_i(x)\,, \tag{5.9}$$

integration by parts with Gauss' theorem

$$\int \mathrm{d}^4x \, f(x)\partial_\mu g(x) = -\int \mathrm{d}^4x \, \left(\partial_\mu f(x)\right) g(x) + \int_{x^{0\,2}+\vec{x}^2=R\to\infty} \mathrm{d}\sigma^\mu(x)\, f(x)g(x) \tag{5.10}$$

[1] Allowing arbitrary high derivatives would break locality, as can be seen from $\mathrm{e}^{a\frac{\mathrm{d}}{\mathrm{d}x}}\phi(x) = \phi(x+a)$.

and the boundary conditions

$$\lim_{x_\nu\to\infty} \partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\phi_i(x) = 0 \tag{5.11}$$

we find

$$\begin{aligned}\delta S(\phi_1,\ldots,\phi_n) = \\ \sum_{i=1}^n \int \mathrm{d}^4x \Biggl(\frac{\partial\mathcal{L}}{\partial\phi_i(x)} &(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots) \\ - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i(x)} &(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots) \\ + \partial_\mu\partial_\nu \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_i(x)} &(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots) + \ldots \Biggr) \delta\phi_i(x)\,,\end{aligned} \tag{5.12}$$

i. e.

$$\begin{aligned}\frac{\delta S}{\delta\phi_i}(\phi_1,\ldots\phi_n,x) = \frac{\partial\mathcal{L}}{\partial\phi_i(x)} &(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots) \\ - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i(x)} &(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots) \\ + \partial_\mu\partial_\nu \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi_i(x)} &(\{\phi_j(x)\}_j, \{\partial_\mu\phi_j(x)\}_{j,\mu},\ldots) + \ldots = 0\,.\end{aligned} \tag{5.13}$$

For example the Lagrangian density for a single real field ϕ

$$\mathcal{L}(\phi(x),\partial_\mu\phi(x)) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)\phi(x) - V(\phi(x)) \tag{5.14}$$

leads to

$$\begin{aligned}0 = \frac{\delta S}{\delta\phi}(\phi,x) = \frac{\partial\mathcal{L}}{\partial\phi(x)}(\phi(x),\partial_\mu\phi(x)) - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi(x)}(\phi(x),\partial_\mu\phi(x)) \\ = -\Box\phi(x) - m^2\phi(x) - V'(\phi(x))\,,\end{aligned} \tag{5.15}$$

i. e. the Klein-Gordon equation with interactions

$$\left(\Box + m^2\right)\phi(x) + V'(\phi(x)) = 0\,. \tag{5.16}$$

This means that we get *second order* PDEs as field equation from Lagrangian densities that contain at most *first order* derivatives of the fields. In fact, all interesting field equations are first or second order in time and space, since higher orders lead to problems with causality. Therefore we will restrict ourselves to Lagrangian densities that contain at most first order derivatives of the fields from now[2].

Form now on, we will write *par abuse de langage* $\mathcal{L}$ both for the Lagrangian density as a function of the fields and as a function of the space-time point:

$$\mathcal{L}(x) = \mathcal{L}\left(\{\phi_j(x)\}_j, \{\partial_\mu \phi_j(x)\}_{j,\mu}\right) . \tag{5.17}$$

The partial derivatives of $\mathcal{L}(x)$ are therefore to be understood as

$$\frac{\partial \mathcal{L}}{\partial \phi(x)}(x) = \frac{\partial \mathcal{L}}{\partial \phi(x)}\left(\{\phi_j(x)\}_j, \{\partial_\mu \phi_j(x)\}_{j,\mu}\right) . \tag{5.18}$$

5.1.2 Charged Fields

Lecture 14: Wed, 06. 12. 2017

So far, we have only written Lagrangian densities for real fields $\phi^*(x) = \phi(x)$ that describe neutral particles. Complex fields can be described by their and imaginary parts

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + \mathrm{i}\phi_2(x)\right) \tag{5.19}$$

with identical masses

$$\mathcal{L} = \sum_{i=1}^{2}\left(\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{m^2}{2}\phi_i^2\right) - V\left((\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}, (\phi_1 - \mathrm{i}\phi_2)/\sqrt{2}\right) \tag{5.20a}$$

or by the complex field and its complex conjugate

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V(\phi,\phi^*) . \tag{5.20b}$$

It's easy to show (cf. exercise) that the two ways lead to equivalent EOMs, provided the variations $\delta\phi$ and $\delta\phi^*$ are formally treated as *independent* in the derivation of the Euler-Lagrange equations.

Dirac fields can be described by the Lagrangian density

$$\mathcal{L} = \bar\psi\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi \tag{5.21}$$

and treating the variations $\delta\psi^\alpha$ and $\delta\bar\psi^\alpha$ as independent. Indeed

$$0 = \frac{\partial \mathcal{L}}{\partial \bar\psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = \left(\mathrm{i}\partial\!\!\!/ - m\right)\psi - 0 = \left(\mathrm{i}\partial\!\!\!/ - m\right)\psi \tag{5.22}$$

[2] Note that integration by parts makes terms that contain two first order derivatives equivalent to terms that contain one second order derivative.

and

$$0 = \frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} = -m\bar\psi - \partial_\mu \mathrm{i}\bar\psi\gamma^\mu = -\left(\mathrm{i}\partial_\mu\bar\psi\gamma^\mu + m\bar\psi\right) . \tag{5.23}$$

We will write the latter equation also as

$$\bar\psi\left(\mathrm{i}\overleftarrow{\partial\!\!\!/} + m\right) = 0\,, \tag{5.24}$$

with the arrow denoting that the derivative acts to the left. If one is unhappy about the unsymmetrical treatment of ψ and $\bar\psi$ in (5.21), one can use

$$\mathcal{L} = \frac{\mathrm{i}}{2}\left(\bar\psi\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar\psi)\gamma^\mu\psi\right) - m\bar\psi\psi \tag{5.25}$$

instead. It's again easy to show (cf. exercise) that the two ways lead to equivalent EOMs.

5.1.3 Canonical Formalism

Second order PDEs can always be reformulated as a larger system of first order PDEs. For example the classical canonical dynamics for one real Klein-Gordon field

$$S = \int \mathrm{d}t\, L(t) \tag{5.26a}$$

$$L(t) = \int_{x^0=t} \mathrm{d}^3\vec x\, \mathcal{L}(x) \tag{5.26b}$$

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) - V(\phi(x)) \tag{5.26c}$$

with canonically conjugate momentum

$$\pi(x) = \frac{\delta S}{\delta(\partial_0\phi(x))}(\phi) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}(x) = \partial^0\phi(x) \tag{5.27}$$

and an Hamiltonian from Legendre transformation

$$\begin{aligned} H(t) &= \int_{x_0=t} \mathrm{d}^3\vec x \Big(\pi(x)\partial_0\phi(x) - \mathcal{L}(x)\Big) \\ &= \int_{x_0=t} \mathrm{d}^3\vec x\, \frac{1}{2}\Big(\pi^2(x) + \vec\nabla\phi(x)\vec\nabla\phi(x) + m^2\phi^2(x) + V(\phi(x))\Big) . \end{aligned} \tag{5.28}$$

The EOMs

$$\dot\phi(t,\vec x) = \{\phi(x), H(t)\} \tag{5.29a}$$

$$\dot{\pi}(t,\vec{x}) = \{\pi(x), H(t)\} \tag{5.29b}$$

with Poisson brackets at equal times[3]

$$\{f,g\} = \sum_i \int \mathrm{d}^3x \left(\frac{\delta f}{\delta\phi_i}(t,\vec{x})\frac{\delta g}{\delta\pi_i}(t,\vec{x}) - \frac{\delta f}{\delta\pi_i}(t,\vec{x})\frac{\delta g}{\delta\phi_i}(t,\vec{x}) \right) . \tag{5.35}$$

are equivalent to the EOMs derived from the Lagrangian.

The first order in time canonical equations of motion (5.29) have a unique solution, if initial conitions for the field ϕ and the momentum π are given on a space-like Cauchy surface.

5.2 *Quantization*

5.2.1 *Canonical Quantization*

Promote fields to operators in a suitable Hilbert space (more precisely: operator valued distributions) and replace Poisson brackets by commutators

$$[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] = \mathrm{i}\delta_{ij}\delta^3(\vec{x}-\vec{y}) \tag{5.36a}$$

[3]Equivalent definition: denote the space of all (nonlinear) functionals of ϕ and π with

$$\mathcal{C} = C^\infty(\mathbf{R}^3) \times C^\infty(\mathbf{R}^3) \to \mathbf{C}\,. \tag{5.30}$$

Then the binary operation

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C}\times\mathcal{C} &\to \mathcal{C} \\ (f,g) &\mapsto \{f,g\} \end{aligned} \tag{5.31}$$

is an antisymmetric *derivation*, i. e.

$$\{f,g\} = -\{g,f\} \tag{5.32a}$$

$$\{f,gh\} = g\{f,h\} + \{f,g\}h \tag{5.32b}$$

$$\{f,\alpha g + \beta h\} = \alpha\{f,g\} + \beta\{f,h\} \tag{5.32c}$$

for $\alpha,\beta\in\mathbf{C}$, and we define

$$\{\phi(t,\vec{x}), \pi(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y}) \tag{5.33a}$$

$$\{\phi(t,\vec{x}), \phi(t,\vec{y})\} = \{\pi(t,\vec{x}), \pi(t,\vec{y})\} = 0\,. \tag{5.33b}$$

The Poisson braket also satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \tag{5.34}$$

and consequently forms a Lie algebra.

$$[\phi_i(t,\vec{x}),\phi_j(t,\vec{y})] = [\pi_i(t,\vec{x}),\pi_j(t,\vec{y})] = 0\,. \tag{5.36b}$$

In perturbation theory one splits the Hamiltonian

$$H = H_0 + V \tag{5.37a}$$

$$H_0 = \sum_i \int_{x_0=t} \mathrm{d}^3\vec{x}\,\frac{1}{2}\Big(\pi_i^2(x) + \vec{\nabla}\phi_i(x)\vec{\nabla}\phi_i(x) + m^2\phi_i^2(x)\Big) \tag{5.37b}$$

and the *linear* Heisenberg picture EOMs resulting from H_0 ("free wave equation")

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,\vec{x}) = \mathrm{i}\,[H_0(t),\phi(t,\vec{x})] \tag{5.38a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t,\vec{x}) = \mathrm{i}\,[H_0(t),\pi(t,\vec{x})] \tag{5.38b}$$

are solved by the free quantum fields constructed in chapter 4:

$$\phi_i(x) = \int\widetilde{\mathrm{d}k}\,\Big(a_i(k)\mathrm{e}^{-\mathrm{i}kx} + a_i^\dagger(k)\mathrm{e}^{\mathrm{i}kx}\Big) \tag{5.39a}$$

$$\pi_i(x) = -\mathrm{i}\int\widetilde{\mathrm{d}k}\,k_0\,\Big(a_i(k)\mathrm{e}^{-\mathrm{i}kx} - a_i^\dagger(k)\mathrm{e}^{\mathrm{i}kx}\Big)\,. \tag{5.39b}$$

Unfortunately, matrix elements of the Hamiltonian

$$H_0 = \sum_i \int\widetilde{\mathrm{d}k}\,\frac{k^0}{2}\,\Big(a_i(k)a_i^\dagger(k) + a_i^\dagger(k)a_i(k)\Big) \tag{5.40}$$

are *not at all* well defined, if attention is payed to operator ordering. Already the vacuum expectation value

$$\langle 0|H_0|0\rangle = \sum_i \int\widetilde{\mathrm{d}k}\,\frac{k^0}{2}\left(\underbrace{\langle 0|a_i(k)a_i^\dagger(k)|0\rangle}_{=\langle k,i|k,i\rangle\,\propto\,\delta^3(\vec{k}-\vec{k})} + \underbrace{\langle 0|a_i^\dagger(k)a_i(k)|0\rangle}_{=0}\right)\,. \tag{5.41}$$

This problem can be solved by *normal ordering*

$$H_0 \to H_0 - \langle 0|H_0|0\rangle = \sum_i \int\widetilde{\mathrm{d}k}\,k_0 a_i^\dagger(k)a_i(k) \tag{5.42}$$

without changing the canonical EOMs, because the commuting constant $\langle 0|H_0|0\rangle$ does not contribute to commutators.

Since the classical theory can not predict operator ordering, we will henceforth always assume normal ordering, i. e. writing all creation operators to the left of all annihilation operators.

5.3 Noether Theorem

As in classical mechanics, the existence of a continuous one-parameter group of invariances of the action leads to conserved quantities.

Consider an infinitesimal transformation of the fields

$$\phi_i(x) \to \phi_i(x) + \delta\phi_i(x) , \tag{5.43}$$

where $\delta\phi_i$ may depend on x explicitely or implicitely via fields and their derivatives, that leaves the action invariant

$$\delta S(\phi) = \sum_i \int \mathrm{d}^4 x \, \frac{\delta S}{\delta \phi_i}(\phi, x)\delta\phi_i(x) = 0 \tag{5.44}$$

not only for solutions of the EOMs, but for *all* field configurations. Then the Lagrangian density is transformed into a total derivative

$$\delta\mathcal{L} = \partial_\mu \Lambda^\mu . \tag{5.45}$$

Noether's theorem tells us that

$$j^\mu = \sum_i \delta\phi_i \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i} - \Lambda^\mu \tag{5.46}$$

is a conserved current

$$\partial_\mu j^\mu = 0 \tag{5.47}$$

for all field configurations that solve the EOMs. The proof of Noether's theorem is a straightforward application of the Euler-Lagrange equations

$$\begin{aligned} \partial_\mu j^\mu &= \sum_i \left(\partial_\mu \delta\phi_i \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i} + \delta\phi_i \partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i} \right) - \partial_\mu \Lambda^\mu \\ &= \sum_i \left(\delta\partial_\mu \phi_i \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i} + \delta\phi_i \frac{\partial \mathcal{L}}{\partial\phi_i} \right) - \partial_\mu \Lambda^\mu = \delta\mathcal{L} - \partial_\mu\Lambda^\mu = 0 , \end{aligned} \tag{5.48}$$

assuming that at most first order partial derivatives of the fields appear in the Lagrangian density $\mathcal{L}$.

Given any conserved current j, we can construct a conserved charge

$$Q = \int_{x_0 = t} \mathrm{d}^3\vec x \, j^0(x) \tag{5.49}$$

as long as there are no contributions from spatial infinity

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{x_0 = t} \mathrm{d}^3\vec x \, \partial_0 j^0(x) = \int_{x_0 = t} \mathrm{d}^3\vec x \, \vec\nabla \vec j(x) = \int_{x_0 = t, |\vec x| \to \infty} \mathrm{d}\vec\sigma(x) \, \vec j(x) \to 0 . \tag{5.50}$$

This current can be written

$$Q = \int_{x_0=t} \mathrm{d}^3\vec{x} \left(\sum_i \delta\phi_i \frac{\partial\mathcal{L}}{\partial\partial_0\phi_i} - \Lambda^\mu \right) = \int_{x_0=t} \mathrm{d}^3\vec{x} \left(\sum_i \delta\phi_i \pi_i - \Lambda^0 \right) \quad (5.51)$$

and we obtain immediately[4]

$$\{\phi_i, Q\} = \delta\phi_i \,. \quad (5.52)$$

5.3.1 Energy Momentum Tensor

Consider a translation

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + \partial_\mu \underbrace{(a^\mu \mathcal{L}(x))}_{\Lambda^\mu(x)} \,. \quad (5.53)$$

This corresponds to field transformations

$$\phi_i(x) \to \phi_i(x+a) = \phi_i(x) + \underbrace{a^\nu \partial_\nu \phi_i(x)}_{=\delta_a\phi_i(x)} + \ldots \quad (5.54)$$

and we find conserved currents j_a for any given four vector a

$$j_a^\mu = \sum_i a^\nu \partial_\nu \phi_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} - a^\mu \mathcal{L} \,. \quad (5.55)$$

Choosing four unit vectors $a_\nu^\mu = \delta_\nu^\mu$ for $\nu = 0, 1, 2, 3$, this reads

$$j_\nu^\mu = \sum_i \partial_\nu \phi_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} - \delta_\nu^\mu \mathcal{L} = \Theta^\mu{}_\nu \,. \quad (5.56)$$

introducing the conserved *energy momentum tensor*

$$\Theta^{\mu\nu} = \sum_i \partial^\nu \phi_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} - g^{\mu\nu} \mathcal{L} \quad (5.57)$$

with

$$\partial_\mu \Theta^{\mu\nu} = 0 \,. \quad (5.58)$$

Since there are four independent conserved currents, we also have four independent conserved charges

$$P^\nu = \int_{x_0=t} \mathrm{d}^3\vec{x}\, \Theta^{0\nu}(x) \,. \quad (5.59)$$

[4] Assuming that $\delta\phi_i$ and Λ don't contain π_i

The temporal component of these is nothing but the Hamiltonian

$$H = P^0 = \int_{x_0=t} \mathrm{d}^3\vec{x}\,\Theta^{00}(x) = \sum_i \int_{x_0=t} \mathrm{d}^3\vec{x} \left(\partial^0\phi_i \frac{\partial\mathcal{L}}{\partial\partial_0\phi_i} - \mathcal{L}\right) = \sum_i \int_{x_0=t} \mathrm{d}^3\vec{x}\, \left(\pi_i\partial^0\phi_i - \mathcal{L}\right) \quad (5.60)$$

and the spacial components

$$P^j = \sum_i \int_{x_0=t} \mathrm{d}^3\vec{x} \left(\partial^j\phi_i \frac{\partial\mathcal{L}}{\partial\partial_0\phi_i}\right) = \sum_i \int_{x_0=t} \mathrm{d}^3\vec{x}\,\pi_i\partial^j\phi_i \quad (5.61)$$

correspond to the total momentum of the field configuration.

Additional material (not discussed during the lectures):
For example, the Lagrangian density for a neutral scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \quad (5.62)$$

leads to the energy momentum tensor

$$\Theta^{\mu\nu} = \partial^\nu\phi\partial^\mu\phi - \frac{1}{2}g^{\mu\nu}\partial_\kappa\phi\partial^\kappa\phi + \frac{1}{2}g^{\mu\nu}m^2\phi^2 + g^{\mu\nu}V(\phi) \quad (5.63)$$

with

$$\Theta^{00} = \partial^0\phi\partial^0\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi) = \frac{1}{2}\partial^0\phi\partial^0\phi + \frac{1}{2}\vec{\nabla}\phi\vec{\nabla}\phi + \frac{1}{2}m^2\phi^2 + V(\phi) = \mathcal{H} \quad (5.64)$$

the Hamiltonian density.

5.3.2 Internal Symmetries

Lecture 15: Tue, 12. 12. 2017

Suppose that we have a multiplet of fields $\{\phi_i\}_i$ transforming under a representation R of some Lie group with generators $\{T_a\}_a$:

$$\delta_a\phi_i(x) = \mathrm{i}\sum_j \left[R(T_a)\right]_{ij}\phi_j(x) \quad (5.65)$$

or compactly

$$\delta_a\phi(x) = \mathrm{i}T_a\phi(x)\,. \quad (5.66)$$

Such symmetries, which commute with all Poincaré generators of spacetime symmetries, are called *internal symmetries*. There is a theorem by Haag, Łopuszański and Sohnius, that shows that, except for *supersymmetries*, all allowed symmetries of a QFT are a direct product of Poincaré symmetries and internal symmetries. If the Lagrangian density is a singlet under these transformations

$$\delta\mathcal{L} = 0\,, \tag{5.67}$$

we obtain conserved currents

$$j_a^\mu = \sum_i \delta\phi_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} = \mathrm{i}\sum_{ij} \left[R(T_a)\right]_{ij} \phi_j \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} = \mathrm{i}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} T_a\phi \tag{5.68}$$

and conserved charges

$$Q_a = \mathrm{i}\sum_{ij} \int \mathrm{d}^3\vec{x}\, \pi_i(x) \left[R(T_a)\right]_{ij} \phi_j(x) \tag{5.69}$$

with Poisson brackets

$$\{Q_a, Q_b\} = \mathrm{i}\sum_c f_{abc} Q_c \tag{5.70}$$

using the structure constants $[T_a, T_b] = \mathrm{i}\sum_c f_{abc}T_c$.

For example, the Lagrangian density for a charged scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V(\phi^*\phi) \tag{5.71}$$

is invariant under phase rotations

$$\phi \to \mathrm{e}^{\mathrm{i}\alpha}\phi = \phi + \mathrm{i}\alpha\phi + \ldots \tag{5.72a}$$

$$\phi^* \to \mathrm{e}^{-\mathrm{i}\alpha}\phi^* = \phi^* - \mathrm{i}\alpha\phi^* + \ldots \tag{5.72b}$$

$$\tag{5.72c}$$

and the corresponding Noether current is

$$j^\mu = \delta\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + \delta\phi^*\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*} = \mathrm{i}\phi\partial^\mu\phi^* - \mathrm{i}\phi^*\partial^\mu\phi = -\mathrm{i}\phi^*\overleftrightarrow{\partial^\mu}\phi\,, \tag{5.73}$$

the electromagnetic current.

5.4 Gauge Principle

We have seen that constant phase rotations

$$\phi(x) \to \mathrm{e}^{\mathrm{i}\alpha}\phi(x) \tag{5.74}$$

leave the action

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V(\phi^*\phi) \tag{5.75}$$

invariant. If we demand invariance under the more general position dependent phase-rotations, a. k. a. *gauge transformations*,

$$\phi(x) \to \mathrm{e}^{\mathrm{i}g\alpha(x)}\phi(x)\,, \tag{5.76}$$

with a coupling constant g added for later convenience, we see that $\phi^*\phi$ remains invariant, but the terms involving derivatives behave differently

$$\partial_\mu\phi \to \mathrm{e}^{\mathrm{i}g\alpha}\left(\partial_\mu\phi\right) + \left(\partial_\mu \mathrm{e}^{\mathrm{i}g\alpha}\right)\phi = \mathrm{e}^{\mathrm{i}g\alpha}\left(\partial_\mu\phi + \mathrm{i}g\left(\partial_\mu\alpha\right)\phi\right) = \mathrm{e}^{\mathrm{i}g\alpha}\left(\partial_\mu + \mathrm{i}g\partial_\mu\alpha\right)\phi\,. \tag{5.77}$$

If there was only the first term, then also $\partial_\mu\phi^*\partial^\mu\phi$ would be invariant, but including the second, we obtain

$$\begin{aligned}\partial_\mu\phi^*\partial^\mu\phi &\to \left(\partial_\mu\phi^* - \mathrm{i}g\left(\partial_\mu\alpha\right)\phi^*\right)\left(\partial^\mu\phi + \mathrm{i}g\left(\partial^\mu\alpha\right)\phi\right)\\ &= \partial_\mu\phi^*\partial^\mu\phi + \mathrm{i}g\left(\partial_\mu\phi^*\right)\phi\left(\partial^\mu\alpha\right) - \mathrm{i}g\phi^*\left(\partial^\mu\phi\right)\left(\partial_\mu\alpha\right) + g^2\left(\partial_\mu\alpha\right)\left(\partial^\mu\alpha\right)\phi^*\phi\\ &= \partial_\mu\phi^*\partial^\mu\phi + gj_\mu\left(\partial^\mu\alpha\right) + g^2\left(\partial_\mu\alpha\right)\left(\partial^\mu\alpha\right)\phi^*\phi\,,\end{aligned} \tag{5.78}$$

where the appearance of the current

$$j^\mu = -\mathrm{i}\phi^*\overleftrightarrow{\partial^\mu}\phi\,, \tag{5.73}$$

is not an accident. Thus

$$\delta\mathcal{L} = gj_\mu\left(\partial^\mu\alpha\right) + g^2\left(\partial_\mu\alpha\right)\left(\partial^\mu\alpha\right)\phi^*\phi\,, \tag{5.79}$$

which can *not* be written as a total derivative.

5.4.1 Covariant Derivative

The reason for the non-invariance of the terms involving derivatives is that they don't transform covariantly

$$\partial_\mu\phi \to \mathrm{e}^{\mathrm{i}g\alpha}\left(\partial_\mu + \mathrm{i}g\left(\partial_\mu\alpha\right)\right)\phi \neq \mathrm{e}^{\mathrm{i}g\alpha}\partial_\mu\phi \tag{5.80a}$$

or

$$\partial_\mu \to \mathrm{e}^{\mathrm{i}g\alpha}\left(\partial_\mu + \mathrm{i}g\left(\partial_\mu\alpha\right)\right)\mathrm{e}^{-\mathrm{i}g\alpha} \to \mathrm{e}^{\mathrm{i}g\alpha}\partial_\mu\mathrm{e}^{-\mathrm{i}g\alpha} + \mathrm{i}g\left(\partial_\mu\alpha\right) \neq \mathrm{e}^{\mathrm{i}g\alpha}\partial_\mu\mathrm{e}^{-\mathrm{i}g\alpha}\,, \tag{5.80b}$$

where ∂_μ is to be treated as an operator that acts on *everything* on its right hand side, unless protected by parentheses[5]. This can be resolved by

[5] I. e. $\partial_\mu f = [\partial_\mu, f]_- + f\partial_\mu = (\partial_\mu f) + f\partial_\mu$.

introducing a *covariant derivative* D_μ that transforms as

$$D_\mu \phi \to \mathrm{e}^{\mathrm{i}g\alpha} D_\mu \phi \tag{5.81a}$$

or

$$D_\mu \to \mathrm{e}^{\mathrm{i}g\alpha} D_\mu \mathrm{e}^{-\mathrm{i}g\alpha} \tag{5.81b}$$

where D_μ is also treated as an operator. In order to achieve this, the shift by $\mathrm{i}g(\partial_\mu \alpha)$ must be compensated somehow. This suggests to introduce a vector field A_μ with the transformation property[6]

$$A_\mu \to \mathrm{e}^{\mathrm{i}g\alpha} \left(A_\mu + (\partial_\mu \alpha)\right) \mathrm{e}^{-\mathrm{i}g\alpha} = A_\mu + (\partial_\mu \alpha) \tag{5.83}$$

and to define

$$D_\mu = \partial_\mu - \mathrm{i}g A_\mu \,, \tag{5.84}$$

where the shifts compensate. Obviously

$$\mathcal{L} = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi - V(\phi^* \phi) \tag{5.85}$$

is then gauge invariant, i. e. invariant under the *simultaneous* transformations

$$\begin{pmatrix} \phi(x) \\ \phi^*(x) \\ A_\mu(x) \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} \mathrm{e}^{\mathrm{i}g\alpha(x)} \phi(x) \\ \mathrm{e}^{-\mathrm{i}g\alpha(x)} \phi^*(x) \\ A_\mu(x) + \partial_\mu \alpha(x) \end{pmatrix} \tag{5.86}$$

by construction, but we are still left with the problem of interpreting A_μ.

5.4.2 Field Strength

The best interpretation of A_μ is by introducing a dynamics and treating it as the field of a vector particle. For this we need a Lagrangian that is quadratic in first order partial derivatives of A_μ and invariant (up to a total derivative) under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \alpha \,. \tag{5.87}$$

[6]Actually, the expression

$$A_\mu \to \mathrm{e}^{\mathrm{i}g\alpha} \left(A_\mu + g\,(\partial_\mu \alpha)\right) \mathrm{e}^{-\mathrm{i}g\alpha} \tag{5.82}$$

would be correct even if α and the components of A_μ were not just commuting numbers, but generators of a non-abelian Lie group. The resulting *non-abelian gauge theories* are discussed in the courses *Advanced QFT* and *Theoretical Particle Physics*. In the present lecture we will restrict ourselves to the abelian case.

Define the *field strength*

$$F_{\mu\nu} = \frac{\mathrm{i}}{g}\left[D_\mu, D_\nu\right] = \frac{\mathrm{i}}{g}\left[\partial_\mu - \mathrm{i}gA_\mu, \partial_\nu - \mathrm{i}gA_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.88}$$

and we see that it is gauge covariant by construction, even invariant

$$F_{\mu\nu} \to \mathrm{i}\left[\mathrm{e}^{\mathrm{i}\alpha} D_\mu \mathrm{e}^{-\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\alpha} D_\nu \mathrm{e}^{-\mathrm{i}\alpha}\right] = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{i}\left[D_\mu, D_\nu\right]\mathrm{e}^{-\mathrm{i}\alpha} = \mathrm{e}^{\mathrm{i}\alpha} F_{\mu\nu} \mathrm{e}^{-\mathrm{i}\alpha} = F_{\mu\nu}\,. \tag{5.89}$$

One can also see this by explicit calculation

$$\begin{aligned} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \partial_\nu\partial_\mu\alpha \\ = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}\,, \end{aligned} \tag{5.90}$$

because the partial derivatives commute. We can then propose a gauge invariant Lagrangian density

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\phi\right)^* D^\mu\phi - m^2\phi^*\phi - V(\phi^*\phi) \\ &= -\frac{1}{2}\left(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\nu A_\mu\partial^\mu A^\nu\right) + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \\ &\qquad + g\mathrm{i}\left(\phi^*\overleftrightarrow{\partial_\mu}\phi\right)A_\mu + g^2A_\mu A^\mu\phi^*\phi - V(\phi^*\phi) \end{aligned} \tag{5.91}$$

and discuss the resulting EOMs.

5.4.3 Minimal Coupling

In general, we can construct gauge invariant Lagrangian densities by taking a Lagrangian density for *matter fields* that is invariant under rigid transformations, replacing

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}gA_\mu \tag{5.92}$$

everywhere[7] and adding the kinetic term for the *gauge field*

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.93}$$

to the Lagrangian density.

[7]Taking care to write $(D_\mu\phi)^*$ instead of $D_\mu\phi^*$!

5.4.4 Equations of Motion

We find Euler-Lagrange equations for the matter field

$$\begin{aligned} 0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*} - \frac{\partial\mathcal{L}}{\partial\phi^*} &= \partial_\mu \left(D^\mu\phi\right) - \mathrm{i}gA_\mu D^\mu\phi + m^2\phi + V'(\phi^*\phi)\phi \\ &= D_\mu \left(D^\mu\phi\right) + m^2\phi + V'(\phi^*\phi)\phi \\ &= \left(\Box + m^2\right)\phi - \mathrm{i}g\phi\partial_\mu A^\mu - 2\mathrm{i}gA^\mu\partial_\mu\phi - g^2 A_\mu A^\mu\phi + \phi V'(\phi^*\phi) \end{aligned} \tag{5.94a}$$

and for the gauge field

$$\begin{aligned} 0 = \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} - \frac{\partial\mathcal{L}}{\partial A_\nu} &= -\partial_\mu F^{\mu\nu} - \mathrm{i}g\phi^* D^\nu\phi + \mathrm{i}g\left(D^\nu\phi\right)^*\phi \\ &= -\Box A^\nu + \partial^\nu\partial_\mu A^\mu + gJ^\nu \end{aligned} \tag{5.94b}$$

with the "covariantized" current

$$J^\mu = -\mathrm{i}\phi^* \overleftrightarrow{D^\mu}\phi = -\mathrm{i}\phi^* D^\nu\phi + \mathrm{i}\left(D^\nu\phi\right)^*\phi = -\mathrm{i}\phi^* \overleftrightarrow{\partial^\mu}\phi - 2gA^\mu\phi^*\phi\,. \tag{5.95}$$

This current turns out to be the Noether current for the rigid transformations

$$\begin{pmatrix} \phi(x) \\ \phi^*(x) \\ A_\mu(x) \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha}\phi(x) \\ \mathrm{e}^{-\mathrm{i}\alpha}\phi^*(x) \\ A_\mu(x) \end{pmatrix} \tag{5.96}$$

for which $\delta\mathcal{L} = 0$ and

$$J^\mu = \delta\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + \delta\phi^*\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*} = \mathrm{i}\phi\left(D^\mu\phi\right)^* - \mathrm{i}\phi^* D^\mu\phi = -\mathrm{i}\phi^*\overleftrightarrow{D^\mu}\phi\,. \tag{5.97}$$

Takin the divergence of the EOM (5.94b) for the vector field

$$0 = -\partial_\nu\Box A^\nu + \partial_\nu\partial^\nu\partial_\mu A^\mu + g\partial_\nu J^\nu = g\partial_\nu J^\nu \tag{5.98}$$

we see that it is only consistent, if J^μ is indeed conserved.

In the limit $g \to 0$, the EOMs (5.94) become homogeneous

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu\partial_\mu A^\mu = 0 \tag{5.99a}$$

$$\left(\Box + m^2\right)\phi = 0\,, \tag{5.99b}$$

corresponding to a free scalar field of mass m and a vector field A^μ. If we separate the latter as before into a spin-0 part $\partial^\mu\chi$ with χ a scalar field and a spin-1 part we find a tautology for χ

$$\Box\partial^\nu\chi = \partial^\nu\partial_\mu\partial^\mu\chi\,. \tag{5.100}$$

If we impose the subsidiary condition $\partial_\mu A^\mu = 0$ on the spin-1 part, we see that each component is a free massless field. If we wanted to describe a massive field, we would have to add a mass term to the Lagrangian density

$$\mathcal{L}_m = \frac{m^2}{2} A_\mu A^\mu \tag{5.101}$$

which is not gauge invariant, because

$$\delta\mathcal{L}_m = m^2 A_\mu \partial^\mu \alpha \tag{5.102}$$

is not a total derivative. Thus we expect a gauge filed to be always massless.

5.4.5 Maxwell Equations

In general, the EOMs for the antisymmetric gauge field strength $F^{\mu\nu} = -F^{\nu\mu}$ can be written[8]

$$\partial_\mu F^{\mu\nu} = g j^\nu \tag{5.104a}$$
$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0\,, \tag{5.104b}$$

where the second equation follows trivially from the commutativity of partial derivatives and the definition of $F^{\mu\nu}$ in terms of A^μ.

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It is useful to separate the temporal and spacial components

$$\sum_{i=1}^{3} \frac{\partial}{\partial x^i} F^{i0} = g j^0 \tag{5.105a}$$
$$\frac{\partial}{\partial x^0} F^{0i} + \sum_{i=1}^{3} \frac{\partial}{\partial x^j} F^{ji} = g j^i \tag{5.105b}$$
$$\sum_{i,j,k=1}^{3} \underbrace{\epsilon_{0ijk}}_{-\epsilon_{ijk}} \frac{\partial}{\partial x_i} F^{jk} = 0 \tag{5.105c}$$

[8] Using the language of differential forms, these equations read

$$\delta F = *\mathrm{d}{*F} = j \tag{5.103a}$$
$$\mathrm{d}F = 0 \tag{5.103b}$$

and the second is solved by $F = \mathrm{d}A$.

$$\sum_{j,k=1}^{3} \underbrace{\epsilon_{i0jk}}_{=\epsilon_{ijk}} \frac{\partial}{\partial x^0} F^{jk} + \sum_{j,k=1}^{3} \underbrace{\epsilon_{ij0k}}_{=-\epsilon_{ijk}} \frac{\partial}{\partial x_j} F^{0k} + \sum_{j,k=1}^{3} \underbrace{\epsilon_{ijk0}}_{=\epsilon_{ijk}} \frac{\partial}{\partial x_j} F^{k0} = 0 \tag{5.105d}$$

and to introduce the notation

$$F^{k0} = E^k = -F^{0k} \tag{5.106a}$$

$$F^{ij} = -\sum_{k=1}^{3} \epsilon_{ijk} B^k \tag{5.106b}$$

$$j^0 = \rho \nabla^i = \frac{\partial}{\partial x^i} = -\frac{\partial}{\partial x_i} = -\partial^i \tag{5.106c}$$

Then (5.104) and (5.105) turn into

$$\vec{\nabla}\vec{E} = g\rho \tag{5.107a}$$

$$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = g\vec{j} \tag{5.107b}$$

$$\vec{\nabla}\vec{B} = 0 \tag{5.107c}$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0\,, \tag{5.107d}$$

i. e. *Maxwell's equations*. Therefore, the six independent components of the antisymmetric rank-2 field strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{5.108}$$

can be interpreted as the electric and magnetic field strengths of classical electrodynamics with the Noether current acting as electric charge and current densities. The gauge invariance is just the gauge invariance of electrodynamics, when potentials Φ and $\vec{A}$ are introduced to solve the homogeneous Maxwell equations. The Lagrangian density can be expressed

$$\mathcal{L} = \frac{1}{2}\vec{E}^2 - \frac{1}{2}\vec{B}^2\,, \tag{5.109}$$

which is a Lorentz scalar, despite the appearance of $\vec{E}$ and $\vec{B}$.

The gauge principle turns out to be a very concise way to construct classical electrodynamics and it should be helpful in the construction of QED as well. However, we still need to face the problem of constructing massless vector fields of spin-1.

5.4.6 Spinor Electrodynamics

Repeating the exercise for Dirac spinors, we obtain a Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar\psi \left(\mathrm{i} \not{D} - m \right) \psi \tag{5.110}$$

and EOMs

$$\partial_\mu F^{\mu\nu} = g j^\nu = g \bar\psi \gamma^\nu \psi \tag{5.111a}$$

$$\left(\mathrm{i} \not\partial - m \right) \psi = -g \not{A} \psi \,. \tag{5.111b}$$

5.4.7 Canonical Quantization

Trying to set up a canonical formalism, we encounter a problem: the conjugate momenta

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 A^\mu} = -F_{0\mu} = F_{\mu 0} \tag{5.112}$$

contain one component π^0 that vanishes identically by construction

$$\pi_0 = 0 \,, \tag{5.113a}$$

which is of course incompatible with canonical poisson brackets

$$\left\{ A_\mu(x), \pi_\nu(y) \right\}_{x^0 = y^0} = -g_{\mu\nu} \delta^3(\vec x - \vec y) + \ldots \neq 0 \tag{5.113b}$$

and CCRs

$$\left[A_\mu(x), \pi_\nu(y) \right]_{-, x^0 = y^0} = -\mathrm{i} g_{\mu\nu} \delta^3(\vec x - \vec y) + \ldots \neq 0 \,. \tag{5.113c}$$

Treating $\partial_\mu A^\mu = 0$ as constraint, we can add

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\xi} \left(\partial_\mu A^\mu \right)^2 \tag{5.114}$$

as *gauge fixing* term to the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu \right)^2 \tag{5.115}$$

Classically, the term $\mathcal{L}_{\text{g.f.}}$ can be viewed as a Lagrange multiplier for the subsidiary condition $\partial_\mu A^\mu = 0$. In the quantum theory, we will show that observable quantities are independent of the parameter ξ. Indeed,

$$\pi_0 = \frac{\partial \mathcal{L}}{\partial \partial_0 A^0} = \frac{\partial \mathcal{L}_{\text{g.f.}}}{\partial \partial_0 A^0} = -\frac{1}{\xi} \partial^\mu A_\mu \,, \tag{5.116}$$

which only vanishes after we impose the condition $\partial_\mu A^\mu = 0$ on physical states in the very end. The simplest case, known as *Feynman gauge* is the choice $\xi = 1$, for which

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\mu A^\mu \partial_\nu A^\nu\right) \\ = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \text{total derivatives}\,.\end{aligned} \tag{5.117}$$

describes four independent massless neutral scalar fields.

Nevertheless, four independent fields are two too many for electrodynamics, where the photons have only two independent polarization states. However, simply enforcing $\partial_\mu A^\mu = 0$ as an operator equation runs afoul of non-trivial commutation relations for π_0.

The way out of this conundrum is to allow

$$\partial_\mu A^\mu \neq 0 \tag{5.118}$$

in a big Hilbert space that may contain unphysical states, but to require

$$\left(\partial_\mu A^\mu\right)^{(+)} |\text{physical}\rangle = 0\,, \tag{5.119}$$

i. e. that the positive energy, a. k. a. annihilation, part of $\partial_\mu A^\mu$ annihilates *all* physical states, not only the vacuum. Adding the adjoint, this is equivalent to

$$\langle\text{physical}|\partial_\mu A^\mu|\text{physical}'\rangle = 0\,. \tag{5.120}$$

We can consistently require this because the EOM for the gauge field in the presence of $\mathcal{L}_{\text{g.f.}}$ is

$$\Box A^\nu - \frac{\xi - 1}{\xi}\partial^\nu \partial_\mu A^\mu = gJ^\nu \tag{5.121}$$

and taking the divergence

$$\frac{1}{\xi}\Box \partial_\mu A^\mu = g\partial_\nu J^\nu = 0 \tag{5.122}$$

we see that $\partial_\mu A^\mu$ is a *free* scalar field, even if coupled to a conserved current. Thus it can be decomposed into independent creation and annihilation parts.

Feynman Gauge

For simplicity, we will choose $\xi = 1$ from now on. The general case can be handled similarly, except for $\xi \to 0$, where special constructions [8] are required. The Lagrangian density and conjugate momenta are

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2 \tag{5.123a}$$

$$\pi^0 = \pi_0 = -\partial_\mu A^\mu \tag{5.123b}$$

$$\pi^i = -\pi_i = F_{0i}\,. \tag{5.123c}$$

The commutation relations for four independent component fields

$$[A_\mu(x), A_\nu(y)]_- = -\mathrm{i}g_{\mu\nu}\Delta(x-y)\,, \tag{5.124}$$

i. e.

$$[A_\mu(x), A_\nu(y)]_{-,x_0=y_0} = [A_\mu(x), \partial_i A_\nu(y)]_{-,x_0=y_0} = 0 \tag{5.125a}$$

$$[A_\mu(x), \partial_0 A_\nu(y)]_{-,x_0=y_0} = -\mathrm{i}g_{\mu\nu}\delta^3(\vec x - \vec y) \tag{5.125b}$$

are compatible with

$$[A_\mu(x), \pi^\nu(y)]_{-,x_0=y_0} = \mathrm{i}\delta_\mu{}^\nu\delta^3(\vec x - \vec y) \tag{5.126}$$

since

$$\left[A_0(x), \pi^0(y)\right]_{-,x_0=y_0} = -\left[A_0(x), \partial_0 A_0(y)\right]_{-,x_0=y_0} + \sum_{i=1}^{3}\underbrace{\left[A_0(x), \partial_i A_i(y)\right]_{-,x_0=y_0}}_{=0} = \mathrm{i}\delta^3(\vec x - \vec y) \tag{5.127a}$$

and

$$\left[A_i(x), \pi^j(y)\right]_{-,x_0=y_0} = \underbrace{\left[A_i(x), \partial^j A^0(y)\right]_{-,x_0=y_0}}_{=0} - \left[A_i(x), \partial^0 A^j(y)\right]_{-,x_0=y_0} = \left[A_i(x), \partial_0 A_j(y)\right]_{-,x_0=y_0} = \mathrm{i}\delta_{ij}\delta^3(\vec x - \vec y)\,. \tag{5.127b}$$

The commutation relations (5.124) are realized by massless vector fields

$$A^\mu(x) = \sum_{\sigma=0}^{3}\int\widetilde{\mathrm{d}p}\,\left(u^\mu_\sigma(p)a_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} + v^\mu_\sigma(p)a^\dagger_\sigma(p)\mathrm{e}^{\mathrm{i}xp}\right) \tag{5.128}$$

with polarization vectors at the reference momentum k

$$u^\mu_\sigma(k) = v^\mu_\sigma(k) = \epsilon^\mu_\sigma(k) = \delta^\mu_\sigma \tag{5.129}$$

and creation- and annihilation operators with CCRs

$$\left[a_\sigma(p), a^\dagger_{\sigma'}(p')\right]_- = -g_{\sigma\sigma'}(2\pi)^3 2|\vec p|\delta^3(\vec p - \vec p') \tag{5.130a}$$

$$[a_\sigma(p), a_{\sigma'}(p')]_- = \left[a^\dagger_\sigma(p), a^\dagger_{\sigma'}(p')\right]_- = 0\,, \tag{5.130b}$$

where the sign of the time-like commutator is dictated by the Lorentz covariance of the commutation relations (5.124).

Gupta-Bleuler Formalism

We have not yet addressed the issue of the two superfluous polarization states. Furthermore, if we compute the norm of the states

$$|f,\sigma\rangle = \int \widetilde{\mathrm{d}p}\, f^*(p) a^\dagger_\sigma(p)\,|0\rangle\ , \tag{5.131}$$

we find

$$\langle f,\sigma|f,\sigma'\rangle = \int \widetilde{\mathrm{d}p}\widetilde{\mathrm{d}p'}\, f(p) f^*(p') \left\langle 0\middle| a_\sigma(p) a^\dagger_{\sigma'}(p')\middle|0\right\rangle = -g_{\sigma\sigma'}\int\widetilde{\mathrm{d}p}\,|f(p)|^2\ , \tag{5.132}$$

i. e.

$$\langle f,0|f,0\rangle < 0\,, \tag{5.133}$$

which is impossible in a proper Hilbert space. The positive energy part of $\partial^\mu A_\mu$

$$\left(\partial_\mu A^\mu\right)^{(+)}(x) = -\mathrm{i}\sum_{\sigma=0}^{3}\int\widetilde{\mathrm{d}p}\, p_\mu \epsilon^\mu_\sigma(p) a_\sigma(p)\mathrm{e}^{-\mathrm{i}px} \tag{5.134}$$

contains only $a_0(p)$ and $a_3(p)$, as can be seen from

$$p_\mu\epsilon^\mu_\sigma(p) = k_\mu\epsilon^\mu_\sigma(k) = \omega\left(\delta_{\sigma0} - \delta_{\sigma3}\right) \tag{5.135}$$

for the reference momentum $k = (\omega, 0, 0, \omega)$. The condition

$$\left(\partial_\mu A^\mu\right)^{(+)}|\text{physical}\rangle = 0 \tag{5.136}$$

is thus equivalent to

$$\forall p, p^2 = 0 : (a_3(p) - a_0(p)) \left|\text{physical}\right\rangle = 0\,. \tag{5.137}$$

Note that

$$\left[a_3(p) - a_0(p), a_1^\dagger(p')\right] = 0 \tag{5.138a}$$

$$\left[a_3(p) - a_0(p), a_2^\dagger(p')\right] = 0 \tag{5.138b}$$

$$\left[a_3(p) - a_0(p), a_3^\dagger(p') - a_0^\dagger(p')\right] = 0 \tag{5.138c}$$

$$\left[a_3(p) - a_0(p), a_3^\dagger(p') + a_0^\dagger(p')\right] = 2(2\pi)^3 2|\vec p|\delta^3(\vec p - \vec p') \tag{5.138d}$$

and physical states are created by the operators $a_1^\dagger(p)$, $a_2^\dagger(p)$ and $a_3^\dagger(p) - a_0^\dagger(p)$, with all states constructed with at least one $a_3^\dagger(p) - a_0^\dagger(p)$ having zero norm and vanishing matrix elements with physical states

$$\left\langle\text{physical}\middle|\left(a_3^\dagger(p) - a_0^\dagger(p)\right)\middle|\Psi\right\rangle = \langle\Psi|(a_3(p) - a_0(p))|\text{physical}\rangle^* = 0\,. \tag{5.139}$$

we can therefore factor these states from the physical Hilbert space

$$\mathcal{H}_{\text{physical}} = \left\{|\Psi\rangle \in \mathcal{H} : (\partial_\mu A^\mu)^{(+)} |\Psi\rangle = 0\right\} \Big/ (\partial_\mu A^\mu)^{(-)}\, \mathcal{H} \tag{5.140}$$

leaving only two physical states per momentum, both corresponding to transversal polarization vectors and having positive norm.

—6—
S-Matrix and Cross Section

Lecture 17: Tue, 19. 12. 2017

The *probability amplitude* for measuring the state $|\text{outgoing}, t\rangle$ after preparing the state $|\text{incoming}, t\rangle$ is

$$A_{\text{incoming}\to\text{outgoing}} = \langle\text{outgoing}, t|\text{incoming}, t\rangle \,, \tag{6.1}$$

where the preparation has been performed at $t \to -\infty$ and the measurement is performed at $t \to +\infty$. Therefore, in order to compute

$$A_{\text{incoming}\to\text{outgoing}} = \langle\text{outgoing}, +\infty|\text{incoming}, +\infty\rangle \,, \tag{6.2}$$

we need to be able to compute $|\text{incoming}, +\infty\rangle$ from $|\text{incoming}, -\infty\rangle$. According to the rules of QM, time evolution is linear and unitary. Thus there has to be a unitary operator S

$$|\text{incoming}, +\infty\rangle = S\,|\text{incoming}, -\infty\rangle \tag{6.3a}$$
$$S^\dagger S = SS^\dagger = \mathbf{1}\,, \tag{6.3b}$$

the *S-matrix*. Furthermore, we will find a prescription for computing S from a given Hamiltonian or Lagrangian density.

Note that we have to be careful when labelling the states $|\text{incoming}, t\rangle$ and $|\text{outgoing}, t\rangle$ with QNs that are not conserved by the interactions. If we prepared a state in the distant past, we can expand it in states

$$|p_1, \alpha_1, p_2, \alpha_2, \ldots; \text{in}; t\rangle$$

corresponding to non-interacting particles with momenta and QNs $\{p_i, \alpha_i\}_i$ in the limit $t \to -\infty$. Similarly, there are states

$$|p_1, \alpha_1, p_2, \alpha_2, \ldots; \text{out}; t\rangle$$

corresponding to non-interacting particles with momenta and QNs $\{p_i, \alpha_i\}_i$ in the limit $t \to +\infty$. These states belong to the same Hilbert space, but in general

$$|p_1, \alpha_1, p_2, \alpha_2, \ldots; \text{in}; t\rangle \neq |p_1, \alpha_1, p_2, \alpha_2, \ldots; \text{out}; t\rangle \;, \tag{6.4}$$

because momenta and QNs may be distributed among the particles differently in the past and in the future. We may only assume that the vacuum state is unique

$$|0; \text{out}\rangle = |0; \text{in}\rangle = |0\rangle \tag{6.5a}$$

and that one-particle states are not affected by the interaction

$$|p, \alpha; \text{out}\rangle = |p, \alpha; \text{in}\rangle \;. \tag{6.5b}$$

Note that the one-particle states may be *stable* bound states of elementary particles, but not unstable resonances.

In addition, we shall assume *asymptotic completeness*, i. e. that both sets of asymptotic states span the whole Hilbert space.

6.1 Schrödinger Picture

The state $|\text{in}, t\rangle$ is the solution of a Schrödinger equation with asymptotic initial condition

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} |\text{in}, t\rangle = H(t) |\text{in}, t\rangle \tag{6.6a}$$

$$\lim_{t \to -\infty} |\text{in}, t\rangle = |\text{in}, -\infty\rangle \;. \tag{6.6b}$$

This Schrödinger equation can be solved by introducing a unitary Schrödinger picture *time evolution operator* $U_S(t, t_0)$ that satisfies

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U_S(t, t_0) = H(t) U_S(t, t_0) \tag{6.7a}$$

$$U_S(t_0, t_0) = \mathbf{1} \tag{6.7b}$$

$$U_S^\dagger(t, t_0) U_S(t, t_0) = U_S(t, t_0) U_S^\dagger(t, t_0) = \mathbf{1} \tag{6.7c}$$

$$U_S(t, t') U_S(t', t_0) = U_S(t, t_0) \tag{6.7d}$$

and we find

$$S = U_S(\infty, -\infty) = \lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} U_S(t, t_0) \,. \tag{6.8}$$

When the Hamiltonian H is time independent, we simply have

$$U_S(t,t_0) = \mathrm{e}^{-\mathrm{i}H\cdot(t-t_0)}\,. \tag{6.9}$$

The total Hamiltonian *is* always constant for autonomous systems, but we will need the general case later.

However, for constant Hamiltonians, the naive limit

$$S = \lim_{\substack{t\to+\infty\\ t_0\to-\infty}} \mathrm{e}^{-\mathrm{i}H\cdot(t-t_0)} \tag{6.10}$$

is not well defined because of the undamped oscillations. One way to solve this problem would be to adiabatically switch off the Hamiltonian with a dampening factor for $t\to\pm\infty$. But this not the correct solution, because even non-interacting one-particle states constructed in chapter 4 contribute undamped oscillatory factors

$$|p,\alpha;t\rangle = \mathrm{e}^{-\mathrm{i}p_0\cdot(t-t_0)}\,|p,\alpha;t_0\rangle\ . \tag{6.11}$$

6.2 Interaction Picture

We should rather take advantage of the fact that we have already solved the dynamical problem for the asymptotic states. Indeed, we can expect

$$U(t,t_0) = \mathrm{e}^{\mathrm{i}H_0t}\mathrm{e}^{-\mathrm{i}H\cdot(t-t_0)}\mathrm{e}^{-\mathrm{i}H_0t_0} = \mathrm{e}^{\mathrm{i}H_0t}\mathrm{e}^{-\mathrm{i}Ht}\mathrm{e}^{\mathrm{i}Ht_0}\mathrm{e}^{-\mathrm{i}H_0t_0} = \Omega^\dagger(t)\Omega(t_0) \tag{6.12}$$

with the *Møller operator*

$$\Omega(t) = \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t} \tag{6.13}$$

to have a well defined limit for $t\to\infty$ and $t_0\to-\infty$, provided H_0 describes the asymptotic dynamics for $t\to\pm\infty$ and $H-H_0$ is negligible for asymptotic states, typically because the free particles are widely separated for $t\to\pm\infty$ and the interaction has a finite range.

The Møller operator $\Omega(t)$ can be interpreted as first evolving a state from $t=0$ to time t using the dynamics derived from the Hamiltonian H_0 and then evolving back to $t=0$ using the dynamics derived from the Hamiltonian H. If H_0 is the Hamiltonian describing the noninteracting asymptotic states and H is the full Hamiltonian, then

$$\Omega(-\infty) = \lim_{t\to-\infty}\Omega(t) \tag{6.14}$$

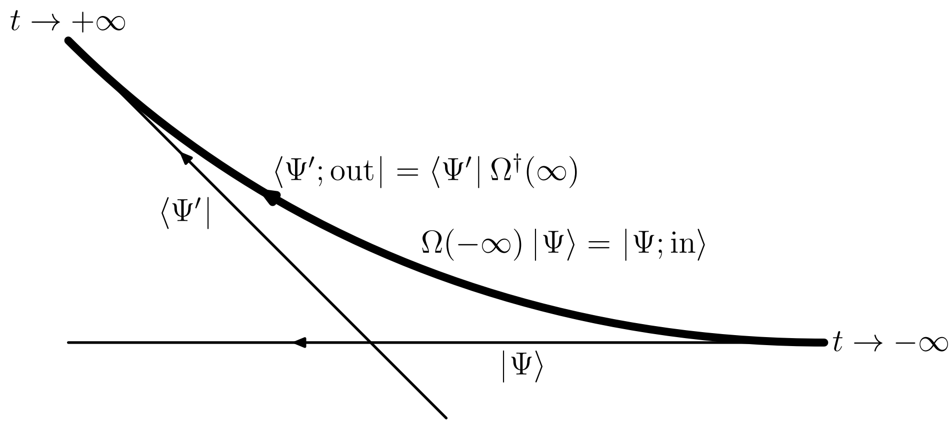


Figure 6.1: *Schematic action of the Møller operators and the S matrix.*

transforms a noninteracting state to the interacting state with matching quantum numbers (energies, momenta, angular momenta, charges, ...) in the past. Then

$$\Omega^\dagger(\infty) \tag{6.15}$$

performs the corresponding transformation for the outgoing state and

$$S = \Omega^\dagger(\infty)\Omega(-\infty) \tag{6.16}$$

is the correct S matrix in the space of noninteracting time-dependent states (cf. figure 6.1).

The time evolution operator $U(t, t_0)$ in (6.12) satisfies the differential equation

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(t,t_0) &= \mathrm{e}^{\mathrm{i}H_0 t} \left(H - H_0 \right) \mathrm{e}^{-\mathrm{i}H\cdot(t-t_0)} \mathrm{e}^{-\mathrm{i}H_0 t_0} \\ &= \mathrm{e}^{\mathrm{i}H_0 t} \left(H - H_0 \right) \mathrm{e}^{-\mathrm{i}H_0 t} \mathrm{e}^{\mathrm{i}H_0 t} \mathrm{e}^{-\mathrm{i}H\cdot(t-t_0)} \mathrm{e}^{-\mathrm{i}H_0 t_0} = H_I(t) U(t,t_0) \end{aligned} \tag{6.17}$$

with the time dependent interaction Hamiltonian

$$H_I(t) = \mathrm{e}^{\mathrm{i}H_0 t} \left(H - H_0 \right) \mathrm{e}^{-\mathrm{i}H_0 t} = \mathrm{e}^{\mathrm{i}H_0 t} H \mathrm{e}^{-\mathrm{i}H_0 t} - H_0 \,. \tag{6.18}$$

The group properties (6.7) remain valid

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} U(t,t_0) = H_I(t) U(t,t_0) \tag{6.19a}$$

$$U(t_0,t_0) = \mathbf{1} \tag{6.19b}$$

$$U^\dagger(t,t_0)U(t,t_0) = U(t,t_0)U^\dagger(t,t_0) = \mathbf{1} \tag{6.19c}$$
$$U(t,t')U(t',t_0) = U(t,t_0) \tag{6.19d}$$

and we can compute the S-matrix by solving the differential equation (6.19). The interaction Hamiltonian (6.22) can be obtained by subtracting the bilinear pieces H_0 in H that describe the time evolution of the asymptotic fields

$$\phi(t,\vec x) = \mathrm{e}^{\mathrm{i}H_0 t}\phi(0,\vec x)\mathrm{e}^{-\mathrm{i}H_0 t)} \tag{6.20}$$

and inserting these time dependent fields in place of the time independent Schrödinger picture fields into

$$H = H_0 - \int \mathrm{d}^3\vec x\,\mathcal{L}_I(\phi(,\vec x)) \tag{6.21}$$

i. e.

$$H_I(t) = -\mathrm{e}^{\mathrm{i}H_0 t}\int \mathrm{d}^3\vec x\,\mathcal{L}_I(\phi(0,\vec x))\mathrm{e}^{-\mathrm{i}H_0 t} = -\int \mathrm{d}^3\vec x\,\mathcal{L}_I(\phi(t,\vec x))\,. \tag{6.22}$$

6.3 Perturbation Theory

Unfortunately, the exact solution of (6.19) can always never be found. Instead, we will use perturbation theory by writing

$$H = H_0 + \lambda H_I = H_0 - \lambda\int \mathrm{d}^3\vec x\,\mathcal{L}_I \tag{6.23}$$

and expanding U in powers of λ. The differential equation with initial condition (6.19) is equivalent to the integral equation

$$U(t,t_0) = \mathbf{1} - \mathrm{i}\lambda\int_{t_0}^t \mathrm{d}t_1\,H_I(t_1)U(t_1,t_0)\,, \tag{6.24}$$

which admits the recursive expansion

$$\begin{aligned} U(t,t_0) &\\ = \mathbf{1} - \mathrm{i}\lambda\int_{t_0}^t \mathrm{d}t_1\,H_I(t_1) - \lambda^2\int_{t_0}^t \mathrm{d}t_2\int_{t_0}^{t_2}\mathrm{d}t_1\,H_I(t_2)H_I(t_1)U(t_1,t_0) &\\ = \mathbf{1} - \mathrm{i}\lambda\int_{t_0}^t \mathrm{d}t_1\,H_I(t_1) - \lambda^2\int_{t_0}^t \mathrm{d}t_2\int_{t_0}^{t_2}\mathrm{d}t_1\,H_I(t_2)H_I(t_1) &\\ + \mathrm{i}\lambda^3\int_{t_0}^t \mathrm{d}t_3\int_{t_0}^{t_3}\mathrm{d}t_2\int_{t_0}^{t_2}\mathrm{d}t_1\,H_I(t_3)H_I(t_2)H_I(t_1) + \mathcal{O}(\lambda^4)\,. \end{aligned} \tag{6.25}$$

The nth order term is obviously

$$
\begin{aligned}
&(-\mathrm{i}\lambda)^n \int_{t_0}^{t}\mathrm{d}t_n \int_{t_0}^{t_n}\mathrm{d}t_{n-1}\ldots\int_{t_0}^{t_2}\mathrm{d}t_1\, H_I(t_n)H_I(t_{n-1})\cdots H_I(t_1)\\
&\quad = (-\mathrm{i}\lambda)^n \int_{t_0}^{t}\prod_{i=1}^{n}\mathrm{d}t_i \left(\prod_{i=1}^{n-1}\Theta(t_{i+1}-t_i)\right) H_I(t_n)H_I(t_{n-1})\cdots H_I(t_1)\,, \quad (6.26)
\end{aligned}
$$

where we have avoided the notation $\prod_{i=1}^n H_I(t_i)$, because operator ordering is important and the time must be decreasing from left to right.

We can introduce *time ordering* of time-dependent operators

$$
\mathrm{T}\left[A(t)B(t')\right] = \Theta(t-t')A(t)B(t') \pm \Theta(t'-t)B(t')A(t)\,, \tag{6.27a}
$$

and also of of local operators

$$
\mathrm{T}\left[A(x)B(y)\right] = \Theta(x_0-y_0)A(x)B(y) \pm \Theta(y_0-x_0)B(y)A(x)\,, \tag{6.27b}
$$

where the lower sign is to be used if and only if both A and B are fermionic. The generalization to n-fold products is obvious and amounts to ordering operators decreasing in time from left to right, keeping track of signs from fermions. Note that (6.27b) is only well defined for local causal fields that (anti-)commute at space-like distances, because the time-ordering is only unambigous for time-like distances. Note that

$$
\mathrm{T}\left[A(x)B(y)\right] = \pm\,\mathrm{T}\left[B(y)A(x)\right]\,, \tag{6.28}
$$

i. e. *all* operators (anti-)commute *under* the time-ordering symbol.

Using this notation, the nth-order term (6.26) can be written more compactly as

$$
\frac{(-\mathrm{i}\lambda)^n}{n!}\int_{t_0}^{t}\prod_{i=1}^{n}\mathrm{d}t_i\;\mathrm{T}\left[\prod_{i=1}^{n}H_I(t_i)\right] = \mathrm{T}\left[\frac{(-\mathrm{i}\lambda)^n}{n!}\prod_{i=1}^{n}\left(\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right)\right]\,, \tag{6.29}
$$

where the $n!$ in the denominator cancels the multiple counting of the same term from $n!$ permutations of the n operators under the time ordering. Now we can write the whole time evolution operator as *Dyson series*

$$
U(t,t_0) = \mathrm{T}\left[\exp\left(-\mathrm{i}\lambda\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right)\right] \tag{6.30}
$$

and the S-matrix

$$
S = \mathrm{T}\left[\exp\left(-\mathrm{i}\lambda\int\mathrm{d}t\,H_I(t)\right)\right] = \mathrm{T}\left[\exp\left(\mathrm{i}\lambda\int\mathrm{d}^4x\,\mathcal{L}_I(x)\right)\right]\,. \tag{6.31}
$$

Note that (6.30) has a very simple interpretation as a solution of (6.19). The initial condition $U(t_0, t_0) = \mathbf{1}$ is obvious, because the integration domain in the exponent vanishes. Since the Hamiltonians are bosonic and commute under the time ordering and operators from to upper limit of the integration domain are always moved to the left hand side

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0) = \mathrm{T}\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(-\mathrm{i}\lambda\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right)\right] \\ = \mathrm{T}\left[\exp\left(-\mathrm{i}\lambda\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right)\lambda H_I(t)\right] \\ = \lambda H_I(t)\,\mathrm{T}\left[\exp\left(-\mathrm{i}\lambda\int_{t_0}^{t}\mathrm{d}t'\,H_I(t')\right)\right] = \lambda H_I(t) U(t,t_0)\,. \end{aligned} \tag{6.32}$$

In the next chapter, we will derive the *Feynman rules* for the efficient perturbative evaluation of matrix elements of (6.31) order by order in λ.

6.4 Poincaré Invariance

Lecture 18: Wed, 20. 12. 2017

Acting on the asymptotic states from chapter 3 with the asymptotic representation U_0 of the Poincaré group, we find

$$\begin{aligned}\langle q_1,\beta_1,q_2,\beta_2,\ldots|U_0^{-1}(\Lambda,0)SU_0(\Lambda,0)|p_1,\alpha_1,p_2,\alpha_2,\ldots\rangle \\ = \langle \Lambda q_1,\beta_1',\Lambda q_2,\beta_2',\ldots|S|\Lambda p_1,\alpha_1',\Lambda p_2,\alpha_2',\ldots\rangle \\ = \langle q_1,\beta_1,q_2,\beta_2,\ldots|S|p_1,\alpha_1,p_2,\alpha_2,\ldots\rangle \end{aligned} \tag{6.33a}$$

and

$$\begin{aligned}\langle q_1,\beta_1,q_2,\beta_2,\ldots|U_0^{-1}(\mathbf{1},a)SU_0(\mathbf{1},a)|p_1,\alpha_1,p_2,\alpha_2,\ldots\rangle \\ = \mathrm{e}^{\mathrm{i}a\left(\sum_i p_i - \sum_i q_i\right)}\langle q_1,\beta_1,q_2,\beta_2,\ldots|S|p_1,\alpha_1,p_2,\alpha_2,\ldots\rangle \\ = \mathrm{e}^{\mathrm{i}\phi}\langle q_1,\beta_1,q_2,\beta_2,\ldots|S|p_1,\alpha_1,p_2,\alpha_2,\ldots\rangle \end{aligned} \tag{6.33b}$$

because the probability amplitudes must not change when the *same* Poincaré transformation is applied to the initial and final states. From (6.33a), we infer

$$U_0^{-1}(\Lambda,0)SU_0(\Lambda,0) = S\,, \tag{6.34a}$$

while for (6.33b), we must also consider the action on wave pakets, where the a-dependence does not cancel in the modulus, leading to

$$\sum_i p_i - \sum_i q_i = 0 \tag{6.34b}$$

and

$$U_0^{-1}(\mathbf{1},a)SU_0(\mathbf{1},a) = S\,. \tag{6.34c}$$

All together, we have overall momentum conservation and

$$U_0^{-1}(\Lambda,a)SU_0(\Lambda,a) = S\,. \tag{6.35}$$

The Dyson series for the S-matrix in the interaction picture

$$S = \mathrm{T}\left[\exp\left(\mathrm{i}\lambda\int \mathrm{d}^4x\,\mathcal{L}_I(x)\right)\right] \tag{6.31}$$

satisfies (6.35), if $\mathcal{L}_I$ is both a scalar

$$U(\Lambda,a)\mathcal{L}_I(x)U^{-1}(\Lambda,a) = \mathcal{L}_I(\Lambda x + a)\,, \tag{6.36a}$$

making the exponent invariant upto operator ordering, and local

$$[\mathcal{L}_I(x),\mathcal{L}_I(x')]_- = 0 \qquad \text{if } (x-x')^2 < 0\,, \tag{6.36b}$$

making the operator ordering independent of the inertial system.

6.5 Cross Section

It is customary to extract the "no-interaction" piece from the S-matrix

$$S = \mathbf{1} + \mathrm{i}T \tag{6.37}$$

and the overall four-momentum conservation allows us to write

$$\langle f|T|i\rangle = (2\pi)^4\delta^4(p_f - p_i)\,\langle f|\mathcal{T}|i\rangle\;, \tag{6.38}$$

where the factors i and $(2\pi)^4$ are conventional. Obviously the δ^4 makes the square of the transition matrix elements ill-defined for plane waves as eigenstates of the four-momentum

$$|A_{i\to f}|^2 \propto \text{“}\delta^4(p_f-p_i)\delta^4(p_f-p_i)\text{”} + \ldots \propto \text{“}\delta^4(0)\text{”} + \ldots$$

and we have to look at wave packets as normalizable states instead.

In practical application we will only consider $2 \to n$ scatterings and $1 \to n$ decays of unstable particles[1], therefore we may be more explicit in the description of the initial state.

[1]Note that there is a tension between "unstable" and thr limit $t \to -\infty$ in th definition of the S-matrix . . .

A normalizable state of two distinguishable scalar particles can be written

$$|g_1,g_2\rangle = \int \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2}\, g_1(p_1)g_2(p_2)\,|p_1,p_2\rangle \\ = \int \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2}\, g_1(p_1)g_2(p_2)a_1^\dagger(p_1)a_2^\dagger(p_2)\,|0\rangle\,. \tag{6.39}$$

The corresponding wave packets

$$\tilde g_i(x) = \int \widetilde{\mathrm{d}p}\,\mathrm{e}^{-\mathrm{i}xp}g_i(p) \tag{6.40}$$

are solutions of the Klein-Gordon equation. Assuming that the final state $|f\rangle$ is a four-momentum eigenstate with eigenvalue p_f and ignoring the part of the initial state that is unaffected by the scattering and will therefore typically not be observed, we have

$$\langle f|S|p_1,p_2\rangle = (2\pi)^4\delta^4(p_f-p_1-p_2)\mathrm{i}\,\langle f|\mathcal{T}|p_1,p_2\rangle \tag{6.41}$$

and

$$\langle f|S|g_1,g_2\rangle = (2\pi)^4\mathrm{i}\int \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2}\,\delta^4(p_f-p_1-p_2)g_1(p_1)g_2(p_2)\,\langle f|\mathcal{T}|p_1,p_2\rangle\,. \tag{6.42}$$

Then

$$\left|\langle f|S|g_1,g_2\rangle\right|^2 = \\ (2\pi)^8\int \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2}\widetilde{\mathrm{d}q_1}\widetilde{\mathrm{d}q_2}\, g_1^*(q_1)g_2^*(q_2)g_1(p_1)g_2(p_2) \\ \underbrace{\delta^4(p_f-q_1-q_2)\delta^4(p_f-p_1-p_2)}_{=\delta^4(p_f-p_1-p_2)\delta^4(q_1+q_2-p_1-p_2)}\langle f|\mathcal{T}|q_1,q_2\rangle^*\,\langle f|\mathcal{T}|p_1,p_2\rangle \\ = \int \mathrm{d}^4x \int \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2}\widetilde{\mathrm{d}q_1}\widetilde{\mathrm{d}q_2}\,\mathrm{e}^{\mathrm{i}x(q_1+q_2-p_1-p_2)}g_1^*(q_1)g_2^*(q_2)g_1(p_1)g_2(p_2) \\ (2\pi)^4\delta^4(p_f-p_1-p_2)\,\langle f|\mathcal{T}|q_1,q_2\rangle^*\,\langle f|\mathcal{T}|p_1,p_2\rangle \tag{6.43}$$

and if we assume that the wave packets g_i are narrowly peaked around a momenta $\bar p_i$, such that $\langle f|\mathcal{T}|p_1,p_2\rangle$ is slowly varying where the g_i are not vanishing

$$\langle f|\mathcal{T}|p_1,p_2\rangle \approx \langle f|\mathcal{T}|\bar p_1,\bar p_2\rangle \approx \langle f|\mathcal{T}|q_1,q_2\rangle \tag{6.44}$$

we find for the transition probability

$$\left|\langle f|S|g_1,g_2\rangle\right|^2 = \int \mathrm{d}^4x\;\left|\tilde g_1(x)\right|^2\left|\tilde g_2(x)\right|^2(2\pi)^4\delta^4(p_f-\bar p_1-\bar p_2)\left|\langle f|\mathcal{T}|\bar p_1,\bar p_2\rangle\right|^2\,. \tag{6.45}$$

Then the integrand can be interpretated as a well defined transition probability density, i. e. a transition rate per volume

$$|\langle f|S|g_1, g_2\rangle|^2 = \int \mathrm{d}^4x\, P_{g_1,g_2\to f}(x) \tag{6.46}$$

and the factor "$(2\pi)^4\delta^4(0)$" appearing in the naive squaring of the matrix element is revealed as the volume of space time that we would recover for plane waves with constant $g_i(x)$.

The Fourier transforms of the g_i that are narrowly peaked in momentum space lead to an approximate probability current density

$$j_i^\mu = \mathrm{i}\tilde g_i^*(x)\overleftrightarrow{\partial_0}\tilde g_i(x) \approx 2\bar p^\mu|\tilde g_i(x)|^2\,. \tag{6.47}$$

Assuming that particle 2 is massive, we can go into the rest frame of the momentum peak

$$\bar p_2 = (m_2, \vec 0) \tag{6.48}$$

and find the target density

$$n_2(x) = 2m_2|\tilde g_2(x)|^2\,. \tag{6.49a}$$

On the other hand, the flux density of incoming particles 1 is

$$\vec j_1(x) = 2\vec{\bar p}_1|\tilde g_1(x)|^2\,. \tag{6.49b}$$

Normalizing the transition probability density by the target density and incoming flux density we find

$$\frac{P_{g_1,g_2\to f}(x)}{|\vec j_1(x)|n_2(x)} = \frac{1}{4|\vec{\bar p}_1|m_2}(2\pi)^4\delta^4(p_f-\bar p_1-\bar p_2)\,|\langle f|\mathcal{T}|\bar p_1,\bar p_2\rangle|^2 \tag{6.50}$$

and using

$$(\bar p_1\bar p_2)^2 - m_1^2m_2^2 = \left(\vec{\bar p}_1^2 + m_1^2\right)m_2^2 - m_1^2m_2^2 = \vec{\bar p}_1^2m_2^2 \tag{6.51}$$

this expression can be expressed in a Lorentz covariant form

$$\frac{P_{g_1,g_2\to f}(x)}{|\vec j_1(x)|n_2(x)} = \frac{1}{4\sqrt{(\bar p_1\bar p_2)^2 - m_1^2m_2^2}}(2\pi)^4\delta^4(p_f-\bar p_1-\bar p_2)\,|\langle f|\mathcal{T}|\bar p_1,\bar p_2\rangle|^2\,. \tag{6.52}$$

Parametrizing the final state $|f\rangle$ by momentum eigenstates and using the proper Lorentz covariant densities of final states in *Fermi's golden rule*, we obtain the *differential cross section*

$$\mathrm{d}\sigma_{2\to n}(p_1,p_2;q_1,q_2,\ldots,q_n) = \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \prod_{i=1}^{n} \widetilde{\mathrm{d}q_i}(2\pi)^4\delta^4(p_f - p_1 - p_2)\left|\langle q_1,\ldots,q_n|T|p_1,p_2\rangle\right|^2 . \tag{6.53}$$

The cross section for a finite phase space volume Δ is obtained from the differential cross section (6.53) by integration

$$\sigma_{2\to n}(p_1,p_2;\Delta) = \int_\Delta \mathrm{d}\sigma_{2\to n}(p_1,p_2;q_1,q_2,\ldots,q_n) \tag{6.54}$$

and corresponds to the rate of events observed in Δ divided by the flux of incoming particles.

In the special case of $2\to 2$ scattering, the phase space is the surface of the unit sphere with area element $\mathrm{d}\Omega = \mathrm{d}\cos\theta\mathrm{d}\phi$ and in the center of mass system with

$$p_1 + p_2 = (E,\vec{0}) \tag{6.55}$$

we find

$$\frac{\mathrm{d}\sigma_{2\to 2}}{\mathrm{d}\Omega_1}(\cos\theta_1,\phi_1) = \frac{1}{32\pi^2 s}\frac{|\vec{q}_1|}{E}\left|\langle q_1,q_2|T|p_1,p_2\rangle\right|^2 \tag{6.56}$$

with θ_2 and ϕ_2 fixed by four-momentum conservation.

The formula (6.53) remains essentially valid for other particles in the initial and final state, but we have to take care of additional quantum numbers and *indistinguishable* particles in the *final* state.

1. if there are additional quantum numbers (spin, polarization, charges) in the final state that are *in principle* distinguishable, but are not observed, the corresponding cross sections are *added up*
2. if there are additional quantum numbers (spin, polarization, charges) in the initial state that are *in principle* distinguishable, but are mixed, the corresponding cross sections are *averaged*
3. if there are *indistinguishable* particles in the *final* state, we *must not* double count identical contributions. There are two ways to ensure this
   (a) restrict the phase space, e. g. in the case of two photons, integrate one over one hemisphere, the other over the other
   (b) apply a symmetry factor

$$\frac{1}{\prod_k n_k!} \tag{6.57}$$

where the product runs over all species k particles and n_k is the number of indistinguishable particles of species k in the final state under consideration.

Usually, the second approach is more convenient.

6.6 Unitarity

The unitarity of the S-matrix implies

$$\mathbf{1} = S^\dagger S = (\mathbf{1} + \mathrm{i}T)^\dagger(\mathbf{1} + \mathrm{i}T) = \mathbf{1} + \mathrm{i}\left(T - T^\dagger\right) + T^\dagger T \tag{6.58}$$

i. e.

$$T^\dagger T = \mathrm{i}\left(T^\dagger - T\right) \tag{6.59}$$

or

$$\sum_n \langle f|T^\dagger|n\rangle \langle n|T|i\rangle = \mathrm{i}\left(\langle f|T^\dagger|i\rangle - \langle f|T|i\rangle\right) . \tag{6.60}$$

Again, we have to be careful with the δ-functions from momentum conservation and find with (6.38)

$$\sum_n (2\pi)^4\delta^4(p_n - p_i) \langle f|\mathcal{T}^\dagger|n\rangle \langle n|\mathcal{T}|i\rangle = \mathrm{i}\left(\langle f|\mathcal{T}^\dagger|i\rangle - \langle f|\mathcal{T}|i\rangle\right) \tag{6.61}$$

or

$$\sum_n (2\pi)^4\delta^4(p_n - p_i) \langle n|\mathcal{T}|f\rangle^* \langle n|\mathcal{T}|i\rangle = \mathrm{i}\left(\langle i|\mathcal{T}|f\rangle^* - \langle f|\mathcal{T}|i\rangle\right) . \tag{6.62}$$

Note that these relations are always violated if T is computed in perturbation theory, because $T^\dagger T$ involves terms of higher order in the coupling than T and $T^\dagger$.

6.6.1 Optical Theorem

In the special case of forward scattering, we obtain the *optical theorem*

$$\sum_n (2\pi)^4\delta^4(p_n - p_i) \left|\langle n|\mathcal{T}|i\rangle\right|^2 = \mathrm{i}\left(\langle i|\mathcal{T}|i\rangle^* - \langle f|\mathcal{T}|i\rangle\right) = 2\,\mathrm{Im}\,\langle i|\mathcal{T}|i\rangle , \tag{6.63}$$

that relates the imaginary part of the *forward scattering amplitude* $\langle i|\mathcal{T}|i\rangle$ to the total cross section

$$\sigma_{2\to X}(p_1,p_2) = \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}} \sum_n (2\pi)^4\delta^4(p_n - p_i) \left|\langle n|\mathcal{T}|p_1,p_2\rangle\right|^2 . \tag{6.64}$$

The intuitive interpretation that the imaginary or absorptive part of the forward scattering amplitude must match the total cross section into other channels to maintain conservation of probability.

Again this mixes different orders in perturbation theory. Assume that

$$T = \mathcal{O}(\lambda)\,. \tag{6.65}$$

then $T^\dagger T = \mathcal{O}(\lambda^2)$ and we find

$$\mathrm{Im}\,\langle i|\mathcal{T}|i\rangle = \mathcal{O}(\lambda^2)\,. \tag{6.66}$$

Thus if we compute T only to first order, we know that $\langle i|\mathcal{T}|i\rangle$ is real, but then $S = \mathbf{1} + \mathrm{i}T$ can not be unitary.

In the plane wave basis, the sum in (6.63) involves integrals over momenta. It turns out to be more helpful to introduce a partial wave basis with good angular momentum quantum numbers J to replace the integrals by discrete sums. Then one obtains matrix equations that can be diagonalized by angular momentum conservation resulting in partial wave unitarity conditions

$$\mathrm{Im}\,T_J(s) = \text{kinematical const.} \cdot |T_J(s)|^2 \tag{6.67}$$

that imply bounds on numbers.

—7— Feynman Rules

Lecture 19: Tue, 09. 01. 2018

Now we can move on to rules for the computation of the matrix elements

$$\langle q_1,\ldots,q_m|S|p_1,\ldots,p_n\rangle = \left\langle q_1,\ldots,q_m\middle|\mathrm{T}\left[\exp\left(\mathrm{i}\lambda\int\mathrm{d}^4x\,\mathcal{L}_I(x)\right)\right]\middle|p_1,\ldots,p_n\right\rangle \tag{7.1}$$

where $\mathcal{L}_I(x)$ can be expressed in terms of creation and annihilation operators of the fields involved. Let's use

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{7.2}$$

with

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{7.3a}$$

$$\mathcal{L}_I = -\frac{\lambda}{3!}\phi^3 \tag{7.3b}$$

for a neutral scalar field with asymptotic representation

$$\phi(x) = \int\widetilde{\mathrm{d}p}\,\left(a(p)\mathrm{e}^{-\mathrm{i}xp} + a^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right)\,. \tag{7.4}$$

In order to get a nonvanishing result, we need a contribution containing m creation and n annihilation operators. Obviously this is only possible in this case with an even or odd power of interactions if $n+m$ is even or odd, respectively, because using the commutation relations can only change the number of creation and annihilation operators by an even number.

Therefore, we find for $2\to 2$-scattering

$$\langle q_1,q_2|S|p_1,p_2\rangle = \langle q_1,q_2|p_1,p_2\rangle \\ - \frac{1}{2!}\frac{\lambda^2}{(3!)^2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \left\langle q_1,q_2\middle|\mathrm{T}\left[\phi^3(x_1)\phi^3(x_2)\right]\middle|p_1,p_2\right\rangle + \mathcal{O}(\lambda^4)\,. \quad (7.5)$$

This can be evaluated directly, but it there is an equivalent, but more simple general approach.

7.1 Two-Point Functions

Obviously, we will need to evaluate many time-ordered products of field operators, the simplest of which is

$$G_2(x-y) = \langle 0|\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle \\ = \Theta(x_0-y_0)\,\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y_0-x_0)\,\langle 0|\phi(y)\phi(x)|0\rangle \\ = \mathrm{i}\Theta(x_0-y_0)\Delta^{(+)}(x-y) + \mathrm{i}\Theta(y_0-x_0)\Delta^{(+)}(y-x)\,. \quad (7.6)$$

It turns out that this can be written compactly in momentum space. First note that the step function has a representation as the boundary value of an analytic function

$$\Theta(t) = -\lim_{\epsilon\to 0+}\int\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega+\mathrm{i}\epsilon}\,, \quad (7.7)$$

where the limit $\epsilon\to 0+$ (i. e. $\epsilon\to 0$ with $\epsilon > 0$) will be implied in all formulae from now on. The formula (7.7) can be verified using the residue theorem with a semicircle in the lower halfplane for $t > 0$ and a semicircle in the upper halfplane for $t < 0$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t) = -\int\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{1}{\omega+\mathrm{i}\epsilon}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\omega t} = \int\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega t} = \delta(t)\,, \quad (7.8)$$

as required. Now, using

$$E(\vec k) = \sqrt{\vec k^2 + m^2} = E(-\vec k) \geq 0 \quad (7.9)$$

we can write

$$\mathrm{i}\Theta(x_0)\Delta^{(+)}(x) + \mathrm{i}\Theta(-x_0)\Delta^{(+)}(-x) \\ = -\int\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\widetilde{\mathrm{d}k}\frac{1}{\omega+\mathrm{i}\epsilon}\left(\mathrm{e}^{-\mathrm{i}\omega x_0 - \mathrm{i}kx} + \mathrm{e}^{\mathrm{i}\omega x_0 + \mathrm{i}kx}\right) \\ = \mathrm{i}\int\frac{\mathrm{d}\omega\mathrm{d}^3\vec k}{(2\pi)^4}\frac{1}{2E(\vec k)}\frac{1}{\omega+\mathrm{i}\epsilon}\left(\mathrm{e}^{-\mathrm{i}(\omega+E(\vec k))x_0+\mathrm{i}\vec k\vec x} + \mathrm{e}^{\mathrm{i}(\omega+E(\vec k))x_0-\mathrm{i}\vec k\vec x}\right)$$

$$\stackrel{\omega=\pm k_0-E(\vec{k})}{=} \mathrm{i}\int\frac{\mathrm{d}k_0\mathrm{d}^3\vec{k}}{(2\pi)^4}\frac{1}{2E(\vec{k})}\left(\frac{1}{k_0-E(\vec{k})+\mathrm{i}\epsilon}+\frac{1}{-k_0-E(\vec{k})+\mathrm{i}\epsilon}\right)\mathrm{e}^{-\mathrm{i}k_0x_0+\mathrm{i}\vec{k}\vec{x}}$$

$$=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2E(\vec{k})}\left(\frac{1}{k_0-E(\vec{k})+\mathrm{i}\epsilon}-\frac{1}{k_0+E(\vec{k})-\mathrm{i}\epsilon}\right)\mathrm{e}^{-\mathrm{i}kx}$$

$$=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k_0^2-(E(\vec{k})-\mathrm{i}\epsilon)^2}\mathrm{e}^{-\mathrm{i}kx}$$

$$=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m^2+2\mathrm{i}\epsilon E(\vec{k}))^2}\mathrm{e}^{-\mathrm{i}kx}$$

$$\stackrel{E(\vec{k})\geq 0}{=}\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m^2+\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}kx}\,, \tag{7.10}$$

i. e.

$$G_2(x-y)=\langle 0|\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k(x-y)}}{k^2-m^2+\mathrm{i}\epsilon}\,. \tag{7.11}$$

Note that

$$\left(\Box+m^2\right)G_2(x)=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k^2+m^2}{k^2-m^2+\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}kx}$$

$$=-\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}kx}=-\mathrm{i}\delta^4(x)\,, \tag{7.12}$$

as required by

$$\left(\Box_x+m^2\right)\langle 0|\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle=-\mathrm{i}\delta^4(x-y)\,, \tag{7.13}$$

in contrast to

$$\left(\Box_x+m^2\right)\langle 0|\phi(x)\phi(y)|0\rangle=0\,. \tag{7.14}$$

In order to understand (7.13), consider

$$\frac{\partial^2}{\partial x_0^2}\mathrm{T}\left[\phi(x)\phi(y)\right]=\frac{\partial^2}{\partial x_0^2}\left(\Theta(x_0-y_0)\phi(x)\phi(y)+\Theta(y_0-x_0)\phi(y)\phi(x)\right)$$

$$=\frac{\partial}{\partial x_0}\Bigg(\delta(x_0-y_0)\phi(x)\phi(y)+\Theta(x_0-y_0)\frac{\partial}{\partial x_0}\phi(x)\phi(y)$$

$$-\delta(x_0-y_0)\phi(y)\phi(x)+\Theta(y_0-x_0)\phi(y)\frac{\partial}{\partial x_0}\phi(x)\Bigg)$$

$$\stackrel{\delta(x_0-y_0)[\phi(x),\phi(y)]=0}{=}\frac{\partial}{\partial x_0}\left(\Theta(x_0-y_0)\frac{\partial}{\partial x_0}\phi(x)\phi(y)+\Theta(y_0-x_0)\phi(y)\frac{\partial}{\partial x_0}\phi(x)\right)$$

$$= \delta(x_0 - y_0)\frac{\partial}{\partial x_0}\phi(x)\phi(y)\Theta(x_0 - y_0)\frac{\partial^2}{\partial x_0^2}\phi(x)\phi(y) \\ - \delta(y_0 - x_0)\phi(y)\frac{\partial}{\partial x_0}\phi(x) + \Theta(y_0 - x_0)\phi(y)\frac{\partial^2}{\partial x_0^2}\phi(x)\,, \quad (7.15)$$

i. e.

$$\Box_x \mathrm{T}\left[\phi(x)\phi(y)\right] = \delta(x_0 - y_0)\left[\frac{\partial}{\partial x_0}\phi(x), \phi(y)\right] + \mathrm{T}\left[\Box_x\phi(x)\phi(y)\right] \\ = \mathrm{T}\left[\Box_x\phi(x)\phi(y)\right] - \mathrm{i}\delta^4(x-y)\,. \quad (7.16)$$

7.2 Wick's Theorem

In section 5.2.1, we have used *normal ordering* of polynomials in the fields in order to make the free hamiltonian well defined. Here we introduce the notation $:\ldots:$ for normal ordering

$$:a_i a_j^\dagger a_k a_l^\dagger: = \pm a_j^\dagger a_l^\dagger a_i a_k\,, \quad (7.17)$$

where the sign keeps track of the number of fermionic permutations required to move all the annihilation operators to the right.

We can use the simplest form of the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} \qquad \text{for } [A,[A,B]] = [B,[A,B]] \quad (7.18)$$

and its inverse

$$\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B\mathrm{e}^{-\frac{1}{2}[A,B]} \qquad \text{for } [A,[A,B]] = [B,[A,B]] \quad (7.19)$$

to normal order exponentials of linear combinations of fields

$$\exp\left(\int \mathrm{d}^4x\,\phi(x)j(x)\right) = \exp\left(\int \mathrm{d}^4x\,\phi^{(+)}(x)j(x) + \int \mathrm{d}^4x\,\phi^{(-)}(x)j(x)\right) \\ = \exp\left(\int \mathrm{d}^4x\,\phi^{(-)}(x)j(x)\right)\exp\left(\int \mathrm{d}^4x\,\phi^{(+)}(x)j(x)\right) \\ \exp\left(\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\,j(x)\left[\phi^{(+)}(x), \phi^{(-)}(y)\right]j(y)\right) \\ = :\exp\left(\int \mathrm{d}^4x\,\phi(x)j(x)\right):\exp\left(\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4y\,j(x)\Delta^{(+)}(x-y)j(y)\right)\,. \quad (7.20)$$

Indeed, expanding to second order in j

$$\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)j(y)\phi(x)\phi(y) =\\ \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)j(y){:}\phi(x)\phi(y){:} + \frac{\mathrm{i}}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)j(y)\Delta^{(+)}(x-y)\,, \quad (7.21)$$

comparing coefficients or taking functional derivatives[1] and taking the limit $j \to 0$ afterwards

$$\phi(x)\phi(y) = {:}\phi(x)\phi(y){:} + \mathrm{i}\Delta^{(+)}(x-y) \quad (7.24)$$

and computing the vacuum expectation value

$$\langle 0|\phi(x)\phi(y)|0\rangle = \mathrm{i}\Delta^{(+)}(x-y)\,. \quad (7.25)$$

Analogously, time ordering and normal ordering of free fields can also differ at most by a function that commutes with every field. Therefore the difference must be equal to its vacuum expectation value, i. e.

$$\mathrm{T}\left[\phi(x)\phi(y)\right] - {:}\phi(x)\phi(y){:}\\ = \langle 0|(\mathrm{T}\left[\phi(x)\phi(y)\right] - {:}\phi(x)\phi(y){:})|0\rangle = \langle 0|\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle \quad (7.26)$$

or

$$\mathrm{T}\left[\phi(x)\phi(y)\right] = {:}\phi(x)\phi(y){:} + \langle 0|\,\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle\,. \quad (7.27)$$

We can again use exponentials to take care of the combinatorical factors

$$\mathrm{T}\left[\exp\left(\int \mathrm{d}^4x\, \phi(x)j(x)\right)\right] = {:}\exp\left(\int \mathrm{d}^4x\, \phi(x)j(x)\right){:}\times\\ \exp\left(\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, j(x)\,\langle 0|\mathrm{T}\left[\phi(x)\phi(y)\right]|0\rangle\, j(y)\right)\,. \quad (7.28)$$

Comparing coefficients or taking functional derivatives[2] we find

$$\mathrm{T}\left[\mathbf{1}\right] = {:}\mathbf{1}{:} \quad (7.29a)$$

[1]NB: (7.24) has some subtleties: comparing coefficients in j or taking functional derivatives w. r. t. j gives us only the *symmetrical* combination

$$\frac{1}{2}\left(\phi(x)\phi(y)+\phi(y)\phi(x)\right) = {:}\phi(x)\phi(y){:} + \frac{\mathrm{i}}{2}\left(\Delta^{(+)}(x-y)+\Delta^{(+)}(y-x)\right)\,, \quad (7.22)$$

but we can rewrite the LHS as

$$\frac{1}{2}\left(\phi(x)\phi(y)+\phi(y)\phi(x)\right) = \phi(x)\phi(y) + \frac{1}{2}\left[\phi(y),\phi(x)\right]\\ = \phi(x)\phi(y) - \frac{\mathrm{i}}{2}\Delta(x-y) = \phi(x)\phi(y) - \frac{\mathrm{i}}{2}\Delta^{(+)}(x-y) + \frac{\mathrm{i}}{2}\Delta^{(+)}(y-x) \quad (7.23)$$

to obtain (7.24).

[2]This time the LHS is symmetrical and needs no special treatment.

$$\mathrm{T}\left[\phi(x_1)\right] = {:}\phi(x_1){:} \tag{7.29b}$$

$$\mathrm{T}\left[\phi(x_1)\phi(x_2)\right] = {:}\phi(x_1)\phi(x_2){:} + \langle 0|\,\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle \tag{7.29c}$$

$$\begin{aligned}\mathrm{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\right] = {}& {:}\phi(x_1)\phi(x_2)\phi(x_3){:} \\ & + \langle 0|\,\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle\, {:}\phi(x_3){:} \\ & + \langle 0|\,\mathrm{T}\left[\phi(x_1)\phi(x_3)\right]|0\rangle\, {:}\phi(x_2){:} \\ & + \langle 0|\,\mathrm{T}\left[\phi(x_2)\phi(x_3)\right]|0\rangle\, {:}\phi(x_1){:}\end{aligned} \tag{7.29d}$$

$$\begin{aligned}\mathrm{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right] = {}& {:}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4){:} \\ & + \langle 0|\,\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle\, {:}\phi(x_3)\phi(x_4){:} + \text{perm.} \\ & + \langle 0|\,\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle \langle 0|\,\mathrm{T}\left[\phi(x_3)\phi(x_4)\right]|0\rangle \\ & \quad + \text{permutations}\end{aligned} \tag{7.29e}$$

$$\ldots\,.$$

This way, we have obtained *Wick's Theorem* that allows us to express the time ordered product of fields as a sum of normal ordered product with all pairs of operators replaced by their *contractions*

$$\wick{\c1\phi(x_1)\c1\phi(x_2)} = \mathrm{T}\left[\phi(x_1)\phi(x_2)\right] - {:}\phi(x_1)\phi(x_2){:} = \langle 0|\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle\ , \tag{7.30}$$

e. g.

$$\begin{aligned}&\mathrm{T}\left[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right] = {:}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4){:} \\ &+ {:}\wick{\c1\phi(x_1)\c1\phi(x_2)}\phi(x_3)\phi(x_4){:} + {:}\wick{\c1\phi(x_1)\phi(x_2)\c1\phi(x_3)}\phi(x_4){:} + {:}\wick{\c1\phi(x_1)\phi(x_2)\phi(x_3)\c1\phi(x_4)}{:} \\ &+ {:}\phi(x_1)\wick{\c1\phi(x_2)\c1\phi(x_3)}\phi(x_4){:} + {:}\phi(x_1)\wick{\c1\phi(x_2)\phi(x_3)\c1\phi(x_4)}{:} + {:}\phi(x_1)\phi(x_2)\wick{\c1\phi(x_3)\c1\phi(x_4)}{:} \\ &+ \wick{\c1\phi(x_1)\c1\phi(x_2)\c2\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c1\phi(x_3)\c2\phi(x_4)} + \wick{\c1\phi(x_1)\c2\phi(x_2)\c2\phi(x_3)\c1\phi(x_4)}\,.\end{aligned} \tag{7.31}$$

7.2.1 Generalizations

Lecture 20: Wed, 10. 01. 2018

There are obvious generalizations of Wick's theorem:

- If there is more than one field, including the case of charged fields, Wick's theorem holds, but only non-vanishing contractions (7.30) have to be taken into account.
- If there are fermions involved, the appropriate signs have to be taken care of.

In addition, we can also evaluate time ordered products of normal ordered products

$$\mathrm{T}\left[:\phi(x_1)\ldots\phi(x_n)::\phi(x_{n+1})\ldots\phi(x_{n+l}):\ldots\right]. \tag{7.32}$$

Again, Wick's theorem holds, if we ignore contractions (7.30) within the *same* normal orderd product. This allows us to compute time ordered products of normal ordered powers such as

$$\begin{aligned}\mathrm{T}\left[:\phi^2(x_1)::\phi^2(x_2):\right] &= :\phi^2(x_1)\phi^2(x_2): \\ &+ 4:\phi(x_1)\overline{\phi(x_1)\phi(x_2)}\phi(x_2): + 2\overline{\phi(x_1)\overline{\phi(x_1)\phi(x_2)}\phi(x_2)}\end{aligned} \tag{7.33}$$

without running into the obviously ill-defined[3]

$$\overline{\phi(x)\phi}(x) = \langle 0|\mathrm{T}\left[\phi(x)\phi(x)\right]|0\rangle = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon} \to \infty\,. \tag{7.35}$$

7.3 Graphical Rules for S-Matrix Elements

The application of the normal ordered annihilation and creation pieces to the incoming

$$\begin{aligned}\phi^{(+)}(x)\,|p\rangle &= \int\widetilde{\mathrm{d}k}\mathrm{e}^{-\mathrm{i}kx}a(k)a^\dagger(p)\,|0\rangle \\ &= \int\widetilde{\mathrm{d}k}\mathrm{e}^{-\mathrm{i}kx}\left[a(k),a^\dagger(p)\right]|0\rangle = \mathrm{e}^{-\mathrm{i}px}\,|0\rangle\Big|_{p_0=\sqrt{\vec p^2+m^2}}\end{aligned} \tag{7.36a}$$

and outgoing states of free particles

$$\langle q|\,\phi^{(-)}(x) = \mathrm{e}^{\mathrm{i}qx}\,\langle 0|\Big|_{q_0=\sqrt{\vec q^2+m^2}} \tag{7.36b}$$

yields only the corresponding plane waves. In the expansion of the S-matrix elements

$$\left\langle q_1,\ldots,q_m\middle|\mathrm{T}\left[\exp\left(\mathrm{i}\lambda\int\mathrm{d}^4x\,:\mathcal{L}_I(x):\right)\right]\middle|p_1,\ldots,p_n\right\rangle \tag{7.37}$$

[3] Note that

$$\overline{\phi(x_1)\overline{\phi(x_1)\phi(x_2)}\phi(x_2)} = \langle 0|\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle^2 \tag{7.34}$$

will turn out to be ill-defined as well, but we will deal with this problem later.

using Wick's theorem, we obtain contributions only from those terms with exactly n annihilation and m creation operators and all other fields contracted. For example

$$\begin{aligned}
&\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \left\langle q_1,q_2\middle|\mathrm{T}\left[:\phi^3(x_1)::\phi^3(x_2):\right]\middle|p_1,p_2\right\rangle = \\
&\qquad\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \left\langle q_1,q_2\middle|\phi^{(-)}(x_1)\phi^{(-)}(x_1)\phi^{(+)}(x_2)\phi^{(+)}(x_2)\middle|p_1,p_2\right\rangle \times \\
&\qquad\qquad \langle 0|\mathrm{T}\left[\phi(x_1)\phi(x_2)\right]|0\rangle + \text{permutations} \\
&= \int \mathrm{d}^4x_1\mathrm{d}^4x_2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}k(x_1-x_2)}\mathrm{e}^{\mathrm{i}q_1x_1}\mathrm{e}^{\mathrm{i}q_2x_1}\mathrm{e}^{-\mathrm{i}p_1x_2}\mathrm{e}^{-\mathrm{i}p_2x_2} \\
&\qquad + \text{permutations} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}(2\pi)^4\delta^4(k-q_1-q_2)(2\pi)^4\delta^4(k-p_1-p_2)\frac{\mathrm{i}}{k^2-m^2+\mathrm{i}\epsilon} + \text{permutations} \\
&= (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\frac{\mathrm{i}}{(p_1+p_2)^2-m^2+\mathrm{i}\epsilon} + \text{permutations}\,, \qquad (7.38)
\end{aligned}$$

where we find the expected overall momentum conservation and a propagator

$$\frac{\mathrm{i}}{(p_1+p_2)^2-m^2+\mathrm{i}\epsilon}\,. \qquad (7.39)$$

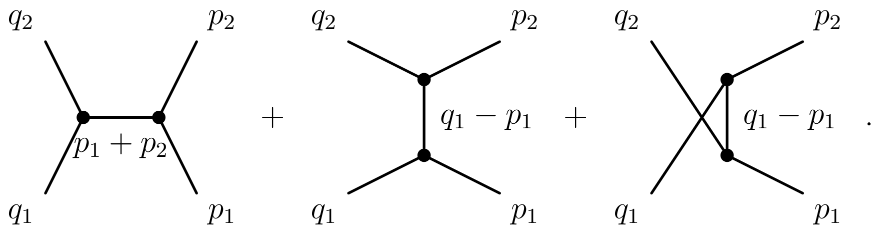
Since there are 3! ways to connect a propagator and two distinct external particles to the three fields in a $\phi^3(x)$-term, the permutations cancel the factors of 1/3! in $\mathcal{L}_I$. The possibility to exchange the integration variables x_1 and x_2 cancels the factor 1/2! from the expansion of the exponential. Finally, the interactions can be connected to all pairs of external particles

$$\begin{aligned}
&\frac{\mathrm{i}^2}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \langle q_1,q_2|\mathrm{T}\left[:\mathcal{L}_I(x_1)::\mathcal{L}_I(x_2):\right]|p_1,p_2\rangle \\
&\qquad = (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\times \\
&(-\lambda^2)\left(\frac{\mathrm{i}}{(p_1+p_2)^2-m^2+\mathrm{i}\epsilon}+\frac{\mathrm{i}}{(q_1-p_1)^2-m^2+\mathrm{i}\epsilon}+\frac{\mathrm{i}}{(q_1-p_2)^2-m^2+\mathrm{i}\epsilon}\right)
\end{aligned} \qquad (7.40)$$

and we obtain the T-matrix element

$$T = -\lambda^2\left(\frac{1}{(p_1+p_2)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(q_1-p_1)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(q_1-p_2)^2-m^2+\mathrm{i}\epsilon}\right)\,, \qquad (7.41)$$

which can be represented by the diagrams



(7.42)

The three diagrams in (7.42) are commonly referred to as s-, t- and u-channel contributions, according to the Lorentz invariant *Mandelstam variables*

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \tag{7.43a}$$
$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \tag{7.43b}$$
$$u = (p_1 - q_2)^2 = (p_2 - q_1)^2 \,, \tag{7.43c}$$

that are related by

$$s + t + u = p_1^2 + p_2^2 + q_1^2 + q_2^2 \,. \tag{7.44}$$

A general contribution is represented by a graph consisting of *interaction vertices* as nodes connected by *propagators* or external lines representing external particles as edges.

7.3.1 Momentum Space

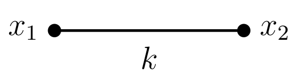
We can express the general rules directly in momentum space. For this, we observe that each interaction vertex contributes a integral over all of space time

$$\int \mathrm{d}^4 x_i \cdots$$

and the fields in each interaction all contribute a factor

$$\mathrm{e}^{\pm \mathrm{i} x_i p_n}$$

either from the action of the operator on an external particle or from the momentum space representation of a contraction



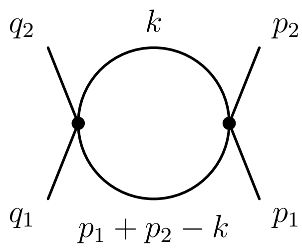
$$\Leftrightarrow \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}kx_i} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{\mathrm{i}kx_j} \,. \tag{7.45}$$

Thus the integral turns into a momentum conservation δ-function at each interaction vertex

$$\int \mathrm{d}^4 x_i \to (2\pi)^4 \delta^4 \left(\sum_n p_n \right) . \tag{7.46}$$

It is easy to see that momentum conservation at each vertex implies overall momentum conservation. The corresponding δ-function which can be extracted to obtain the iT-matrix.

If the graph is a tree graph, i. e. contains no closed loops, momentum conservation at each vertex fixes the momentum of each propagator uniquely. On The Other Hand (OTOH), if there are closed loops, e. g.



(7.47)

the momentum conservation at the vertices does not constrain the *loop momentum*, k in (7.47), at an integral[4] over the whole momentum space

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \cdots$$

remains.

7.3.2 Combinatorics

If there are n fields corresponding to identical particles in a given vertex, there are $n!$ ways to attach the propagators that must all be counted, but give the same contribution. Therefore it is customary to write

$$\mathcal{L}_I = \frac{\lambda}{n!} \phi^n \tag{7.48}$$

so that the resulting vertex factor is just iλ. In the case of diagrams like (7.47), this leads to an overcounting, because the an exchange of the two lines in the loop leaves the diagram unchanged. Therefore the diagram has to be divided by the size of the automorphism group of the graph, e. g. 2! in the case of (7.47).

[4] More often than not, these integrals will not converge and we will be forced to perform *renormalization* to handle the divergencies.

7.3.3 Vertex Factors

In general, the vertex factors what remains after removing the fields from the corresponding term in $\mathrm{i}\mathcal{L}_I$. The combinatorical factors can be obtained unambigously by talking functional derivatives w. r. t. the fields. If $\mathrm{i}\mathcal{L}_I$ contains derivatives ∂_μ they are to be replaced by $-\mathrm{i}p_\mu$, with p the *incoming* momentum of the field on which the derivative acts.

7.3.4 Charged Fields

In the case of charged fields, we must distinguish incoming anti particles

$$a^{c\dagger}(p)\left|0\right\rangle$$

from particles

$$a^{\dagger}(p)\left|0\right\rangle$$

and act with $\phi^\dagger(x)$ on the former and $\phi(x)$ on the latter. For the outgoing particles, the roles are reversed.

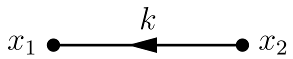
Also, we must distinguish

$$\overline{\phi(x)\phi}^\dagger(y) = \left\langle 0\middle|\mathrm{T}\left[\phi(x)\phi^\dagger(y)\right]\middle|0\right\rangle \tag{7.49a}$$

from

$$\overline{\phi^\dagger(x)\phi}(y) = \left\langle 0\middle|\mathrm{T}\left[\phi^\dagger(x)\phi(y)\right]\middle|0\right\rangle \tag{7.49b}$$

and add an arrow to the corresponding edges



$$\Leftrightarrow \overline{\phi(x_1)\phi}^\dagger(x_2) = \left\langle 0\middle|\mathrm{T}\left[\phi(x_1)\phi^\dagger(x_2)\right]\middle|0\right\rangle .$$

7.3.5 Spin

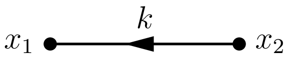
In the case of particles with spin, we have to take into account the coefficient functions, e. g. u and v for spin 1/2

$$\psi(x) = \sum_\sigma \int \widetilde{\mathrm{d}p}\,\left(u_\sigma(p)c_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} + v_\sigma(p)d_\sigma^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right) . \tag{7.50}$$

Then

$$\psi^{(+)}(x)c_\sigma^\dagger(p)\left|0\right\rangle = \left|0\right\rangle u_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} \tag{7.51}$$

etc. and



$\Leftrightarrow \psi(x_1)\bar{\psi}(x_2)$ (contracted)

$$= \langle 0 | \mathrm{T}\left[\psi(x_1)\bar{\psi}(x_2)\right] | 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{i} \frac{\not{k} + m}{k^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}k(x_1 - x_2)} \tag{7.52}$$

Additional material (not discussed during the lectures):

There is one subtlety that we have to discuss

$$\begin{aligned}\langle 0 | \mathrm{T}\left[\psi_\alpha(x)\bar{\psi}_\beta(y)\right] | 0 \rangle &= \Theta(x_0 - y_0) \langle 0 | \psi_\alpha(x)\bar{\psi}_\beta(y) | 0 \rangle - \Theta(y_0 - x_0) \langle 0 | \bar{\psi}_\beta(y)\psi_\alpha(x) | 0 \rangle \\ &= \Theta(x_0 - y_0) \int \widetilde{\mathrm{d}p}\, u_\alpha(p)\bar{u}_\beta(p) e^{-\mathrm{i}p(x-y)} - \Theta(y_0 - x_0) \int \widetilde{\mathrm{d}p}\, v_\alpha(p)\bar{v}_\beta(p) e^{\mathrm{i}p(x-y)}\end{aligned} \tag{7.53}$$

i. e.

$$\begin{aligned}&\langle 0 | \mathrm{T}\left[\psi(x)\bar{\psi}(y)\right] | 0 \rangle \\ &\quad = \Theta(x_0 - y_0) \int \widetilde{\mathrm{d}p}\, u(p)\bar{u}(p) e^{-\mathrm{i}p(x-y)} - \Theta(y_0 - x_0) \int \widetilde{\mathrm{d}p}\, v(p)\bar{v}(p) e^{\mathrm{i}p(x-y)} \\ &\quad = \Theta(x_0 - y_0) \int \widetilde{\mathrm{d}p}\, (\not{p} + m) e^{-\mathrm{i}p(x-y)} - \Theta(y_0 - x_0) \int \widetilde{\mathrm{d}p}\, (\not{p} - m) e^{\mathrm{i}p(x-y)} \\ &\quad = \Theta(x_0 - y_0)(\mathrm{i}\not{\partial} + m) \int \widetilde{\mathrm{d}p}\, e^{-\mathrm{i}p(x-y)} + \Theta(y_0 - x_0)(\mathrm{i}\not{\partial} + m) \int \widetilde{\mathrm{d}p}\, e^{\mathrm{i}p(x-y)} \\ &\quad = \Theta(x_0 - y_0)(\mathrm{i}\not{\partial} + m)\mathrm{i}\Delta^+(x - y) + \Theta(y_0 - x_0)(\mathrm{i}\not{\partial} + m)\mathrm{i}\Delta^+(y - x) \\ &\quad = (\mathrm{i}\not{\partial} + m)\left(\Theta(x_0 - y_0)\mathrm{i}\Delta^+(x - y) + \Theta(y_0 - x_0)\mathrm{i}\Delta^+(y - x)\right) \\ &\qquad + \gamma_0\delta(x_0 - y_0)\Delta^+(x - y) - \gamma_0\delta(x_0 - y_0)\Delta^+(y - x) \\ &\quad = (\mathrm{i}\not{\partial} + m)G_2(x - y) + \gamma_0\delta(x_0 - y_0)\Delta(x - y) = (\mathrm{i}\not{\partial} + m)G_2(x - y)\end{aligned} \tag{7.54}$$

since

$$\Delta(x)\big|_{x_0 = 0} = 0\,. \tag{7.55}$$

In a momentum space computation of

$$\Theta(x_0) \int \widetilde{\mathrm{d}p}\, (\not{p} + m) e^{-\mathrm{i}px} - \Theta(-x_0) \int \widetilde{\mathrm{d}p}\, (\not{p} - m) e^{\mathrm{i}px} \tag{7.56}$$

one needs to be careful when substituting from $\hat{p} = (E(\vec{p}), \vec{p})$ to $(p_0, \vec{p})$ with the unconstrained integration variable p_0. In essence, one needs to add a momentum space representation of $\gamma_0\delta(x_0 - y_0)\Delta^{(+)}(x - y)$ to the first term and that of $-\gamma_0\delta(x_0 - y_0)\Delta^{(+)}(x - y)$ to the second term, knowing that they cancel.

—8— Quantum Electrodynamics in Born Approximation

Lecture 21: Wed, 17. 01. 2018

Recall the gauge invariant Lagrangian for a charged spin-1/2 field and a photon

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 + \bar\psi \left(\mathrm{i} \not{D} - m\right) \psi \,. \tag{8.1}$$

We split it in the quadratic free part and an interaction part

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 + \bar\psi \left(\mathrm{i} \not\partial - m\right) \psi \tag{8.2a}$$

$$\mathcal{L}_I = e \bar\psi \not{A} \psi \,. \tag{8.2b}$$

8.1 Propagators and External States

We may simplify the calculation of the propagator for the photon field by looking at the free equation of motion for the field

$$\left(-g_{\mu\nu} \Box + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right) A^\nu(x) = 0 \tag{8.3}$$

and the corresponding equation for the propagator

$$\left(-g_{\mu\nu} \Box + \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right) \langle 0 | \mathrm{T} \left[A^\nu(x) A^\rho(y) \right] | 0 \rangle = -\mathrm{i} \delta_\mu{}^\rho \delta^4(x-y) \,, \tag{8.4}$$

where the sign has been fixed so that the spatial components behave like scalar fields for $\xi = 1$. We can now make the ansatz

$$\langle 0 | \mathrm{T} \left[A_\mu(x) A_\nu(y) \right] | 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(g_{\mu\nu} D_1(k^2) + k_\mu k_\nu D_2(k^2) \right) \mathrm{e}^{-\mathrm{i}k(x-y)} \tag{8.5}$$

since there are only two ways we can build rank-2 tensor from a single momentum vector. Then

$$-\mathrm{i}\delta_\mu{}^\rho = \left(g_{\mu\nu}k^2 - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right)\left(g^{\nu\rho}D_1(k^2) + k^\nu k^\rho D_2(k^2)\right) = \\ k^2\delta_\mu{}^\rho D_1(k^2) + k^2 k_\mu k^\rho D_2(k^2) - \left(1-\frac{1}{\xi}\right)k_\mu k^\rho D_1(k^2) - \left(1-\frac{1}{\xi}\right)k^2 k_\mu k^\rho D_2(k^2) \tag{8.6}$$

i. e.

$$k^2 D_1(k^2) = -\mathrm{i} \tag{8.7a}$$

$$k^2 D_2(k^2) - \left(1-\frac{1}{\xi}\right)D_1(k^2) - \left(1-\frac{1}{\xi}\right)k^2 D_2(k^2) = 0 \tag{8.7b}$$

with the solution

$$D_1(k^2) = -\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon} \tag{8.8a}$$

$$D_2(k^2) = -(1-\xi)\frac{1}{k^2+\mathrm{i}\epsilon}D_1(k^2)\,, \tag{8.8b}$$

where we have again use the analogy with the scalar case to Thus we find the propagator

$$\langle 0|\mathrm{T}\left[A_\mu(x)A_\nu(y)\right]|0\rangle = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}\right)\mathrm{e}^{-\mathrm{i}k(x-y)}\,, \tag{8.9}$$

for the photon field with a arbitrary gauge parameter ξ that must cancel in physical results.

The action of the field operators in the normal ordered S-matrix on the external particle and anti-particle states is

$$\psi^{(+)}(x)\,|p,\sigma;e^-\rangle = |0\rangle\,\mathrm{e}^{-\mathrm{i}px}u_\sigma(p) \tag{8.10a}$$

$$\langle p,\sigma;e^-|\,\bar\psi^{(-)}(x) = \mathrm{e}^{\mathrm{i}px}\bar u_\sigma(p)\,\langle 0| \tag{8.10b}$$

$$\bar\psi^{(+)}(x)\,|p,\sigma;e^+\rangle = |0\rangle\,\mathrm{e}^{-\mathrm{i}px}\bar v_\sigma(p) \tag{8.10c}$$

$$\langle p,\sigma;e^+|\,\psi^{(-)}(x) = \mathrm{e}^{\mathrm{i}px}v_\sigma(p)\,\langle 0|\;. \tag{8.10d}$$

Hence

$$\left\langle p,\sigma;e^+\right|{:}\bar\psi(x)\gamma^\mu\psi(x){:}\left|p',\sigma;e^+\right\rangle \\ = \sum_{\alpha\beta}\gamma^\mu_{\alpha_\beta}\left\langle 0\middle|d_\sigma(p)\psi^{(-)}_\beta(x)\bar\psi^{(+)}_\alpha(x)d^\dagger_{\sigma'}(p')\middle|0\right\rangle$$

$$= \sum_{\alpha\beta} \gamma^{\mu}_{\alpha_\beta} \mathrm{e}^{\mathrm{i}x(p-p')} v_{\sigma,\beta}(p) \bar{v}_{\sigma',\alpha}(p') = \mathrm{e}^{\mathrm{i}x(p-p')} \bar{v}_{\sigma'}(p') \gamma^{\mu} v_{\sigma}(p) \quad (8.11)$$

and we see that incoming *anti*particles contribute a $\bar{v}$ on the *left* side of the expressions, while incoming particles contribute a u on the *right* side of the expressions. Analoguously for outgoing particles and antiparticles.

8.2 The Feynman Rules

Using the customary convention, that incoming particles and antiparticles are drawn on the LHS and outgoing particles and antiparticles on the RHS of diagrams, the comprehensive set of rules for the computation of iT in spinor QED is

$|p, \sigma; e^-\rangle$  $= u_\sigma(p)$ (8.12a)

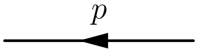
 $\langle p, \sigma; e^-|$ $= \bar{u}_\sigma(p)$ (8.12b)

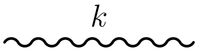
$|p, \sigma; e^+\rangle$  $= \bar{v}_\sigma(p)$ (8.12c)

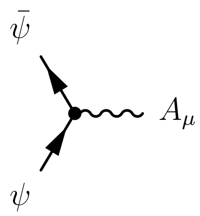
 $\langle p, \sigma; e^+|$ $= v_\sigma(p)$ (8.12d)

$|k, \lambda; \gamma\rangle$  $= \epsilon_\lambda(k)$ (8.12e)

 $\langle k, \lambda; \gamma|$ $= \epsilon^*_\lambda(k)$ (8.12f)

$\bar{\psi}$  ψ $\quad = \mathrm{i}\dfrac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon} = \dfrac{\mathrm{i}}{\not{p} - m + \mathrm{i}\epsilon}$ (8.12g)

A_μ  A_ν $\quad = \dfrac{\mathrm{i}}{k^2 + \mathrm{i}\epsilon}\left(-g_{\mu\nu} + (1-\xi)\dfrac{k_\mu k_\nu}{k^2 + \mathrm{i}\epsilon}\right)$ (8.12h)



$$= \mathrm{i}e\gamma^\mu \tag{8.12i}$$

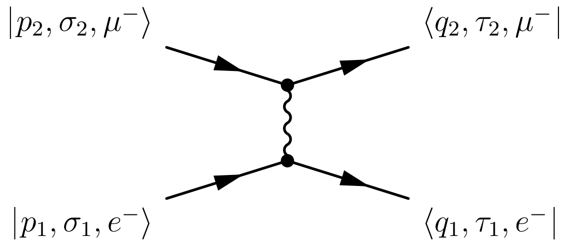
where the overall momemtum conservation factor $(2\pi)^4\delta^4(\sum q_{\text{out}} - \sum p_{\text{in}})$ is to be factored out.

The strength of the interaction e is a free parameter that can take any real value. If there is more than one species of particles, e can be replaced by eQ_i with a different charge for each species without spoiling gauge invariance. Note that there is only one vertex and one propagator per species, which accounts for particles and antiparticles at the same time. There is not a second vertex with opposite charge for the antiparticles. The antiparticles are taken care of by the direction of the momentum flow relative to the arrows in the propagators.

Also note that it is essential that the species of particle *never* changes at the vertex.

8.3 $e^-\mu^- \to e^-\mu^-$

The simplest example is the elastic[1] scattering of two distinguishable particles, taken to be an electron e^- and a muon μ^-. There is only one tree Feynman diagram



(8.13)

and the corresponding expression for iT reads

$$\mathrm{i}T = \bar u_{\tau_1}(q_1)\mathrm{i}e\gamma_\mu u_{\sigma_1}(p_1)\bar u_{\tau_2}(q_2)\mathrm{i}e\gamma_\nu u_{\sigma_2}(p_2)\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2+\mathrm{i}\epsilon}\right) \tag{8.14}$$

[1] 'Elastic' is the term for scattering where target and projectile just change their momentum and are not modified or excited internally.

with $k = q_1 - p_1$. Noting that

$$k^\mu \bar u_\tau(q_1)\gamma_\mu u_\sigma(p_1) = \bar u_\tau(q_1)\left(\not q_1 - \not p_1\right) u_\sigma(p_1) = \bar u_\tau(q_1)\left(m - m\right) u_\sigma(p_1) = 0\,, \tag{8.15}$$

which is the momentum space version of current conservation

$$\partial^\mu\left(\bar\psi(x)\gamma_\mu\psi(x)\right) = 0\,, \tag{8.16}$$

we find the ξ-independent amplitude for polarized scattering

$$T = e^2\frac{1}{(q_1-p_1)^2 + \mathrm{i}\epsilon}\bar u_{\tau_1}(q_1)\gamma_\mu u_{\sigma_1}(p_1)\bar u_{\tau_2}(q_2)\gamma^\mu u_{\sigma_2}(p_2)\,. \tag{8.17}$$

This can be squared to obtained the transition probability. Instead, we will look at the sum or average of polarizations

$$\begin{aligned}\sum_{\text{polarizations}} |T|^2 = e^4 \left|\frac{1}{(q_1-p_1)^2+\mathrm{i}\epsilon}\right|^2 \times \\ \sum_{\tau_1,\sigma_1=1}^{2}\left(\bar u_{\tau_1}(q_1)\gamma_\mu u_{\sigma_1}(p_1)\right)^* \bar u_{\tau_1}(q_1)\gamma_\nu u_{\sigma_1}(p_1)\times \\ \sum_{\tau_2,\sigma_2=1}^{2}\left(\bar u_{\tau_2}(q_2)\gamma^\mu u_{\sigma_2}(p_2)\right)^* \bar u_{\tau_2}(q_2)\gamma^\nu u_{\sigma_2}(p_2)\,.\end{aligned} \tag{8.18}$$

Using

$$\begin{aligned}\left(\bar u_\tau(q)\gamma_\mu u_\sigma(p)\right)^* = \left(u_\tau^\dagger(q)\gamma_0\gamma_\mu u_\sigma(p)\right)^* = u_\sigma^\dagger(p)\gamma_\mu^\dagger\gamma_0^\dagger u_\tau(q) \\ = u_\sigma^\dagger(p)\gamma_0\gamma_\mu u_\tau(q) = \bar u_\sigma(p)\gamma_\mu u_\tau(q)\,,\end{aligned} \tag{8.19}$$

we find

$$\begin{aligned}\sum_{\tau,\sigma=1}^{2}\left(\bar u_\tau(q)\gamma_\mu u_\sigma(p)\right)^* \bar u_\tau(q)\gamma_\nu u_\sigma(p) = \sum_{\tau,\sigma=1}^{2}\bar u_\sigma(p)\gamma_\mu u_\tau(q)\bar u_\tau(q)\gamma_\nu u_\sigma(p) \\ = \sum_{\tau,\sigma=1}^{2}\mathrm{tr}\left(\gamma_\mu u_\tau(q)\otimes\bar u_\tau(q)\gamma_\nu u_\sigma(p)\otimes\bar u_\sigma(p)\right) = \mathrm{tr}\left(\gamma_\mu\left(\not q + m\right)\gamma_\nu\left(\not p + m\right)\right)\end{aligned} \tag{8.20}$$

from (4.136)

$$\sum_\sigma u_\sigma(p)\otimes\bar u_\sigma(p) = \not p + m\,. \tag{8.21}$$

8.3.1 Trace Theorems

It turns out that we can compute the traces from the Dirac algebra (4.85) without committing to a concrete realization.

From the cyclic invariance of the trace

$$\mathrm{tr}\,(ABC) = \mathrm{tr}\,(BCA) \tag{8.22}$$

we conclude

$$\mathrm{tr}\,(\not{p}\not{q}) = \frac{1}{2}\,\mathrm{tr}\,(\not{p}\not{q} + \not{q}\not{p}) = \frac{1}{2} 2pq\,\mathrm{tr}\,(\mathbf{1}) = 4pq \tag{8.23}$$

or, equivalently

$$\mathrm{tr}\,(\gamma_\mu\gamma_\nu) = 4g_{\mu\nu} \tag{8.24}$$

$$\mathrm{tr}\,(\gamma_\mu\not{p}) = 4p_\mu\,. \tag{8.25}$$

Also

$$\begin{aligned}\mathrm{tr}\,(\not{a}\not{b}\not{c}) = \mathrm{tr}\,(\not{a}\not{b}\not{c}\gamma_5\gamma_5) = \mathrm{tr}\,(\gamma_5\not{a}\not{b}\not{c}\gamma_5) = -\,\mathrm{tr}\,(\not{a}\gamma_5\not{b}\not{c}\gamma_5)\\ = \ldots = -\,\mathrm{tr}\,(\not{a}\not{b}\not{c}\gamma_5\gamma_5) = -\,\mathrm{tr}\,(\not{p})\;,\end{aligned} \tag{8.26}$$

i. e.

$$\mathrm{tr}\,(\not{a}\not{b}\not{c}) = 0\,. \tag{8.27}$$

The same argument works for any product of an *odd* number of Dirac matrices. Finally (for our purposes)

$$\mathrm{tr}\,(\not{a}\not{b}\not{c}\not{d}) = 4\left((ab)(cd) - (ac)(bd) + (ad)(cb)\right) \tag{8.28}$$

and therefore

$$\begin{aligned}\mathrm{tr}\,(\gamma_\mu\,(\not{q}+m)\,\gamma_\nu\,(\not{p}+m)) = \mathrm{tr}\,(\gamma_\mu\not{q}\gamma_\nu\not{p}) + m^2\,\mathrm{tr}\,(\gamma_\mu\gamma_\nu)\\ = 4\,(q_\mu p_\nu - g_{\mu\nu}pq + q_\nu p_\mu) + 4m^2 g_{\mu\nu} = 4\left(q_\mu p_\nu + q_\nu p_\mu + (m^2 - pq)g_{\mu\nu}\right)\end{aligned} \tag{8.29}$$

8.3.2 Squared Amplitude

Now it is a straightforward exercise to compute

$$\sum_{\text{polarizations}} |T|^2 = e^4 \left|\frac{1}{(q_1-p_1)^2 + \mathrm{i}\epsilon}\right|^2 \times$$

$$\begin{aligned}
&\operatorname{tr}\left(\gamma_\mu \left(\not{q}_1 + m_e\right) \gamma_\nu \left(\not{p}_2 + m_e\right)\right) \operatorname{tr}\left(\gamma^\mu \left(\not{q}_2 + m_\mu\right) \gamma^\nu \left(\not{p}_2 + m_\mu\right)\right) \\
&= \frac{16e^4}{(q_1 - p_1)^4 + \epsilon^2} \left(q_{1,\mu} p_{1,\nu} + q_{1,\nu} p_{1,\mu} + (m_e^2 - p_1 q_1) g_{\mu\nu}\right) \\
&\qquad\qquad \left(q_2^\mu p_2^\nu + q_2^\nu p_2^\mu + (m_\mu^2 - p_2 q_2) g^{\mu\nu}\right) \\
&= \frac{32e^4}{(q_1 - p_1)^4 + \epsilon^2} \times \\
&\quad \left((p_1p_2)(q_1q_2) + (p_1q_2)(p_2q_1) - (p_1q_1)m_\mu^2 - (p_2q_2)m_e^2 + 2m_e^2 m_\mu^2\right) \\
&= \frac{8e^4}{t^2 + \epsilon^2} \left(s^2 + u^2 - 4(u+s)(m_e^2 + m_\mu^2) + 6(m_e^2 + m_\mu^2)^2\right) ,
\end{aligned} \tag{8.30}$$

using

$$2p_1p_2 = (p_1 + p_2)^2 - p_1^2 - p_2^2 = s - m_e^2 - m_\mu^2 \tag{8.31a}$$
$$2q_1q_2 = (q_1 + q_2)^2 - q_1^2 - q_2^2 = s - m_e^2 - m_\mu^2 \tag{8.31b}$$
$$2p_1q_1 = -(p_1 - q_1)^2 + p_1^2 + q_1^2 = -t + 2m_e^2 \tag{8.31c}$$
$$2p_2q_2 = -(p_2 - q_2)^2 + p_2^2 + q_2^2 = -t + 2m_\mu^2 \tag{8.31d}$$
$$2p_1q_2 = -(p_1 - q_2)^2 + p_1^2 + q_2^2 = -u + m_e^2 + m_\mu^2 \tag{8.31e}$$
$$2p_2q_1 = -(p_2 - q_1)^2 + p_2^2 + q_1^2 = -u + m_e^2 + m_\mu^2 . \tag{8.31f}$$

From kinematics (cf. exercise), we know that

$$-s \le t \le 0 \tag{8.32}$$

and $t = 0$ corresponds to forward scattering. Therefore we may assume $t < 0$ and take the limit $\epsilon \to 0$:

$$\sum_{\text{polarizations}} |T|^2 = 8e^4 \frac{s^2 + u^2 - 4(u+s)(m_e^2 + m_\mu^2) + 6(m_e^2 + m_\mu^2)^2}{t^2} . \tag{8.33}$$

8.3.3 "Old Fashioned" Perturbation Theory

Lecture 22: Tue, 23. 01. 2018

Note that there are two contribution in nonrelativistic second order perturbation theory

$$\sum_\Psi \langle \text{out} | H_I | \Psi \rangle \frac{1}{E_\Psi - E_0 + \mathrm{i}\epsilon} \langle \Psi | H_I | \text{in} \rangle ,$$

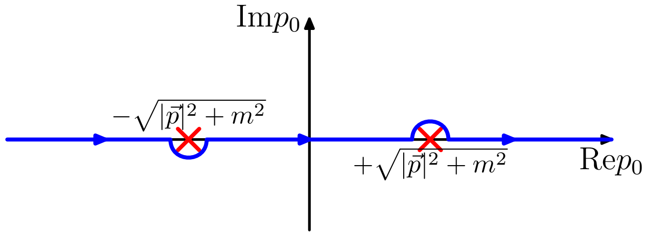
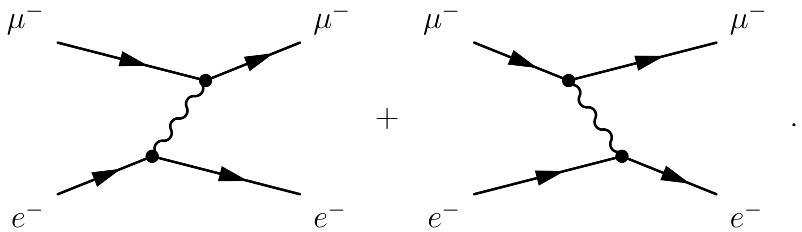


Figure 8.1: *Integration contour in the complex p_0-plane for the Feynman propagator.*

one corresponding to the virtual photon being first emitted by the electron and later absorbed by the muon and the other to the opposite sequence of events



(8.34)

Both of these terms are taken care of simultaneously by the Feynman propagator. This can be seen by performing the p_0-integration in the propagator

$$\int \frac{\mathrm{d}^3\vec p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\vec p\vec x} \int \frac{\mathrm{d}p_0}{2\pi}\,\mathrm{e}^{-\mathrm{i}p_0x_0} \frac{\mathrm{i}}{p_0^2 - \vec p^2 - m^2 + \mathrm{i}\epsilon} \tag{8.35}$$

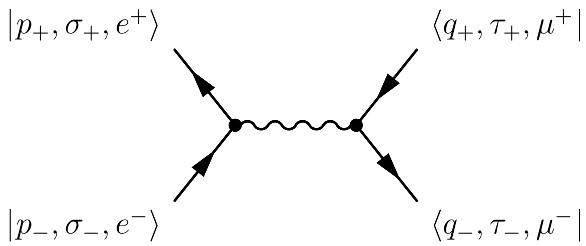
where the ϵ-prescription corresponds to the contour depicted in figure 8.1. Due to the exponential $e^{-\mathrm{i}p_0x_0}$, the contour can be closed with a semicircle at infinity on the lower halfplane for $x_0 > 0$ and on the upper halfplane for $x_0 < 0$.

Thus we pick up the residue at $p_0 = +\sqrt{|\vec p|^2 + m^2}$ for $x_0 > 0$ and the residue at $p_0 = -\sqrt{|\vec p|^2 + m^2}$ for $x_0 < 0$. This means that positive energy contributions are propagated into the future, while negative energy contributions are propagated into the past.

The violation of energy conservation in the sum over all virtual intermediate states in nonrelativistic perturbation theory is replaced by the violation of the mass-shell conditions $p^2 = m^2$ in the propagators of internal lines in Feynman diagrams.

8.4 $e^+e^- \to \mu^+\mu^-$

For the production of muon-antimuon pairs in the annihilation of electron-positron pairs, we have again a single tree level diagram



(8.36)

and a similar expression for the scattering amplitude

$$\mathrm{i}T = \bar{v}_{\sigma_+}(p_+)\mathrm{i}e\gamma_\mu u_{\sigma_-}(p_-)\bar{u}_{\tau_-}(q_-)\mathrm{i}e\gamma_\nu v_{\sigma_+}(q_+)\frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(-g^{\mu\nu}+(1-\xi)\frac{k^\mu k^\nu}{k^2+\mathrm{i}\epsilon}\right) \tag{8.37}$$

with $k = p_+ + p_- = q_+ + q_-$. Again

$$k^\mu \bar{v}_\tau(p_+)\gamma_\mu u_\sigma(p_-) = 0 \tag{8.38}$$

and we find[2]

$$T = \frac{e^2}{s}\bar{v}_{\sigma_+}(p_+)\gamma_\mu u_{\sigma_-}(p_-)\bar{u}_{\tau_-}(q_-)\gamma^\nu v_{\sigma_+}(q_+)\,. \tag{8.39}$$

with

$$\sum_{\text{polarizations}} |T|^2 = 8e^4\frac{t^2+u^2-4(t+u)(m_e^2+m_\mu^2)+6(m_e^2+m_\mu^2)^2}{s^2}\,, \tag{8.40}$$

using

$$s = (p_+ + p_-)^2 = (q_+ + q_-)^2 \tag{8.41a}$$
$$t = (q_- - p_-)^2 = (q_+ - p_+)^2 \tag{8.41b}$$
$$u = (q_- - p_+)^2 = (q_+ - p_-)^2\,. \tag{8.41c}$$

We notice that this is the *same* expression as (8.33) with s and t interchanged or equivalently subject to the cyclic shift $s \to u \to t \to s$.

[2] NB: $k^2 = s \geq 4m_\mu^2 > 0$!

8.4.1 Crossing Symmetry

The relation between (8.33) and (8.40) is not an accident. Comparing the Mandelstam variables (8.41) with (7.43)

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \tag{8.42a}$$
$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \tag{8.42b}$$
$$u = (p_1 - q_2)^2 = (p_2 - q_1)^2 \,, \tag{8.42c}$$

we observe that flipping the lines leaving the diagrams and inverting the fourmomenta simultaneously

$$\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} p_- \\ -q_+ \\ -p_+ \\ q_- \end{pmatrix} \tag{8.43}$$

results in a cyclic shift

$$\begin{pmatrix} s \\ t \\ u \end{pmatrix} = \begin{pmatrix} (p_1 + p_2)^2 \\ (q_1 - p_1)^2 \\ (q_2 - p_1)^2 \end{pmatrix} \to \begin{pmatrix} (p_- - q_+)^2 \\ (p_- + p_+)^2 \\ (q_- - p_-)^2 \end{pmatrix} = \begin{pmatrix} u \\ s \\ t \end{pmatrix} \,, \tag{8.44}$$

that leads to

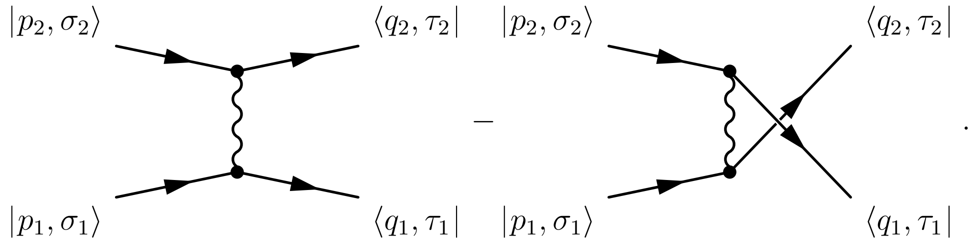
$$\frac{s^2 + u^2}{t^2} \to \frac{u^2 + t^2}{s^2} \,. \tag{8.45}$$

It is hardly surprising, because in (8.10) we have seen that $\mathrm{e}^{-\mathrm{i}px} u_\sigma(p)$ and $\mathrm{e}^{\mathrm{i}px} v_\sigma(p)$ arise from acting with $\psi(x)$ on external states, while $\mathrm{e}^{\mathrm{i}px} \bar u_\sigma(p)$ and $\mathrm{e}^{-\mathrm{i}px} \bar v_\sigma(p)$ arise from acting with $\bar\psi(x)$ on external states. Therefore we can treat antiparticles as particles with reversed fourmomenta.

The corresponding relations among amplitudes are called *crossing relations*. Note, however, that the amplitudes are not the same, they are rather analytic continuations of each other because the physical regions for s, t and u are different.

8.5 $e^-e^- \to e^-e^-$

If we're scattering electrons on electrons, a. k. a. *Møller Scattering*, there is a second Feynman diagram with the outgoing electrons interchanged. According to Fermi statistics, we must subtract these diagrams in order to ensure antisymmetry



(8.46)

The overall sign is not observable and can be chosen by convention. The ξ-dependence cancels again and the scattering amplitude is

$$T = T_t - T_u \tag{8.47a}$$

$$T_t = \frac{e^2}{t+\mathrm{i}\epsilon}\bar u_{\tau_1}(q_1)\gamma_\mu u_{\sigma_1}(p_1)\bar u_{\tau_2}(q_2)\gamma^\mu u_{\sigma_2}(p_2) \tag{8.47b}$$

$$T_u = \frac{e^2}{u+\mathrm{i}\epsilon}\bar u_{\tau_2}(q_2)\gamma_\mu u_{\sigma_1}(p_1)\bar u_{\tau_1}(q_1)\gamma^\mu u_{\sigma_2}(p_2)\,. \tag{8.47c}$$

In the squared amplitude, we will not only have the squares of the individual diagrams, but also interference terms

$$|T|^2 = |T_t - T_u|^2 = |T_t|^2 - T_t^*T_u - T_u^*T_t + |T_u|^2\,. \tag{8.48}$$

The interference terms produce more complicated traces, as can be seen from

$$T_u^*T_t = \frac{e^4}{tu}\underbrace{\bar u_{\sigma_1}(p_1)\gamma_\nu u_{\tau_2}(q_2)\bar u_{\sigma_2}(p_2)\gamma^\nu u_{\tau_1}(q_1)\bar u_{\tau_1}(q_1)\gamma_\mu u_{\sigma_1}(p_1)\bar u_{\tau_2}(q_2)\gamma^\mu u_{\sigma_2}(p_2)}_{=\bar u_{\sigma_1}(p_1)\gamma_\nu u_{\tau_2}(q_2)\bar u_{\tau_2}(q_2)\gamma^\mu u_{\sigma_2}(p_2)\bar u_{\sigma_2}(p_2)\gamma^\nu u_{\tau_1}(q_1)\bar u_{\tau_1}(q_1)\gamma_\mu u_{\sigma_1}(p_1)} \tag{8.49}$$

and the sum over polarizations

$$\sum_{\text{polarization}} T_u^*T_t = \frac{e^4}{tu}\operatorname{tr}\left(\gamma_\nu(\not q_2 + m_e)\gamma^\mu(\not p_2 + m_e)\gamma^\nu(\not q_1 + m_e)\gamma_\mu(\not p_1 + m_e)\right) \tag{8.50}$$

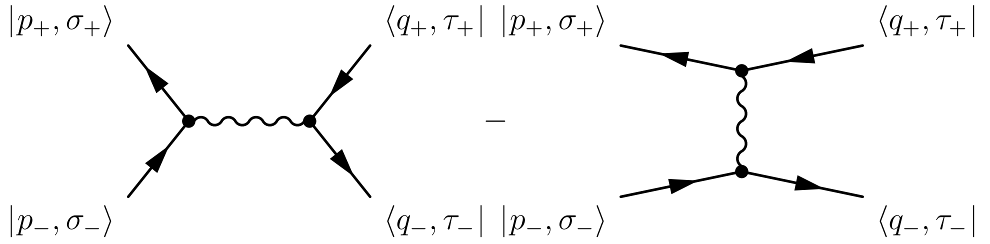
appears to contain traces of up to eight γ-matrices. Fortunately, four of these can be removed by using the contraction identities and we only need to compute traces of four γ-matrices, as before. The final result is remarkably simple

$$\sum_{\text{polarizations}} |T|^2 = 8e^4\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu}\right) + \mathcal{O}(m_e^2)\,. \tag{8.51}$$

Since there are two identical particles in the final state, we have to apply the symmetry factor 1/2! in order to avoid double counting in the differential cross section.

8.6 $e^+e^- \to e^+e^-$

Bhabha Scattering is the crossed process of Møller scattering



(8.52)

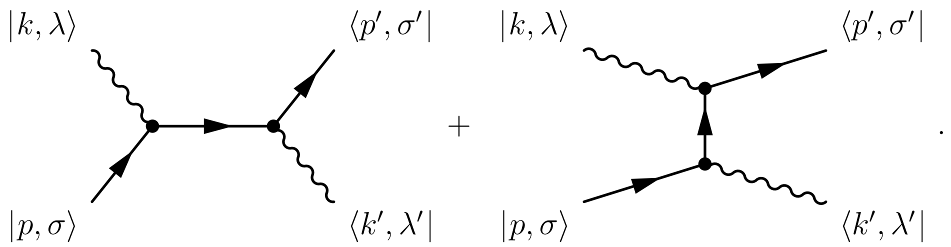
and we get again a relative minus sign and the summed squared matrix element.

$$\sum_{\text{polarizations}} |T|^2 = 8e^4 \left(\frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{st} \right) + \mathcal{O}(m_e^2) \tag{8.53}$$

8.7 Compton Scattering

Lecture 23: Wed, 24. 01. 2018

In the scattering of photons on electrons, a. k. a. *Compton Scattering*, the incoming photon can connect to the incoming electron or to the outgoing electron. Since the diagrams are related by crossing of bosons, they are to be added.



(8.54)

The scattering amplitude can be written

$$\mathrm{i}T = \bar u_{\sigma'}(p')\mathrm{i}e\gamma_\mu \frac{\mathrm{i}}{\not p + \not k - m + \mathrm{i}\epsilon}\mathrm{i}e\gamma_\nu u_\sigma(p)\epsilon^{*,\mu}_{\lambda'}(k')\epsilon^\nu_\lambda(k) \\ + \bar u_{\sigma'}(p')\mathrm{i}e\gamma_\mu \frac{\mathrm{i}}{\not p - \not k' - m + \mathrm{i}\epsilon}\mathrm{i}e\gamma_\nu u_\sigma(p)\epsilon^{*,\nu}_{\lambda'}(k')\epsilon^\mu_\lambda(k) \quad (8.55)$$

i. e., writing $\not\epsilon^* = \gamma^\mu\epsilon^*_\mu$, which is different from $(\not\epsilon)^*$,

$$T = -e^2\bar u_{\sigma'}(p')\left(\frac{\not\epsilon^*_{\lambda'}(k')\,(\not p + \not k + m)\,\not\epsilon_\lambda(k)}{2pk} - \frac{\not\epsilon_\lambda(k)\,(\not p - \not k' + m)\,\not\epsilon^*_{\lambda'}(k')}{2pk'}\right)u_\sigma(p) \quad (8.56)$$

using

$$(p+k)^2 - m^2 = 2pk \quad (8.57a)$$
$$(p-k')^2 - m^2 = -2pk' \quad (8.57b)$$

since $p^2 = p'^2 = m^2$ and $k^2 = k'^2 = 0$. Using the Dirac equation for $u(p)$ and the tranversality of ϵ, we find

$$(\not p + \not k + m)\,\not\epsilon_\lambda(k)u_\sigma(p) = \\ \not\epsilon_\lambda(k)\,(-\not p - \not k + m)\,u_\sigma(p) + 2p\epsilon_\lambda(k) + 2k\epsilon_\lambda(k) \\ = -\not\epsilon_\lambda(k)\not k u_\sigma(p) + 2p\epsilon_\lambda(k) \quad (8.58)$$

and

$$(\not p - \not k' + m)\,\not\epsilon^*_{\lambda'}(k')u_\sigma(p) = \\ \not\epsilon^*_{\lambda'}(k')\,(-\not p + \not k' + m)\,u_\sigma(p) + 2p\epsilon^*_{\lambda'}(k') - 2k'\epsilon^*_{\lambda'}(k') \\ = \not\epsilon^*_{\lambda'}(k')\not k' u_\sigma(p) + 2p\epsilon^*_{\lambda'}(k')\,. \quad (8.59)$$

Furthermore, we may choose our gauge such that

$$p\epsilon_\lambda(k) = p\epsilon^*_{\lambda'}(k') = 0 \quad (8.60)$$

to find

$$T = e^2\bar u_{\sigma'}(p')\left(\frac{\not\epsilon^*_{\lambda'}(k')\not\epsilon_\lambda(k)\not k}{2pk} + \frac{\not\epsilon_\lambda(k)\not\epsilon^*_{\lambda'}(k')\not k'}{2pk'}\right)u_\sigma(p)\,. \quad (8.61)$$

Summing over the spins, but keeping the photon polarizations

$$\sum_{\text{spins}}|T|^2 = e^4\,\mathrm{tr}\left(\left(\frac{\not\epsilon^*_{\lambda'}(k')\not\epsilon_\lambda(k)\not k}{2pk} + \frac{\not\epsilon_\lambda(k)\not\epsilon^*_{\lambda'}(k')\not k'}{2pk'}\right)(\not p + m)\right.$$

$$\left(\frac{\not{k}\not{\epsilon}_\lambda(k)\not{\epsilon}^*_{\lambda'}(k')}{2pk} + \frac{\not{k}'\not{\epsilon}^*_{\lambda'}(k')\not{\epsilon}_\lambda(k)}{2pk'} \right) (\not{p}' + m) \Bigg) \,, \quad (8.62)$$

which still contains terms with up to 8 Dirac matrices. However one can use the Dirac algebra

$$\not{a}\not{b} = -\not{b}\not{a} + 2ab \quad (8.63a)$$
$$\not{a}\not{a} = a^2 \quad (8.63b)$$

together with

$$k^2 = k'^2 = 0 \quad (8.64a)$$
$$(\epsilon_\lambda(k))^2 = (\epsilon^*_{\lambda'}(k'))^2 = -1 \,, \quad (8.64b)$$

transversality, energy-momentum conservation and our choice of gauge (8.60) to avoid traces of more than four Dirac matrices.

8.7.1 Ward Identity

Using an unphysical polarization vector $\epsilon(k') = k'$ for the outgoing photon in the Compton amplitude, we find

$$\tilde{T} = -e^2 \bar{u}_{\sigma'}(p')\not{k}' \frac{1}{\not{p}' + \not{k}' - m + \mathrm{i}\epsilon} \not{\epsilon}_\lambda(k) u_\sigma(p) - e^2 \bar{u}_{\sigma'}(p')\not{\epsilon}_\lambda(k) \frac{1}{\not{p} - \not{k}' - m + \mathrm{i}\epsilon} \not{k}' u_\sigma(p) \quad (8.65)$$

using momentum conservation $p + k = p' + k'$. Then we can use the Dirac equation in the form

$$\bar{u}_{\sigma'}(p')\not{k}' = \bar{u}_{\sigma'}(p') \left(\not{p}' + \not{k}' - m \right) \quad (8.66a)$$
$$\not{k}' u_\sigma(p) = - \left(\not{p} - \not{k}' - m \right) u_\sigma(p) \quad (8.66b)$$

to find

$$\begin{aligned}\tilde{T} &= -e^2 \bar{u}_{\sigma'}(p') \left(\not{p}' + \not{k}' - m \right) \frac{1}{\not{p}' + \not{k}' - m + \mathrm{i}\epsilon} \not{\epsilon}_\lambda(k) u_\sigma(p) \\ &\quad + e^2 \bar{u}_{\sigma'}(p')\not{\epsilon}_\lambda(k) \frac{1}{\not{p} - \not{k}' - m + \mathrm{i}\epsilon} \left(\not{p} - \not{k}' - m \right) u_\sigma(p) \\ &= -e^2 \bar{u}_{\sigma'}(p')\not{\epsilon}_\lambda(k) u_\sigma(p) + e^2 \bar{u}_{\sigma'}(p')\not{\epsilon}_\lambda(k) u_\sigma(p) = 0 \,, \end{aligned} \quad (8.67)$$

i. e. that the amplitude for producing unphysical photons with polarization vector $\epsilon(k) = k$ *vanishes*, iff *all* other external particles satisfy the equations of motion. This is called the *Ward Identity* and can be shown to hold for all processes and can be maintained in higher orders of perturbation theory.

It is important that the Ward Identity does *not* hold diagram by diagram, but only for complete scattering amplitudes. Thus it can be used to guard against some sources of errors.

Note that the Ward Identity is nothing but the statement that matrix elements of the current operator are conserved

$$\partial^\mu \left\langle \text{out} \middle| {:}\bar\psi(x)\gamma_\mu\psi(x){:} \middle| \text{in} \right\rangle = 0\,. \tag{8.68}$$

In fact, in the case of QED treated here, we do not need to put the other photons on the mass-shell or to require physical polarizations for them. This is *not* the case for more general (non-abelian) gauge theories, but will be useful in the next section.

8.7.2 Polarization Sum

A possible expression for the polarization sum for a photon with momentum k is given by

$$\sum_{\sigma=-1,1} \epsilon^\mu_\sigma(k)\epsilon^{*,\nu}_\sigma(k) = \Pi^{\mu\nu}(k,c) = -g^{\mu\nu} + \frac{k^\mu c^\nu + k^\nu c^\mu}{kc} = \Pi^{\nu\mu}(k,c) \tag{8.69}$$

with any fourvector c such that $ck \neq 0$. Indeed, using $k^2 = 0$, the sum is transversal

$$k_\mu \Pi^{\mu\nu}(k,c) = 0 \tag{8.70}$$

and for $k = (\omega, 0, 0, \omega)$ and $c = (1, 0, 0, -1)$ we find

$$\Pi^{\mu\nu}(k,c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{8.71}$$

Other choices for c correspond to different gauge choices.

Using the Ward identity (8.68), we observe that the term

$$\frac{k_\mu c_\nu + k_\nu c_\mu}{kc}$$

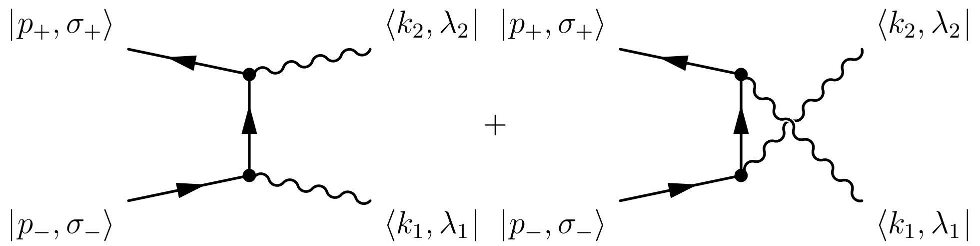
gives no contribution to the polarization sum and we can make the more convenient choice

$$\sum_{\sigma=-1,1} \epsilon^\mu_\sigma(k)\epsilon^{*,\nu}_\sigma(k) = -g_{\mu\nu}\,. \tag{8.72}$$

We can use the simplified for more than one photon in QED only because the Ward identity for one photon does not require a physical polarization for the other photons. This is not true for non-abelian gauge theories with self interactions among gauge bosons and more complicated procedures are required.

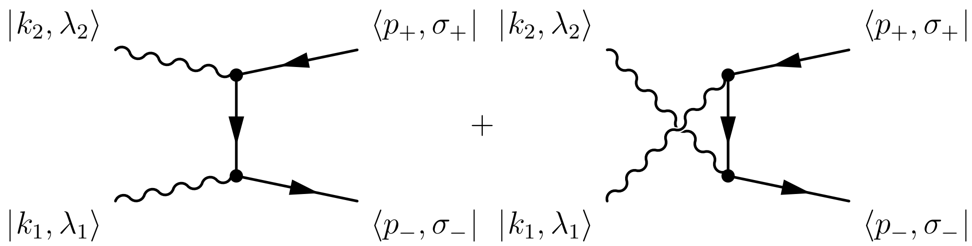
8.8 *Pair Creation and Annihilation*

The processes $e^+e^- \to \gamma\gamma$



(8.73)

and $\gamma\gamma \to e^+e^-$



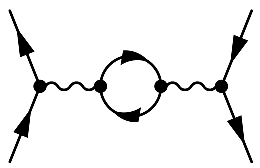
(8.74)

can be obtained by crossing Compton scattering.

—9— Radiative Corrections

9.1 Example

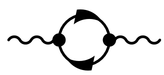
In $e^+e^- \to \mu^+\mu^-$ there is, among others, a contribution with a fermion loop



$$. \tag{9.1}$$

If we isolate the loop

$$\mathrm{i}\Gamma^{(2)}_{\mu\nu}(p) = \quad A_\mu(p)$$



$$A_\nu(-p)$$

$$= e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{tr}\left(\gamma_\mu \left(\not{k} + m\right) \gamma_\nu \left(\not{k} + \not{p} + m\right)\right)}{\left(k^2 - m^2 + \mathrm{i}\epsilon\right)\left((k+p)^2 - m^2 + \mathrm{i}\epsilon\right)}, \tag{9.2}$$

we observe that we have to learn how to evaluate integrals of rational functions of fourmomenta over all of $\mathbf{R}^4$.

9.2 General Tensor Integrals (1-Loop)

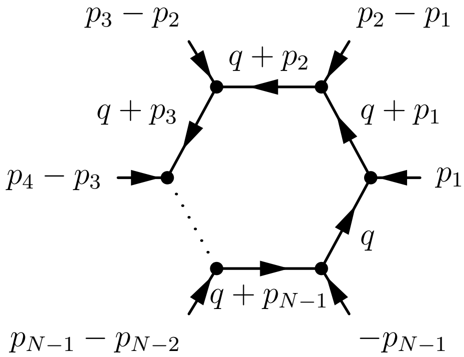
Lecture 24: Tue, 30. 01. 2018

The most general integral with a single loop can be written as a linear combination of tensors

$$\tilde{T}^{(N)}_{\mu_1\mu_2\ldots\mu_M}(p_1,p_2,\ldots,p_{N-1};m_0,m_1,\ldots,m_{N-1}) = \int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_M}}{(q^2-m_0^2+\mathrm{i}\epsilon)\left((q+p_1)^2-m_1^2+\mathrm{i}\epsilon\right)\cdots\left((q+p_{N-1})^2-m_{N-1}^2+\mathrm{i}\epsilon\right)} \tag{9.3}$$

and can be represented graphically as

$$\tilde{T}^{(N)} = \tag{9.4}$$



with the arrows just denoting the flow of momenta. For later convenience, we generalize from 4 to D dimensions and extract a prefactor

$$T^{(N)}_{\mu_1\mu_2\ldots\mu_M}(p_1,p_2,\ldots,p_{N-1};m_0,m_1,\ldots,m_{N-1};D,\mu) = \frac{16\pi^2}{\mathrm{i}}\mu^{4-D}\tilde{T}^{(N)}_{\mu_1\mu_2\ldots\mu_M}(p_1,p_2,\ldots,p_{N-1};m_0,m_1,\ldots,m_{N-1})\Big|_{\text{“}4\to D\text{”}} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int\mathrm{d}^Dq\,\frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_M}}{(q^2-m_0^2+\mathrm{i}\epsilon)\left((q+p_1)^2-m_1^2+\mathrm{i}\epsilon\right)\cdots\left((q+p_{N-1})^2-m_{N-1}^2+\mathrm{i}\epsilon\right)}\,. \tag{9.5}$$

There are notational conventions for one, two, three and four point integrals:

- $T^{(1)}_{\mu_1\mu_2\ldots\mu_M} = A_{\mu_1\mu_2\ldots\mu_M}(m_0)$,
- $T^{(2)}_{\mu_1\mu_2\ldots\mu_M} = B_{\mu_1\mu_2\ldots\mu_M}(p_1;m_0,m_1)$,
- $T^{(3)}_{\mu_1\mu_2\ldots\mu_M} = C_{\mu_1\mu_2\ldots\mu_M}(p_1,p_2;m_0,m_1,m_2)$,
- $T^{(4)}_{\mu_1\mu_2\ldots\mu_M} = D_{\mu_1\mu_2\ldots\mu_M}(p_1,p_2,p_3;m_0,m_1,m_2,m_3)$

and in particular for the *scalar integrals* (i. e. $M=0$)

- $A(m_0) = A_0(m_0)$,
- $B(p_1;m_0,m_1) = B_0(p_1;m_0,m_1)$,
- $C(p_1,p_2;m_0,m_1,m_2) = C_0(p_1,p_2;m_0,m_1,m_2)$,
- $D(p_1,p_2,p_3;m_0,m_1,m_2,m_3) = D_0(p_1,p_2,p_3;m_0,m_1,m_2,m_3)\,.$

9.2.1 Tensor Decomposition

The only vectors and tensors that can appear in the result are the *external* momenta p_i and the metric g. Since the integrand is totally symmetric, the result must be totally symmetric as well and the totally antisymmetric ϵ-tensor can not appear. Therefore we can expand the tensor integrals in covariants

$$B^\mu(p_1;m_0,m_1) = p_1^\mu B_1(p_1;m_0,m_1) \tag{9.6a}$$

$$C^\mu(p_1,p_2;m_0,m_1,m_2) = p_1^\mu C_1(p_1,p_2;m_0,m_1,m_2) + p_2^\mu C_2(p_1,p_2;m_0,m_1,m_2) \tag{9.6b}$$

$$\ldots = \ldots \tag{9.6c}$$

and

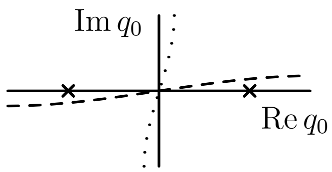
$$B^{\mu\nu}(p_1;m_0,m_1) = p_1^\mu p_1^\nu B_{11}(p_1;m_0,m_1) + g^{\mu\nu} B_{00}(p_1;m_0,m_1) \tag{9.7a}$$

$$\begin{aligned} C^{\mu\nu}(p_1,p_2;m_0,m_1,m_2) = {} & p_1^\mu p_1^\nu C_{11}(p_1,p_2;m_0,m_1,m_2) \\ & + \left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu\right) C_{12}(p_1,p_2;m_0,m_1,m_2) \\ & + p_2^\mu p_2^\nu C_{22}(p_1,p_2;m_0,m_1,m_2) \\ & + g^{\mu\nu} C_{00}(p_1,p_2;m_0,m_1,m_2) \end{aligned} \tag{9.7b}$$

$$\ldots = \ldots \tag{9.7c}$$

9.2.2 Wick Rotation

The q_0-integration contour in the loop integrals can be deformed from the dashed curves to the dotted curve



without crossing poles or cuts. With the subsequent substitution

$$(q^0, \vec{q}) \to (iq_E^0, \vec{q}_E)\,, \tag{9.8}$$

the Minkowski-"length" becomes a *euclidean* length

$$q^2 = (q^0)^2 - \vec{q}^2 = -(q_E^0)^2 - \vec{q}^2 = -q_E^2\,. \tag{9.9}$$

9.2.3 D−Dimensional Integration

In the following, we will assume

$$n > \max\left\{1, \frac{D}{2}\right\} \tag{9.10a}$$

$$a > 0 \tag{9.10b}$$

and attempt to continue analytically in D and a, if necessary. Using the Wick rotation we can the rewrite the integral

$$\begin{aligned} I_n(a) &= \int \frac{\mathrm{d}^D q}{(q^2 - a + \mathrm{i}\epsilon)^n} = \int_{-\infty}^{\infty} \mathrm{d}q_0 \int \frac{\mathrm{d}^{(D-1)}\vec q}{(q_0^2 - \vec q^2 - a + \mathrm{i}\epsilon)^n} \\ &= \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}q_0 \int \frac{\mathrm{d}^{(D-1)}\vec q}{(q_0^2 - \vec q^2 - a + \mathrm{i}\epsilon)^n} = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}q_{E,0} \int \frac{\mathrm{d}^{(D-1)}\vec q_E}{(-q_{E,0}^2 - \vec q_E^2 - a + \mathrm{i}\epsilon)^n} \\ &= (-1)^n \mathrm{i} \int \frac{\mathrm{d}^D q_E}{(q_E^2 + a - \mathrm{i}\epsilon)^n} \end{aligned} \tag{9.11}$$

and introducing D-dimensional polar coordinates

$$\int \mathrm{d}^D q_E = \int \mathrm{d}\Omega_D \int_0^\infty |q_E|^{D-1} \mathrm{d}|q_E| = \frac{1}{2} \int \mathrm{d}\Omega_D \int_0^\infty \left(q_E^2\right)^{\frac{D}{2}-1} \mathrm{d}q_E^2 \tag{9.12}$$

with

$$\int \mathrm{d}\Omega_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \tag{9.13}$$

we find[1]

$$\begin{aligned} I_n(a) &= (-1)^n \mathrm{i} \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \mathrm{d}q_E^2 \frac{(q_E^2)^{\frac{D}{2}-1}}{(q_E^2 + a - \mathrm{i}\epsilon)^n} \\ &= (-1)^n \mathrm{i} \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} (a - \mathrm{i}\epsilon)^{\frac{D}{2}-n} \int_0^\infty \mathrm{d}x \frac{x^{\frac{D}{2}-1}}{(x+1)^n} \\ &= (-1)^n \mathrm{i} \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} (a - \mathrm{i}\epsilon)^{\frac{D}{2}-n} B\left(\frac{D}{2}, n - \frac{D}{2}\right) \\ &= (-1)^n \mathrm{i} \pi^{\frac{D}{2}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} (a - \mathrm{i}\epsilon)^{\frac{D}{2}-n} . \end{aligned} \tag{9.14}$$

From the properties of Euler's Γ-function

[1] Euler's Beta-function:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} .$$

- $\Gamma(z)$ is analytical everywhere, except for simple poles at $0, -1, -2, \ldots$
- $1/\Gamma(z)$ is analytical everywhere
- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(n+1) = n!$ for $n \in \mathbf{N_0}$
- $\Gamma(1/2) = \sqrt{\pi}$
- Laurent expansion at the origin

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{9.15}$$

  with $\gamma_E = 0.5772\ldots$,

we can derive the analytical continuation of $I_n(a)$ in D and a and we find that logarithmic UV, i. e. large energy and momentum, divergencies appear as poles in $\epsilon = 2 - \frac{D}{2}$ and quadratic divergencies UV as poles in $2 - D$.

9.2.4 Scalar Integrals

A_0

Using these formulae, we find

$$\begin{aligned} A_0(m_0) &= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{1}{q^2 - m_0^2 + \mathrm{i}\varepsilon} = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} I_1(m_0^2) \\ &= -m_0^2 \left(\frac{m_0^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right) = -m_0^2 \left(\frac{4\pi\mu^2}{m_0^2}\right)^{\epsilon} \Gamma(\epsilon - 1) \end{aligned} \tag{9.16}$$

with the conventional definition

$$D = 4 - 2\epsilon\,. \tag{9.17}$$

For $D \to 4$, i. e. $\epsilon \to 0$, we can expand

$$\left(\frac{4\pi\mu^2}{m_0^2}\right)^{\epsilon} = 1 + \epsilon \ln \frac{4\pi\mu^2}{m_0^2} + \mathcal{O}(\epsilon^2) \tag{9.18a}$$

$$\begin{aligned} \Gamma(\epsilon - 1) &= \frac{1}{\epsilon - 1}\Gamma(\epsilon) = -\left(1 + \epsilon + \mathcal{O}(\epsilon^2)\right)\left(\frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right) \\ &= -\left(\frac{1}{\epsilon} - \gamma_E + 1\right) + \mathcal{O}(\epsilon) \end{aligned} \tag{9.18b}$$

and therefore

$$A_0(m_0; D, \mu) = m_0^2 \left(\underbrace{\frac{1}{\epsilon} - \gamma_E + \ln(4\pi)}_{\Delta} + \ln\frac{\mu^2}{m_0^2} + 1 \right) + \mathcal{O}(\epsilon)$$
$$= m_0^2 \left(\Delta + \ln\frac{\mu^2}{m_0^2} + 1 \right) + \mathcal{O}(\epsilon) \quad (9.19)$$

B_0

Lecture 25: Wed, 31. 01. 2018

Similarly

$$B_0(p_1; m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \mathrm{d}^D q \, \frac{1}{(q^2 - m_0^2 + \mathrm{i}\varepsilon)\left((q+p_1)^2 - m_1^2 + \mathrm{i}\varepsilon\right)} \quad (9.20)$$

but before we can use (9.14) we need to combine denominators using *Feynman parameters*

$$\frac{1}{xy} = \int_0^1 \frac{\mathrm{d}\xi}{\left((1-\xi)x + \xi y\right)^2} \,. \quad (9.21)$$

Completing the square

$$\frac{1}{(q^2 - m_0^2 + \mathrm{i}\varepsilon)\left((q+p_1)^2 - m_1^2 + \mathrm{i}\varepsilon\right)} =$$
$$\int_0^1 \frac{\mathrm{d}\xi}{\left((1-\xi)(q^2 - m_0^2 + \mathrm{i}\varepsilon) + \xi\left((q+p_1)^2 - m_1^2 + \mathrm{i}\varepsilon\right)\right)^2}$$
$$= \int_0^1 \frac{\mathrm{d}\xi}{\left(q^2 + \xi 2qp_1 + \xi(p_1^2 - m_1^2 + m_0^2) - m_0^2 + \mathrm{i}\varepsilon\right)^2}$$
$$= \int_0^1 \frac{\mathrm{d}\xi}{\left((\underbrace{q + \xi p_1}_{q'})^2 - \underbrace{\left(\xi^2 p_1^2 - \xi(p_1^2 - m_1^2 + m_0^2) + m_0^2\right)}_{a} + \mathrm{i}\varepsilon \right)^2}$$
$$= \int_0^1 \frac{\mathrm{d}\xi}{\left((q')^2 - a + \mathrm{i}\varepsilon\right)^2} \quad (9.22)$$

we can substitute q' for q with unit Jacobian

$$B_0(p_1; m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int_0^1 \mathrm{d}\xi \int \mathrm{d}^D q' \, \frac{1}{\left((q')^2 - a(\xi) + \mathrm{i}\varepsilon\right)^2}$$

$$= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int_0^1 \mathrm{d}\xi\, I_2(a(\xi))$$
$$= (4\pi)^\epsilon \Gamma(\epsilon) \int_0^1 \mathrm{d}\xi \left(\frac{\xi^2 p_1^2 - \xi(p_1^2 - m_1^2 + m_0^2) + m_0^2 - \mathrm{i}\varepsilon}{\mu^2} \right)^{-\epsilon} \tag{9.23}$$

and we can again expand for $\epsilon \to 0$:

$$B_0(p_1; m_0, m_1) = \Delta - \int_0^1 \mathrm{d}\xi \ln \frac{\xi^2 p_1^2 - \xi(p_1^2 - m_1^2 + m_0^2) + m_0^2 - \mathrm{i}\varepsilon}{\mu^2} + \mathcal{O}(\epsilon)\,. \tag{9.24}$$

Observations:

- $B_0(p_1; m_0, m_1)$ depends on p_1 only through p_1^2 and we could write

$$B_0(p_1; m_0, m_1) = \tilde{B}_0(p_1^2; m_0, m_1) \tag{9.25}$$

- we have

$$B_0(p_1; m_0, m_1) = B_0(p_1; m_1, m_0) \tag{9.26}$$

  because we could have shifted the loop momentum $q \to q - p_1$.

9.3 Tensor Reduction

Observation: since

$$\underbrace{p_i^\mu q_\mu}_{T_M^{(N)}} = \underbrace{\underbrace{\frac{1}{2}\left[(q+p_i)^2 - m_i^2\right]}_{i\text{th denominator}}}_{T_{M-1}^{(N-1)}} - \underbrace{\underbrace{\frac{1}{2}\left[q^2 - m_0^2\right]}_{0\text{th denominator}}}_{T_{M-1}^{(N-1)}} - \underbrace{\underbrace{\frac{1}{2}\left[p_i^2 - m_i^2 + m_0^2\right]}_{\text{lower rank tensor}}}_{T_{M-1}^{(N)}} \tag{9.27a}$$

$$\underbrace{g^{\mu\nu} q_\mu q_\nu}_{T_M^{(N)}} = \underbrace{\underbrace{q^2 - m_0^2}_{0\text{th denominator}}}_{T_{M-2}^{(N-1)}} + \underbrace{\underbrace{m_0^2}_{\text{lower rank tensor}}}_{T_{M-2}^{(N)}} \tag{9.27b}$$

all contractions of tensor integrals can be expressed by tensor integrals of strictly lower rank and/or strictly lower number of denominators. Likewise, contracting the expansion in covariants (9.6) results in linear combinations of the coefficient functions. Therefore, we obtain a hierarchy of systems of linear equations that can be solved recursively (provided we can avoid singularities).

9.3.1 B_μ

Notational shorthand

$$\langle f(q;\ldots)\rangle_q = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D q\, f(q;\ldots) \tag{9.28}$$

and all $+\mathrm{i}\epsilon$ in the denominators implied.

$$B^\mu(p_1;m_0,m_1) = p_1^\mu B_1(p_1;m_0,m_1) = \left\langle \frac{q^\mu}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)} \right\rangle_q . \tag{9.29}$$

Since there is only one invariant B_1, a single contraction suffices. Contracting both sides with $p_{1,\mu}$

$$\begin{aligned}
p_1^2 B_1(p_1;m_0,m_1) &= \left\langle \frac{p_1 q}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)} \right\rangle_q \\
&= \frac{1}{2}\left\langle \frac{\left((q+p_1)^2-m_1^2\right)-(q^2-m_0^2)-(p_1^2-m_1^2+m_0^2)}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)} \right\rangle_q \\
&= \frac{1}{2}\left\langle \frac{1}{q^2-m_0^2} \right\rangle_q - \frac{1}{2}\left\langle \frac{1}{(q+p_1)^2-m_1^2} \right\rangle_q \\
&\quad - \frac{p_1^2-m_1^2+m_0^2}{2}\left\langle \frac{1}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)} \right\rangle_q \\
&= \frac{1}{2}A_0(m_0^2) - \frac{1}{2}A_0(m_1^2) - \frac{p_1^2-m_1^2+m_0^2}{2}B_0(p_1;m_0,m_1)
\end{aligned} \tag{9.30}$$

i. e.

$$B_1(p_1;m_0,m_1) = \frac{1}{2p_1^2}\left(A_0(m_0^2) - A_0(m_1^2) - (p_1^2-m_1^2+m_0^2)B_0(p_1;m_0,m_1)\right) . \tag{9.31}$$

9.3.2 $B_{\mu\nu}$

Expand in available tensors with new scalar coefficient functions:

$$\begin{aligned}
B^{\mu\nu}(p_1;m_0,m_1) &= p_1^\mu p_1^\nu B_{11}(p_1;m_0,m_1) + g^{\mu\nu}B_{00}(p_1;m_0,m_1) \\
&= \left\langle \frac{q^\mu q^\nu}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)} \right\rangle_q .
\end{aligned} \tag{9.32}$$

In the following

$$B_{\ldots} = B_{\ldots}(p_1;m_0,m_1) \tag{9.33}$$

will be implied. We need two contractions for two invariants: first with $g^{\mu\nu}$

$$p_1^2 B_{11} + D B_{00} = \left\langle \frac{(q^2 - m_0^2) + m_0^2}{(q^2 - m_0^2)\left((q+p_1)^2 - m_1^2\right)} \right\rangle_q = A_0(m_1) + m_0^2 B_0 \quad (9.34a)$$

(note that $g^{\mu\nu} g_{\mu\nu} = D$) and

$$\begin{aligned} p_{1,\nu} p_1^2 B_{11} + p_{1,\nu} B_{00} = & \\ & \frac{1}{2} \left\langle \frac{q_\nu}{(q^2 - m_0^2)} \right\rangle_q - \frac{1}{2} \left\langle \frac{q_\nu}{((q+p_1)^2 - m_1^2)} \right\rangle_q \\ & - \frac{p_1^2 - m_1^2 + m_0^2}{2} \left\langle \frac{q_\nu}{(q^2 - m_0^2)\left((q+p_1)^2 - m_1^2\right)} \right\rangle_q \\ & = 0 - \frac{1}{2} \left\langle \frac{q'_\nu - p_{1,\nu}}{(q')^2 - m_1^2} \right\rangle_{q'} - \frac{p_1^2 - m_1^2 + m_0^2}{2} B_\nu \\ & = \frac{1}{2} p_{1,\nu} A_0(m_1) - \frac{p_1^2 - m_1^2 + m_0^2}{2} p_{1,\nu} B_1 \end{aligned} \quad (9.34b)$$

where we have made use of *symmetric integration*

$$\left\langle q_\mu f(q^2) \right\rangle_q = 0\,. \quad (9.35)$$

Thus we obtain a linear equation for B_{00} and B_{11}:

$$\begin{pmatrix} D & p_1^2 \\ 1 & p_1^2 \end{pmatrix} \begin{pmatrix} B_{00} \\ B_{11} \end{pmatrix} = \begin{pmatrix} A_0(m_1) + m_0^2 B_0 \\ \frac{1}{2} A_0(m_1) - \frac{p_1^2 - m_1^2 + m_0^2}{2} B_1 \end{pmatrix}, \quad (9.36)$$

with solution

$$B_{00} = \frac{A_0(m_1) + 2m_0^2 B_0 + (p_1^2 - m_1^2 + m_0^2) B_1}{2(D-1)} \quad (9.37a)$$

$$B_{11} = \frac{(D-2) A_0(m_1) - 2m_0^2 B_0 - D(p_1^2 - m_1^2 + m_0^2) B_1}{2(D-1) p_1^2} \quad (9.37b)$$

and divergent pieces

$$B_{00} = -\frac{1}{12} \left(p_1^2 - 3(m_0^2 + m_1^2) \right) \Delta + \text{finite} \quad (9.38a)$$

$$B_{11} = \frac{1}{3} \Delta + \text{finite} \quad (9.38b)$$

9.3.3 C_μ

$$\begin{aligned} C^\mu(p_1,p_2;m_0,m_1,m_2) &= p_1^\mu C_1(p_1,p_2;m_0,m_1,m_2)+p_2^\mu C_2(p_1,p_2;m_0,m_1,m_2) \\ &= \left\langle \frac{q^\mu}{(q^2-m_0^2)\left((q+p_1)^2-m_1^2\right)\left((q+p_2)^2-m_2^2\right)} \right\rangle_q . \end{aligned} \tag{9.39}$$

A simple exercise yields

$$\begin{aligned} &\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix}^{-1} \times \\ &\begin{pmatrix} B_0(p_2^2;m_0,m_2) - B_0((p_1-p_2)^2;m_1,m_2) - (p_1^2-m_1^2+m_0^2)C_0 \\ B_0(p_1^2;m_0,m_1) - B_0((p_1-p_2)^2;m_1,m_2) - (p_2^2-m_2^2+m_0^2)C_0 \end{pmatrix} . \end{aligned} \tag{9.40}$$

The divergent part of B_0 is independent of masses and momenta

$$B_0 = \Delta + \text{finite}\,, \tag{9.41}$$

therefore the divergencies cancel in $C_{1,2}$.

9.3.4 Gram Determinants

Lecture 26: Tue, 06. 02. 2018

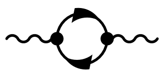
However, whenever the Gram determinant

$$G(p_1,p_2,\ldots,p_n) = \begin{vmatrix} p_1^2 & p_1p_2 & \cdots & p_1p_n \\ p_2p_1 & p_2^2 & \cdots & p_2p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_np_1 & p_np_2 & \cdots & p_n^2 \end{vmatrix} \tag{9.42}$$

vanishes, the expressions for the invariants become ill defined. This is easily understood geometrically, because it means that the momenta are not linearly independent. Fundamentally, this is no problem, because the values on the singular submanifolds can be obtained by continuity. Unfortunately, this complicates the numerical evaluation significantly and other, potentially better behaved, methods are being studied.

9.3.5 Vacuum Polarization

Returning to the fermion loop

$$\mathrm{i}\Sigma_{\mu\nu}(p) = - \quad A_\mu(p)$$  $$A_\nu(-p)$$

$$= e^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathrm{tr}\left[\gamma_\mu \left(\not{q} + m\right) \gamma_\nu \left(\not{q} + \not{p} + m\right)\right]}{\left(q^2 - m^2 + \mathrm{i}\epsilon\right)\left((q+p)^2 - m^2 + \mathrm{i}\epsilon\right)} \qquad (9.1')$$

and performing the traces

$$\begin{aligned}
&\mathrm{tr}\left[\gamma_\mu \left(\not{q} + m\right) \gamma_\nu \left(\not{q} + \not{p} + m\right)\right] \\
&\qquad = \mathrm{tr}\left[\gamma_\mu \not{q} \gamma_\nu \not{q}\right] + \mathrm{tr}\left[\gamma_\mu \not{q} \gamma_\nu \not{p}\right] + m^2\, \mathrm{tr}\left[\gamma_\mu \gamma_\nu\right] \\
&\qquad = \left(2q_\mu q_\nu - q^2 g_{\mu\nu}\right) \mathrm{tr}\,\mathbf{1} + \left(q_\mu p_\nu + p_\mu q_\nu - qp g_{\mu\nu}\right) \mathrm{tr}\,\mathbf{1} + m^2 g_{\mu\nu}\, \mathrm{tr}\,\mathbf{1} \\
&\qquad = \left(2q_\mu q_\nu + q_\mu p_\nu + p_\mu q_\nu - \left(q^2 + qp - m^2\right) g_{\mu\nu}\right) \mathrm{tr}\,\mathbf{1} \\
&= \left(2q_\mu q_\nu + q_\mu p_\nu + p_\mu q_\nu\right) \mathrm{tr}\,\mathbf{1} - \left((q^2 - m^2) + ((q+p)^2 - m^2) - p^2\right) g_{\mu\nu} \frac{\mathrm{tr}\,\mathbf{1}}{2}\,,
\end{aligned} \qquad (9.43)$$

we find ($+\mathrm{i}\epsilon$ implied, again):

$$\begin{aligned}
\Sigma_{\mu\nu}(p) &= \frac{\alpha}{\pi} \frac{\mathrm{tr}\,\mathbf{1}}{4} \left\langle \frac{2q_\mu q_\nu + q_\mu p_\nu + p_\mu q_\nu}{\left(q^2 - m^2\right)\left((q+p)^2 - m^2\right)} \right\rangle_q \\
&\quad + \frac{\alpha}{\pi} \frac{\mathrm{tr}\,\mathbf{1}}{8} g_{\mu\nu} \Bigg(\left\langle \frac{-1}{\left((q+p)^2 - m^2\right)} \right\rangle_q + \left\langle \frac{-1}{\left(q^2 - m^2\right)} \right\rangle_q \\
&\qquad + \left\langle \frac{p^2}{\left(q^2 - m^2\right)\left((q+p)^2 - m^2\right)} \right\rangle_q \Bigg) \\
&= \frac{\alpha}{\pi} \frac{\mathrm{tr}\,\mathbf{1}}{4} \left(2B_{\mu\nu}(p;m,m) + p_\mu B_\nu(p;m,m) + p_\nu B_\mu(p;m,m)\right) \\
&\quad - \frac{\alpha}{\pi} \frac{\mathrm{tr}\,\mathbf{1}}{8} g_{\mu\nu} \left(A_0(m) + A_0(m) - p^2 B_0(p;m,m)\right) \\
&= \frac{\alpha}{\pi} \frac{\mathrm{tr}\,\mathbf{1}}{4} \left(2g_{\mu\nu} B_{00}(p;m,m) + 2p_\mu p_\nu B_{11}(p;m,m) + 2p_\mu p_\nu B_{00}(p;m,m)\right) \\
&\quad - \frac{\alpha}{\pi} \frac{\mathrm{tr}\,\mathbf{1}}{4} g_{\mu\nu} \left(A_0(m) - \frac{p^2}{2} B_0(p;m,m)\right)
\end{aligned}$$

$$= \frac{\alpha}{\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\Biggl(p_\mu p_\nu\left(2B_{11}(p;m,m) + 2B_1(p;m,m)\right) \\ + g_{\mu\nu}\left(2B_{00}(p;m,m) - A_0(m) + \frac{p^2}{2}B_0(p;m,m)\right)\Biggr) \tag{9.44}$$

Useful decomposition

$$\Sigma_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Sigma_T(p^2) + \frac{p_\mu p_\nu}{p^2}\Sigma_L(p^2) \tag{9.45}$$

with

$$\Sigma_T(p^2) = \frac{\alpha}{\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\left(2B_{00}(p;m,m) + \frac{p^2}{2}B_0(p;m,m) - A_0(m)\right) \tag{9.46a}$$

$$\Sigma_L(p^2) = \frac{\alpha}{\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\Biggl(2B_{00}(p;m,m) + 2p^2(B_{11}(p;m,m) + B_1(p;m,m)) \\ + \frac{p^2}{2}B_0(p;m,m) - A_0(m)\Biggr) \tag{9.46b}$$

and ultimately

$$\Sigma_T(p^2) = \frac{\alpha}{3\pi}\frac{\operatorname{tr}\mathbf{1}}{4}\left((p^2+2m^2)B_0(p;m,m) - \frac{p^2}{3} - 2m^2B_0(0;m,m)\right) \tag{9.47a}$$

$$\Sigma_L(p^2) = 0 \tag{9.47b}$$

(see exercise) using $A_0(m) = m^2B_0(0;m,m) + m^2$ etc. This is *not* an accident, but the result of gauge invariance[2] and required for the consistency of the theory.

Remark #1

What is $\operatorname{tr}\mathbf{1}$?

- in fourdimensional Dirac algebra, the smallest faithful representation is also fourdimensional, thus $\operatorname{tr}\mathbf{1}\big|_{D=4} = 4$.
- in D-dimensional Dirac algebra, the smallest faithful representation is $2^{\lfloor D/2\rfloor}$-dimensional, thus $\operatorname{tr}\mathbf{1} = 2^{\lfloor D/2\rfloor}$.

[2] Provided we use a gauge invariant regularization scheme such as dimensional regularization.

In any case,

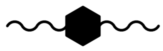
$$\operatorname{tr}\mathbf{1} = 4 + \mathcal{O}(D-4) \tag{9.48}$$

and any difference can be absorbed in the definition of

$$\Delta = \frac{1}{\epsilon} - \gamma_{\mathrm{E}} + \ln(4\pi)\ . \tag{9.49}$$

Remark #2

What is the Feynman rule



(9.50)

corresponding to

$$\mathcal{L}_I = -\frac{c}{4}F_{\mu\nu}F^{\mu\nu}\,? \tag{9.51}$$

Up to boundary terms

$$\begin{aligned}\mathcal{L}_I = -\frac{c}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{c}{2}\partial_\mu A_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\\ &= \frac{c}{2}A_\mu\left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu \to \frac{c}{2}A_\mu\left(p^\mu p^\nu - p^2 g^{\mu\nu}\right)A_\nu\end{aligned} \tag{9.52}$$

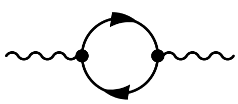
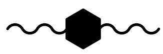
this is proportional to the *transversal* vacuum polarization

$$\Sigma_{\mu\nu}(p) = \frac{\alpha}{3\pi}\Delta\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) + \text{finite}\,. \tag{9.53}$$

Thus, we can add the *counter term*

$$\mathcal{L}_{\text{c.t.}} = -\frac{\alpha}{6\pi}\Delta F_{\mu\nu}F^{\mu\nu} \tag{9.54}$$

to the interaction Lagrangian in order to obtain a *finite* correction for the photon propagator:

 $+$  $=$ finite. (9.55)

Since we can also write

$$\mathcal{L}_0 + \mathcal{L}_{\text{c.t.}} = -\frac{Z_A}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.56}$$

with

$$Z_A = 1 + \delta Z_A = 1 + \frac{2\alpha}{3\pi}\Delta\,, \tag{9.57}$$

adding the counterterm amounts to a *renormalization* of the photon field with the factor

$$\sqrt{Z_A} = \sqrt{1+\delta Z_A} = 1 + \frac{\delta Z_A}{2} + \mathcal{O}(\alpha^2) = 1 + \frac{\alpha}{3\pi}\Delta + \mathcal{O}(\alpha^2)\,. \tag{9.58}$$

In order for this to work, we also have to renormalize the gauge fixing parameter ξ

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \to -\frac{Z_A}{2Z_\xi\xi}\left(\partial_\mu A^\mu\right)^2 = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 \tag{9.59}$$

with $Z_\xi = Z_A$.

Remark #3

Naively, on would expect that

$$\Sigma_{\mu\nu}(p) \approx \frac{\alpha}{3\pi}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{1}{q^2} \tag{9.60}$$

diverges quadratically as a function of the upper limits in momentum space. A more careful consideration reveals, however that

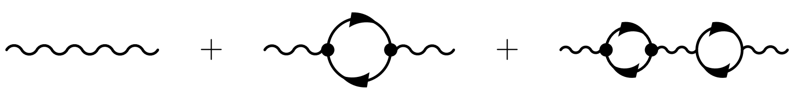
$$\Sigma_{\mu\nu}(p) \approx \frac{\alpha}{3\pi}\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{1}{q^4}\,, \tag{9.61}$$

where the additional powers in the denominator are dictated by dimensional analysis. Therefore the vacuum polarization diverges only logarithmically.

Remark #4

Lecture 27: Wed, 07. 02. 2018

One can compute the series

 $+\ldots$ (9.62)

as

$$D_{\mu\nu}(k) = D^{(0)}_{\mu\nu}(k) + D^{(0)}_{\mu\rho}(k)(-\mathrm{i})\Sigma^{\rho\sigma}(k)D^{(0)}_{\sigma\nu}(k) + \ldots \tag{9.63}$$

with

$$D^{(0)}_{\mu\nu}(k) = D^{(0,T)}_{\mu\nu}(k) + D^{(0,L)}_{\mu\nu}(k) \tag{9.64a}$$

$$D^{(0,T)}_{\mu\nu}(k) = \frac{-\mathrm{i}g_{\mu\nu} + \mathrm{i}\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}}{k^2+\mathrm{i}\epsilon} \tag{9.64b}$$

$$D^{(0,L)}_{\mu\nu}(k) = -\xi\frac{\mathrm{i}k_\mu k_\nu}{(k^2+\mathrm{i}\epsilon)^2}\,. \tag{9.64c}$$

Then

$$D^{(0,L)}_{\mu\nu}(k)\Sigma^{\nu\rho}(k) = -\xi\frac{\mathrm{i}k_\mu k_\nu}{(k^2+\mathrm{i}\epsilon)^2}\left(g^{\nu\rho} - \frac{k^\nu k^\rho}{k^2}\right)\Sigma_T(k^2) = 0 \tag{9.65}$$

and

$$\begin{aligned} D^{(0,T)}_{\mu\nu}(k)\Sigma^{\nu\rho}(k) &= \frac{-\mathrm{i}g_{\mu\nu} + \mathrm{i}\frac{k_\mu k_\nu}{k^2+\mathrm{i}\epsilon}}{k^2+\mathrm{i}\epsilon}\left(g^{\nu\rho} - \frac{k^\nu k^\rho}{k^2}\right)\Sigma_T(k^2) \\ &= \frac{-\mathrm{i}g_\mu{}^\rho + \mathrm{i}\frac{k_\mu k^\rho}{k^2+\mathrm{i}\epsilon}}{k^2+\mathrm{i}\epsilon}\Sigma_T(k^2) = D^{(0,T)\,\rho}_{\mu}(k)\Sigma_T(k^2)\,. \end{aligned} \tag{9.66}$$

Therefore

$$D^{(0)}_{\mu\rho}(k)(-\mathrm{i})\Sigma^{\rho\sigma}(k)D^{(0)}_{\sigma\nu}(k) = D^{(0,T)}_{\mu\nu}(k)(-\mathrm{i})\Sigma_T(k^2)\frac{-\mathrm{i}}{k^2+\mathrm{i}\epsilon} = -D^{(0,T)}_{\mu\nu}(k)\Pi(k^2) \tag{9.67}$$

with the vacuum polarization

$$\Pi(k^2) = \frac{\Sigma_T(k^2)}{k^2}\,. \tag{9.68}$$

The series becomes

$$D_{\mu\nu}(k) = D^{(0,T)}_{\mu\nu}(k) + D^{(0,L)}_{\mu\nu}(k) - D^{(0,T)}_{\mu\nu}(k)\Pi(k^2) + D^{(0,T)}_{\mu\nu}(k)\Pi^2(k^2) + \ldots \tag{9.69}$$

and can be resummed as a geometric series

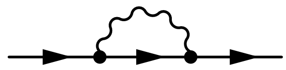
$$D_{\mu\nu}(k) = \frac{D^{(0,T)}_{\mu\nu}(k)}{1+\Pi(k^2)} + D^{(0,L)}_{\mu\nu}(k)\,. \tag{9.70}$$

We wee that the pole remains at $k^2 = 0$ and the photons remains massless!

9.3.6 Self Energy

The divergent contribution to the electron's *self energy*

$$\mathrm{i}\Sigma(p) = \tag{9.71}$$



consists of a piece proportional to $\not{p}$ and a piece proportional to $m\mathbf{1}$

$$\Sigma(p) = \frac{\alpha}{4\pi}\Delta\not{p} - \frac{\alpha}{\pi}\Delta m + \text{finite}\,. \tag{9.72}$$

Remark #1

The divergent piece can be absorbed in a counterterm of the form

$$\mathcal{L}_{\text{c.t.}} = c_1\bar\psi\mathrm{i}\not{\partial}\psi - c_2 m\bar\psi\psi\,, \tag{9.73}$$

but since $c_1 \neq c_2$, a renormalization of ψ and $\bar\psi$ does *not* suffice. Instead, an additive renormalization of the mass $m \to m + \delta m$ is also required.

Remark #2

The mass renormalization of massless electrons vanishes. Unlike the case of photons, this is not enforced by gauge invariance, but by the discrete γ_5 symmetry

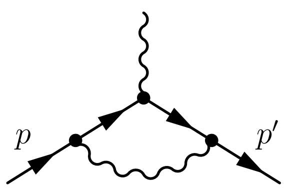
$$\psi \to \gamma_5\psi \tag{9.74}$$

of the Lagrangian for $m = 0$.

9.3.7 Vertex Correction

The divergent piece in the vertex correction

$$\mathrm{i}\Lambda_\mu(p,p') = \tag{9.75}$$



is proportional to $e\gamma_\mu$

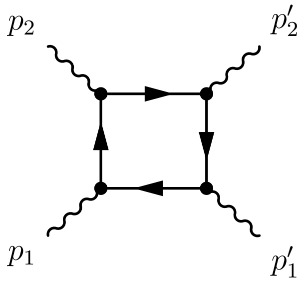
$$\Lambda_\mu(p,p') = \frac{\alpha}{4\pi}\Delta e\gamma_\mu + \text{finite} \tag{9.76}$$

and can be absorbed by a counterterm of the form

$$\mathcal{L}_{\text{c.t.}} = ce\bar\psi\not{A}\psi\,. \tag{9.77}$$

9.3.8 Photon-Photon Scattering

The one-loop diagram for photon-photon scattering



(9.78)

appears to be logarithmically divergent

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4}, \tag{9.79}$$

but gauge invariance dictates that the photon field A_μ must only appear in the combinations $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, allowing to pull out one power of each external momentum and dimensional analysis shows that we get four more powers of the loop momentum in the denominator

$$p_1^{\mu_1} p_2^{\mu_2} p_1'^{\nu_1} p_2'^{\nu_2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^8}, \tag{9.80}$$

rendering the loop integral *finite*. This we will not need a counterterm A^4 that is not in the original QED Lagrangian.

9.4 Power Counting and Dimensional Analysis

Consider free fields for scalars ϕ, spin-1/2 fermions ψ and vectors A_μ

$$S_0^\phi = \int d^4x \left(\frac{1}{2} \frac{\partial\phi(x)}{\partial x_\mu} \frac{\partial\phi(x)}{\partial x^\mu} - \frac{m_\phi^2}{2} \phi^2(x) \right) \tag{9.81a}$$

$$S_0^\psi = \int d^4x \left(\bar\psi(x) i\gamma_\mu \frac{\partial}{\partial x_\mu} \psi(x) - m_\psi \bar\psi(x)\psi(x) \right) \tag{9.81b}$$

$$S_0^A = \int d^4x \frac{-1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x), \tag{9.81c}$$

$$F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu}. \tag{9.81d}$$

actions are dimensionless. The *mass dimension*

$$\dim(m) = 1 \tag{9.82}$$

of the fields follows with

$$\dim\left(\mathrm{d}^4x\right) = -4 \tag{9.83}$$

$$\dim\left(\frac{\partial}{\partial x_\mu}\right) = 1 \tag{9.84}$$

as

$$\dim\left(\phi(x)\right) = 1 \tag{9.85a}$$

$$\dim\left(\psi(x)\right) = \dim\left(\bar\psi(x)\right) = \frac{3}{2} \tag{9.85b}$$

$$\dim\left(A_\mu(x)\right) = 1\,. \tag{9.85c}$$

One can equivalently use the fact that the Lagrangian densities must have mass dimension 4.

As a result, the high energy asymptotics of the propagators is $p^{2\dim-4}$

$$\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \left\langle 0\middle|\mathrm{T}\phi(x)\phi(0)\middle|0\right\rangle = \frac{\mathrm{i}}{p^2 - m_\phi^2 + \mathrm{i}\epsilon} \tag{9.86a}$$

$$\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \left\langle 0\middle|\mathrm{T}\psi(x)\bar\psi(0)\middle|0\right\rangle = \mathrm{i}\frac{\not p + m_\psi}{p^2 - m_\psi^2 + \mathrm{i}\epsilon} \tag{9.86b}$$

$$\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} \left\langle 0\middle|\mathrm{T}A_\mu(x)A_\mu(0)\middle|0\right\rangle = \frac{-ig_{\mu\nu}}{p^2 + \mathrm{i}\epsilon} \tag{9.86c}$$

and the high energy asymptotics of integrands in Feynman loop diagrams is determined by dimensional analysis.

This exercise allows us to determine the dimension of coupling constants. Since the lagrangian densities have mass dimension 4 the coupling constant of a vertex must have mass dimension

$$4 - \#\text{bosons} - \frac{3}{2}\#\text{bosons} - \#\text{derivatives}\,. \tag{9.87}$$

If a coupling constant has a positive mass dimension, adding the corresponding vertex to a diagram requires more powers of momenta in the denominator, making the diagram more convergent Vice versa, adding a vertex with a negative mass dimension will make the diagram more divergent This analysis can be made precise by proving that the *superficial degree of divergence* $\omega(G)$ of a diagram G

$$\int \frac{\mathrm{d}k}{k}\, k^{\omega(G)} \tag{9.88}$$

is given by

$$\omega(G) = 4 + \sum_v (\omega_v - 4) - \frac{3}{2}E_F - E_B - \delta\,, \tag{9.89}$$

where ω_v is the mass dimension of the vertex v and $\delta > 0$ is the number of *external* momenta that can be factored out. As long as $\omega_v \leq 4$ for all v, we find

$$\omega(G) \leq 4 - \frac{3}{2}E_F - E_B - \delta\,, \tag{9.90}$$

i. e. that all diagrams with $3E_F/2 + E_B + \delta > 4$ are superficially convergent. Thus, we will never need a counterterm of dimension greater than 4 and the renornmalization procedure with counterterms can be iterated indefinitely.

9.5 Renormalization

We have seen that it is possible to absorb *all* divergencies in renormalization constants for the fields, couplings and masses. In QED, we write

$$\mathcal{L} = -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + Z_2\bar\psi\left(\mathrm{i}\partial\!\!\!/ - (m - \delta m)\right)\psi + Z_1 e\bar\psi A\!\!\!/\psi \tag{9.91}$$

with $Z_1 = Z_2$ as a result of gauge invariance.

We obtain finite predictions for physical quantities at $D = 4$ by adjusting the renormalization constants to cancel all terms proportional to

$$\Delta = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) \tag{9.92}$$

for $\epsilon \neq 0$ and taking the limit $\epsilon \to 0$. This introduces ambiguities, because we have only fixed the infinite part proportional to $1/\epsilon$ this way. We still have the freedom to perform additional finite renormalizations.

However, it turns out that we *need* this freedom in order to make sure that the masses and couplings that appear in the predictions correspond to the physical values.

—A— Acronyms

CAR Canonical Anticommutation Relation

CCR Canonical Commutation Relation

EFT Effective (Quantum) Field Theory

EOM Equation of Motion

LT Lorentz Transformation

PDE Partial Differential Equation

PT Perturbation Theory

OTOH On The Other Hand

QED Quantum Electrodynamics

QFT Quantum Field Theory

QM Quantum Mechanics

QN Quantum Number

SSB Spontaneous Symmetry Breaking

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