11. Problemset Relativistic Quantum Field Theory
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QED

11.1 Kinematics

Consider the scattering of two particles with masses $m_1$ and $m_2$ into two particles with masses $m_3$ and $m_4$. The respective fourmomenta are $\{p_i\}_{i=1,2,3,4}$ with $p_i^2 = m_i^2$. The Mandelstam variables are

\begin{align}
  s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
  t &= (p_3 - p_1)^2 = (p_4 - p_2)^2 \\
  u &= (p_4 - p_1)^2 = (p_3 - p_2)^2.
\end{align}

Choose the 3-axis in the center of mass system according to

\begin{align}
  \vec{p}_1 &= (0, 0, |\vec{p}_1|) \\
  \vec{p}_2 &= (0, 0, -|\vec{p}_2|) = -\vec{p}_1
\end{align}

1. Show that

$$s + t + u = \sum_{i=1}^{4} m_i^2.$$  \hspace{1cm} (3)

2. Parametrize the Mandelstam variables by masses, total energy and scattering angle in the center of mass system.

3. Determine the allowed region for the Mandelstam variables.

11.2 $e^- \mu^- \rightarrow e^- \mu^-$

Consider the special case $m_1 = m_3 = m_e$ and $m_2 = m_4 = m_\mu$ of the previous problem.

1. Compute the differential cross section

$$\frac{d\sigma}{d\omega_e}(\cos \theta_e, s)$$ \hspace{1cm} (4)
for unpolarized $e^-\mu^- \rightarrow e^-\mu^-$ scattering, i.e. averaged over the initial spins and summed over the final spins, using the result from the lecture

$$\sum_{\text{polarizations}} |T|^2 = 8e^4 s^2 + u^2 - 4(u + s)(m_e^2 + m_\mu^2) + 6(m_e^2 + m_\mu^2)^2. \quad (5)$$

2. Does the integrated cross section

$$\sigma(s) = \int d\Omega_e \frac{d\sigma}{d\omega_e}(\cos \theta_e, s) \quad (6)$$

exist?

(a) if yes: compute it!

(b) if no: why not?

11.3 Scalar Particle Exchange

Instead of QED, consider a theory of two spin-$1/2$ fermions $\psi_1$ and $\psi_2$ coupled to a scalar $\phi$ with strengths $\lambda_1$ and $\lambda_2$, respectively

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \quad (7a)$$

$$\mathcal{L}_0 = \bar{\psi}_1(i\not\partial - m_1)\psi_1 + \bar{\psi}_2(i\not\partial - m_2)\psi_2 + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \quad (7b)$$

$$\mathcal{L}_I = -\lambda_1 \phi \bar{\psi}_1 \psi_1 - \lambda_2 \phi \bar{\psi}_2 \psi_2. \quad (7c)$$

1. Compute the scattering amplitude for $\psi_1 \psi_2 \rightarrow \psi_1 \psi_2$.

2. Compute the spin summed/averaged squared scattering amplitude.

3. Compute the unpolarized differential cross section.

4. Compute the unpolarized total cross section.

5. Compare with $e^-\mu^- \rightarrow e^-\mu^-$ in QED: is there an observable difference?
11.4 More Traces

1. Prove
\[ \text{tr} (\phi \phi \phi) = 4 ((ab)(cd) - (ac)(bd) + (ad)(cb)) \]  (8)

2. Compute
\[ \text{tr} (\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6) \]  (9)

3. Using the Lorentz transformation properties of \( \gamma_5 \), argue that
\[ \text{tr} (\gamma_5) = \text{tr} (\phi_1 \gamma_5) = \text{tr} (\phi_1 \phi_2 \gamma_5) = \text{tr} (\phi_1 \phi_2 \phi_3 \gamma_5) = 0 \]  (10)

4. Which result (up to normalization) do you expect for
\[ \text{tr} (\phi_1 \phi_2 \phi_3 \phi_4 \gamma_5) ? \]  (11)

5. Can you compute the normalization factor?