8. Problemset Relativistic Quantum Field Theory
December 13, 2017

Photons

8.1 Prolegomena

Show that the non-interacting real scalar field

$$\phi(x) = \int \widetilde{\mathrm{d}p}\, \left(a(p)\mathrm{e}^{-\mathrm{i}xp} + a^\dagger(p)\mathrm{e}^{\mathrm{i}xp}\right) \tag{1}$$

satisfies the equations of motion in the Heisenberg picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,\vec{x}) = \mathrm{i}\,[H, \phi(t,\vec{x})] \tag{2}$$

with the normal ordered Hamiltonian derived in section 5.2.1 of the lecture

$$H = \int \widetilde{\mathrm{d}p}\, p_0 a^\dagger(p) a(p)\,. \tag{3}$$

8.2 Hamiltonian

Consider the gauge-fixed Lagrangian density for photons in Feynman gauge

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2 \tag{4}$$

and the fields

$$A^\mu(x) = \sum_{\sigma=0}^{3} \int \widetilde{\mathrm{d}p}\, \left(u^\mu_\sigma(p) a_\sigma(p)\mathrm{e}^{-\mathrm{i}xp} + v^\mu_\sigma(p) a^\dagger_\sigma(p)\mathrm{e}^{\mathrm{i}xp}\right) \tag{5}$$

with polarization vectors at the reference momentum k

$$u^\mu_\sigma(k) = v^\mu_\sigma(k) = \epsilon^\mu_\sigma(k) = \delta^\mu_\sigma \tag{6}$$

and creation- and annihilation operators with canonical commutation relations

$$\left[a_\sigma(p), a^\dagger_{\sigma'}(p')\right]_- = -g_{\sigma\sigma'}(2\pi)^3 2|\vec{p}|\delta^3(\vec{p}-\vec{p}') \tag{7a}$$

$$\left[a_\sigma(p), a_{\sigma'}(p')\right]_- = \left[a^\dagger_\sigma(p), a^\dagger_{\sigma'}(p')\right]_- = 0\,. \tag{7b}$$

1. Determine the momenta π_μ conjugate to A^μ (has been done in the lecture).
2. Determine the Hamiltonian H with a Legendre transform.
3. Express the Hamiltonian H in terms of the $a_\sigma(p)$ and $a^\dagger_\sigma(p)$ and normal order it, if necessary.
4. Do the $A^\mu(x)$ satisfy the equations of motion in the Heisenberg picture

$$\frac{\mathrm{d}}{\mathrm{d}t} A^\mu(t,\vec{x}) = \mathrm{i}\,[H, A^\mu(t,\vec{x})]\,? \tag{8}$$

8.3 Prolegomena 2

Consider the quantum mechanical Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{\omega^2}{2m}q^2 + \lambda q \tag{9}$$

with a external real perturbation λ that commutes with everything. Determine the eigenvalues and eigenstates of H.

8.4 External Fields

Add an interaction

$$H_I = \int \mathrm{d}^3\vec{x}\phi(x)j(x) \tag{10}$$

with a square integrable external scalar and real source $j(x)$ to the Hamiltonian H of the first problem. Inspired by the third problem, determine the eigenvalues and eigenstates of $H + H_I$.